ON CROSSED PRODUCT RINGS WITH TWISTED INVOLUTIONS, THEIR MODULE CATEGORIES AND L-THEORY

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Abstract. We study the Farrell-Jones Conjecture with coefficients in an additive G-category with involution. This is a variant of the L-theoretic Farrell-Jones Conjecture which originally deals with group rings with the standard involution. We show that this formulation of the conjecture can be applied to crossed product rings $R * G$ equipped with twisted involutions and automatically implies the a priori more general fibered version.

Introduction

The *Farrell-Jones Conjecture for algebraic L-theory* predicts for a group G and a ring R with involution $r \mapsto \overline{r}$ that the so called *assembly map*

$$\operatorname{asmb}_n^{G,R} \colon H_n^G\big(E_{\mathcal{VC}\mathrm{yc}}(G); \mathbf{L}_R^{\langle -\infty\rangle}\big) \to L_n^{\langle -\infty\rangle}(RG) \tag{0.1}$$

is bijective for all $n \in \mathbb{Z}$. Here the target is the *L-theory* of the group ring RG with the standard involution sending $\sum_{g\in G} r_g \cdot g$ to $\sum_{g\in G} \overline{r_g} \cdot g^{-1}$. This is the group one wants to understand. It is a crucial ingredient in the surgery program for the classification of closed manifolds. The source is a much easier to handle term, namely, a G-homology theory applied to the *the classifying space $E_{\mathcal{VC}yc}(G)$ of the family $\mathcal{VC}yc$ of virtually cyclic subgroups of G*. There is also a K-theory version of the Farrell-Jones Conjecture. The original source for the (Fibered) Farrell-Jones Conjecture is the paper by Farrell-Jones [6, 1.6 on page 257 and 1.7 on page 262]. More information can be found for instance in the survey article [10].

In this paper we study the Farrell-Jones Conjecture with coefficients in an additive G-category with involution. We show that this more general formulation of the conjecture allows to consider instead of the group ring RG the crossed product ring with involution $R *_{c,\tau,w} G$ (see Section 4), which is a generalization of the twisted group ring, and to use more general involutions, for instance the one given by twisting the standard involution with a group homomorphism $w_1 \colon G \to \{\pm 1\}$. The data describing $R *_{c,\tau} G$ and more general involutions are pretty complicated. It turns out that it is convenient to put these into a more general but easier to handle context, where the coefficients are given by an additive G-categories $\mathcal{A}$ with involution (see Definition 4.22).

Definition 0.2 (L-theoretic Farrell-Jones Conjecture)**.** A group G together with an additive G-category with involution $\mathcal{A}$ satisfy the *L-theoretic Farrell-Jones Conjecture with coefficients in $\mathcal{A}$* if the assembly map

$$\operatorname{asmb}_n^{G,\mathcal{A}} \colon H_n^G\big(E_{\mathcal{VC}\mathrm{yc}}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}\big) \to H_n^G\big(\mathrm{pt}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}\big) = L_n^{\langle -\infty\rangle}\left(\textstyle\int_G \mathcal{A}\right). \tag{0.3}$$

induced by the projection $E_{\mathcal{VC}\mathrm{yc}}(G) \to \mathrm{pt}$ is bijective for all $n \in \mathbb{Z}$..

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A group G satisfies the *L-theoretic Farrell-Jones Conjecture* if for any additive G-category with involution $\mathcal{A}$ the *L-theoretic Farrell-Jones Conjecture with coefficients in $\mathcal{A}$* is true.

Here $\int_G \mathcal{A}$ is a certain homotopy colimit which yields an additive category with involution and we use the L-theory associated to an additive category with involution due to Ranicki (see [12], [13] and [14]). The G-homology theory $H_n^G\big(-;\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}\big)$ is briefly recalled in Section 9. If R is a ring with involution, $\mathcal{A}$ is the additive category with involution given by finitely generated free R-modules and we equip $\mathcal{A}$ with the trivial G-action, then the assembly map (0.3) agrees with the one for RG in (0.1) (see Theorem 0.4 below). This general setup is also a very useful framework when one is dealing with categories appearing in controlled topology, which is an important tool for proving the Farrell-Jones Conjecture for certain groups.

Next we state the main results of this paper.

Theorem 0.4. *Suppose that G satisfies the L-theoretic Farrell-Jones Conjecture in the sense of Definition 0.2. Let R be ring with the data (c,τ,w) and $R*_{c,\tau,w}G$ be the associated crossed product ring with involution as explained in Section 4. Then the assembly map*

$$\operatorname{asmb}_n^{G,R_{c,\tau,w}}\colon H_n^G\big(E_{\mathcal{VC}yc}(G);\mathbf{L}_{R,c,\tau,w}^{\langle -\infty\rangle}\big)\to L_n^{\langle -\infty\rangle}(R*_{c,\tau,w}G) \tag{0.5}$$

is bijective.

Here $\mathbf{L}_{R,c,\tau,w}^{\langle -\infty\rangle}$ is a functor from the orbit category $\mathsf{Or}(G)$ to the category of spectra such that $\pi_n\big(\mathbf{L}_{R,c,\tau,w}^{\langle -\infty\rangle}(G/H)\big)$ for $H\subseteq G$ agrees with $L_n^{\langle -\infty\rangle}(R*_{c|_H,\tau|_N,w|_H}H)$.

Another important feature is that in this setting the (unfibered) Farrell-Jones Conjecture does already imply the fibered version.

