GNS constructions, and functions of positive type

Based on Appendix C of “Kazhdan’s Property T” by Bekka, Harpe, and Valette.

Cyclic representations

Definition:
A unitary representation \((\pi, \mathcal{H})\) is cyclic if there exists \(\xi \in \mathcal{H}\) s.t. the span of the orbit of \(\xi\) is dense. Call such \(\xi\) a cyclic vector.

Remark:
If \((\pi, \mathcal{H})\) is a representation and \(\xi \in \mathcal{H}\), then the closure of the span of the orbit of \(\xi\), \(<G\xi>\), is an invariant closed subspace, so yields a cyclic subrepresentation. So irreducible \(\Rightarrow\) cyclic (any \(\xi \neq 0\) is cyclic), and any unitary representation \((\pi, \mathcal{H})\) can be decomposed as an orthogonal direct sum of cyclic subrepresentations. (Proof: take a maximal such direct sum; if not \(\mathcal{H}\), consider closure of orbit of an element of the orthogonal complement for a contradiction.)

Functions of positive type

Definition:
(i) For \(X\) a topological space, a kernel of positive type is a continuous function \(\Phi : X^2 \to \mathbb{C}\) such that for any finite tuple \((x_1, ..., x_n)\) from \(X\) and any \((a_1, ..., a_n) \in \mathbb{C}^n\),
\[
\sum_{i,j} a_i \Phi(x_i, x_j) \overline{a_j} \geq 0.
\]
(ii) For \(G\) a topological group, a function of positive type is a continuous function \(\phi : G \to \mathbb{C}\) such that \((g, h) \mapsto \overline{\phi(h^{-1}g)}\) is a kernel of positive type.

Remark:
\(\Phi\) is of positive type iff for any finite subset \(\{x_1, ..., x_n\} \subseteq X\), the matrix \((a_{ij})_{ij}\) with \(a_{ij} = \Phi(x_i, x_j)\) is positive semidefinite.

Example:
\(G := \langle \mathbb{R}; + \rangle\), \(\phi(z) := e^{iz}\).

Lemma:
(i) If \(\Phi : X^2 \to \mathbb{C}\) is of +ve type,
(a) \(\overline{\Phi(x, y)} = \Phi(y, x)\) (so \(\Phi(x, x) \in \mathbb{R}\))
(b) \(|\Phi(x, y)|^2 \leq \Phi(x, x)\Phi(y, y)\)
(ii) If \(\phi : G \to \mathbb{C}\) is of +ve type,
(a) \(\overline{\phi(g)} = \phi(g^{-1})\) (so \(\phi(e) \in \mathbb{R}\))
(b) \(|\phi(g)| \leq \phi(e)\)

Proof:
(i) The matrix
\[
\begin{bmatrix}
\Phi(x, x) & \Phi(x, y) \\
\Phi(y, x) & \Phi(y, y)
\end{bmatrix}
\]
is positive semidefinite, so is Hermitian with non-negative determinant.

(ii) Follows, by considering \(\Phi(g, e)\).
Remark:
Let $(\pi, \mathcal{H})$ be a unitary representation of $G$, let $\xi \in \mathcal{H}$.
Define $\phi_{\pi, \xi} : g \mapsto \langle g\xi, \xi \rangle$.
Then $\phi_{\pi, \xi}$ is of +ve type.
Indeed, $(g, h) \mapsto \langle h^{-1}g\xi, \xi \rangle = \langle g\xi, h\xi \rangle$ is of +ve type,
since $\sum_{ij} a_i \langle g_i \xi, j \xi \rangle \pi_j = \sum_{ij} (a_i g_i \xi, a_j j \xi) = \langle \sum_i a_i g_i \xi, \sum_i a_i j \xi \rangle \geq 0$.

