# Model completeness and o-minimality of $\mathbb{R}_{\mathrm{an}}$

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Notes on the Denef - van den Dries quantifier elimination for $\mathbb{R}_{\mathrm{an}}$ and its consequences, written for a pair of seminars in Muenster, January 2018.

These notes are based in large part on some lecture notes by Alex Wilkie on the topic, currently available at: `http://www.logique.jussieu.fr/modnet/Publications/Introductory\%20Notes\%20and\%20surveys/Wilkie.pdf`

A reader who has stumbled upon these notes and has not read Wilkie's notes should stop reading now and seek out Wilkie's notes instead. In these notes I attempt to spell out the odd thing left implicit in Wilkie's notes, but essentially these notes cover a proper subset of the material covered in Wilkie's notes, skip over many details handled nicely there, and indubitably add errors and imprecisions of their own.

# 1 Analytic functions

## 1.1 Formal power series

For $R$ a ring,

$$R[[X]] := \{\sum_{i\in\mathbb{N}} a_i X^i : a_i \in R\}$$

$$R[[\overline{X}]] = R[[X_1, \ldots, X_n]] = (\ldots((R[[X_1]])[[X_2]])\ldots)[[X_n]]$$

If $F \in R[[\overline{X}]]$, we can write

$$F = \sum_{\nu\in\mathbb{N}^n} a_\nu \overline{X}^\nu$$

where $\overline{X}^\nu := X_1^{\nu_1} \ldots X_n^{\nu_n}$.

$|\nu| := \max_i \nu_i$.

$F(0) :=$ constant term of $F = a_0$

For $R$ a ring, $R^* := \{x \in R : \exists y \in R.\ xy = 1\} =$ multiplicative group of units.

**Lemma 1.1.** *For $F \in R[[\overline{X}]]$, $F \in R[[\overline{X}]]^*$ iff $F(0) \in R^*$.*

*Proof.* By induction, suffices to show when $\overline{X} = X$ is a single variable.

If $FG = 1$ then $F(0)G(0) = 1$.

If $F = \sum a_i X^i$ If $a_0 b_0 = 1$, then define recursively for $n \geq 0$

$$b_{n+1} := -a_0^{-1}(a_1 b_n + a_2 b_{n-1} + \ldots a_{n+1} b_0).$$

Then $\sum_{0\leq i\leq n+1} a_i b_{n+1-i} = 0$, so $(\sum a_i X^i)(\sum b_i X^i) = 1$. □

## 1.2 Convergent power series

For $F \in \mathbb{R}[[\overline{X}]]$, define

$$\begin{aligned}\mathrm{dom}(\sum a_\nu \overline{X}^\nu) &:= \mathrm{int}(\{\overline{x} \in \mathbb{R}^n : \sum a_\nu \overline{x}^\nu \text{ converges }\}) \\ &= \mathrm{int}(\{\overline{x} : \{a_\nu \overline{x}^\nu : \nu\} \text{ is bounded }\})\end{aligned}$$

e.g. $\mathrm{dom}(\sum_\nu X^\nu Y^\nu) = \{(x, y) : xy < 1\}$.
Let $\mathbb{R}\left\langle \overline{X} \right\rangle := \{F : 0 \in \mathrm{dom}(F)\}$.

**Fact 1.2.** *$\mathbb{R}\left\langle \overline{X} \right\rangle$ is a subring of $\mathbb{R}[[\overline{X}]]$.*

Now $F = \sum a_\nu \overline{X}^\nu \in \mathbb{R}\left\langle \overline{X} \right\rangle$ defines a function $\widetilde{F} : \mathrm{dom}(F) \to \mathbb{R}$; $\widetilde{F}(\overline{x}) := \sum a_\nu \overline{x}^\nu$.

**Fact 1.3.** *$\widetilde{F} = 0$ iff $F = 0$. $\widetilde{F} \in C^\infty(\mathrm{dom}(F))$.*

Say $f : U \to \mathbb{R}$ is **analytic** at $\overline{b} \in U$ if $f(\overline{X} - \overline{b}) = \widetilde{F}(\overline{X})$ on $U_{\overline{b}}$, some $\overline{b} \in U_{\overline{b}} \subseteq_{\mathrm{op}} U$ and some $F \in \mathbb{R}\left\langle \overline{X} \right\rangle$.