Definition 0.6 (Fibered L-theoretic Farrell-Jones Conjecture)**.** A group G satisfies the *fibered L-theoretic Farrell-Jones Conjecture* if for any group homomorphism $\phi\colon K\to G$ and additive G-category with involution $\mathcal{A}$ the assembly map

$$\operatorname{asmb}_n^{\phi,\mathcal{A}}\colon H_*^K\big(E_{\phi^*\mathcal{VC}\mathrm{yc}}(G);\mathbf{L}_{\phi^*\mathcal{A}}^{\langle -\infty\rangle}\big)\to L_n^{\langle -\infty\rangle}\left(\textstyle\int_K \phi^*\mathcal{A}\right).$$

is bijective for all $n\in\mathbb{Z}$, where the family $\phi^*\mathcal{VC}\mathrm{yc}$ of subgroups of K consists of subgroups $L\subseteq K$ with $\phi(L)$ virtually cyclic and $\phi^*\mathcal{A}$ is the additive K-category with involution obtained from $\mathcal{A}$ by restriction with ϕ.

Obviously the fibered version for the group G of Definition 0.6 implies the version for the group G of Definition 0.2, take $\phi=\mathrm{id}$ in Definition 0.6. The converse is also true.

Theorem 0.7. *Let G be a group. Then G satisfies the fibered L-theoretic Farrell-Jones Conjecture if and only if G satisfies the L-theoretic Farrell-Jones Conjecture of Definition 0.2.*

A general statement of a *Fibered Isomorphism Conjecture* and the discussion of its inheritance properties under subgroups and colimits of groups can be found in [1, Section 4] (see also [6, Appendix], [7, Theorem 7.1]). These very useful inheritance properties do not hold for the unfibered version of Definition 0.2. The next three corollaries are immediate consequences of Theorem 0.7 and [1, Theorem 3.3, Lemma 4.4, Lemma 4.5 and Lemma 4.6].

Corollary 0.8. *Let $\{G_i\mid i\in I\}$ be a directed system (with not necessarily injective) structure maps and let G be its colimit $\operatorname{colim}_{i\in I}G_i$. Suppose that G_i satisfy the Farrell-Jones Conjecture of Definition 0.2 for every $i\in I$.*

Then G satisfies the Farrell-Jones Conjecture of Definition 0.2.

Corollary 0.9. *Let $1 \to K \to G \xrightarrow{p} Q \to 1$ be an extension of groups. Suppose that the group Q and for any virtually cyclic subgroup $V \subseteq Q$ the group $p^{-1}(V)$ satisfy the Farrell-Jones Conjecture of Definition 0.2.*

Then the group G satisfies the Farrell-Jones Conjecture of Definition 0.2.

Corollary 0.10. *If G satisfies the Farrell-Jones Conjecture of Definition 0.2, then any subgroup $H \subseteq G$ satisfies the Farrell-Jones Conjecture of Definition 0.2.*

Corollary 0.9 and Corollary 0.10 have also been proved in [8].

Remark 0.11. Suppose that the Farrell-Jones Conjecture of Definition 0.2 has been proved for the product of two virtually cyclic subgroups.

Then Corollary 0.9 and Corollary 0.10 imply that $G \times H$ satisfy the Farrell-Jones Conjecture of Definition 0.2 if and only if both G and H satisfy the Farrell-Jones Conjecture of Definition 0.2

It is sometimes useful to have strict structures on $\mathcal{A}$, e.g., the involution is desired to be strict and there should be a (strictly associative) functorial direct sum. The functorial direct sum is actually needed in some proofs in order to guarantee good functoriality properties of certain categories arising from controlled topology. We will show

Theorem 0.12. *The group G satisfies the L-theoretic Farrell-Jones Conjecture of Definition 0.2 if it satisfies the obvious version of it, where one only considers additive G-category with (strictly associative) functorial direct sum and strict involution (see Definition 10.6).*

The Farrell-Jones Conjecture with coefficients (in K- and L-theory) has been introduced in [4]. Our treatment here is more general in that we allow involutions that are not necessarily strict and also deal with twisted involutions on the crossed product ring.

All results mentioned here have obvious analogues for K-theory whose proof is actually easier since one does not have to deal with the involutions.

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The paper is organized as follows:

1. Additive categories with involution

In this section we will review the notion of an additive category with involution as it appears and is used in the literature. This will be one of our main examples.

Let $\mathcal{A}$ be an *additive category*, i.e., a small category $\mathcal{A}$ such that for two objects A and B the morphism set $\operatorname{mor}_{\mathcal{A}}(A, B)$ has the structure of an abelian group and the direct sum $A \oplus B$ of two objects A and B exists and the obvious compatibility conditions hold. A covariant *functor of additive categories* $F\colon \mathcal{A}_0 \to \mathcal{A}_1$ is a covariant functor such that for two objects A and B in $\mathcal{A}_0$ the map $\mathrm{mor}_{\mathcal{A}_0}(A,B) \to \mathrm{mor}_{\mathcal{A}_1}(F(A),F(B))$ sending f to $F(f)$ respects the abelian group structures and $F(A \oplus B)$ is a model for $F(A) \oplus F(B)$. The notion of a contravariant functor of additive categories is defined analogously.

