Manin kernels and exponentials

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Let A be an abelian variety over the differential function field K of a complex algebraic curve S with a rational vector field.

Recall the "Manin kernel", a DCF-definable/K subset $A^{\#}$ of A admitting a number of descriptions, one being that it is the smallest Zariski-dense DCF-definable subset of A.

Let T be the theory of $A^{\#}$ with all induced structure (over K).

This is a rigid divisible commutative group of finite Morley rank.

Let $\widehat{T}$ be the theory of the universal cover of $A^{\#}$, in the sense of [covers-fRM]. The aim of this note is to exhibit a natural analytic model of $\widehat{T}$. We first describe this model.

Let

$$0 \longrightarrow H \longrightarrow G \xrightarrow{p} A \longrightarrow 0$$

be the universal vector extension of A.

G is equipped with a canonical D-structure, i.e. a rational section

$$s_G : G \to \tau G$$

where $\tau G \to G$ is the twisted tangent bundle (aka first prolongation).

Let G^{δ} be the subgroup of "horizontal" points, i.e. for a differential field extension $K' \geq K$,

$$G^{\delta}(K') = \{x \in G(K') \mid (x, \delta x) = s_G(x)\}.$$

Then for $\mathcal{U} \models DCF$, $A^{\#}(\mathcal{U}) = p(G^{\delta}(\mathcal{U}))$ [Marker-maninKernels].

Taking the Lie algebras of these algebraic groups, we have

$$0 \longrightarrow LH \longrightarrow LG \xrightarrow{Lp} LA \longrightarrow 0$$

Now LG also has a natural D-structure induced from that on G,

$$Ls_G : LG \to L\tau G \cong \tau LG$$

(c.f. [BP]), so we have subgroups LG^{δ} and $LA^{\#} := Lp(LG^{\delta})$.

Now let $S' \subseteq S$ be a disc (or in fact any simply-connected domain) in S, suppose S' avoids the finitely many $s \in S$ for which A_s is not an abelian variety, and let $L \geq K$ be the differential field of meromorphic functions on S'. We consider L-points, where we define

$$A^{\#}(L) := p(G^{\delta}(L))$$
$$LA^{\#}(L) := p(LG^{\delta}(L)).$$

As in [BP Appendix], we have relative exponential maps

$$\begin{CD} LG(L) @>>> LA(L) \\ @V{\exp_G}VV @V{\exp_A}VV \\ G(L) @>>> A(L) \end{CD}$$

which respect the D-structures, hence restrict to

$$\begin{CD} LG^\delta(L) @>>> LA^\#(L) \\ @V{\exp_G}VV @V{\exp_A}VV \\ G^\delta(L) @>>> A^\#(L) \end{CD} .$$

Now our claim is that,

$$\exp_A : LA^\#(L) \to A^\#(L),$$

when considered as a structure in the language of $\widehat{T}$ (we discuss below exactly how it may be so considered), is a model of $\widehat{T}$.

Let us remark that in the constant case, i.e. when A is over $\mathbb{C}$, the D-structures on G and LG are trivial, and $A^\#(L) = A(\mathbb{C})$, and $LA^\#(L) = LA^\#(\mathbb{C})$, and $\exp_A$ is the usual complex exponential map; so we are reduced to the case of [covers-fRM Corollary 4.2.1].

We begin collecting some facts, (I)-(III) below, which we will need in the proof our structure satisfies the axioms of $\widehat{T}$.

By a remark credited to Hamm ([BuiumDiffAlgDiophGeom p.143]), over S', G analytically descends to the constants. In terms of L-points, this has the following consequence:

Fact 0.1. *Let $s_0 \in S'$. Let $G_0 := G_{s_0}$ be the fibre of G over s_0, a complex Lie group, and let*

$$\exp_{G_0} : LG_0 \to G_0$$

be its exponential map. There exists an isomorphism

$$\theta_G : G(L) \to G_0(L)$$

and a corresponding $\mathbb{C}$-linear isomorphism

$$L\theta_G : LG(L) \to LG_0(L)$$

such that $\theta_G(G^\delta(L)) = G_0(\mathbb{C})$, and $L\theta_G(LG^\delta(L)) = LG_0(\mathbb{C})$, and $\exp_G \circ L\theta_G = \theta_G \circ \exp_{G_0}$.

