Selected topics in representation theory

– Torsion theories and tilting modules –

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This lecture will be about a result of Hoshino [2] relating general torsion theories and those given by tilting modules.

1 Torsion theories

Definition. A torsion theory is a pair $(\mathcal{T}, \mathcal{F})$ consisting of subcategories of a module category $\operatorname{mod} A$ with the following properties:

- $\operatorname{Hom}_A(X, Y) = 0$ for all $X \in \mathcal{T}$ and all $Y \in \mathcal{F}$.
- If $\operatorname{Hom}_A(X, Y) = 0$ for all $X \in \mathcal{T}$, then $Y \in \mathcal{F}$.
- If $\operatorname{Hom}_A(X, Y) = 0$ for all $Y \in \mathcal{F}$, then $X \in \mathcal{T}$.

By $t(M)$ we denote the maximal torsion submodule of module $M \in \operatorname{mod} A$. Then we get the *canonical exact sequence*

$$0 \to t(M) \to M \to M/t(M) \to 0,$$

and $t(M/t(M)) = 0$.

Lemma. *Every tilting module $T \in \operatorname{mod} A$ gives rise to a torsion theory $(\mathcal{T}(T), \mathcal{F}(T))$ by setting $\mathcal{T}(T) := \{M \in \operatorname{mod} A \mid M$ is generated by $T\}$ and $\mathcal{F}(T) := \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, Y) = 0\} = \{M \in \operatorname{mod} A \mid M$ is cogenerated by $\tau T\}$.*

A natural question is the following: Given a torsion theory on $\operatorname{mod} A$, are there conditions such that there exists a tilting module inducing the torsion theory?

Definition. Let $\mathcal{C}$ be a full subcategory of $\operatorname{mod} A$. A module $M \in \mathcal{C}$ is called *Ext-projective* (resp. *Ext-injective*) (in $\mathcal{C}$) if $\operatorname{Ext}^1_A(M, C) = 0$ (resp. $\operatorname{Ext}^1_A(C, M) = 0$) for all $C \in \mathcal{C}$.

We are now going to prove several Lemmas which will guarantee the existence of a tilting module inducing a given torsion theory in special cases.

Here are two Lemmas characterising Ext-projective (resp. Ext-injective) modules.

Lemma A. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory on $\operatorname{mod} A$. Then $M \in \mathcal{F}$ is Ext-projective if and only if $M \cong P/t(P)$ for some projective module $P \in \operatorname{mod} A$.*

Proof. First, let $P \in \operatorname{mod} A$ be projective. Consider the canonical exact sequence

$$0 \to t(P) \to P \to P/t(P) \to 0.$$

Now take any $Y \in \mathcal{F}$ and apply $\operatorname{Hom}_A(-, Y)$ to the sequence. We get the following exact sequence in the corresponding long exact sequence:

$$\operatorname{Hom}_A(t(P), Y) \to \operatorname{Ext}^1_A(P/t(P), Y) \to \operatorname{Ext}^1_A(P, Y).$$

Both $\operatorname{Hom}_A(t(P), Y)$ and $\operatorname{Ext}^1_A(P, Y)$ are zero, because $Y \in \mathcal{F}$ and P is projective. So $\operatorname{Ext}^1_A(P/t(P), Y) = 0$, and, therefore, $P/t(P)$ is Ext-projective.

On the other hand, take an Ext-projective module $M \in \mathcal{F}$. Consider its projective cover $\varepsilon : P \to M$. The induced map $\overline{\varepsilon} : P/t(P) \to M$ is still surjective, because $M \in \mathcal{F}$ (and, therefore, $t(P)$ is mapped to 0 by ε).

Let

$$0 \to \overline{K} \to P/t(P) \xrightarrow{\overline{\varepsilon}} M \to 0$$

be the exact sequence for $\overline{\varepsilon}$. This sequence splits, because $\overline{K} \in \mathcal{F}$, (since $P/t(P) \in \mathcal{F}$,) and $M \in \mathcal{F}$ is Ext-projective.