**Theorem** ["GNS construction"]: 
$G$ topological group, $\phi : G \to \mathbb{C}$ of +ve type.
Then $\phi = \phi_{\pi, \xi, \phi}$ for some unitary representation $(\pi, \mathcal{H}_\phi)$ and cyclic vector $\xi, \phi \in \mathcal{H}_\phi$.
Moreover, the pair $(\pi, \xi, \phi)$ associated to $\phi$ is unique up to isomorphism, in the sense that if $\phi = \phi_{\pi', \xi'}$ where $\xi'$ is a cyclic vector in another unitary representation $(\pi', \mathcal{H}')$,
then there is an intertwining isomorphism $\mathcal{H}_\phi \xrightarrow{\sim} \mathcal{H}'; \xi, \phi \mapsto \xi'$.

Remark:
So $(\pi, \xi) \mapsto \phi_{\pi, \xi}$ induces a bijective correspondence between +ve functions and cyclic vectors in unitary representations up to isomorphism.
The zero function corresponds to the trivial representation.

**Cone of functions of +ve type**

$\mathcal{P}(G) := \{ \phi : G \to \mathbb{C} \text{ of } +\text{ve type} \}$.
$\mathcal{P}(G)$ is a cone (i.e. closed under taking positive linear combinations).
$\mathcal{P}_1(G) := \{ \phi \in \mathcal{P}(G) \mid \phi(e) = 1 \}$.
So $\mathcal{P}_1(G)$ is a convex subset of $\mathcal{P}(G)$.
Note $\phi_{\pi, \xi} \in \mathcal{P}_1(G)$ iff $\| \xi \| = 1$.

**Definition:**
A pure function of +ve type is an extreme point of $\mathcal{P}_1(G)$;
i.e. $\phi \in \mathcal{P}_1(G)$ such that whenever $\phi = t\psi_1 + (1-t)\psi_2$ with $t \in [0,1]$ and $\psi_1 \in \mathcal{P}_1(G)$,
actually $t \in \{0,1\}$.

**Theorem:**
$\phi \in \mathcal{P}_1(G)$ is pure iff the GNS representation $\pi, \phi$ is irreducible.

**Proof:**
We have $\phi = \phi_{\pi, \xi, \phi}$, with $\xi, \phi$ cyclic.
Suppose $\pi, \phi$ is reducible.
Say $\mathcal{H}_\phi = \mathcal{K} \oplus \mathcal{K}^\perp$ is a non-trivial invariant orthogonal decomposition;
decompose $\xi = \xi, \phi \oplus \xi, \phi^\perp$.
Set $s := \| \xi, \phi \|$, $t := \| \xi, \phi^\perp \|$. 
So $s^2 + t^2 = \| \xi, \phi \|^2 = \phi(e) = 1$.

Now one may calculate that $\phi = s^2 \phi_{\pi, \xi, \phi} + t^2 \phi_{\pi, \xi, \phi^\perp}$, but $s \neq 0 \neq t$ since $\xi$ is cyclic.
So $\phi$ is not pure.

Conversely, suppose $\phi$ is not pure,
say $\phi = s^2 \psi_1 + t^2 \psi_2$, $s^2 + t^2 = 1$, $s \neq 0 \neq t$, $\psi_i \in \mathcal{P}_1(G)$, 
and suppose $\pi, \phi$ is irreducible.

By GNS, say $\phi_1 = \phi_{\pi_1, \xi_1}$, and $\phi_2 = \phi_{\pi_2, \xi_2}$;
consider the direct sum representation $\pi_1 \oplus \pi_2$ of $G$ on $\mathcal{H}_1 \oplus \mathcal{H}_2$,
and let $\xi' := \xi_1 + t \xi_2$.

Then $\phi_{\pi_1 \oplus \pi_2, \xi'}(g) = \langle s g \xi_1 + t g \xi_2, s \xi_1 + t \xi_2 \rangle = s^2 \phi_1(g) + t^2 \phi_2(g) = \phi(g)$.
Consider the subrepresentation $\pi_0$ on $\mathcal{H}' := \langle \xi' \rangle$.
By the uniqueness of GNS, this is equivalent to $\pi, \phi$.
Since $\xi_1$ is cyclic,
the projection $T : \mathcal{H} \to \mathcal{H}_1; \xi' \mapsto \xi_1$ has dense image,
so by irreducibility of $\pi_\phi$, $T$ is an isomorphism
lest $(\ker T)^\perp$ be a proper subrepresentation).

