Group rings and topological rigidity

Arthur Bartels

WWU Münster

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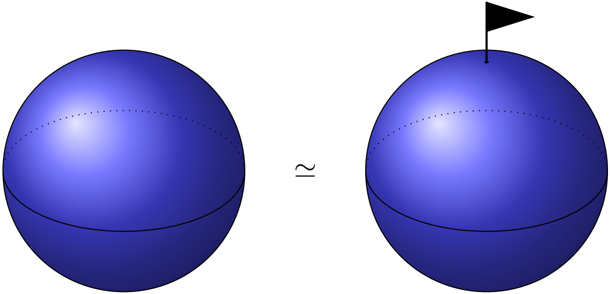
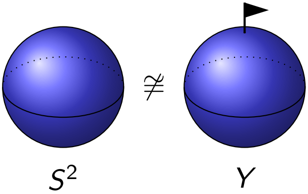
Two topological spaces



$S^2 :=$ 2-sphere $\qquad\qquad$ $Y := S^2 \cup$ little flag

We can deform Y to S^2 by shrinking the flag, so Y and S^2 are homotopy equivalent: $S^2 \simeq Y$.
But they are not homeomorphic: $S^2 \ncong Y$.

Manifolds



In fact S^2 and Y are even locally different:

- Every point in S^2 has a neighborhood that is homeomorphic to $\mathbb{R}^2$.
- The base of the flag in Y has no such neighborhood.

Definition

A compact topological space is called a closed *n*-manifold if every point has a neighborhood homeomorphic to $\mathbb{R}^n$.

- S^2 is a 2-manifold.
- All n-manifolds are locally homeomorphic.

Topological rigidity

In general: $X \simeq Y \not\Longrightarrow X \cong Y$.

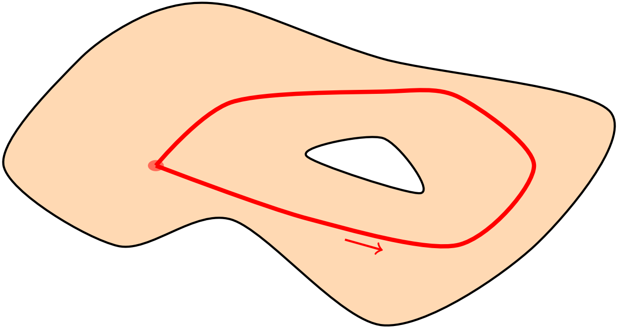
Definition

A closed manifold M is said to be topologically rigid if any other closed manifold N which is homotopy equivalent to M is even homeomorphic to M. (So: $\simeq \Longrightarrow \cong$.)

- The n-sphere S^n is topologically rigid. (Poincaré conjecture.)
- The n-sphere S^n is in general not smoothly rigid. (Exotic spheres.)
- All 1- and all 2-dimensional manifolds are rigid.
- Not all closed manifolds are rigid:
  - Lens spaces are in general not rigid. (Reidemeister torsion.)
  - Products of spheres are in general not rigid. (Rational Pontrjagin classes.)

The fundamental group

The fundamental group $\pi_1(X)$ of a topological space measures how many homotopically different maps $S^1 \to X$ there are.



There are also higher homotopy groups $\pi_n(X)$ that measure how many homotopically different maps $S^n \to X$ there are.

Aspherical manifolds

Definition

A connected topological space X is said to be aspherical if every continuous map $S^n \to X$, $n \geq 2$ is homotopic to a constant map, i.e., if $\pi_n(X) = 0$ for all $n \geq 2$.

- X aspherical $\iff$ universal cover $\widetilde{X}$ is contractible ($\simeq$ pt).
- M closed n-Riemannian manifold of non-positive sectional curvature
  $\implies \widetilde{M} \cong \mathbb{R}^n \quad \implies M$ aspherical.
  (The converse fails.)
- All surfaces of genus ≥ 1 are aspherical.
- For X and Y aspherical we have: $X \simeq Y \iff \pi_1(X) \cong \pi_1(Y)$.
- For any group G, there is an aspherical space BG, whose fundamental group is G.

The Borel conjecture

Conjecture

Closed aspherical manifolds are topologically rigid.

