SQUEEZING AND HIGHER ALGEBRAIC K-THEORY

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Abstract. We prove that the Assembly map in algebraic K-theory is split injective for groups of finite asymptotic dimension admitting a finite classifying space.

1. Introduction

It is well known that the Novikov conjecture on the homotopy invariance of higher signatures is equivalent to rational injectivity of the assembly map $H_*(B\Gamma; \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}\Gamma)$. However, there are also other important assembly maps, e.g. in algebraic K-theory and the Baum-Connes map for topological K-theory. A technique that has been very successful in studying these assembly maps is controlled topology. Yu [Yu98] used a C^*-algebra version of this technique to prove the Novikov conjecture for groups of finite asymptotic dimension (cf. Section 6) admitting a finite classifying space. In fact, he proved injectivity of the Baum-Connes map for this class of groups, which also implies the Novikov conjecture. The purpose of this paper is to give a proof of the corresponding result in algebraic K-theory.

Theorem 1.1. *Let R be an associative ring with unit and Γ be a group of finite asymptotic dimension admitting a finite $B\Gamma$. Then the assembly map*
$$H_*(B\Gamma; \mathbb{K}^{-\infty}R) \to K_*(R\Gamma)$$
is split injective.

In fact, the result holds for coefficients in any additive category (see 6.5). This is very much in the spirit of [CP95] and there is also an L-theory version (7.2). For more information on groups of finite asymptotic dimension see for example [DJ99] and [BD01].

Very roughly, Yu proceeds as follows to prove his result. Controlled constructions are used to set up an obstruction group to the injectivity of the Baum-Connes map. This obstruction group comes with the additional notion of control, i.e. elements are r-controlled for some

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$r > 0$. The crucial step is then to prove that there is some $\epsilon > 0$ such that all r-controlled elements with $r < \epsilon$ in the obstruction group vanish. Such a result is often referred to as a squeezing result. Finite asymptotic dimension is then used to prove that every element in the obstruction group is arbitrarily small controlled.

Our proof of 1.1 is parallel to Yu's approach. The main difference is, that while there are squeezing theorems for K_1 (cf. [Qui82]) there is a priori no obvious analogue for higher K-theory. The main contribution of this paper is the formulation of a somewhat different result (4.1), that works for higher K-theory and can be used similarly to squeezing (cf. 4.3) to prove vanishing results.

It should be noted that Yu proved the Novikov conjecture in [Yu00] for the class of groups admitting a uniform embedding into Hilbert space and a finite $B\Gamma$. Here a version of Bott periodicity is used and it is at present not clear if this result can also be carried over to algebraic K-theory. Groups of finite asymptotic dimension admit such a uniform embedding by [HR00]. Later on the Novikov conjecture was established for groups of finite asymptotic dimension ([Hig00]) and groups admitting a uniform embedding ([STY02]), irrespective of finiteness or otherwise of $B\Gamma$. Thus it seems to be an important question whether the finite-$B\Gamma$ hypothesis can be removed from 1.1. Without any geometric assumptions rational injectivity of the algebraic K-theory assembly map for the ring $\mathbb{Z}$ is known under the rather weak finiteness assumption that the homology of Γ is finitely generated in every degree by [BHM93]. The referee informed me that the recent PhD thesis of Wright [Wri02] contains another proof of Yu's result along lines similar to those of this paper.

This paper is organized as follows. Section 2 briefly recalls the properties of K-theory needed in this paper. Section 3 reviews controlled algebra. We collect various results from the literature and slightly expand some of them. Using the abstract language of coarse structures from [HPR97] will be useful in formulating and proving the squeezing result in Section 4. In Section 5 we recall the descent principle from [CP95]. This identifies $K_*\mathcal{A}(\mathbb{J}_b(E\Gamma))$ as the obstruction group in question. Section 6 contains the proof of our main result. The paper concludes with a very brief discussion of L-theory.

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2. K-theory

We will use the non-connected K-theory functor $\mathbb{K}^{-\infty}$ from (small) additive categories to spectra. Applied to the category of finitely generated free R-modules its homotopy groups give the higher K-theory K_*R of Quillen in positive degrees and Bass' negative K-theory in negative degrees. This functor has been constructed by Pedersen and Weibel in [PW85]. A crucial property of this functor is its behavior with respect to Karoubi filtrations ([Kar70, 1.5]). For a proof of the following result see for example [CP97].

Theorem 2.1. *If $\mathcal{A}$ is a Karoubi filtration of the the category $\mathcal{B}$, then there is a fibration sequence of spectra*

$$\mathbb{K}^{-\infty}\mathcal{A} \to \mathbb{K}^{-\infty}\mathcal{B} \to \mathbb{K}^{-\infty}\mathcal{A}/\mathcal{B}.$$

□

We will mostly be interested in the homotopy groups of $\mathbb{K}^{-\infty}$. We state further well known properties of K-theory used in this paper.

Theorem 2.2.

(i) Eilenberg swindle. *If $\mathcal{A}$ is flasque, i.e. there is an additive functor $S : \mathcal{A} \to \mathcal{A}$ together with a natural equivalence* $\mathrm{id} \oplus S \cong S$*, then $K_*\mathcal{A} = 0$.*

(ii) Equivalence of categories. *Naturally equivalent functors induce the same maps of K-groups.*

(iii) Colimits. *If $\mathcal{A}$ is the union of subcategories $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ then $K_*(\mathcal{A}) = \mathrm{colim}\ K_*(\mathcal{A}_i)$.*

□

3. Controlled algebra

We will use the concept of a coarse structure from [HPR97].

Definition 3.1 (Coarse structure)**.** By a *coarse structure* on a topological space X we mean a collection $\mathbb{E}$ of subsets of $X \times X$ satisfying the following conditions

(i) For $E, E' \in \mathbb{E}$ their union $E \cup E'$ is contained in some $E'' \in \mathbb{E}$.

(ii) For $E, E' \in \mathbb{E}$ their composition (as relations) $E \circ E'$ is contained in some $E'' \in \mathbb{E}$.

(iii) For $E \in \mathbb{E}$ its inverse $E^{op} := \{(x, y) \mid (y, x) \in E\}$ is contained in some $E' \in \mathbb{E}$.

(iv) For $E \in \mathbb{E}$ and $K \subset X$ compact the closure of

$$\{x \mid (k, x) \text{ or } (x, k) \in E \text{ for some } k \in K\}$$

is also compact.

The sets E are called entourages in [HPR97].

Definition 3.2. Let $\mathcal{S}$ be a collection of subsets of a topological space X that is closed under finite unions. Let $\mathbb{E}$ be a coarse structure on X. Let $p_1, p_2 : X \times X \to X$ denote the projections onto the first and second factor. Let $\Delta : X \to X \times X$ denote the diagonal embedding.