So $\phi = \phi_{\pi_0, \xi'} = \phi_{\pi_1, \xi_1} = \phi_1$, contradicting impurity.

\[ \square \]

**Fact:**
If $G$ is locally compact, the convex hull of $\text{ext}(\mathcal{P}_1(G))$ is dense in $\mathcal{P}_1(G)$ for the topology of uniform convergence on compact subsets of $G$.

**Proof of GNS**
Let $\phi$ be a non-zero function of +ve type on $G$.
Consider the translates of $\phi$, $\phi_h(g) := \phi(h^{-1}g)$.
Let $V$ be the subspace of $C(G)$ they span.
Let $W$ be the vector space freely generated by $G$.
$\langle g, h \rangle := \phi(h^{-1}g)$ extends to a conjugate-symmetric sesquilinear form on $W$,
$\langle \sum_{i,j} a_{ij} b_{ij}, \sum_{i,j} c_{ij} d_{ij} \rangle = \sum_{i,j} a_{ij} \phi(h_{ij}^{-1} b_{ij}) b_{ij}$.
Now $\langle \sum_{i,j} a_{ij} b_{ij}, \sum_{i,j} c_{ij} d_{ij} \rangle = \sum_{i,j} a_{ij} \phi(h_{ij}^{-1} b_{ij}) b_{ij} \geq 0$ by definition of a positive function, so $\langle \cdot, \cdot \rangle$ is a semi-inner product on $W$.
The kernel is $\{ \sum_{i} b_{ih_i} | \sum_{i} \phi_{h_i} = 0 \}$, so it induces a well-defined inner product $\langle \cdot, \cdot \rangle$ on $V$.

Let $\mathcal{H}$ be the Hilbert space completion of $V$ with respect to $\langle \cdot, \cdot \rangle$;
so $\mathcal{H}$ is the space of functions on $G$ which are pointwise limits of Cauchy sequences in $V$ with respect to the norm induced by $\langle \cdot, \cdot \rangle$.

Let $\pi$ be the representation of $G$ acting by translation on $\mathcal{H}$,
$\pi(h)f(g) := f(h^{-1}g)$, and let $\xi := \phi \in \mathcal{H}$.
This is unitary, since it preserves $\langle \cdot, \cdot \rangle$ on the $\phi_g$,
and is strongly continuous since $\phi$ is continuous
(and so then $g \mapsto gf$ is continuous for $f = \phi$, hence for any $f \in V$, and hence for $f \in \mathcal{H}$ since a uniform limit of continuous functions is continuous),
and $\xi$ is cyclic, and $\phi_{\pi, \xi}(h) = (\phi_h, \phi_\xi) = \phi(h)$.

It remains to show the uniqueness.
Suppose $\phi_{\pi', \xi'} = \phi_{\pi, \xi}$,
with $\xi'$ cyclic for $\pi'$: $G \to \mathcal{H}'$.

(*) $\| \sum_{i,j} a_{ij} g_i \xi_j \|^2 = \sum_{i,j} a_{ij} \langle g_i \xi_j, g_j \xi_j \rangle \pi_j = \sum_{i,j} a_{ij} \langle g_i \xi', g_j \xi' \rangle \pi_j = \| \sum_{i,j} a_{ij} g_i \xi' \|^2$

Define an isomorphism $T : \mathcal{H} \xrightarrow{\sim} \mathcal{H}'$ by $T(g\xi) := g\xi'$ and extending linearly and continuously; this is well-defined by (*).
$T$ intertwines $\pi$ and $\pi'$, and $T(\xi) = \xi'$.

\[ \square \]

**Example:**
$G := (\mathbb{R}; +)$, $\phi(z) := e^{iz}$.
$\phi_a(z) = e^{i(z-a)} = e^{-ia} \phi(z)$,
so $\mathcal{H}$ is 1-dimensional,
and the representation is $a \mapsto (e^{-ia})$.  