An *involution* (I,E) *on an additive category* $\mathcal{A}$ is contravariant functor

$$I\colon \mathcal{A} \to \mathcal{A} \tag{1.1}$$

of additive categories together with a natural equivalence of such functors

$$E\colon \mathrm{id}_{\mathcal{A}} \to I^2 := I \circ I \tag{1.2}$$

such that for every object A we have the equality of morphisms

$$E(I(A)) \quad = \quad I(E(A)^{-1})\colon I(A) \to I^3(A). \tag{1.3}$$

In the sequel we often write $I(A) = A^*$ and $I(f) = f^*$ for a morphism $f\colon A \to B$ in $\mathcal{A}$. If $I^2 = \mathrm{id}_{\mathcal{A}}$ and $E(A) = \mathrm{id}_A$ for all objects A, then we call $I = (I, \mathrm{id})$ a *strict involution.*

Definition 1.4 (Additive category with involution)**.** An *additive category with involution* is an additive category together with an involution (I,E).

An *additive category with strict involution* is an additive category together with a strict involution I.

The following example is a key example and illustrates why one cannot expect in concrete situation that the involution is strict.

Example 1.5. Let R be a ring. Let R-FGP be the category of finitely generated projective R-modules. This becomes an additive category by the direct sum of R-modules and the elementwise addition of R-homomorphisms.

A *ring with involution* is a ring R together with a map $R \to R,\ r \mapsto \overline{r}$ satisfying $\overline{1} = 1$, $\overline{r+s} = \overline{r} + \overline{s}$ and $\overline{r \cdot s} = \overline{s} \cdot \overline{r}$ for $r,s \in R$. Given a ring with involution R, define an involution I on the additive category R-FGP as follows. Given a finitely generated projective R-module P, let $I(P) = P^*$ be the finitely generated projective $\hom_R(P,R)$,where for $r \in R$ and $f \in \hom_R(P,R)$ the element $rf \in \hom_R(P,R)$ is defined by $rf(x) = f(x) \cdot \overline{r}$ for $x \in P$. The desired natural transformation

$$E\colon \mathrm{id}_{R\text{-}\mathsf{FGP}} \to I^2$$

assigns to a finitely generated projective R-module P the R-isomorphism $P \xrightarrow{\cong} (P^*)^*$ sending $x \in P$ to $\hom_R(P,R) \to R,\ f \mapsto \overline{f(x)}$.

A *functor of additive categories with involution* $(F,T)\colon \mathcal{A} \to \mathcal{B}$ consists of a covariant functor F of the underlying additive categories together with a natural equivalence $T\colon F \circ I_{\mathcal{A}} \to I_{\mathcal{B}} \circ F$ such that for every object A in $\mathcal{A}$ the following diagram commutes

$$\begin{tikzcd} F(A) \arrow[r, "F(E_{\mathcal{A}}(A))"] \arrow[d, "E_{\mathcal{B}}(F(A))"] & F(A^{**}) \arrow[d, "T(A^*)"] \\ F(A)^{**} \arrow[r, "T(A)^*"] & F(A^*)^* \end{tikzcd} \tag{1.6}$$

If $T(A) = \mathrm{id}_A$ for all objects A, then we call F a *strict* functor of additive categories with involution.

The *composite* of functors of additive categories with involution $(F_1,T_1)\colon \mathcal{A}_1 \to \mathcal{A}_2$ and $(F_2,T_2)\colon \mathcal{A}_2 \to \mathcal{A}_3$ is defined to be $(F_2 \circ F_1, T_2 \circ T_1)$, where $F_2 \circ F_1$ is the composite of functors of additive categories and the natural equivalence $T_2 \circ T_1$ assigns to an object $A \in \mathcal{A}_1$ the isomorphism in $\mathcal{A}_3$

$$F_2 \circ F_1 \circ I_{\mathcal{A}_1}(A) \xrightarrow{F_2(T_1(A))} F_2 \circ I_{\mathcal{A}_2} \circ F_1(A) \xrightarrow{T_2(F_1(A))} I_{\mathcal{A}_3} \circ F_2 \circ F_1(A).$$

A *natural transformation* $S\colon (F_1, T_1) \to (F_2, T_2)$ *of functors* $\mathcal{A}_1 \to \mathcal{A}_2$ *of additive categories with involutions* is a natural transformation $S\colon F_1 \to F_2$ of functors of additive categories such that for every object A in $\mathcal{A}$ the following diagram commutes

$$\begin{tikzcd} F_1(I_{\mathcal{A}_1}(A)) \arrow[r, "T_1(A)"] \arrow[d, "S(I_{\mathcal{A}_1}(A))"] & I_{\mathcal{A}_2}(F_1(A)) \\ F_2(I_{\mathcal{A}_1}(A)) \arrow[r, "T_2(A)"] & I_{\mathcal{A}_2}F_2(A)) \arrow[u, "I_{\mathcal{A}_2}(S(A))"'] \end{tikzcd} \tag{1.7}$$

2. Additive categories with weak (G, v)-action

In the sequel G is a group and $v\colon G \to \{\pm 1\}$ is a group homomorphism to the multiplicative group $\{\pm 1\}$. In this section we want to introduce the notion of an additive category with weak (G, v)-action such that the notion of an additive category with involution is the special case of an additive category with weak $(\mathbb{Z}/2, v)$-action for $v\colon \mathbb{Z}/2 \to \{\pm 1\}$ the unique group isomorphism and we can also treat G-actions up to natural equivalence. Notice that this will force us to deal with covariant and contravariant functors simultaneously. The homomorphism v will take care of that.