(i) We define the *domain* of $\mathbb{E}$ as

$$dom(\mathbb{E}) := \{F \subset X \mid \Delta(F) \text{ is contained in some } E \in \mathbb{E}\}.$$

$\mathbb{E}$ is called *unital* if $X \in dom(\mathbb{E})$.

(ii) The *restriction of* $\mathbb{E}$ *to* $\mathcal{S}$ is defined by

$$\mathbb{E}_{\mathcal{S}} := \{E \mid E \subset E', p_1(E), p_2(E) \subset F \text{ for some } E' \in \mathbb{E}, F \in \mathcal{S}\}.$$

It is again a coarse structure.

(iii) We define the $\mathbb{E}$-*enlargement of* $\mathcal{S}$ by

$$\langle \mathcal{S} \rangle_{\mathbb{E}} := \{p_1(E \circ \Delta(F)) \mid E \in \mathbb{E}, F \in \mathcal{S}\}.$$

Definition 3.3 (Coarse map)**.** Let $\mathbb{E}_X$ and $\mathbb{E}_Y$ be coarse structures on topological spaces X and Y. A map $f : X \to Y$ is said to be *coarse* (w.r.t. $\mathbb{E}_X, \mathbb{E}_Y$) if the following two conditions are satisfied.

(i) For every $E_X \in \mathbb{E}_X$ there is $E_Y \in \mathbb{E}_Y$ such that $f(E_X) \subset E_Y$.

(ii) For $F \in \mathcal{S}(\mathbb{E}_X)$ and $K \subset Y$ compact, the closure of $F \cap f^{-1}(K)$ is compact.

Geometric modules are a useful tool from controlled topology. In our case the control conditions will come from a coarse structure as in [HPR97].

Definition 3.4 (Geometric modules and morphisms over X)**.** Let $\mathcal{A}$ be a small additive category and X be a topological space. Let $\mathbb{E}$ be a coarse structure on X.

(i) A *geometric* $\mathcal{A}$*-module over* X consists of a collection of objects $M_x \in \mathcal{A}$ for $x \in X$, such that the *support*

$$\text{supp}(M) := \{x \mid M_x \neq 0\}$$

is a locally finite subset of X.

(ii) A morphism $\phi : M \to N$ between geometric $\mathcal{A}$-modules over X is given by a collection of morphisms $\phi_{x,y} : M_y \to N_x$ such that for fixed x (resp. fixed y) $\phi_{x,y} \neq 0$ for only a finite number of y (resp. x). Define

$$\text{supp}(\phi) := \{(x, y) \mid \phi_{x,y} \neq 0\} \subset X \times X.$$

Composition is matrix multiplication, i.e.

$$(\phi \circ \psi)_{x,y} = \sum_z \phi_{x,z} \circ \psi_{z,y}.$$

(iii) The category $\mathcal{A}(\mathbb{E})$ has as objects geometric modules M with $\mathrm{supp}(M)$ contained in some $F \in \mathcal{S}(\mathbb{E})$. Morphisms in $\mathcal{A}(\mathbb{E})$ are required to have support contained in some $E \in \mathbb{E}$.

Remark 3.5 (Functoriality). Let $f : X \to Y$ be a coarse map (w.r.t. coarse structures $\mathbb{E}_X, \mathbb{E}_Y$). This induces a functor $f_* : \mathcal{A}(\mathbb{E}_X) \to \mathcal{A}(\mathbb{E}_Y)$ as follows. Given a module M in $\mathcal{A}(\mathbb{E}_X)$, we define the module f_*M by $(f_*M)_y := \oplus_{x \in f^{-1}(y)} M_x$. Note that this sum is finite by 3.3 (ii). Given a morphism ϕ in $\mathcal{A}(\mathbb{E}_X)$, the morphism $f_*\phi$ is defined by $\phi_{y,y'} = \oplus \phi_{x,x'}$ where we sum over all $(x, x') \in (f \times f)^{-1}(y, y')$.

This defines the functor f_* only up natural equivalence (cf. [CP95, 1.16]). Following Weiss [Wei, Section 2], the problem can be solved by equipping every module over a space by choices of all these finite sums, i.e. a module is then a functor defined on the category of finite subsets of the space. We will simply ignore this matter to simplify the presentation.

Karoubi filtrations appear very naturally with geometric modules, cf. [CP95, 1.29]. Here they will relate $\mathcal{A}(\mathbb{E})$ and $\mathcal{A}(\mathbb{E}_\mathcal{S})$. The proof of the following lemma is immediate from the definitions.

Lemma 3.6. *Let $\mathbb{E}$ be a coarse structure on X. Let $\mathcal{S}$ be a family of subsets of X that is closed under finite unions. Denote $\langle \mathcal{S} \rangle_\mathbb{E}$ by $\overline{\mathcal{S}}$.*

(i) We can consider $\mathcal{A}(\mathbb{E}_{\overline{\mathcal{S}}})$ in an obvious way as a subcategory of $\mathcal{A}(\mathbb{E})$. This is a Karoubi filtration. We denote the quotient by $\mathcal{A}(\mathbb{E})/\mathcal{S}$.

(ii) The canonical inclusion $\mathcal{A}(\mathbb{E}_\mathcal{S}) \to \mathcal{A}(\mathbb{E}_{\overline{\mathcal{S}}})$ is an equivalence of categories.

□

Remark 3.7. Let us describe the quotient category $\mathcal{A}(\mathbb{E})/\mathcal{S}$. It has the same set of objects as $\mathcal{A}(\mathbb{E})$. Let M and N be geometric modules in $\mathcal{A}(\mathbb{E})$. Then $\mathrm{Hom}_{\mathcal{A}(\mathbb{E})/\mathcal{S}}(M, N)$ is the quotient of $\mathrm{Hom}_{\mathcal{A}(\mathbb{E})}(M, N)$ by the following equivalence relation: morphisms $\phi, \psi : M \to N$ are identified whenever their difference factors over an object in $\mathcal{A}(\mathbb{E}_{\langle \mathcal{S} \rangle_\mathbb{E}})$. Equivalently: $\mathrm{supp}(\phi - \psi)$ is contained in $F \times F$ for some $F \in \langle \mathcal{S} \rangle_\mathbb{E}$.

Remark 3.8. In the situation of 3.6 the sequence

$$\mathcal{A}(\mathbb{E}_\mathcal{S}) \to \mathcal{A}(\mathbb{E}) \to \mathcal{A}(\mathbb{E})/\mathcal{S}$$

induces a long exact sequence of K-groups (using 2.1 and 2.2(ii)). We will call such a sequence a *Karoubi sequence*.

Next we define the coarse structures that will be relevant in this paper.