**Fact 1.4.** *$\widetilde{F}$ is analytic at every $b \in \mathrm{dom}(F)$.*

*Idea of proof.* show the Taylor series at $b$,

$$\sum_\nu 1/((\sum \nu)!)\widetilde{D^\nu F}(b) X^\nu$$

where $D^\nu = D_1^{\nu_1} \dots D_n^{\nu_n}$ where $D_i = d/dX_i$ is formal derivation, converges near $b$. See [Krantz-Park "A primer of Real Analytic Functions" Proposition 2.2.7]. □

**Fact 1.5.** *if $f$ is analytic at $b$ and $g$ is analytic at $f(b)$, then $(g \circ f)$ is analytic at $b$.*

**Corollary 1.6.**

$$\mathbb{R}\left\langle \overline{X} \right\rangle^* = \mathbb{R}[[\overline{X}]]^* \cap \mathbb{R}\left\langle \overline{X} \right\rangle$$

*Proof.* Suppose $F \in \mathbb{R}\left\langle \overline{X} \right\rangle$ has non-zero constant term; WTS $F$ is invertible in $\mathbb{R}\left\langle \overline{X} \right\rangle$. Multiplying $F$ by a constant, WMA $F(0) = 1$. So say $F = 1 - G$ where $G \in \mathbb{R}[[\overline{X}]]$ with $G(0) = 0$. Let $H(X) := \sum X^i \in \mathbb{R}\langle X \rangle$. Then $(1 - X)H(X) = 1$, so $F(H \circ G) = F(\overline{X})(H(G(\overline{X})) = (1 + G(\overline{X}))(H(G(\overline{X}))) = 1$ and $H \circ G \in \mathbb{R}\left\langle \overline{X} \right\rangle$ by the Fact. □

## 1.3 Weierstrass Preparation

Say $F \in \mathbb{R}[[\overline{X}, Y]]$ is **regular** in $Y$ if some term $aY^p$ with $p \geq 0$ and $a \neq 0$ occurs in the power series $F$, i.e. if "$F(0, Y) \neq 0$".

**Fact 1.7** (Weierstrass Preparation Theorem)**.** *If $F \in \mathbb{R}\left\langle \overline{X}, Y \right\rangle$ is regular in $Y$, then exists $Q \in \mathbb{R}\left\langle \overline{X}, Y \right\rangle^*$ and $L \in \mathbb{R}\left\langle \overline{X} \right\rangle [Y]$, such that $F = QL$.*

## 1.4 Denef - van den Dries Preparation

**Fact 1.8.** *$\mathbb{R}\left\langle\overline{X}\right\rangle$ is Noetherian; that is, any ideal is finitely generated.*

**Fact 1.9.** *The embedding $\mathbb{R}\left\langle\overline{X}\right\rangle \leq \mathbb{R}[[X]]$ is faithfully flat.*

I omit the definition of faithful flatness, because in fact we need only the following consequence.

**Fact 1.10.** *If $F_i$,$F \in \mathbb{R}\left\langle\overline{X}\right\rangle$ and the linear equation $F_1x_1 + \dots F_nx_n = F$ has a solution in $\mathbb{R}[[\overline{X}]]$, then it already has a solution in $\mathbb{R}\left\langle\overline{X}\right\rangle$.*

**Corollary 1.11.** *If $F_i$,$F \in \mathbb{R}\left\langle\overline{X}\right\rangle$ and the linear equation $F_1x_1 + \dots F_nx_n = F$ has a solution in $\mathbb{R}[[\overline{X}]]^*$, then it already has a solution in $\mathbb{R}\left\langle\overline{X}\right\rangle^*$.*

*Proof.* Say $U_i \in \mathbb{R}[[\overline{X}]]^*$ is a solution. Then $\sum_i (F_i)(U_i(0) + \sum_j x_i^j X_j) = F$ has a solution in $\mathbb{R}[[\overline{X}]]$, hence in $\mathbb{R}\left\langle\overline{X}\right\rangle$. So since $U_i(0) \neq 0$, the original equation has a solution in $\mathbb{R}\left\langle\overline{X}\right\rangle^*$. □

The following consequence of these facts is what we will need in the QE proof; I follow Wilkie in naming it as follows.