We call a functor $+1$-variant if it is covariant and -1-variant if it is contravariant. If $F_1\colon \mathcal{C}_0 \to \mathcal{C}_1$ is an ϵ_1-variant functor and $F_2\colon \mathcal{C}_1 \to \mathcal{C}_2$ is an ϵ_2-variant functor, then the composite $F_2 \circ F_1\colon \mathcal{C}_0 \to \mathcal{C}_2$ is $\epsilon_1\epsilon_2$-variant functor. If $f\colon x_0 \to x_1$ is an isomorphism and $\epsilon \in \{\pm 1\}$, then define $f^\epsilon\colon x_0 \to x_1$ to be f if $\epsilon = 1$ and $f^\epsilon\colon x_1 \to x_0$ to be the inverse of f if $\epsilon = -1$. If $F\colon \mathcal{C}_0 \to \mathcal{C}_1$ is ϵ-variant and $f\colon x_0 \xrightarrow{\cong} y_0$ is an isomorphism in $\mathcal{C}_0$, then $F(f)^\epsilon\colon F(x_0) \to F(x_1)$ is an isomorphism in $\mathcal{C}_1$.

Definition 2.1 (Additive category with weak (G, v)-action)**.** Let G be a group together with a group homomorphism $v\colon G \to \{\pm 1\}$. An *additive category with weak* (G, v)*-action* $\mathcal{A}$ is an additive category together with the following data:

- For every $g \in G$ we have a $v(g)$-variant functor $R_g\colon \mathcal{A} \to \mathcal{A}$ of additive categories;
- For every two elements $g, h \in G$ there is a natural equivalence of $v(gh)$-variant functors of additive categories
$$L_{g,h}\colon R_{gh} \to R_h \circ R_g.$$

We require:

(i) $R_e = \mathrm{id}$ for $e \in G$ the unit element;
(ii) $L_{g,e} = L_{e,g} = \mathrm{id}$ for all $g \in G$;
(iii) The following diagram commutes for all $g, h, k \in G$ and objects A in $\mathcal{A}$

$$\begin{tikzcd} R_{ghk}(A) \arrow[r, "L_{gh,k}(A)"] \arrow[d, "L_{g,hk}(A)"] & R_k(R_{gh}(A)) \arrow[d, "R_k(L_{g,h}(A))^{v(k)}"] \\ R_{hk}(R_g(A)) \arrow[r, "L_{h,k}(R_g(A))"] & R_k(R_h(R_g(A))) \end{tikzcd}$$

If for every two elements $g, h \in G$ we have $L_{g,h} = \mathrm{id}$ and in particular $R_{gh} = R_h R_g$, we call $\mathcal{A}$ with these data an *additive category with strict* (G, v)*-action* or briefly a *additive* (G, v)*-category.* If v is trivial, we will omit it from the notation.

Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories with weak (G,v)-action and let $\epsilon \in \{\pm 1\}$. An *ϵ-variant functor $(F,T)\colon \mathcal{A} \to \mathcal{B}$ of additive categories with weak (G,v)-action* is a ϵ-variant functor $F\colon \mathcal{A} \to \mathcal{B}$ of additive categories together with a collection $\{T_g \mid g \in G\}$ of natural transformations of $\epsilon v(g)$-variant functors of additive categories $T_g\colon F \circ R_g^{\mathcal{A}} \to R_g^{\mathcal{B}} \circ F$. We require that for all $g,h \in G$ and all objects A in $\mathcal{A}$ the following diagram commutes

$$\begin{tikzcd}
F(R_{hg}^{\mathcal{A}}(A)) \arrow[r, "F(L_{h,g}^{\mathcal{A}}(A))^{\epsilon}"] \arrow[d, "T_{hg}(A)"] & F(R_g^{\mathcal{A}}(R_h^{\mathcal{A}}(A))) \arrow[r, "T_g(R_h(A))"] & R_g^{\mathcal{B}}(F(R_h^{\mathcal{A}}(A))) \arrow[d, "R_g^{\mathcal{B}}(T_h(A))^{v(g)}"] \\
R_{hg}^{\mathcal{B}}(F(A)) \arrow[rr, "L_{hg}^{\mathcal{B}}(F(A))"] & & R_g^{\mathcal{B}}(R_h^{\mathcal{B}}(F(A)))
\end{tikzcd} \tag{2.2}$$

The composite $(F_2,T_2) \circ (F_1,T_1)\colon \mathcal{A}_1 \to \mathcal{A}_3$ of an ϵ_1-variant functor of additive categories with weak (G,v)-action $(F_1,T_1)\colon \mathcal{A}_1 \to \mathcal{A}_2$ and an ϵ_2-variant functor of additive categories with weak (G,v)-action $(F_2,T_2)\colon \mathcal{A}_2 \to \mathcal{A}_3$ is the $\epsilon_1\epsilon_2$-variant functor of additive categories with weak (G,v)-action whose underlying $\epsilon_1\epsilon_2$-variant functor of additive categories is $F_2 \circ F_1\colon \mathcal{A}_1 \to \mathcal{A}_3$ and whose required natural transformations for $g \in G$ are given for an object A in $\mathcal{A}_1$ by

$$F_2 \circ F_1 \circ R_g^{\mathcal{A}_1}(A) \xrightarrow{F_2((T_2)_g(A))^{\epsilon_2}} F_2 \circ R_g^{\mathcal{A}_2} \circ F_1(A) \xrightarrow{(T_2)_g(F_1(A))} R_g^{\mathcal{A}_3} \circ F_2 \circ F_1(A).$$