So, M is a direct summand of $P/t(P)$, therefore, of the form $\tilde{P}/t(\tilde{P})$ for some projective module $\tilde{P} \in \operatorname{mod} A$. □

Lemma B. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory on $\operatorname{mod} A$. Then $M \in \mathcal{T}$ is Ext-projective if and only if $\tau M \in \mathcal{F}$, where τ denotes the Auslander-Reiten translation. (Dually, $M \in \mathcal{F}$ is Ext-injective if and only if $\tau^- M \in \mathcal{T}$.)*

Proof. Let $M \in \mathcal{T}$ with $\tau M \in \mathcal{F}$, and take any $N \in \mathcal{T}$. The map $D\operatorname{Hom}_A(N, \tau M) \to \operatorname{Ext}^1_A(M, N)$ is surjective.

Since $\operatorname{Hom}_A(N, \tau M) = 0$, because $\tau M \in \mathcal{F}$ and $N \in \mathcal{T}$, $D\operatorname{Hom}_A(N, \tau M)$ and, therefore, $\operatorname{Ext}^1_A(M, N)$ are both zero. So $M \in \mathcal{T}$ is Ext-projective.

Let now $M \in \mathcal{T}$ be Ext-projective. If M is projective, then $\tau M = 0 \in \mathcal{F}$. Otherwise, consider the canonical sequence for τM:

$$0 \to t(\tau M) \to \tau M \to \tau M/t(\tau M) \to 0.$$

Apply $\operatorname{Hom}_A(M, -)$ to this sequence. We obtain the following exact sequence in the corresponding long exact sequence:

$$\operatorname{Ext}^1_A(M, t(\tau M)) \to \operatorname{Ext}^1_A(M, \tau M) \to \operatorname{Ext}^1_A(M, \tau M/t(\tau M)).$$

Since $M \in \mathcal{T}$ is Ext-projective, $\operatorname{Ext}^1_A(M, t(\tau M)) = 0$. But $\operatorname{Ext}^1_A(M, \tau M) \neq 0$, because the AR-sequence

$$0 \to \tau M \to E \to M \to 0$$

does not split. Since $\operatorname{Ext}^1_A(M, \tau M) \to \operatorname{Ext}^1_A(M, \tau M/t(\tau M))$ is injective, the AR-sequence

$$0 \to \tau M \to E \to M \to 0$$

is mapped to a non split exact sequence. We get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau M & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \tau M/t(\tau M) & \longrightarrow & \tilde{E} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

But if $t(\tau M) \neq 0$, the map $\tau M \to \tau M/t(\tau M)$ would not be a split monomorphism, and the sequence

$$0 \to \tau M/t(\tau M) \to \tilde{E} \to M \to 0$$

would split. So, $t(\tau M) = 0$, and $\tau M \in \mathcal{F}$. □

Lemma C. *Let* $(\mathcal{T},\mathcal{F})$ *be a torsion theory on* $\operatorname{mod} A$ *such that* $D(A_A) \in \mathcal{T}$*. If* $M \in \mathcal{T}$ *is Ext-projective, then* $\operatorname{pd} M \leq 1$*.*

Proof. If M is projective, then, clearly, $\operatorname{pd} M \leq 1$.

So assume that M be not projective. We are going to construct a projective resolution of M.

Let $0 \to \tau M \to I_0 \to I_1$ be the minimal injective resolution of τM. By definition of τ^-, we obtain a projective resolution ending with

$$\operatorname{Hom}_A(D(I_0), A) \xrightarrow{p_1} \operatorname{Hom}_A(D(I_1), A) \to M \to 0.$$

Now, $\tau M \in \mathcal{F}$ and $D(A_A) \in \mathcal{T}$, and we get

$$\ker p_1 \cong \operatorname{Hom}_A(D(\tau M), A) \cong \operatorname{Hom}_A(D(A), \tau M) = 0.$$

Therefore, $\operatorname{pd} M \leq 1$. □

Corollary. *Let* $(\mathcal{T},\mathcal{F})$ *be a torsion theory on* $\operatorname{mod} A$ *such that* $D(A_A) \in \mathcal{T}$*. Let* M *be the sum of all Ext-projective modules belonging to* $\mathcal{T}$*. Then* M *is a partial tilting module, i. e.* $\operatorname{pd} M \leq 1$ *and* $\operatorname{Ext}^1_A(M, M) = 0$*.*

Remark. A partial tilting module $M \in \operatorname{mod} A$ is a tilting module if and only if the number of indecomposable direct summands of M equals the number of simple modules in $\operatorname{mod} A$.