**Theorem 1.12** (Denef - van den Dries Preparation Theorem). *If $F \in \mathbb{R}\left\langle\overline{X},\overline{Y}\right\rangle$ then $F(\overline{X},\overline{Y}) = \sum_{|\nu|<d} a_\nu(\overline{X})\overline{Y}^\nu u_\nu(\overline{X},\overline{Y})$ for some $d \in \mathbb{N}$, $a_\nu \in \mathbb{R}\left\langle\overline{X}\right\rangle$, $u_\nu \in \mathbb{R}\left\langle\overline{X},\overline{Y}\right\rangle^*$.*

*Proof.* By induction on the length of $\overline{Y}$. So suppose for $\mathbb{R}\left\langle\overline{X},\overline{Y}\right\rangle$; we prove it for $\mathbb{R}\left\langle\overline{X},\overline{Y},Z\right\rangle$. Let $F = \sum_{i\in\mathbb{N}} a_iZ^i \in \mathbb{R}\left\langle\overline{X},\overline{Y},Z\right\rangle$, $a_i \in \mathbb{R}\left\langle\overline{X},\overline{Y}\right\rangle$. $\mathbb{R}\left\langle\overline{X},\overline{Y}\right\rangle$ is Noetherian, so there exist $d \in \mathbb{N}$ and $b_{i,j} \in \mathbb{R}\left\langle\overline{X},\overline{Y}\right\rangle$ such that $a_{d+i} = \sum_{i<d} b_{i,j}a_i$. So

$$\begin{aligned} F &= \sum_{i<d} a_iZ^i + \sum_{j\geq 0}\sum_{i<d} b_{i,j}a_iZ^{d+j} \\ &= \sum_{i<d} a_iZ^i + \sum_{i<d} a_iZ^iZ^{d-i}\sum_{j\geq 0} b_{i,j}Z^j \\ &= \sum_{i<d}(a_iZ^i)(1 + Z^{d-i}\sum_{j\geq 0} b_{i,j}Z^j) \\ &= \sum_{i<d} a_iZ^iu_i \end{aligned}$$

Here we have $u_i \in \mathbb{R}[[\overline{X},\overline{Y},Z]]^*$, and then by Corollary 1.11, we can find such $u_i \in \mathbb{R}\left\langle\overline{X},\overline{Y},Z\right\rangle^*$.

Now apply the inductive hypothesis to the $a_i$. □

# 2 $\mathbb{R}_{\text{an}}$

$$\|\overline{x}\| := max_i|x_i|$$

$$B_{<r} := \{\overline{x} \in \mathbb{R}^n : \|\overline{x}\| < r\}$$

$$B_{\leq r} := \{\overline{x} \in \mathbb{R}^n : \|\overline{x}\| \leq r\}$$

For $r > 0$,

$$\mathbb{R}\{\overline{X}\}_r := \{F \in \mathbb{R}[[\overline{X}]] : B_{\leq r} \subseteq \text{dom}(F)\},$$

so $\mathbb{R}\left\langle \overline{X} \right\rangle = \cap_{r \in \mathbb{R}_{>0}} \mathbb{R}\{\overline{X}\}_r$.

If $F \in \mathbb{R}\{\overline{X}\}_r$,

$$\widetilde{F}\restriction_r (\overline{x}) := \begin{cases} \widetilde{F}(\overline{x}) & \text{if } \overline{x} \in B_{<r} \\ 0 & \text{else} \end{cases}$$

"restricted analytic function".

Say $(r, F)$ is **acceptable** if $r \in \mathbb{R}_{>0}$ and $F \in \mathbb{R}\{X\}_r$.

$$\mathbb{R}_{\text{an}} := \{\mathbb{R}; +, -, \cdot, <, (a)_{a \in \mathbb{R}}, (\widetilde{F}\restriction_r)_{(r,F) acceptable}\}$$

structure in language $L_{\text{an}}$.