Definition 3.9. Let X be a topological space and Z be a proper metric space.

(i) *Continuous control.* A subset E of $(X \times [0,1))^{\times 2}$ is said to be continuously controlled if for every $x \in X$ and every neighborhood U of $(x,1)$ in $X \times [0,1]$ there is a neighborhood V such that $(X \times [0,1] - U) \times V$ and $V \times (X \times [0,1] - U)$ do not intersect E. The set of all continuously controlled E that satisfy also 3.1 (iv) form the continuously controlled coarse structure $\mathbb{J}(X)$ on $X \times [0,1)$.

(ii) *Bounded control.* Let $\mathbb{B}(Z)$ consist of all subsets $E \subset Z \times Z$ satisfying the following: there is $R = R(E)$ such that $d(x,y) < R$ whenever $(x,y) \in E$.

(iii) *Continuous control with bounded control in Z-direction.* The coarse structure $\mathbb{J}_b(Z)$ (on $Z \times [0,1)$) consists of all $E \in \mathbb{J}(Z)$ that satisfy in addition the following: there is $R = R(E)$ such that $d(x,y) < R$ whenever $((x,t),(y,s)) \in E$.

Continuous control was introduced in [ACFP94] to study homology with coefficients in the K-theory spectrum of $\mathcal{A}$ (cf. 3.12). Bounded control only captures large scale properties of Z. In particular, $\mathcal{A}(\mathbb{B}(Z))$ is equivalent to $\mathcal{A}$ whenever Z is compact. The mixture of continuous control with bounded control in (iii) is particular suited for non-compact Z. Let us note the functorial behavior of our different notions of control.

Remark 3.10 (Functoriality). Each example in 3.9 describes a functor from an appropriate category of topological or metric spaces to the category of coarse spaces. We fix first some terminology. We will call a map $f : X \to Y$ proper if the closure of $f^{-1}(K)$ in X is compact for any compact subset K of Y. (Here proper maps are not necessary continuous.) A metric space is proper if any closed ball of finite radius is compact. A proper map $f : X \to Y$ between metric spaces will be called *metrically coarse* if it satisfies the following growth condition: for all $R > 0$ there is $S > 0$ such that

$$d_X(x,y) < R \quad \Longrightarrow \quad d_Y(f(x), f(y)) < S.$$

Using this terminology, the construction 3.9(i) is functorial on the category whose objects are locally compact Hausdorff spaces and whose morphisms are proper, continuous maps; construction 3.9(ii) is functorial on the category whose objects are proper metric spaces and whose morphisms are metrically coarse maps; construction 3.9(iii) is functorial on the category whose objects are proper metric spaces and whose morphisms are metrically coarse, continuous maps.

In order to obtain a homology theory from the category $\mathcal{A}(\mathbb{J}(X))$ we have to introduce the germ category.

Notation 3.11. For a topological space X we abbreviate

$$\mathcal{A}(\mathbb{J}(X))_\infty = \mathcal{A}(\mathbb{J}(X))/\langle\{(X \times \{0\})\}\rangle_{\mathbb{J}(X)}.$$

This category can be thought of, as obtained from $\mathcal{A}(\mathbb{J}(X))$ by taking germs at $X \times \{1\}$. Similar, for a metric space Z we abbreviate

$$\mathcal{A}(\mathbb{J}_b(Z))_\infty = \mathcal{A}(\mathbb{J}_b(Z))/\langle\{(Z \times \{0\})\}\rangle_{\mathbb{J}_b(Z)}.$$

The first version of the following result is [PW89, 3.1]. In this form it is proven in [Wei, 3.1,4.2].

Theorem 3.12. *The functor $X \mapsto K_*\mathcal{A}(\mathbb{J}(X))_\infty$ is a generalized locally finite homology theory on the category of second countable, locally compact Hausdorff spaces.* □

In particular, the functor $X \mapsto K_*\mathcal{A}(\mathbb{J}(X))_\infty$ is homotopy invariant. Next we study corresponding invariance results for $\mathbb{B}(X)$ and $\mathbb{J}_b(X)$.

Remark 3.13. Let $f, g : X \to Y$ be metrically coarse maps between proper metric spaces. If

$$\exists C > 0 \quad \text{such that} \quad d(f(x), g(x)) < C \quad \forall x \in X \tag{3.14}$$

then it is not hard to see that the functors $f_*, g_* : \mathcal{A}(\mathbb{B}(X)) \to \mathcal{A}(\mathbb{B}(Y))$ are natural equivalent and induce the same map on K-theory. Maps satisfying (3.14) are called *bornotopic* in [HR95]. A bornotopy equivalence $f : X \to Y$ is a metrically coarse map that is invertible up to bornotopy. Clearly, such a bornotopy equivalence induces an isomorphism $K_*\mathcal{A}(\mathbb{B}(X)) \to K_*\mathcal{A}(\mathbb{B}(Y))$.

We will mostly use the category $\mathcal{A}(\mathbb{J}_b(X))$ not $\mathcal{A}(\mathbb{J}(X))$. However, by the next lemma the germ categories agree.

Lemma 3.15. *Let X be a proper metric space. The canonical inclusion $\mathcal{A}(\mathbb{J}_b(X)) \to \mathcal{A}(\mathbb{J}(X))$ induces an isomorphism of categories*

$$\mathcal{A}(\mathbb{J}_b(X))_\infty \cong \mathcal{A}(\mathbb{J}(X))_\infty.$$

In particular, with respect to the equivalence relation that defines the quotient category $\mathcal{A}(\mathbb{J}(X))_\infty$, every morphism in $\mathcal{A}(\mathbb{J}(X))$ is equivalent to a morphism in $\mathcal{A}(\mathbb{J}_b(X))$.

Proof. We only prove the second statement. The isomorphism follows easily from this. Let ϕ be a morphism in $\mathcal{A}(\mathbb{J}(X))$. Let

$$W = \{(x,t) \quad | \quad \exists (y,s) \text{ such that } d(x,y) > 1, \\ \phi_{(x,t),(y,s)} \neq 0 \text{ or } \phi_{(y,s),(x,t)} \neq 0\}.$$

Let ψ be defined by

$$\psi_{(x,t),(y,s)} = \begin{cases} \phi_{(x,t),(y,s)} & \text{for } (x,t) \notin W \\ 0 & \text{otherwise} \end{cases}.$$

This is a morphism in $\mathcal{A}(\mathbb{J}_b(X))$.