A *natural transformation $S\colon (F_1,T_1) \to (F_2,T_2)$ of functors $\mathcal{A}_1 \to \mathcal{A}_2$ of additive categories with weak (G,v)-action* is a natural transformation $S\colon F_1 \to F_2$ of functors of additive categories such that for all $g \in G$ and objects A in $\mathcal{A}_1$ the following diagram commutes

$$\begin{tikzcd}
F_1(R_g^{\mathcal{A}_1}(A)) \arrow[r, "(T_1)_g(A)"] \arrow[d, "S(R_g^{\mathcal{A}_1}(A))"] & R_g^{\mathcal{A}_2}(F_1(A)) \arrow[d, "\left(R_g^{\mathcal{A}_2}(S(A))\right)^{v(g)}"] \\
F_2(R_g^{\mathcal{A}_1}(A)) \arrow[r, "(T_2)_g(A)"] & R_g^{\mathcal{A}_2}(F_2(A))
\end{tikzcd} \tag{2.3}$$

An *ϵ-variant functor $F\colon \mathcal{A} \to \mathcal{B}$ of additive categories with strict (G,v)-action* is an ϵ-variant functor $F\colon \mathcal{A} \to \mathcal{B}$ of additive categories satisfying $F \circ R_g^{\mathcal{A}_1} = R_g^{\mathcal{A}_2} \circ F$ for all $g \in G$. A *natural transformation $S\colon F_1 \to F_2$ of ϵ-variant functors $\mathcal{A}_1 \to \mathcal{A}_2$ of additive categories with strict (G,v)-action* is a natural transformation $S\colon F_1 \to F_2$ of additive categories satisfying $S(R_g^{\mathcal{A}_1}(A)) = R_g^{\mathcal{A}_2}(S(A))^{v(g)}$ for all $g \in G$ and objects A in $\mathcal{S}_1$.

Example 2.4 (Additive categories with involution)**.** Given an additive category A, the structure of an additive category with weak $(\mathbb{Z}/2,v)$-action for $v\colon \mathbb{Z}/2 \to \{\pm 1\}$ the unique group isomorphism is the same as an additive category with involution. Namely, let $t \in \mathbb{Z}/2$ be the generator. Given an involution (I,E) in the sense of Definition 1.4, define the structure of an additive category with weak $(\mathbb{Z}/2,v)$-action in the sense of Definition 2.1 by putting $R_e = \mathrm{id}$, $R_t = I$, $L_{e,e} = L_{t,e} = L_{t,e} = \mathrm{id}$ and $L_{t,t} = E$. Condition (iii) in Definition 2.1 follows from condition (1.3). Given the structure of an additive category with weak $(\mathbb{Z}/2,v)$-action, define the involution (E,I) by $E = R_t$ and $I = L_{t,t}$. The corresponding statement is true for functors of additive categories with weak $(\mathbb{Z}/2,v)$-action and natural transformations between them, where diagram (1.6) corresponds to diagram (2.2).

Analogously we get that the structure of a additive category with strict $(\mathbb{Z}/2,v)$-action is the same as an additive category with strict involution.

3. Making an additive categories with weak (G, v)-action strict

Many interesting examples occur as additive categories with weak (G, v)-action which are not necessarily strict. On the other hand additive categories with strict (G, v)-action are easier to handle. We explain how we can turn an additive category with weak (G, v)-action $\mathcal{A}$ to an additive category with strict (G, v)-action which we will denote by $\mathcal{S}(\mathcal{A})$.

Definition 3.1 ($\mathcal{S}(\mathcal{A})$)**.** An object in $\mathcal{S}(\mathcal{A})$ is a pair (A, g) consisting of an object $A \in \mathcal{A}$ and an element $g \in G$. A morphism (A, g) to (B, h) is a morphism $\phi\colon R_g(A) \to R_h(B)$ in $\mathcal{A}$. The composition of morphisms is given by the one in $\mathcal{A}$. The category $\mathcal{S}(\mathcal{A})$ inherits the structure of an additive category from $\mathcal{A}$ in the obvious way.

Next we define the structure of an additive category with strict (G, v)-action on $\mathcal{S}(\mathcal{A})$. Define for $g \in G$ a functor $R_g^{\mathcal{S}}\colon \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{A})$ of additive categories as follows. Given an object (A, h), define

$$R_g^{\mathcal{S}}(A, h) = (A, hg).$$

Given a morphism $\phi\colon (A, h) \to (B, k)$ define

$$R_g^{\mathcal{S}}(\phi)\colon R_g^{\mathcal{S}}(A, h) = (A, hg) \to R_g^{\mathcal{S}}(B, k) = (B, kg)$$

by the composite of morphisms in $\mathcal{A}$

$$R_{hg}(A) \xrightarrow{L_{h,g}(A)} R_g\left(R_h(A)\right) \xrightarrow{R_g(\phi)} R_g\left(R_k(B)\right) \xrightarrow{L_{k,g}(B)^{-1}} R_{kg}(B).$$

if $v(g) = 1$ and

$$R_g^{\mathcal{S}}(\phi)\colon R_g^{\mathcal{S}}(B, k) = (B, kg) \to R_g^{\mathcal{S}}(A, h) = (A, hg)$$

by the composite of morphisms in $\mathcal{A}$

$$R_{kg}(B) \xrightarrow{L_{k,g}(B)} R_g\left(R_k(B)\right) \xrightarrow{R_g(\phi)} R_g\left(R_h(A)\right) \xrightarrow{L_{h,g}(A)^{-1}} R_{hg}(A)$$