Let $L^D_{\text{an}} := L_{\text{an}} \cup \{D\}$, and $\mathbb{R}^D_{\text{an}} :=$ expansion of $\mathbb{R}_{\text{an}}$ interpreting $D$ by

$$D(x, y) := \begin{cases} x/y & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

**Theorem 2.1** (Denef - van den Dries)**.** *$T^D_{\text{an}} := Th(\mathbb{R}^D_{\text{an}})$ has QE.*

**Corollary 2.2.** *$T_{\text{an}} := Th(\mathbb{R}_{\text{an}})$ is model complete.*

*Proof.* Any qf $L^D_{\text{an}}$-formula is equivalent to an existential $L_{\text{an}}$-formula.

For example, if $t_1, t_2$ are terms, then

$$\phi(D(t_1(x), t_2(x))) \Leftrightarrow \exists z. \, (\phi(z) \wedge ((t_2(x) = 0 \wedge z = 0) \vee (t_2(x) \neq 0 \wedge z \cdot t_2(x) = t_1(x)))).$$

□

*Remark* 2.3. Model completeness of $\mathbb{R}_{\text{an}}$ was previously proven (expressed in different but equivalent terms) by Gabrielov in the 1960s ("theorem of the complement" for subanalytic sets).

## 2.1 o-minimality of $\mathbb{R}_{\text{an}}$

**Lemma 2.4.** *Suppose $0 \neq F \in \mathbb{R}\langle X \rangle$. Then $F = X^m G$ for some unique $m \in \mathbb{N}$ and $G \in F[[X]]^*$, and $\text{dom}(G) = \text{dom}(F)$, and for some $\epsilon > 0$, $\widetilde{F}$ has constant non-zero sign on $(0, \epsilon)$.*

*Proof.* Existence and uniqueness of $m$,$G$ is immediate from the description of $F[[X]]^*$. For $\text{dom}(G) = \text{dom}(X^m G)$: given $x \in \mathbb{R}$, $\{a_i x^i\}$ is bounded iff $\{a_i x^{m+i} = x^m(a_i x^i)\}$ is.

Now $G$ has non-zero constant term and $\widetilde{G}$ is continuous, so $\widetilde{G}$ and hence $\widetilde{F}$ has constant non-zero sign on some $(0, \epsilon)$. □

**Lemma 2.5.** *Let $t(x)$ be an $L^D_{\text{an}}$-term in 1 variable. Exists $\epsilon > 0$ s.t. $t(x) = 0$ on $(0, \epsilon)$ or $t(x) = (x^m \widetilde{F}(x))$ on $(0, \epsilon)$ for some $m \in \overline{Z}$ and $F \in \mathbb{R}[[X]]^* \cap \mathbb{R}\{X\}_\epsilon$. In particular, $t(x)$ has constant sign on $(0, \epsilon)$.*

*Proof.* By induction on terms. Obvious for $t(x) = x$ or $t(x) = b$.

Suppose $t(x) = t_1(x) + t_2(x)$ and $t_i(x)$ are as required. If either $t_i(x) = 0$ on $(0, \epsilon_i)$, this is clear. Else, say $t_i(x) = x^{m_i}\widetilde{F}_i(x)$ on $(0, \epsilon_i)$ with $F_i \in \mathbb{R}[[X]]^* \cap \mathbb{R}\{X\}_{\epsilon_i}$. Say $m_1 \leq m_2$. then $t_1(x)+t_2(x) = x^{m_1}(\widetilde{F}_1(x)+x^{m_2-m_1}\widetilde{F}_2(x))$; but by Lemma 2.4, $F_1(X) + X^{m_2-m_1}F_2(X) = X^k G(X)$ say, and then $t_1(x) + t_2(x) = x^{m_1+k}\widetilde{G}(x)$ is as required with $\epsilon := \min_i \epsilon_i$.

Similarly for $-$. Similarly for $*$ and $D$, since the product or ratio of two units is a unit.