For $z \in X$ let U_z be the product of the open ball with radius 1 around z with $[0,1]$. Then there is an open subset $V_z \subset X \times [0,1]$ containing $(z,1)$ such that $\phi_{(x,t),(y,s)} = 0$ whenever $(x,t) \in V_z$ and $(y,s) \notin U_z$. In particular, $W \cap V_z = \emptyset$. We can choose a locally finite set $Z \subset X$ such that

$$X \times \{1\} \subset \bigcup_{z \in Z} V_z.$$

Then

$$E = \{((x,t),(x,0)) \mid (x,t) \notin V_z \ \ \forall z \in Z\}$$

is an entourage in $\mathbb{J}(X)$. Now $\{((x,t),(x,0)) \mid (x,t) \in W\} \subset E$ and therefore $W \subset p_1(E \circ \Delta(X \times \{0\}))$. This implies that ϕ and ψ define the same morphism in $\mathcal{A}(\mathbb{J}(X))_\infty$. □

In order to get invariance results for the functor $X \mapsto K_*\mathcal{A}(\mathbb{J}_b(X))$ the notion of homotopy has to strengthened to continuous Lipschitz homotopy as follows.

Definition 3.16. Let X and Y be proper metric spaces. Let $f, g : X \to Y$ be two metrically coarse maps. A metrically coarse map $H : X \times [0,\infty) \to Y$ is called a Lipschitz homotopy (from f to g), if the following conditions are satisfied.

(i) $H(x,0) = f(x)$.
(ii) For every compact $K \subset X$ there is t_K such that $H(k,t) = g(k)$ for $k \in K, t > t_K$.
(iii) If $K \subset Y$ is compact, then the set $\{x \mid H(x,t) \in K \text{ for some } t\}$ is also compact.

If f, g and H are continuous maps, then we call H a continuous Lipschitz homotopy.

The following result is from [HPR97, 11.3].

Proposition 3.17. *Let X be a proper path-length metric space. Lipschitz homotopic maps induce the same maps*

$$K_*\mathcal{A}(\mathbb{B}(X)) \to K_*\mathcal{A}(\mathbb{B}(Y)).$$

□

Remark 3.18. In [HPR97], this result is stated for a weaker notion of Lipschitz homotopy. However, this is incorrect, and the proof given in [HPR97] in fact only works for the stronger notion formulated above. (Condition (iii) is formulated weaker, to the effect that the identity and the absolute value map $\mathbb{R} \to \mathbb{R}$ are Lipschitz homotopic. But $\mathcal{A}(\mathbb{B}(-))$ applied to the absolute value map is trivial in K-theory.)

Corollary 3.19. *Let X be a proper geodesic space. Continuously Lipschitz homotopic maps induce the same maps*

$$K_*\mathcal{A}(\mathbb{J}_b(X)) \to K_*\mathcal{A}(\mathbb{J}_b(Y)).$$

Proof. First assume that f is a continuous Lipschitz homotopy equivalence (in the obvious sense). Consider the Karoubi sequence

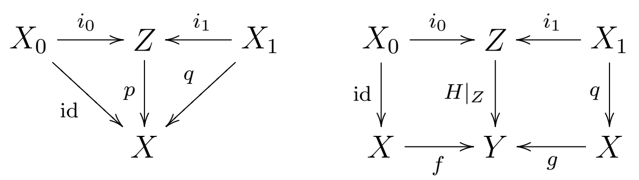
$$\mathcal{A}(\mathbb{B}(-)) \to \mathcal{A}(\mathbb{J}_b(-)) \to \mathcal{A}(\mathbb{J}_b(-))_\infty$$

for $- = X, Y$. The functor f_* induces an isomorphism on the K-theory of the first term by 3.17 and on the third term by 3.12 and 3.15. Now the long exact sequence 3.8 and the 5-Lemma imply that f_* induces an isomorphism on $K_*\mathcal{A}(\mathbb{J}_b(-))$.

For the general case we use the notation from 3.16. Let $\varphi : X \to [0, \infty)$ be a continuous map such that $\varphi(x) > t_K$ for $x \in K$. Let

$$\begin{array}{rcl} X_0 & = & X \times \{0\} \\ X_1 & = & \{(x, \varphi(x)) \mid x \in X\} \\ Z & = & \{(x,t) \mid x \in X, 0 \leq t \leq \varphi(x)\} \end{array}$$

be equipped with the induced path-length metric from $X \times [0, \infty)$. Consider the following commutative diagrams (cf. [HPR97, 11.2]).



Observe now that i_0 and i_1 are both continuous Lipschitz homotopy equivalences. The general case follows, since p_* is the inverse of $(i_0)_*$.

□

It is often easy to see that categories of geometric modules are flasque. We review a well known example.

Remark 3.20. Let $*$ denote the one point space. Let $f : [0,1) \to [0,1)$ be defined by

$$f(t) = t + (1-t)/2.$$

Then f induces a functor $Sh : \mathcal{A}(\mathbb{J}(*)) \to \mathcal{A}(\mathbb{J}(*))$. Clearly Sh is naturally equivalent to id. Moreover, the functor

$$S = \bigoplus_{i=1}^{\infty} Sh^i$$

is well defined and S is natural equivalent to $S \oplus \mathrm{id}$. Therefore, $\mathcal{A}(\mathbb{J}(*))$ has trivial K-theory. Cf. 2.2 (i).

4. Squeezing

The metric spaces we will consider are usually simplicial complexes with the spherical metric. Let us review the definition of this metric, cf. [HR95, 3.1]. We consider the standard n-simplex Δ^n as the set of points of $S^n \subset \mathbb{R}^{n+1}$ with nonnegative coordinates. The Riemannian metric on S^n induces the standard spherical metric on Δ^n. The spherical metric d_Q on a simplicial complex Q is the path metric whose restriction to each simplex is the standard spherical metric. (Thus, the distance between points in different path components is defined to be ∞.) Let us agree that all simplicial complexes in this section are assumed to be locally finite.

The main result of this section is the following proposition. We will discuss its relation to squeezing later on.

Proposition 4.1. *Let Q_n be a sequence of simplicial complexes of uniformly bounded dimension. Let Y be the disjoint union*

$$Q_1 \amalg Q_2 \amalg Q_3 \amalg \dots \ .$$

Equip Y with the metric d that restrict to n times the spherical metric on Q_n and satisfies $d(Q_n, Q_m) = \infty$ for $n \neq m$. Let $\mathcal{S}$ be the set consisting of all finite unions $Y_n := Q_1 \times [0,1) \amalg \dots \amalg Q_n \times [0,1) \subset Y \times [0,1)$. Then the inclusion

$$\mathcal{A}(\mathbb{J}_b(Y)_{\mathcal{S}}) \to \mathcal{A}(\mathbb{J}_b(Y))$$

induces an isomorphism on K-theory.

Of course, it is crucial here, that we blow up the metric on the Q_n as n increases. We discuss a special case before giving the proof of 4.1.