Lemma D. *Let* $(\mathcal{T},\mathcal{F})$ *be a torsion theory on* $\operatorname{mod} A$*.*

- *Let* $M \in \mathcal{T}$ *be not Ext-projective and* $g : E \to M$ *be a sink map in* $\mathcal{T}$*, then* $\ker g \in \mathcal{T}$*.*
- *Let* $M \in \mathcal{F}$ *be not Ext-injective and* $f : M \to E$ *be a source map in* $\mathcal{F}$*, then* $\operatorname{cok} f \in \mathcal{F}$*.*

Proof. The proof can be found (for example) in [1], Chapter III, Section 4. □

Now, we can prove the following Theorem:

Theorem. *Let* $(\mathcal{T},\mathcal{F})$ *be a torsion theory on* $\operatorname{mod} A$ *such that* $D(A_A) \in \mathcal{T}$ *and either* $\mathcal{T}$ *or* $\mathcal{F}$ *contain only finitely many isomorphism classes of indecomposable A-modules. Let* T *be the sum of all Ext-projectives belonging to* $\mathcal{T}$*. Then* T *is a tilting module such that* $(\mathcal{T}(T), \mathcal{F}(T)) = (\mathcal{T},\mathcal{F})$*.*

Proof. By the remark and the corollary above, we only have to show that T contains the correct number of indecomposable direct summands.

First of all, let us note that $\mathcal{F}$ and $\mathcal{T}$ are both KRS-categories. Since finite KRS-categories have sink and source maps (see e. g. [3], Section 2.2, Lemma 2), we can apply the lemmas above.

Let n be the number of non isomorphic simple modules in $\operatorname{mod} A$.

Case 1. $\mathcal{T}$ is finite.

Since $D(A_A) \in \mathcal{T}$, we have n indecomposable non isomorphic injective modules in $\mathcal{T}$. Now, we apply the AR-translation several times to each of the indecomposable injective modules, and using Lemma D, we reach a sink map whose kernel is contained in $\mathcal{F}$. (This is possible, since $\mathcal{T}$ is finite.) So we get n non isomorphic indecomposable Ext-projective modules in $\mathcal{T}$ (by Lemma B).

Case 2. $\mathcal{F}$ is finite.

Let s be the number of indecomposable projective modules in $\mathcal{T}$. Then $\mathcal{F}$ has $n-s$ indecomposable projective modules. We apply τ^{-} to these modules, and we obtain $n-s$ non isomorphic indecomposable Ext-injective modules in $\mathcal{F}$ (using Lemma D, again). (This is possible, since $\mathcal{F}$ is finite.) These $n-s$ indecomposable Ext-injective modules give rise to $n-s$ non isomorphic indecomposable Ext-projective modules in $\mathcal{T}$ (by Lemma B). Since the s non isomorphic indecomposable projective modules from $\mathcal{T}$ are, in particular, Ext-projective (and not isomorphic to any of the other Ext-projective modules constructed as above), we obtain $n-s+s$ non isomorphic indecomposable Ext-projective modules in $\mathcal{T}$.

And T induces the torsion theory, we wanted to have. □

References

[1] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras.* London Mathematical Society Lecture Note Series, 119. Cambridge University Press, Cambridge, 1988. x+208 pp.

[2] M. Hoshino, *Tilting modules and torsion theories*, Bull. London Math. Soc. **14** (1982), 334–336

[3] C .M. Ringel, *Tame algebras and integral quadratic forms.* Lecture Notes in Mathematics, 1099. Springer-Verlag, Berlin, 1984. xiii+376 pp.