Finally, suppose $(F, r)$ is acceptable and $t(x) = \widetilde{F}\restriction_r (t_1(x), \ldots, t_n(x))$ Pick common $\epsilon$ for the $t_i$. If $t_i = 0$ on $(0, \epsilon)$, replace $F$ with $F(x_1, \ldots, 0, \ldots, x_n)$. So WMA $t_i(x) = f_i(x)$ on $(0, \epsilon)$, where $f_i(x) = x_i^{m_i}\widetilde{F}_i(x)$ for some $F_i \in \mathbb{R}[[X]]^* \cap \mathbb{R}\{X\}_\epsilon$. If any $m_i < 0$, then $t(x) = 0$ near 0 so done. Reducing $\epsilon$ further, WMA $r - |f_i(x)|$ has constant sign on $(0, \epsilon)$ by Lemma 2.4. If any such sign is not positive, again $F = 0$ so done. Else, $|f_i(0)| \leq r$, so $(f_1(0), \ldots, f_n(0)) \in B_{\leq r} \subseteq \mathrm{dom}(F)$. Then $f(x) := \widetilde{F}(f_1(x), \ldots, f_n(x))$ is analytic at 0, so done by Lemma 2.4. $\square$

**Corollary 2.6** (Corollary of QE)**.** *$T_{\mathrm{an}}$ is o-minimal.*

*Proof.* Any qf formula in $\mathbb{R}^D_{\mathrm{an}}$ is equivalent to a boolean combination of $\{t(x) > 0\}$ for $t$ an $L^D_{\mathrm{an}}$-term in one variable, so STS for any such $t$ there is a partition of $\mathbb{R}$ into finitely many points and intervals such that $\mathrm{sign}(t(x))$ is constant on each.

Applying Lemma 2.5 to $t(D(1, x))$ and $t(-D(1, x))$, $t(x)$ has constant sign on some $(-\infty, a)$ and $(b, \infty)$. Then for $c \in [a, b]$, by Lemma 2.5 applied to $t(x - c)$ and $t(x - (-c))$, for some $\epsilon > 0$, $t(x)$ has constant sign on $(c - \epsilon, c)$, $\{c\}$, and $(c, c + \epsilon)$.

We conclude by compactness of $[a, b]$. $\square$

# 3 Proof of QE

## 3.1 QE criterion

**Fact 3.1.** *$T$ has QE iff: if $M_1, M_2 \models T$ are $\omega$-saturated with a common f.g. substructure $A$, and if $b \in M_1$, then there exists an embedding $\langle Ab\rangle^{M_1} \hookrightarrow M_2$ extending $\mathrm{id}_A$.*

So let $M_1, M_2 \models T^D_{\mathrm{an}}$ be $\omega$-saturated with a common f.g. substructure $K$, and $b \in M_1$. Note $K \leq M_1$ is a subfield and $\mathbb{R}^D_{\mathrm{an}}$ is a substructure of $K$.

**Lemma 3.2.** *If there exists an embedding $\langle Kb\rangle^{M_1} \hookrightarrow M_2' \succ M_2$ extending $\mathrm{id}_K$, then there exists an embedding $\langle Kb\rangle^{M_1} \hookrightarrow M_2$ extending $\mathrm{id}_K$.*

*Proof.* Say $b \mapsto b' \in M_2'$. Then realise $\mathrm{tp}(b'/K)$ in $M_2$ by $\omega$-saturation. $\square$

So we may freely replace $M_2$ with an elementary extension.

## 3.2 I: local $\to$ global

For $M \models T_{\mathrm{an}}$, let $\mu(M) := \{\eta \in M : |\eta| < r \text{ for all } r \in \mathbb{R}_{>0}\}$. For $\alpha \in M$, there is at most one $s \in \mathbb{R}$ s.t. $\alpha \in s + \mu(M)$. Let $\mathrm{st}(\alpha) := s$ if such $s$ exists, else $\mathrm{st}(\alpha) := \infty$.

*Remark* 3.3. for $\alpha \neq 0$, $\mathrm{st}(\alpha) = \infty$ iff $1/\alpha \in \mu$.

*Remark* 3.4. If $\overline{a} \in \mu^n$ and $F \in \mathbb{R}\left\langle\overline{X}\right\rangle$, then $\widetilde{F}\restriction_r(\overline{a})$ is independent of the choice of $r$. We write $\widetilde{F}(\overline{a})$ for the common value. Note $\widetilde{F}(\overline{a}) \in \mu$, by continuity.