Lemma 4.2. *Suppose that in 4.1 each Q_n is a disjoint union of j-simplices. Then $\mathcal{A}(\mathbb{J}_b(Y))$ and $\mathcal{A}(\mathbb{J}_b(Y)_{\mathcal{S}})$ have vanishing K-theory.*

Proof. Different simplices have infinite distance. Therefore morphisms in $\mathcal{A}(\mathbb{J}_b(Y))$ and $\mathcal{A}(\mathbb{J}_b(Y)_\mathcal{S})$ cannot be nontrivial between different j-simplices Δ, Δ', i.e. for $\phi \in \mathcal{A}(\mathbb{J}_b(Y))$ we have $\operatorname{supp}(\phi) \cap \Delta \times \Delta' = \emptyset$. If we assume for a moment $j = 0$, then $\mathcal{A}(\mathbb{J}_b(Y))$ and $\mathcal{A}(\mathbb{J}_b(Y)_\mathcal{S})$ are flasque. This can be seen by pushing modules over $Y \times [0,1)$ along $[0,1)$ towards 1, cf. 3.20.

For the general case, pick a point on each simplex and let $p : Y \to Y$ the map that projects each simplex to this point. Then p is Lipschitz homotopic to the identity. On the other hand, p induces the trivial map in K-theory by the case $j = 0$. The claim follows now from 3.19. $\square$

Proof of 4.1. We proceed by induction over the skeleta. Let $Q_i^{(j)}$ denote the j-skeleton of Q_i. Let $Y^{(j)} \subset Y$ denote the disjoint union of the $Q_i^{(j)}$. Let $\mathcal{S}^{(j)} := \{(Y^{(j)} \cap Y_n) \times [0,1) \mid n \in \mathbb{N}\}$. We abbreviate $\mathbb{J}_b^{(j)} := \mathbb{J}_b(Y^{(j)})$ and $\mathbb{J}_{b,f}^{(j)} := \mathbb{J}_b(Y^{(j)})_{\mathcal{S}^{(j)}}$.

Consider first the Karoubi sequence

$$\mathcal{A}(\mathbb{J}_{b,f}^{(0)}) \to \mathcal{A}(\mathbb{J}_b^{(0)}) \to \mathcal{A}(\mathbb{J}_b^{(0)})/\mathcal{S}^{(0)}.$$

Note the following: for fixed R there is n such that pairs of different points in $Y^{(0)}$ with distance less than R must lie in $Q_1 \amalg \cdots \amalg Q_n$. This has the consequence that the quotient category $\mathcal{A}(\mathbb{J}_b^{(0)})/\mathcal{S}^{(0)}$ remains unchanged if we replace the metric $d|_{Y^{(0)}}$ with a metric d_∞ that gives different points always infinite distance, i.e we may assume that each Q_i is just a collection of points. (Use the description in 3.7 of $\mathcal{A}(\mathbb{J}_b^{(0)})/\mathcal{S}^{(0)}$ to see this). Thus, by 4.2 and the long exact sequence 3.8 the isomorphism follows for $j = 0$.

Now consider

$$\begin{array}{ccccc}
\mathcal{A}(\mathbb{J}_{b,f}^{(j)}) & \longrightarrow & \mathcal{A}(\mathbb{J}_{b,f}^{(j+1)}) & \longrightarrow & \mathcal{A}(\mathbb{J}_{b,f}^{(j+1)})/\mathcal{S}^{(j)} \\
{\scriptstyle F_1}\downarrow & & {\scriptstyle F_2}\downarrow & & {\scriptstyle F_3}\downarrow \\
\mathcal{A}(\mathbb{J}_b^{(j)}) & \longrightarrow & \mathcal{A}(\mathbb{J}_b^{(j+1)}) & \longrightarrow & \mathcal{A}(\mathbb{J}_b^{(j+1)})/\{Y^{(j)}\}.
\end{array}$$

The two rows are Karoubi sequences. By induction we may assume that F_1 gives an isomorphism on K-theory. It will therefore suffice to show the same for F_3 (using 3.8). In the two middle categories morphisms can be non trivial between different $j+1$-simplices. However, as we move towards $Y \times \{1\}$ this can only happen close to the boundary of those simplices. More precisely, this can only happen over some $F \in \langle \mathcal{S}^{(j)} \rangle_{\mathbb{J}_{b,f}^{(j+1)}}$ (resp. $F \in \langle \{Y^{(j)}\} \rangle_{\mathbb{J}_b^{(j+1)}}$). In the quotient categories on the right hand side morphisms that factor over objects with support in such F are identified with the trivial morphism (cf. 3.7). Therefore, we can assume that morphisms in the quotient categories are trivial between different $j+1$-simplices. Thus, the quotient categories remain unchanged if we replace each Q_n by the disjoint union of its $j+1$ simplices. In this case the two middle terms have vanishing K-theory by 4.2. By induction we can also in this case assume that F_1 is an K-theory isomorphism and we can conclude that the same holds for F_3. □

The following corollary is the squeezing result we will use in Section 6.

Corollary 4.3. *Let Y, Q_n be as in 4.1. Let $F : \mathcal{B} \to \mathcal{A}(\mathbb{J}_b(Y))$ be a functor of additive categories. Denote by $F_n : \mathcal{B} \to \mathcal{A}(\mathbb{J}_b(Q_n))$ the composition of F with the projection $\mathcal{A}(\mathbb{J}_b(Y)) \to \mathcal{A}(\mathbb{J}_b(Q_n))$. Let $a \in K_i\mathcal{B}$. Then there is N such that $(F_n)_*(a) = 0$ for all $n > N$.*

Remark 4.4. The projection functor $\mathcal{A}(\mathbb{J}_b(Y)) \to \mathcal{A}(\mathbb{J}_b(Q_n))$ in 4.3 is not induced by a map $Y \to Q_i$. It is given by restricting modules M over $Y \times [0,1)$ to $Q_i \times [0,1)$. This gives indeed a well defined functor, since Q_i and $Y - Q_i$ have infinite distance and morphism in $\mathcal{A}(\mathbb{J}_b(Y))$ are therefore always trivial between Q_i and $Y - Q_i$.