This conjecture holds for example if

- $\dim M \geq 5$ and M is flat (Farrell-Hsiang),
- $\dim M \geq 5$ and M has non-positive sectional curvature (Farrell-Jones),

and

Theorem (B-Lück)

Let M be a closed aspherical manifold of dimension ≥ 5. If $\pi_1(M)$ is Gromov-hyperbolic or a $CAT(0)$-group, then M is topologically rigid.

Knowlege about K- and L-theory of the group ring $\mathbb{Z}[G]$

$+$

Surgery theory

Browder-Novikov-Sullivan-Wall

Kirby-Siebenmann-…-Ranicki-…

$\Downarrow$

Classification of manifolds of dimension ≥ 5

with fundamental group G

Group rings

Let R be a ring and G be a group.
The group ring $R[G]$ is obtained by adding a unit to R for every element of G. Formally

$$R[G] = \left\{ \sum_{\text{finite}} r_i \cdot g_i \;\middle|\; r_i \in R, g_i \in G \right\},$$

multiplication is defined by $(r \cdot g) \cdot (s \cdot h) := (rs) \cdot (gh)$.

Examples

- G infinite cyclic. Then $R[G] \cong R[t, t^{-1}]$.
  This ring contains for $n \in \mathbb{Z}$ the unit t^n.
- G cyclic of order n. Then $R[G] \cong R[t]/(t^n - 1)$.

Units in group rings

- $r \mapsto r \cdot e_G$ defines an inclusion $R \hookrightarrow R[G]$ of rings. Thus $R^\times \subseteq R[G]^\times$.
- $g \mapsto 1_R \cdot g$ defines an inclusion $G \hookrightarrow R[G]^\times$. $((1_R \cdot g)^{-1} = (1_R \cdot g^{-1}).)$
- If $v \in R$ is nilpotent ($v^n = 0$, say) and $g \in G$, then
$$(1 - v \cdot g)^{-1} = 1 + v \cdot g + \cdots + (v \cdot g)^{n-1}.$$
- If $g \in G$ and $g^5 = e_G$, then
$$(1 - g - g^4)^{-1} = (1 - g^2 - g^3).$$

Units of the form $u \cdot g$, with $u \in R^\times$, $g \in G$ are said to be canonical.

Unit question

Let G be a torsion-free group and R be an integral domain.
Are then all units in $R[G]$ canonical?

The Whitehead group

Definition

For a ring define $K_1(R) := \mathrm{GL}(R)_{ab}$.

There is a canonical map $R^\times \to K_1(R)$, that sends a unit $u \in R^\times$ to the class of the $1{\times}1$-matrix whose entry is u.

Definition (Whitehead group)

$\mathrm{Wh}(G) := K_1(\mathbb{Z}[G])/\{[\pm g] \mid g \in G\}$.

Conjecture

If G is torsion-free, then $\mathrm{Wh}(G) = 0$.

Via the s-cobordism theorem the Whitehead group plays a crucial role in topology and in particular in the classifiction of manifolds.

Separation of variables

$$
\begin{array}{ccc}
 & K_*(R[G]) & \\
\swarrow & & \searrow \\
K_*(R) & & H_*(G)
\end{array}
$$

More precisely, there is the assembly map:

$$\alpha^K \colon H_*(BG; \mathbf{K}_R) \to K_*(R[G])$$

Example

If $R = \mathbb{Z}$, $* = 1$ then

$$\begin{aligned} H_1(BG; \mathbf{K}_{\mathbb{Z}}) &\cong H_1(BG; K_0(\mathbb{Z})) \oplus H_0(BG; K_1(\mathbb{Z})) \\ &\cong H_1(BG) \otimes K_0(\mathbb{Z}) \oplus H_0(BG) \otimes K_1(\mathbb{Z}) \\ &\cong G_{ab} \oplus \mathbb{Z}^{\times} \\ &\cong \{[\pm g] \mid g \in G\}. \end{aligned}$$

- In fact, $\mathrm{Wh}(G)$ is the cokernel of the assembly map $\alpha^K \colon H_1(BG; \mathbf{K}_{\mathbb{Z}}) \to K_1(\mathbb{Z}[G])$.
- Since, for example $\mathrm{Wh}(\mathbb{Z}/5\mathbb{Z}) \neq 0$, this assembly map is in general not surjective.