**Definition 3.5.** If $A \subseteq \mu(M_1)$, a map $e : A \to \mu(M_2)$ is a **partial** $T_{\mathrm{an}}$**-**$\mu$**-embedding** if for $\overline{a} \in A^{<\omega}$ and $F \in \mathbb{R}\left\langle\overline{X}\right\rangle$,

$$\widetilde{F}(\overline{a}) > 0 \Leftrightarrow \widetilde{F}(e(\overline{a})) > 0.$$

**Lemma 3.6.** *Any partial* $T_{\mathrm{an}}$*-*$\mu$*-embedding* $e : \langle Kb\rangle \cap \mu(M_1) \to \mu(M_2)$ *extends to an* $L^D_{\mathrm{an}}$*-embedding* $e' : \langle Kb\rangle \longrightarrow M_2$.

*Proof.*

$$e'(\alpha) := \begin{cases} \mathrm{st}(\alpha) + e(\alpha - \mathrm{st}(\alpha)) & \text{if } \mathrm{st}(\alpha) \in \mathbb{R} \\ 1/e(1/\alpha) & \text{if } \mathrm{st}(\alpha) = \infty \end{cases}$$

**Claim 3.7.** $e'$ *is an ordered field embedding.*

Proof: First we show that $e'$ is order-preserving. STS $e$ is order-preserving. Consider $F := X - Y \in \mathbb{R}\{X, Y\}_1$.

$$T_{\mathrm{an}} \vDash \forall x, y.\ (|x| < 1 \wedge |y| < 1 \to \widetilde{F}\restriction_1(x, y) = x - y),$$

so for $\eta, \eta' \in \langle Kb\rangle \cap \mu$,

$$\begin{aligned} &\eta > \eta' \\ &\quad \Leftrightarrow \widetilde{F}(\eta, \eta') > 0 \\ &\quad \Leftrightarrow \widetilde{F}(e(\eta), e(\eta')) > 0 \\ &\quad \Leftrightarrow e(\eta) > e(\eta'). \end{aligned}$$

Now if $f(X, Y), g(X, Y) \in \mathbb{R}[X, Y]$ and $\eta, \eta', \eta'' \in \langle Kb\rangle \cap \mu$ and $g(\eta, \eta') \neq 0$ and $\eta'' = f(\eta, \eta')/g(\eta, \eta')$, then by considering $F := Zg(X, Y) - f(X, Y)$, $e(\eta'') = f(e(\eta), e(\eta'))/g(e(\eta), e(\eta'))$.

Preservation by $e'$ of $+, *$ follows.

For example, suppose $\eta, \eta' \in \langle Kb\rangle \cap \mu$, we claim $e'(1/\eta + 1/\eta') = e'(1/\eta) + e'(1/\eta')$. Let $s := \mathrm{st}((1/\eta + 1/\eta')) \in \mathbb{R} \cup \infty$.

If $s = \infty$, then

$$\eta'' := 1/(1/\eta + 1/\eta') \in \langle Kb\rangle \cap \mu.$$

Then by the above applied to the rational function $1/(1/X + 1/Y)$,

$$e(\eta'') = 1/(1/e(\eta) + 1/e(\eta')).$$

So

$$\begin{aligned} e'(1/\eta + 1/\eta') &= 1/e(\eta'') \\ &= 1/(1/(1/e(\eta) + 1/e(\eta'))) \\ &= e'(1/\eta) + e'(1/\eta'). \end{aligned}$$

If $s \in \mathbb{R}$, then

$$\eta''' := (1/\eta + 1/\eta') - s \in \langle Kb\rangle \cap \mu.$$

Then

$$\begin{aligned} e'(1/\eta + 1/\eta') &= s + e(\eta''') \\ &= s + ((1/e(\eta) + 1/e(\eta')) - s) \\ &= e'(1/\eta) + e'(1/\eta'). \end{aligned}$$

The other cases (e.g. $e'((s+\eta)(1/\eta')))$ can be handled similarly.

□

It remains to see that for acceptable $(r, F)$, and $\overline{\alpha} \in \langle Kb \rangle^n$, $e'(\widetilde{F}\restriction_r (\overline{\alpha})) = \widetilde{F}\restriction_r (e'(\overline{\alpha}))$.