It follows that $LG^\delta(L)$ is a $2g$-dimensional $\mathbb{C}$-vector space, and $\ker \exp_G \leq LG^\delta(L)$, and hence

$$\ker \exp_A \leq LA^\# \tag{I}$$

Since $\exp_{G_0} : LG_0(\mathbb{C}) \twoheadrightarrow G_0(\mathbb{C})$, it also follows that $\exp_G : LG^\delta(L) \twoheadrightarrow G^\delta(L)$, and it follows by diagram-chase that

$$\exp_A : LA^\#(L) \twoheadrightarrow A^\#(L). \tag{II}$$

So we have:

$$\begin{array}{ccc} LG^{\delta}(L) & \twoheadrightarrow & LA^{\#}(L)\ . \\ {\scriptstyle \exp_G}\big\downarrow\!\!\!\!\downarrow & & {\scriptstyle \exp_A}\big\downarrow\!\!\!\!\downarrow \\ G^{\delta}(L) & \twoheadrightarrow & A^{\#}(L) \end{array}$$

Lemma 0.2.

$$A^{\#}(L^{\mathrm{diff}}) = A^{\#}(L). \tag{III}$$

Proof. We first show $LG^{\delta}(L^{\mathrm{diff}}) = LG^{\delta}(L)$.

Let X be a $\mathbb{C}$-basis of LG^{δ}. For any subdisc $S'' \subseteq S'$ and corresponding field $L' \geq L$ of meromorphic functions, by the above Fact, $LG^{\delta}(L')$ is still a 2g-dimensional $\mathbb{C}$-vector space, so $LG^{\delta}(L') = LG^{\delta}(L) = \langle X \rangle_{\mathbb{C}}$.

So by the (proof of) the Seidenberg Embedding Theorem [MarkerDCF Lemma A.1], for any $y \in LG^{\delta}(L^{\mathrm{diff}})$, we have $y \in \langle X \rangle_{\mathbb{C}(L^{\mathrm{diff}})} = \langle X \rangle_{\mathbb{C}} = LG^{\delta}(L)$.

So $LG^{\delta}(L^{\mathrm{diff}}) = LG^{\delta}(L)$, and hence $LA^{\#}(L^{\mathrm{diff}}) = LA^{\#}(L)$.

Now $G^{\delta}(L') = \exp_G(LG^{\delta}(L')) = \exp_G(LG^{\delta}(L)) = G^{\delta}(L)$. By another Seidenberg argument applied to algebraic extensions, we therefore have $G^{\delta}(L^{\mathrm{alg}}) = G^{\delta}(L)$.

Hence $A^{\#}(L^{\mathrm{alg}}) = A^{\#}(L)$. But it follows from [Wagner FieldsFRM] that $A^{\#}(L^{\mathrm{diff}}) = A^{\#}(L^{\mathrm{alg}})$; this is discussed in [BBP-MLMM], Corollary 1.11 and proof of Theorem 1.1(i). □

So $A^{\#}(L) \models T$.

From now on, we mostly omit explicit mention of L, writing $A^{\#}$ for $A^{\#}(L)$ and so on.

To make

$$\exp_A : LA^{\#} \to A^{\#},$$

a structure in the language of $\widehat{T}$, it remains to define $\widehat{H} \leq (LA^{\#})^n$ for each connected definable subgroup $H \leq (A^{\#})^n$.

If B is a connected algebraic subgroup of A, by [BBP 4.9] we have

$$A^{\#} \cap B = B^{\#}. \tag{*}$$

Now $LB^{\#} \subseteq LA^{\#}$, and by (I) and (*) we have

$$LA^{\#} \cap LB = LB^{\#}. \tag{IV}$$

Now if H is a connected definable subgroup of $A^{\#}$, then $H = B^{\#}$ where B is the Zariski closure of H; indeed, by (*) we have $H \leq A^{\#} \cap B = B^{\#}$, and meanwhile $B^{\#} \leq H$ since $B^{\#}$ is the smallest Zariski-dense definable subgroup of B.

Note that $(A^n)^{\#} = (A^{\#})^n$, and $L(A^n)^{\#} = (LA^{\#})^n$.

So for B a connected algebraic subgroup of A^n, we interpret $\widehat{B^{\#}}$ as $LB^{\#}$.

Proposition 0.3. *With the structure described above,*

$$\exp_A : LA^{\#}(L) \to A^{\#}(L)$$

is a model of $\widehat{T}$.

Proof. (A1) is by (I).

(A2)-(A5) are clear from the setup and (IV).

(A6) is by (II).

(A7) and (A8) follow from (I).

(A9)(I): by (IV), the exact sequence

$$0 \to L(K^o) \to LG \to LH \to 0$$

remains exact on applying $(\cdot^{\#})$.

(A9)(II) is by (I). □

References