if $v(g) = -1$

A direct computation shows that $R_g^{\mathcal{S}}$ is indeed a functor of additive categories. We conclude $R_e^{\mathcal{S}} = \mathrm{id}_{\mathcal{S}(\mathcal{A})}$ from the conditions $R_e = \mathrm{id}$ and $L_{g,e} = L_{e,g} = \mathrm{id}$. We have to check $R_{g_2}^{\mathcal{S}} \circ R_{g_1}^{\mathcal{S}} = R_{g_1 g_2}^{\mathcal{S}}$. We will do this for simplicity only in the case $v(g_1) = v(g_2) = 1$, the other cases are analogous. Given a morphism $\phi\colon (A, h) \to (B, k)$, the morphism $R_{g_1 g_2}^{\mathcal{S}}(\phi)$ is given by the composite in $\mathcal{A}$

$$\begin{aligned} R_{hg_1g_2}(A) \xrightarrow{L_{h,g_1g_2}(A)} R_{g_1g_2}\left(R_h A\right)\right) \xrightarrow{R_{g_1g_2}(\phi)} R_{g_1g_2}\left(R_k(B)\right) & \\ \xrightarrow{L_{k,g_1g_2}(B)^{-1}} R_{kg_1g_2}(B). & \end{aligned}$$

The morphism $R_{g_2}^{\mathcal{S}} \circ R_{g_1}^{\mathcal{S}}(\phi)$ is given by the composite in $\mathcal{A}$

$$\begin{aligned} R_{hg_1g_2}(A) \xrightarrow{L_{hg_1,g_2}(A)} & R_{g_2}\left(R_{hg_1}(A)\right) \\ & \xrightarrow{R_{g_2}(L_{h,g_1}(A))} R_{g_2}\left(R_{g_1}(R_h(A))\right) \xrightarrow{R_{g_2}\left(R_{g_1}(\phi)\right)} R_{g_2}\left(R_{g_1}(R_k(B))\right) \\ & \qquad \xrightarrow{R_{g_2}\left(L_{k,g_1}(B)^{-1}\right)} R_{g_2}\left(R_{kg_1}(B)\right) \xrightarrow{L_{kg_1,g_2}(B)^{-1}} R_{kg_1g_2}(B). \end{aligned}$$

Next we compute that these two morphisms agree. Because of condition (iii) in Definition 2.1 have

$$\begin{aligned} R_{g_2}(L_{h,g_1}(A)) \circ L_{hg_1,g_2}(A) &= L_{g_1,g_2}(R_h(A)) \circ L_{h,g_1g_2}(A); \\ R_{g_2}(L_{k,g_1}(B)) \circ L_{kg_1,g_2}(B) &= L_{g_1,g_2}(R_k(B)) \circ L_{k,g_1g_2}(B). \end{aligned}$$

Hence it suffices to show that the composite

$$R_{g_1g_2}(R_h(A)) \xrightarrow{L_{g_1,g_2}(R_h(A))} R_{g_2}(R_{g_1}(R_h(A)))$$
$$\xrightarrow{R_{g_2}(R_{g_1}(\phi))} R_{g_2}(R_{g_1}(R_k(B))) \xrightarrow{L_{g_1,g_2}(R_k(B))^{-1}} R_{g_1g_2}(R_k(B))$$

agrees with

$$R_{g_1g_2}(R_k(B)) \xrightarrow{R_{g_1g_2}(\phi)} R_{g_1g_2}(R_k(B)).$$

This follows from the fact that $L_{g_1,g_2}\colon R_{g_1g_2} \to R_{g_2} \circ R_{g_2}$ is a natural equivalence.

Let $(F,T)\colon \mathcal{A} \to \mathcal{B}$ be an ϵ-variant functor of additive categories with weak (G,v)-action. It induces an ϵ-variant functor $\mathcal{S}(F,T)\colon \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{B})$ of additive categories with strict (G,v)-action as follows. For simplicity we will only treat the case $\epsilon = 1$, the other case $\epsilon = -1$ is analogous. The functor $\mathcal{S}(F,T)$ sends an object (A,h) in $\mathcal{S}(\mathcal{A})$ to the object $(F(A),h)$ in $\mathcal{S}(\mathcal{B})$. It sends a morphism $\phi\colon (A,h) \to (B,k)$ in $\mathcal{S}(\mathcal{A})$ which is given by a morphism $\phi\colon R_h^{\mathcal{A}}(A) \to R_k^{\mathcal{A}}(B)$ in $\mathcal{A}$ to the morphism $\mathcal{S}(F,T)(\phi)\colon (F(A),h) \to (F(B),k)$ in $\mathcal{S}(\mathcal{B})$ which is given by the following composite of morphisms in $\mathcal{B}$

$$R_h^{\mathcal{B}}(F(A)) \xrightarrow{T_h(A)^{-1}} F(R_h^{\mathcal{A}}(A)) \xrightarrow{F(\phi)} F(R_k^{\mathcal{A}}(B)) \xrightarrow{T_k(B)} R_k^{\mathcal{A}}(F(B)).$$