Separation of variables (up to finite subgroups)

$K_*(R[G])$

$K_*(R[F])$ $H_*(G)$

$F \leq G$ finite

More precisely, there is the assembly map relative to the family of finite subgroups:

$$\alpha_{\mathrm{Fin}}^{K} \colon H^{G}{}_{*}(E_{\mathrm{Fin}}G; \mathbf{K}_R) \to K_*(R[G])$$

The Bass-Heller-Swan formula

If $G = \mathbb{Z}$ is infinite cyclic and R is regular, then

$$\begin{aligned} K_1(R[\mathbb{Z}]) &\cong K_0(R) \oplus K_1(R) \\ &\cong H_1(B\mathbb{Z}; \mathbf{K}_R), \end{aligned}$$

but for arbitrary R,

$$K_1(R[\mathbb{Z}]) \cong K_0(R) \oplus K_1(R) \oplus \mathrm{Nil}(R) \oplus \mathrm{Nil}(R).$$

Thus, if $\mathrm{Nil}(R) \neq 0$,
then $\alpha^K_{\mathrm{Fin}} \colon H^{\mathbb{Z}}{}_1(E_{\mathrm{Fin}}\mathbb{Z}; \mathbf{K}_R) \to K_1(R[\mathbb{Z}])$ is not surjective.

Separation of variables (up to virtually cyclic subgroups)

$$K_*(R[G])$$

$$K_*(R[V]) \qquad\qquad H_*(G)$$

$V \leq G$ virtually cyclic

More precisely, there is the assembly map relative to the family of virtually cyclic subgroups:

$$\alpha^K_{\mathrm{VCyc}} \colon H^G{}_*(E_{\mathrm{VCyc}}G; \mathbf{K}_R) \to K_*(R[G])$$

It is no longer easy to find examples for which this map is not an isomorphism.

L-theory

Everything said so far has (more or less) an analog in L-theory.

The Farrell-Jones Conjecture

Let G be a group and R be a ring. Then the assembly maps

$$\begin{aligned} \alpha^K_{\mathsf{VCyc}} &: \quad H^G_*(E_{\mathsf{VCyc}}G; \mathbf{K}_R) \to K_*(R[G]) \\ \alpha^L_{\mathsf{VCyc}} &: \quad H^G_*(E_{\mathsf{VCyc}}G; \mathbf{L}_R) \to L_*(R[G]) \end{aligned}$$

are isomorphisms.

- If G is torsion-free and R is regular, then $\alpha_{\mathsf{VCyc}} \cong \alpha$.
- In particular, the Farrell-Jones Conjecture implies that $\mathrm{Wh}(G) = 0$ for torsion-free G.

The Farrell-Jones Conjecture has applications to the following:

- The Borel conjecture (assuming $\dim M \geq 5$).
- Classification of h-Cobordisms.
- Wall's finiteness obstruction.
- The Novikov Conjecture on the homotopy invariance of higher signatures.
- The Bass Conjecture on the Hattori-Stallings rank of finitely generated projective $R[G]$-modules, for R a commutative integral domain.
- Moody's induction theorem.
- Kaplansky's conjecture on idempotents in group rings.

Kaplansky's conjecture

Conjecture

Let R be an integral domain and G be a torsion-free group. If $p = p^2 \in R[G]$ then $p \in \{0, 1\}$.

Theorem (B-Lück-Reich)

Let F be a skew-field and let G be a group for which α^K_{VCyc} is an isomorphism. Assume that one of the following conditions is satisfied:

- *F is commutative and has characteristic zero and G is torsionfree,*
- *G is torsionfree and sofic,*
- *the characteristic of F is p, all finite subgroups of G are p-groups and G is sofic.*

Then 0 and 1 are the only idempotents in $F[G]$.

Theorem (B-Farrell-Lück-Reich)

- *If G is Gromov-hyperbolic or poly-cyclic, then the Farrell-Jones Conjecture holds for G.*
- *If G is a $CAT(0)$-group or a discrete cocompact subgroup of a virtually connected Lie group then*
  - α^L_{VCyc} *is an isomorphism;*
  - α^K_{VCyc} *is an isomorphism for $* \leq 0$ and surjective for $* = 1$.*

Inheritance properties of the Farrell-Jones Conjecture

- The class of groups for which the Farrell-Jones (with coefficients) holds is closed under taking subgroups, finite direct products, free products and directed colimits.
- There are many constructions of groups with exotic properties which arise as directed colimits of hyperbolic groups. An example are counterexamples to the Baum-Connes Conjecture with coefficients (Gromov, Higson-Lafforgue-Skandalis).