Proof of 4.3. Observe first that $\mathcal{A}(\mathbb{J}_b(Y)_{\mathcal{S}}) \cong \operatorname{colim} \mathcal{A}(\mathbb{J}_b(Q_1 \amalg \cdots \amalg Q_n))$. In particular, by 4.1 there is $b \in K_i\mathcal{A}(\mathbb{J}_b(Q_1 \amalg \cdots \amalg Q_N))$ for some N, that maps to $F_*(a) \in \mathcal{A}(\mathbb{J}_b(Y))$. The composition

$$\mathcal{A}(\mathbb{J}_b(Q_1 \amalg \cdots \amalg Q_N)) \to \mathcal{A}(\mathbb{J}_b(Y)) \to \mathcal{A}(\mathbb{J}_b(Q_n))$$

is the trivial functor provided $n > N$ and the claim follows. □

In the remainder of this section we will discuss the relation of Proposition 4.1 and Corollary 4.3 to the classical squeezing of automorphisms of geometric modules, cf. [Qui82, 4.5]. Let X be a proper metric space. Recall that an ϵ-automorphism ϕ is an automorphism in $\mathcal{A}(\mathbb{B}(X))$ such that the support of ϕ and ϕ^{-1} are contained in

$$E_\epsilon = \{(x,y) \mid d(x,y) \leq \epsilon\}.$$

Recall also, that elements in K_1 of an additive category are equivalence classes of automorphisms. Let us denote by $K_1^{(\epsilon)}\mathcal{A}(\mathbb{B}(X))$ the subgroup

$$\{[\phi] \mid \phi \text{ is an } \epsilon\text{-automorphism}\} \subset K_1\mathcal{A}(\mathbb{B}(X)).$$

The classical squeezing result of Quinn says that there is an ϵ such that every ϵ-automorphism can be deformed to an δ-automorphism for every $\delta > 0$. Let us phrase this as follows.

Theorem 4.5. *Let Q be a finite dimensional simplicial complex. Consider the spherical metric on Q. Then there is $\epsilon > 0$ (depending only on the dimension of Q), such that*

$$K_1^{(\delta)}\mathcal{A}(\mathbb{B}(Q)) = K_1^{(\epsilon)}\mathcal{A}(\mathbb{B}(Q)),$$

for all $0 \leq \delta \leq \epsilon$.

This is immediate from [Qui82, 4.5]. See also [Ped00, 3.7]. (Both statements give in fact more information.) Using 4.1 we can give a very simple proof of the following analogue to 4.5. Here $K_1^{(\epsilon)}\mathcal{A}(\mathbb{J}_b(X))$ is the obvious analogue to $K_1^{(\epsilon)}\mathcal{A}(\mathbb{B}(X))$. The difference in the statements can be attributed to the fact that $K_1\mathcal{A}(\mathbb{J}_b(*))$ vanishes, while $K_1\mathcal{A}(\mathbb{B}(*))$ is $K_1\mathcal{A}$.

Corollary 4.6. *Let Q be a finite dimensional simplicial complex. Consider the spherical metric on Q. Then there is $\epsilon > 0$ (depending only on the dimension of Q), such that*

$$K_1^{(\delta)}\mathcal{A}(\mathbb{J}_b(Q)) = 0,$$

for all $0 \leq \delta \leq \epsilon$.

Proof. We proceed by contradiction and assume that there is a sequence Q_n of simplicial complexes of dimension $\leq d$ and $1/n$-automorphisms ϕ_n in $\mathcal{A}(\mathbb{J}_b(Q_n))$ representing nontrivial elements in $K_1\mathcal{A}(\mathbb{J}_b(Q_n))$. Let $Y = Q_1 \amalg Q_2 \amalg \ldots$ be equipped with the metric from 4.1. Then $\phi_1 \oplus \phi_2 \oplus \ldots$ can be viewed as an automorphism in $\mathcal{A}(\mathbb{J}_b(Y))$. It is now a consequence of 4.1 that $[\phi_n] = 0$ for all but finitely many n. □

Remark 4.7. It is a consequence of 4.6 that $K_1^{(\epsilon)}\mathcal{A}(\mathbb{B}(Q))$ is contained in the image of the boundary map

$$\partial : K_2\mathcal{A}(\mathbb{J}_b(Q)_\infty) \to K_1\mathcal{A}(\mathbb{B}(Q))$$

associated to the Karoubi sequence (5.1). This is an analogue to a result of Pedersen [Ped00, 3.6] and he deduces the squeezing theorem from this in [Ped00, 3.7]. Now 4.3 is a higher K-theory version of 4.6. Hence our squeezing result 4.3 can be viewed as generalizing [Ped00, 3.6]) to higher K-theory.

5. The descent principle

Let X be a proper metric space. We denote the boundary map in the long exact sequence associate to the Karoubi sequence (cf. 3.8 and 3.11)

$$\mathcal{A}(\mathbb{B}(X)) \to \mathcal{A}(\mathbb{J}_b(X)) \to \mathcal{A}(\mathbb{J}_b(X))_\infty \tag{5.1}$$

by

$$CA : K_n\mathcal{A}(\mathbb{J}_b(X))_\infty \to K_{n-1}\mathcal{A}(\mathbb{B}(X)). \tag{5.2}$$

In analogy to the coarse Baum-Connes conjecture [Roe93] we can ask for which metric spaces CA is an isomorphism. This question is relevant for the assembly map in algebraic K-theory, because of the following result from [CP95]. For a group Γ let $\mathcal{A}[\Gamma]$ denote the category that has the same objects as $\mathcal{A}$, but $\hom_{\mathcal{A}[\Gamma]}(-,-) = \hom_{\mathcal{A}}(-,-)[\Gamma]$. In particular, $\mathcal{A}[\Gamma]$ is equivalent to the category of finitely generated free $R[\Gamma]$-modules, provided $\mathcal{A}$ is the category of finitely generated free R modules.

Theorem 5.3 (Descent principle)**.** *Let Γ be a discrete group and let $EG \to BG$ be a model for the universal Γ-bundle. Assume that $B\Gamma$ is equipped with a path-length metric inducing a metric on $E\Gamma$. If*

$$CA : K_*\mathcal{A}(\mathbb{J}_b(E\Gamma))_\infty \to K_{*-1}\mathcal{A}(\mathbb{B}(E\Gamma))$$

is an isomorphism, then the assembly map

$$H_*(B\Gamma; \mathbb{K}^{-\infty}\mathcal{A}) \to K_*\mathcal{A}[\Gamma]$$

in algebraic K-theory is split injective, provided that $B\Gamma$ is a finite CW-complex.

Sketch of proof. In [CP95, Section 2] the map CA is constructed as a Γ-equivariant map of spectra with Γ-action $S \to T$ such that restriction to Γ-fix points gives the above assembly map. The assumption implies now that $S \to T$ is a homotopy equivalence. Thus we also get a homotopy equivalence on *homotopy fix points* $S^{h\Gamma} \to T^{h\Gamma}$. Also the map $S^\Gamma \to S^{h\Gamma}$ can be seen to be an isomorphism (using that $B\Gamma$ is finite). At this point another property of K-theory is used: it commutes with infinite products [Car95]. In [CP95] slightly different control conditions are used: instead of a bounded control assumption a compactification of $E\Gamma$ is used. This does not affect the argument. For more details see [CP95]. □

One way to pass from (possibly discrete) metric spaces to topology is given by the Rips-complex. Related is the notion of an anti-Čech system, due to Roe. It gives a systematic way of looking at larger and larger parts of a metric space. This will be a useful tool to study the map CA.