We have to show $R_g^{\mathcal{S}(\mathcal{B})} \circ \mathcal{S}(F) = \mathcal{S}(F) \circ R_g^{\mathcal{S}(\mathcal{A})}$ for every $g \in G$. We only treat the case $v(g) = 1$. This is obvious on objects since both composites send an object (A,h) to $(F(A),hg)$. Let $\phi\colon (A,h) \to (B,k)$ be a morphism in $\mathcal{S}(\mathcal{A})$ which is given by a morphism $\phi\colon R_h^{\mathcal{A}}(A) \to R_k^{\mathcal{A}}(B)$ in $\mathcal{A}$. Then $R_g^{\mathcal{S}(\mathcal{B})} \circ \mathcal{S}(F)(\phi)$ is the morphism $(F(A),hg) \to (F(B),kg)$ in $\mathcal{S}(\mathcal{B})$ which is given by the composite in $\mathcal{B}$

$$R_{hg}^{\mathcal{B}}(F(A)) \xrightarrow{L_{h,g}^{\mathcal{B}}(F(A))} R_g^{\mathcal{B}}(R_h^{\mathcal{B}}(F(A))) \xrightarrow{R_g^{\mathcal{B}}(T_h(A)^{-1})} R_g^{\mathcal{B}}(F(R_h^{\mathcal{A}}(A)))$$
$$\xrightarrow{R_g^{\mathcal{B}}(F(\phi))} R_g^{\mathcal{B}}(F(R_k^{\mathcal{A}}(B))) \xrightarrow{R_g^{\mathcal{B}}(T_k(B))} R_g^{\mathcal{B}}(R_k^{\mathcal{B}}(F(B))) \xrightarrow{L_{k,g}^{\mathcal{B}}(F(B))^{-1}} R_{kg}^{\mathcal{B}}(F(B))$$

and $\mathcal{S}(F) \circ R_g^{\mathcal{S}(\mathcal{A})}(\phi)$ is the morphism $(F(A),hg) \to (F(B),kg)$ in $\mathcal{S}(\mathcal{B})$ which is given by the composite in $\mathcal{B}$

$$R_{hg}^{\mathcal{B}}(F(A)) \xrightarrow{T_{hg}(A)^{-1}} F(R_{hg}^{\mathcal{A}}(A)) \xrightarrow{F(L_{h,g}^{\mathcal{A}}(A))} F(R_g^{\mathcal{A}}(R_h^{\mathcal{A}}(A)))$$
$$\xrightarrow{F(R_g^{\mathcal{A}}(\phi))} F(R_g^{\mathcal{A}}(R_k^{\mathcal{A}}(B))) \xrightarrow{F(L_{k,g}^{\mathcal{A}}(B)^{-1})} F(R_{kg}^{\mathcal{A}}(B)) \xrightarrow{T_{kg}(A)} R_{kg}^{\mathcal{B}}(F(B)).$$

Since $T_g\colon F \circ R_g^{\mathcal{A}} \to R_g^{\mathcal{B}} \circ F$ is a natural transformation, the following diagram commutes

$$\begin{tikzcd}
F(R_g^{\mathcal{A}}(R_h^{\mathcal{A}}(A))) \arrow[r, "F(R_g^{\mathcal{A}}(\phi))"] \arrow[d, "T_g(R_h^{\mathcal{A}}(A))"] & F(R_g^{\mathcal{A}}(R_k^{\mathcal{A}}(B))) \arrow[d, "T_g(R_k^{\mathcal{A}}(B))"] \\
R_g^{\mathcal{B}}(F(R_h^{\mathcal{A}}(A))) \arrow[r, "R_g^{\mathcal{B}}(F(\phi))"] & R_g^{\mathcal{B}}(F(R_k^{\mathcal{A}}(B)))
\end{tikzcd}$$

Hence it suffices to show that the composite

$$R_{hg}^{\mathcal{B}}(F(A)) \xrightarrow{L_{h,g}^{\mathcal{B}}(F(A))} R_g^{\mathcal{B}}(R_h^{\mathcal{B}}(F(A))) \xrightarrow{R_g^{\mathcal{B}}(T_h(A)^{-1})} R_g^{\mathcal{B}}(F(R_h^{\mathcal{A}}(A)))$$
$$\xrightarrow{T_g(R_h^{\mathcal{A}}(A))^{-1}} F(R_g^{\mathcal{A}}(R_h^{\mathcal{A}}(A)))$$

agrees with the composite

$$R_{hg}^{\mathcal{B}}(F(A)) \xrightarrow{T_{hg}(A)^{-1}} F(R_{hg}^{\mathcal{A}}(A)) \xrightarrow{F(L_{h,g}^{\mathcal{A}}(A))} F(R_g^{\mathcal{A}}(R_h^{\mathcal{A}}(A)))$$

and that the composite

$$F(R_g^{\mathcal{A}}(R_k^{\mathcal{A}}(B))) \xrightarrow{T_g(R_k^{\mathcal{A}}(B))} R_g^{\mathcal{B}}(F(R_k^{\mathcal{A}}(B))) \xrightarrow{R_g^{\mathcal{B}}(T_k(B))} R_g^{\mathcal{B}}(R_k^{\mathcal{B}}(F(B))) \xrightarrow{L_{k,g}^{\mathcal{B}}(F(B))^{-1}} R_{kg}^{\mathcal{B}}(F(B))$$

agrees with the composite

$$F(R_g^{\mathcal{A}}(R_k^{\mathcal{A}}(B))) \xrightarrow{F(L_{k,g}^{\mathcal{A}}(B)^{-1})} F(R_{kg}^{\mathcal{A}}(B)) \xrightarrow{T_{kg}(A)} R_{kg}^{\mathcal{B}}(F(B)).$$

This follows in both cases from the commutativity of the diagram (2.2). This finishes the proof that $\mathcal{S}(F,T)$ is a functor of additive categories with strict (G,v)-action.