If $\|\overline{\alpha}\| \geq r$ then also $\|e'(\overline{\alpha})\| \geq r$, so the equality is clear. Else, $\overline{s} := \mathrm{st}(\overline{\alpha}) \in \mathrm{dom}(F)$, so $\widetilde{F}$ is analytic at $\overline{s}$. Let $s' := \widetilde{F}(\overline{s})$, and say $(\epsilon, G)$ is acceptable s.t. $\widetilde{F} \in \mathbb{R}\{\overline{X}\}_{r+\epsilon}$ and $\widetilde{G}(\overline{x}) = \widetilde{F}(\overline{s} + \overline{x}) - s'$ for $\overline{x} \in \mathbb{R}^n$, $\|\overline{x}\| < \epsilon$. Then

$$T_{\mathrm{an}}^D \vDash \forall \overline{x}. \, (\|\overline{x}\| < \epsilon \to \widetilde{G}\restriction_\epsilon (\overline{x}) = \widetilde{F}\restriction_{r+\epsilon} (\overline{s} + \overline{x}) - s').$$

So

$$\begin{aligned} \widetilde{F}\restriction_r (e'(\overline{\alpha})) &= s' + \widetilde{F}\restriction_{r+\epsilon} (\overline{s} + e(\overline{\eta})) - s' \text{ (where } \overline{\eta} := \overline{\alpha} - \overline{s}) \\ &= s' + \widetilde{G}(e(\overline{\eta})) \\ &= s' + e(\widetilde{G}(\overline{\eta})) \text{ (considering } Y - G(X)) \\ &= s' + e(\widetilde{F}\restriction_{r+\epsilon} (\overline{s} + \overline{\eta}) - s') \\ &= e'(\widetilde{F}\restriction_{r+\epsilon} (\overline{\alpha})) \\ &= e'(\widetilde{F}\restriction_r (\overline{\alpha})) \text{ (since } \|\overline{\alpha}\| < r) \end{aligned}$$

as required.

## 3.3 II: finding a $T_{\mathrm{an}}$-$\mu$-embedding

It remains to show that such an $e$ exists.

Since we may replace $M_2$ by an elementary extension, it suffices to show that the corresponding (long) type over $K \cap \mu$ is consistent, as follows:

**Lemma 3.8.** *Let* $m,n \in \mathbb{N}$, $\overline{X} = (X_1, \ldots, X_m)$, $\overline{Y} = (Y_1, \ldots, Y_n)$, *and* $S \subseteq \mathbb{R}\langle \overline{X}, \overline{Y} \rangle$ *finite, and* $\overline{c} \in (K \cap \mu)^m$, *and* $\overline{b} \in \mu(M_1)^n$.

*Then exists* $\overline{b}' \in \mu(M_2)^n$ *s.t. for* $F \in S$,

$$M_1 \vDash \widetilde{F}(\overline{c}, \overline{b}) > 0 \Leftrightarrow M_2 \vDash \widetilde{F}(\overline{c}, \overline{b}') > 0.$$

*Proof.* By induction on $n$, the $n = 0$ case being trivial.

First we show how to handle a single $F$. So suppose $S = \{F\}$. WMA $F \neq 0$.

By Denef - van den Dries Preparation,

$$F(\overline{X}, \overline{Y}) = \sum_{|\nu| < d} a_\nu(\overline{X}) \overline{Y}^\nu u_\nu(\overline{X}, \overline{Y})$$

for some $d \in \mathbb{N}$, $a_\nu \in \mathbb{R}\langle \overline{X} \rangle$, $u_\nu \in \mathbb{R}\langle \overline{X}, \overline{Y} \rangle^*$.

Let $\nu_0$ s.t. $M := |a_{\nu_0}(\overline{c})| \geq |a_\nu(\overline{c})| \forall \nu$. Define

$$s_\nu := \mathrm{st}(a_\nu(\overline{c})/M) \in \mathbb{R}$$

$$k_\nu := a_\nu(\overline{c})/M - s_\nu \in K$$

(Remark: here it is crucial that $K$ is a $L_{\mathrm{an}}^D$-substructure, and hence a subfield, rather than merely a $L_{\mathrm{an}}$-substructure.)

Let $\overline{Z} = (Z_\nu)_{|\nu|<d}$, and define

$$G(\overline{X}, \overline{Z}, \overline{Y}) := \sum_{|\nu|<d} (Z_\nu + s_\nu)\overline{Y}^\nu u_\nu(\overline{X}, \overline{Y}).$$

Then $G \in \mathbb{R}\left\langle \overline{X}, \overline{Z}, \overline{Y} \right\rangle$, and $\widetilde{F}(\overline{c}, \overline{y}) = M\widetilde{G}(\overline{c}, \overline{k}, \overline{y})$ for $\overline{y} \in \mu^n$.