Definition 5.4.

(i) ([Roe93, 3.13]) An *anti-Čech system* for a metric space X is a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X such that there is a sequence of numbers R_n tending to infinity with the property that the diameter of every set in $\mathcal{U}_n$ is bounded by R_n and the Lebesgue number of $\mathcal{U}_n$ is at least R_{n-1}. The nerve $|\mathcal{U}_n|$ of $\mathcal{U}_n$ is the simplicial complex with a vertex for each $U \in \mathcal{U}_n$ and a p-simplex for every $p+1$-tuple $U_0, \ldots, U_p \in \mathcal{U}_n$ having nonempty intersection. For each set U in $\mathcal{U}_n$ we can choose a set in $\mathcal{U}_{n+1}$ containing U. These choices are part of the structure of an anti-Čech system and determine maps

$$|\mathcal{U}_1| \xrightarrow{i_1} |\mathcal{U}_2| \xrightarrow{i_2} |\mathcal{U}_3| \xrightarrow{i_3} \cdots .$$

(ii) ([HR95, Section 3]) Given a locally finite homology theory h_*, the associated functor hx_* on metric spaces (a 'coarse homology theory) is given by

$$hx_*(X) := \operatorname{colim}\ h_*(|\mathcal{U}_n|),$$

where the limit is taken over an anti-Čech system for X. A partition of unity subordinate to the cover $\mathcal{U}_1$ determines a map $f : X \to |\mathcal{U}_1|$ and therefore a *coarsening map*

$$c : h_*(X) \to hx_*(X).$$

Note that the map f is by construction metrically coarse.

We recall a result of Higson and Roe, that will be used in Section 6.

Proposition 5.5 ([HR95, 3.9])**.** *Assume that X is a uniformly contractible, bounded geometry complex. Then the coarsening map $c : h_*(X) \to hx_*(X)$ is an isomorphism.* □

Remark 5.6. For the definition of uniformly contractible, bounded geometry complex see [HR95, Section 3]. Let Q be simplicial complex endowed with the spherical metric. Assume that $Q = E\Gamma$ for some discrete group Γ such that Q/Γ is a finite complex. Then it is not hard to see that Q is a uniformly contractible, bounded geometry complex.

6. Finite asymptotic dimension

Let X be a proper geodesic space. Let us abbreviate

$$h_*(X) = K_*\mathcal{A}(\mathbb{J}_b(X))_\infty.$$

Recall from 3.12 that h_* is a locally finite homology theory. We have the coarsening map $c : h_*(X) \to hx_*(X)$ from 5.4 (ii). The main goal of this section is the proof of the following result.

Theorem 6.1. *Suppose that X is a proper geodesic space of finite asymptotic dimension m. Then CA from (5.2) is an isomorphism, provided the coarsening map c is an isomorphism.*

Recall from [Gro93, p.28] that the asymptotic dimension of X is the smallest integer n such that for any $R > 0$, there exists a cover $\mathcal{U}$ of X with the property that the diameter of members in $\mathcal{U}$ is uniformly bounded and every ball of radius R in X intersects at most $n + 1$ members of $\mathcal{U}$.

A key ingredient in Yu's proof of the coarse Baum-Connes version of 6.1 is [Yu98, Lemma 6.3]. We will need the following reformulation of Yu's lemma.

Lemma 6.2. *Let X be a proper metric space of asymptotic dimension m. Then there is an anti-Čech system $\mathcal{U}_n$ for X and maps $G_n : |\mathcal{U}_1| \to |\mathcal{U}_n|$ such that the following holds*

(i) G_n is metrically coarse, proper and continuous.
(ii) G_n is properly homotopic to $i_1 \circ \cdots \circ i_{n-1}$, where the i_j come from the anti-Čech system $\mathcal{U}_n$ (cf. 5.4 (i)).
(iii) For all $S > 0$ there is $T = T(S) > 0$ such that for $x, y \in |\mathcal{U}_1|$ with $d(x, y) \leq S$ we have $d(G_n(x), G_n(y)) < T/n$.
(iv) The dimension of the $|\mathcal{U}_n|$ is uniformly bounded by m.

Proof. It follows easily from [Yu98, 6.3] that there are $\mathcal{U}'_n, G'_n$ satisfying (i),(ii),(iv) and the following version of (iii).

> For $R > 0$ there is $K = K(R)$ such that for $x, y \in |\mathcal{U}'_1|$ with $d(x, y) \leq R$ we have $d(G'_n(x), G'_n(y)) < 1/R$ if $n > K(R)$.

Let now $j_1, j_2, \ldots$ be a strictly increasing sequence of integers such that $j_R > K(R)$ for all R. Then $\mathcal{U}_n = \mathcal{U}'_{j_n}$ and $G_n = G'_{j_n}$ satisfy our claim. □

Lemma 6.2 allows us to use the squeezing result 4.3 to prove the following vanishing result for elements in algebraic K-theory. This will be the decisive point in the proof of 6.1.

Proposition 6.3. *Let $\mathcal{U}_m, G_m$ be as in* 6.2. *For $a \in K_*\mathcal{A}(\mathbb{J}_b(|\mathcal{U}_1|))$ there is N such that*

$$(G_n)_*(a) = 0 \in K_*\mathcal{A}(\mathbb{J}_b(|\mathcal{U}_n|))$$

for all $n \geq N$.

Proof. Let

$$Y = Q_1 \amalg Q_2 \amalg Q_3 \amalg \ldots$$

where $Q_n = |\mathcal{U}_n|$. We use the path metric on Y that restricts to n times the spherical metric on Q_n. By 6.2 (iii) the functors $(G_n)_* : \mathcal{A}(\mathbb{J}_b(Q_1)) \to \mathcal{A}(\mathbb{J}_b(Q_n))$ can be assembled to a functor $F : \mathcal{A}(\mathbb{J}_b(Q_1)) \to$

$\mathcal{A}(\mathbb{J}_b(Y))$ such that $(G_n)_*$ is the composition of F with the projection $\mathcal{A}(\mathbb{J}_b(Y)) \to \mathcal{A}(\mathbb{J}_b(Q_n))$. The claim follows now from 4.3. $\square$

Proof of 6.1. We use the anti-Čech system $\mathcal{U}_m$ for X and the maps G_n from 6.2. Let $f_1 : X \to |\mathcal{U}_1|$ induce the coarsening map and set $f_m = i_{m-1} \circ \cdots \circ i_1 \circ f_1$. Compare the following long exact sequences, cf. (5.1).

