Torsion in Groups of Finite Morley Rank

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Zilber Geometric Model Theory Conference
March 25-28
I Connected Groups
- Structure

II Permutation Groups
- Bounds on Rank

III Torsion
- Centralizers
- Semisimplicity
- Sylow Theorem
- Weyl Group
1 Structure Theory

2 Permutation Groups

3 Torsion
Essential Notions—Generalities

- Morley rank \((\text{rk}(X))\)
- Connected group
  \[
  [G : H] < \infty \implies G = H.
  \]
  \[
  X, Y \subseteq G \text{ generic} \implies X \cap Y \text{ generic}
  \]
- \(d(X)\): definable subgroup generated by \(X\).
- **Fubini:** Zilber-Lascar-Borovik-Poizat
The Algebraicity Conjecture

**Conjecture (Algebraicity)**

\[ G: \text{finite Morley rank, connected.} \]
\[ H: \text{maximal connected solvable normal, definable.} \]

\[ 1 \rightarrow H \rightarrow G \rightarrow \tilde{G} \rightarrow 1 \]

\( \tilde{G} \): *a central product of algebraic groups.*

Equivalently: The simple groups are algebraic.
Conjecture (Algebraicity)

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Theorem (ABC, 2008)

\[ 1 \to U_2(G) \to G \to \bar{G} \to 1 \]

\( U_2(G) \): \[ 1 \to O_2(G) \to \prod_i L_i \] (char 2, Altınel’s Jugendtraum - and his habilitation - and Wagner’s good tori)

\( \bar{G} \): Connected 2-Sylow divisible abelian. (“odd type”)
Odd Type: Torsion

Theorem (Degenerate Type)
If there is no nontrivial connected abelian $p$-subgroup, then there is no $p$-torsion.

Theorem (Altınel-Burdges)
The centralizer of a divisible torsion subgroup is connected.
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Theorem (Degenerate Type)

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Theorem (Altınel-Burdges)

*The centralizer of a divisible torsion subgroup is connected.*

Corollary

*If there are no p-unipotent subgroups, then any p-element which centralizes a maximal divisible p-subgroup $T$ lies in $T$.***
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Corollary

If there are no \( p \)-unipotent subgroups, then any \( p \)-element which centralizes a maximal divisible \( p \)-subgroup \( T \) lies in \( T \).

Proof.

\( T \) the definable hull of a maximal divisible \( p \)-subgroup.
\( H = C(T)/T \) connected.
\( H \) has no \( p \)-torsion.
1. Structure Theory
2. Permutation Groups
3. Torsion
Definably primitive: no nontrivial $G$-invariant definable equivalence relation.

Theorem (BC)\((G, X)\) definably primitive. Then $\text{rk} (G)$ is bounded by a function of $\text{rk} (X)$.

MPOSA = Macpherson-Pillay/O’Nan-Scott-Aschbacher
A description of the socle of a primitive permutation group, and the stabilizer of a point in that socle.

- **Affine**: The socle $A$ is abelian and can be identified with the set $X$ on which $G$ acts.
- **Non-affine**: The socle is a product of copies of one simple group.
Generic multiple transitivity

Theorem

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Generic transitivity: one large orbit.

Generic \(t\)-transitivity: on \(X^t\).
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Generic transitivity: one large orbit.

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**Proposition**

\((G, X)\) definably primitive. Then the degree of multiple transitivity of \(G\) is bounded by a function of \(rk(X)\).

(Special case of the theorem, but sufficient.)
Bounds on $t$

Proposition

$(G, X)$ definably primitive, generically $t$-transitive. Then $t$ is bounded by a function of $rk(X)$. 
Bounds on $t$

**Proposition**

$(G, X)$ definably primitive, generically $t$-transitive. Then $t$ is bounded by a function of $rk(X)$.

**Strategy:** Let $T$ be a maximal 2-torus.

1. Derive an upper bound on the complexity of $T$ from $rk(X)$;
2. Derive a lower bound on the complexity of $T$ from $t$. 
Bounds on $t$

Proposition

$(G, X)$ definably primitive, generically $t$-transitive. Then $t$ is bounded by a function of $\text{rk}(X)$.

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The upper bound: $\text{rk}(T/T_\infty) \leq \text{rk}(X)$. This is because the stabilizer of a generic element of $X$ is torsion-free.
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The upper bound: $\text{rk} (T/T_\infty) \leq \text{rk} (X)$. This is because the stabilizer of a generic element of $X$ is torsion-free.

But the lower bound requires attention.
We want to show that a large degree of generic transitivity ($t$ large) blows up $\text{rk} \left( \frac{T}{T_{\infty}} \right)$ for $T$ the definable hull of a 2-torus.
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The group \(G\) will induce the action of \(Sym_t\) on any \(t\) independent generic points.

Trading \(T\) in for a smaller torus, and trading \(t\) in for a smaller value as well (but not too small) we can set this up so that we have:

- a finite group \(\Sigma\) operating on \(T\), covering \(Sym_t\),
- — and sitting inside a connected group \(H\) —
- such that \(T\) is the definable hull of a maximal 2-torus in \(H\).

Let us simplify considerably.
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There is a glaring hole in this argument.
Plugging a hole

The Setup

$T$ inside $G$, $G$ connected, $Sym_t$ acts on $T$, $t$ large, and $T$ is the definable hull of a maximal 2-torus.

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But since this configuration is in a connected subgroup of $G$, and $T$ is a maximal 2-torus, the 2-elements of $\text{Sym}_t$ act nontrivially on $T$, and the action of $\text{Alt}_t$ is faithful.
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So we are done.
Torsion in Groups of Finite Morley Rank

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1. Structure Theory
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More results on torsion

Assume no $p$-unipotents.
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- **Semisimplicity**
  If $G$ is connected, then every $p$-element is in a torus.

- **Sylow theory**
  For all primes $p$

- **Weyl groups $N(T)/T$.**
  If the Weyl group is nontrivial, it contains an involution.
  (Burdges-Deloro) If the group is minimal simple, the Weyl group is cyclic
Applications

1. Permutation Groups
2. Classification in odd type and low 2-rank
3. Bounds on 2-rank revisited?
Other aspects

1. The Borovik Program: Signalizer functor theory, strong embedding, black box group theory . . .
2. Burdges unipotence theory and the Bender method
3. Generix strikes back [Nesin, Jaligot]
4. Conjugacy of Carter subgroups [Frécon]
5. Quasithin methods
   1. Amalgam method, representation theory (even type)
   2. Component analysis (odd type) [Borovik, Altseimer, Burdges]
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Desiderata

$L^*$-group theory in odd type (absolute bounds on 2-rank)
Control of actions of 2-tori on degenerate type groups. and
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$L^*$-group theory in odd type (absolute bounds on 2-rank)
Control of actions of 2-tori on degenerate type groups. and
Bad groups and non-commutative geometry . . . ?