Define $\Lambda : \mu^n \to \mu^n$;

$$\overline{y} \mapsto (y_1 + y_n^{d^{n-1}}, y_2 + y_n^{d^{n-2}}, \ldots, y_{n-1} + y_n^d, y_n).$$

Let $H(\overline{X}, \overline{Z}, \overline{Y}) := G(\overline{X}, \overline{Z}, \Lambda(\overline{Y})) \in \mathbb{R}\left\langle \overline{X}, \overline{Z}, \overline{Y} \right\rangle$ (considering $\Lambda$ as a tuple of (polynomial) formal power series).

**Claim 3.9.** *$H$ is regular in $Y_n$.*

*Proof.*

$$\begin{aligned} H(0,0,0,Y_n) &= \sum_{|\nu|<d} s_\nu \Lambda(0, Y_n)^\nu u_\nu(0, \Lambda(0, Y_n)) \\ &= \sum_{|\nu|<d} s_\nu Y_n^{\sum \nu_i d^{n-i}} u_\nu(0, \Lambda(0, Y_n)) \end{aligned}$$

Now for $\nu$ with $|\nu| < d$, the exponents $\sum \nu_i d^{n-i}$ are distinct for distinct $\nu$, and are ordered according to the lexicographic order on the $\nu$. So taking $\nu$ lexicographically minimal s.t. $s_\nu \neq 0$, which exists since $s_{\nu_0} = 1$, witnesses regularity of $H$. □

So by Weierstrass preparation,

$$\widetilde{F}(\overline{c}, \Lambda(\overline{y})) = M\widetilde{G}(\overline{c}, \overline{k}, \overline{y}) = M\widetilde{Q}(\overline{c}, \overline{k}, \overline{y})\widetilde{L}(\overline{c}, \overline{k}, \overline{y})$$

where $Q \in \mathbb{R}\left\langle \overline{X}, \overline{Z}, \overline{Y} \right\rangle^*$ and $L \in \mathbb{R}\left\langle \overline{X}, \overline{Z}, \overline{Y}_{<n} \right\rangle [Y_n]$, where $\overline{Y}_{<n} := (Y_1, \ldots, Y_{n-1})$. WLOG $Q(0) > 0$.

Say $L(\overline{X}, \overline{Z}, \overline{Y}) = \sum_{i=0}^p L_i(\overline{X}, \overline{Z}, \overline{Y}_{<n})Y_n^i$ with $L_i \in \mathbb{R}\left\langle \overline{X}, \overline{Z}, \overline{Y}_{<n} \right\rangle$.

By QE for RCF, for $\epsilon \in \mathbb{R}_{>0}$,

$$\exists y_n \in (-\epsilon, \epsilon). \sum_{i=0}^p w_i y_n^i > 0$$

is equivalent modulo RCF to a qf ordered ring formula $\psi_\epsilon(\overline{w})$, which is a boolean combination of atomic formulae $f(\overline{w}) > 0$ where $f$ is a polynomial over $\mathbb{Z}$. So by the inductive hypothesis, and since $\mathbb{R}\left\langle \overline{X}, \overline{Z}, \overline{Y}_{<n} \right\rangle$ is a ring, there exists $\overline{b}''_\epsilon$ in $\mu(M_2)^{n-1}$ s.t.

$$\vDash \psi_\epsilon(\widetilde{L}_i(\overline{c}, \overline{k}, \overline{b}''_\epsilon))$$

So by $\omega$-saturation, exists $\overline{b}''$ in $\mu(M_2)^n$ s.t. $\widetilde{L}(\overline{c}, \overline{k}, \overline{b}'') > 0$. Hence $\widetilde{F}(\overline{c}, \Lambda(\overline{b}'')) > 0$, so $\overline{b}' := \Lambda(\overline{b}'')$ is as required.

For general finite $S$, we may take a $d$ common to all $F \in S$, proceed as above (so using the same $\Lambda$ for all $F$), and apply RCF QE to the corresponding conjunction of inequalities. □