$$\begin{array}{ccccc} K_j\mathcal{A}(\mathbb{J}_b(X)) & \longrightarrow & K_j\mathcal{A}(\mathbb{J}_b(X))_\infty & \xrightarrow{CA} & K_{j-1}\mathcal{A}(\mathbb{B}(X)) \\ \downarrow{\scriptstyle (f_n)_*} & & \downarrow{\scriptstyle (f_n)_*} & & \downarrow{\scriptstyle (f_n)_*} \\ K_j\mathcal{A}(\mathbb{J}_b(|\mathcal{U}_n|)) & \longrightarrow & K_j\mathcal{A}(\mathbb{J}_b(|\mathcal{U}_n|))_\infty & \xrightarrow{CA} & K_{j-1}\mathcal{A}(\mathbb{B}(|\mathcal{U}_n|)) \end{array}$$

Now take the colimit over the maps i_m in the second row. Then the second and third vertical arrows become isomorphisms: The second one is the coarsening map c and an isomorphism by assumption. For the third arrow observe that f_1 and the i_m are bornotopy equivalences (see 3.13) and are therefore isomorphisms. The colimit preserves exactness and using the 5-Lemma we get an isomorphism

$$\text{colim}(f_n)_* : K_j\mathcal{A}(\mathbb{J}_b(X)) \cong \text{colim}\ K_j\mathcal{A}(\mathbb{J}_b(|\mathcal{U}_n|)). \tag{6.4}$$

On the other hand, by 3.19 and 6.2 (ii) we have

$$(i_n)_* = (G_n)_* : K_j\mathcal{A}(\mathbb{J}_b(|\mathcal{U}_1|)) \to K_j\mathcal{A}(\mathbb{J}_b(|\mathcal{U}_n|)).$$

Therefore 6.3 implies that the map in (6.4) is the trivial map. This can only happen if $K_j\mathcal{A}(\mathbb{J}_b(X))$ vanishes and CA is an isomorphism as claimed. $\square$

The above argument depends on the assumption that X is geodesic: it has been pointed out by Wright [Wri02] that f_1 may not be coarse for general X.

We can now prove our main result. If we take for $\mathcal{A}$ the category of finitely generated free R-modules we obtain Theorem 1.1 from the introduction.

Theorem 6.5. *Let Γ be a group that is equipped with a word length metric of finite asymptotic dimension. Assume moreover, that $B\Gamma$ can be realized as finite CW-complex. Then the assembly map*

$$H_*(B\Gamma; \mathbb{K}^{-\infty}\mathcal{A}) \to K_*\mathcal{A}[\Gamma]$$

in algebraic K-theory is split injective.

Proof. Let $B\Gamma$ be realized as a finite simplicial complex. Then the universal cover $E\Gamma$ of $B\Gamma$ is quasi isometric to Γ equipped with any word length metric. In particular $E\Gamma$ has finite asymptotic dimension.

Now 5.5 allows us to apply 6.1 and our claim becomes a consequence of the descent principle 5.3. □

7. L-theory

Results in algebraic K-theory have very often analogues in L-theory. This is also the case for the results of this paper. For an additive category $\mathcal{A}$ with involution Ranicki [Ran92] defines an L-theory spectrum with various decorations. Section 4 of [CP95] contains a review of L-theory that is sufficient for our purposes here. In this sections all additive categories will have an involution, even if this is not specifically mentioned. We will mostly be interested in the functor $\mathbb{L}^{-\infty}$ from additive categories (with involutions) to spectra, cf. [CP95, 4.16]. This functor has properties completely analogous to the properties of K-theory stated in Section 2. For the fibration sequence associated to a Karoubi filtration see [CP95, 4.2]. This allows the extension of results from the previous sections to L-theory. Given an involution on $\mathcal{A}$ it is not hard to construct an involution on $\mathcal{A}(\mathbb{E})$, cf. [CP95, Section 5]. We will denote the homotopy groups of $\mathbb{L}^{-\infty}\mathcal{A}$ by $L_*^{-\infty}(\mathcal{A})$. The squeezing result 4.3 has the following L-theory analogue. The proof is completely parallel to the K-theory case.

Proposition 7.1. *Let $Y, Q_n, \mathcal{B}, F, F_n$ be as in 4.3. Let $a \in L_i^{-\infty}(\mathcal{B})$. Then there is N such that $(F_n)_*(a) = 0$ for all $n > N$.* □

This in turn can be used to prove the following analogue of 6.5. Note that this is a corollary to Yu's result [Yu98] if $\mathcal{A}$ is the category of finitely generated free modules over $\mathbb{Z}$. The assumption on vanishing of lower K-theory guarantees the analogue of [Car95] for L-theory, that is needed for the descent principle 5.3.

Theorem 7.2. *Let Γ be a group that is equipped with a word length metric of finite asymptotic dimension. Assume moreover, that $B\Gamma$ can be realized as finite CW-complex and that $K_{-j}\mathcal{A} = 0$ for all sufficiently large j. Then the assembly map*

$$H_*(B\Gamma; \mathbb{L}^{-\infty}\mathcal{A}) \to L_*^{-\infty}\mathcal{A}[\Gamma]$$

in L-theory is split injective. □

It is explained in [CP95, Section 5] how this can in certain cases be used to derive splitting results with other decorations than $-\infty$.

There is also an analogue to 4.6. Recall that $L_n^h\mathcal{A}$ can be defined as the bordism classes of n-dimensional quadratic Poincaré complexes over $\mathcal{A}$, cf. [Ran89]. We will say that such a Poincaré complex (C, ψ)

over $\mathcal{A}(\mathbb{J}_b(X))$ is *ϵ-controlled* if all involved morphism (in particular the homotopy that proves Poincaré duality) have support contained in

$$\{((x,t),(y,s)) \mid d(x,y) \leq \epsilon\}.$$

Proposition 7.3. *Let Q be a finite dimensional simplicial complex equipped with the spherical metric. Then there is $\epsilon > 0$ (depending only on the dimension of Q), such that every ϵ-controlled n-dimensional quadratic Poincaré complex over $\mathcal{A}(\mathbb{J}_b(X))$ represents an element in the kernel of*

$$L_n^h\mathcal{A}(\mathbb{J}_b(X)) \to L_n^{-\infty}\mathcal{A}(\mathbb{J}_b(X)).$$

□

Remark 7.4. There are two slightly different definitions of n-dimensional quadratic Poincaré complexes (C,ψ) in the literature. In Ranicki's original definition it was required that the chain complex C is concentrated between dimensions 0 and n. This was given up in later definitions. However, above we have to use the original definition: in the proof of 4.6 we viewed an infinite direct sum of automorphisms ϕ_n as an automorphism. The corresponding construction for Poincaré complexes works only if chain complexes involved are concentrated in uniformly bounded dimensions.

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