Dr. M. Bays
Dr. T. Zou

Universität Münster
Wintersemester 2022/23

Geometric Group Theory I
Exercise Sheet 3

Exercise 1. (Lemma 2.25 in the lecture notes)
Show that if $p : X \to T$ is a locally injective morphism from a connected graph X to a tree T, then p is injective and X is a tree.

(4 Points)

Exercise 2.

a) Draw the following Cayley graphs (as oriented graphs):

   i) $\Gamma(\mathbb{Z}/6\mathbb{Z}, \{1, 5\})$;
   ii) $\Gamma(S_3, \{(12), (23)\})$.

b) Let $\Gamma(G, S)$ be a Cayley graph of a group G. An endomorphism $p : \Gamma(G, S) \to \Gamma(G, S)$ is called *label preserving* if for all $(g, s) \in \Gamma(G, S)^+$ there is some $h \in G$ such that $p(g, s) = (h, s)$.[1] Show that the group of label preserving automorphisms of $\Gamma(G, S)$ is isomorphic to G.

(4 Points)

Exercise 3. Let T be a finite tree and $\sigma : T \to T$ be an automorphism.

a) Show that for any $x \in X$ there is $n \geq 1$ such that $\sigma^n(x) = x$.
b) Suppose the distance $d(x, \sigma(x))$ between x and $\sigma(x)$ is 1 for some $x \in X$. Show that σ inverts an edge, i.e. $\sigma(e) = \bar{e}$ for some edge e.
c) Show that σ either fixes a vertex or inverts an edge.
Hint: Induction on $d(x, \sigma(x))$. Let p be the geodesic from x to $\sigma(x)$ and consider the composition of $p, \sigma(p), \sigma^2(p), \cdots$.

(4 Points)

Exercise 4. Show that a free group F of rank ≥ 2 has a free subgroup of rank r for any $r \in \mathbb{N} \cup \{\aleph_0\}$, where $\aleph_0 = |\mathbb{N}|$.
Hint: Given $x, y \in F$ consider the set $\{x, y^{-1}xy, y^{-2}xy^2, \cdots\}$.

(4 Points)

*Submission by **Wednesday** morning 11:00, 02.11.2022, in Briefkasten 161.*
The exercise sheets should be solved and submitted in pairs.
Tutorial: Fridays 12:00-14:00, in room SR1d.
If you have questions about the problem sheet, please write to Tingxiang: tingxiangzou@gmail.com.

[1] Given a positive edge (g, s), we could assign s as the label of (g, s). Then label preserving morphisms are morphisms that preserve labels.