! The Farrell-Jones Conjecture holds for these groups.

Controlled topology

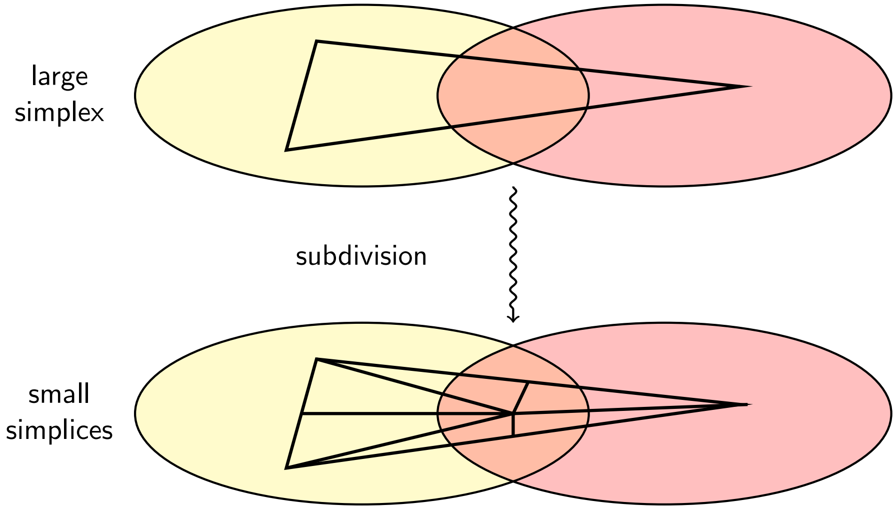
Consider again the assembly map

$$\alpha^K \colon H_*(BG; \mathbf{K}_R) \to K_*(R[G])$$

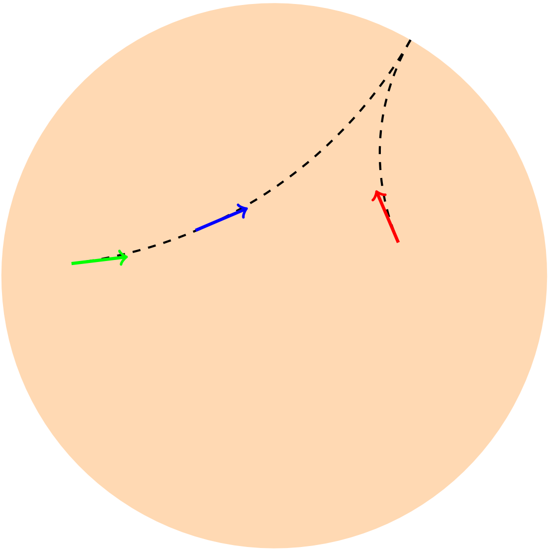
- The homology group $H_*(BG; \mathbf{K}_R)$ is local in BG.
- The group $K_*(R[G])$ is not local in BG. ($G = \pi_1(BG)$.)
- Controlled topology (Quinn-Pedersen-...) can be used to descibe $H_*(BG; \mathbf{K}_R)$ using small (or controlled) cycles, and to describe $K_*(R[G])$ using bounded cycles.
- The assembly map α^K is then described as a 'forget-control'-map.

Need a procedure to gain control.

Digression: singular homology



Dynamics of the geodesic flow on $\mathbb{H}^2$



- Farrell-Jones exploited this dynamic to prove their conjecture for fundamental groups of non-positively curved manifolds.
- Mineyev constructed a flow space for Gromov-hyperbolic groups whose dynamics is exploited in the proof of the Farrell-Jones Conjecture in this case. This flow space is no longer a manifold.
- For $CAT(0)$-groups a different flow space is used. In this situation the flow has weaker contracting properties.
- For poly-cyclic groups, the existence of finite but very large index subgroups is exploited.