Let $S\colon (F_1,T_1) \to (F_2,T_2)$ be a natural transformation of ϵ-variant functors of additive categories with weak (G,v)-action $(F_1,T_1)\colon \mathcal{A}_1 \to \mathcal{A}_2$ and $(F_2,T_2)\colon \mathcal{A}_1 \to \mathcal{A}_2$. It induces a natural transformation $\mathcal{S}(S)\colon \mathcal{S}(F_1,T_1) \to \mathcal{S}(F_2,T_2)$ of functors of additive categories with strict (G,v)-action as follows. Given an object (A,g) in $\mathcal{S}(\mathcal{A})$, we have to specify a morphism in $\mathcal{S}(\mathcal{A})$

$$\mathcal{S}(S)(A)\colon \mathcal{S}(F_1,T_1)(A,g) = (F_1(A),g) \to \mathcal{S}(F_2,T_2)(A,g) = (F_2(A),g),$$

i.e., a morphism $R_g^{\mathcal{A}}(F_1(A)) \to R_g^{\mathcal{A}}(F_2(A))$ in $\mathcal{A}$. We take $R_g^{\mathcal{A}}(S(A))^{v(g)}$. We leave it to the reader to check that this is indeed a natural transformation of ϵ-variant functors of additive categories with strict (G,v)-action using the commutativity of the diagram (2.3).

Let (G,v)-$\mathsf{Add\text{-}Cat}^{\epsilon}$ be the category of additive categories with weak (G,v)-action with ϵ-variant functors as morphisms and let strict-(G,v)-$\mathsf{Add\text{-}Cat}^{\epsilon}$ be the category of additive categories with strict (G,v)-action with ϵ-variant functors as morphisms. There is the forgetful functor

$$\mathrm{forget}\colon \mathsf{strict}\text{-}(G,v)\text{-}\mathsf{Add\text{-}Cat}^{\epsilon} \to (G,v)\text{-}\mathsf{Add\text{-}Cat}^{\epsilon}$$

and the functor constructed above

$$\mathcal{S}\colon (G,v)\text{-}\mathsf{Add\text{-}Cat}^{\epsilon} \to \mathsf{strict}\text{-}(G,v)\text{-}\mathsf{Add\text{-}Cat}^{\epsilon}.$$

Lemma 3.2. *(i) We obtain an adjoint pair of functors* $(\mathcal{S}, \mathrm{forget})$.

(ii) We get for every additive category $\mathcal{A}$ with weak (G,v)-action a functor of additive categories with weak (G,v)-action

$$P_{\mathcal{A}}\colon \mathcal{A} \to \mathrm{forget}(\mathcal{S}(\mathcal{A}))$$

which is natural in $\mathcal{A}$ and whose underlying functor of additive categories is an equivalence of additive categories.

Proof. We will only treat the case, where v is trivial and $\epsilon = 1$, the other cases are analogous.

(i) We have to construct for any additive category $\mathcal{A}$ with weak G-action and any additive category $\mathcal{B}$ with strict G-action to one another inverse maps

$$\alpha\colon \mathrm{func}_{\mathsf{strict}\text{-}(G,v)\text{-}\mathsf{Add\text{-}Cat}}(\mathcal{S}(\mathcal{A}),\mathcal{B}) \to \mathrm{func}_{(G,v)\text{-}\mathsf{Add\text{-}Cat}}(\mathcal{A},\mathrm{forget}(\mathcal{B}))$$

and

$$\beta\colon \mathrm{func}_{(G,v)\text{-}\mathsf{Add\text{-}Cat}}(\mathcal{A},\mathrm{forget}(\mathcal{B})) \to \mathrm{func}_{\mathsf{strict}\text{-}(G,v)\text{-}\mathsf{Add\text{-}Cat}}(\mathcal{S}(\mathcal{A}),\mathcal{B}).$$

For a functor of additive categories with strict G-action $F\colon \mathcal{S}(\mathcal{A}) \to \mathcal{B}$, the functor of additive categories with weak G-action, $\alpha(F)\colon \mathcal{A} \to \mathrm{forget}(\mathcal{B})$ is given by a functor $\alpha(F)\colon \mathcal{A} \to \mathrm{forget}(\mathcal{B})$ of additive categories and a collection of natural transformations $T(F)_g\colon \alpha(F) \circ R_g^{\mathcal{A}} \to R_g^{\mathcal{B}} \circ \alpha(F)$ satisfying certain compatibility conditions. We first explain the functor $\alpha(F)\colon \mathcal{A} \to \mathrm{forget}(\mathcal{B})$. It sends a morphism

$f\colon A \to B$ in $\mathcal{A}$ to the morphism in $\mathcal{B}$ which is given by the value of F on the morphism $(A,e) \to (B,e)$ in $\mathcal{S}(\mathcal{A})$ defined by f. For $g \in G$ the transformation $T(F)_g$ evaluated at an object A in $\mathcal{A}$ is the morphism

$$\alpha(F)(R_g^{\mathcal{A}}(A)) = F(R_g^{\mathcal{A}}(A),e) \to R_g^{\mathcal{B}}(\alpha(F)(A)) = R_g^{\mathcal{B}}(F(A,e))$$

defined as follows. It is given by the composite of the image under F of the morphism $(R_g^{\mathcal{A}}(A),e) \to R_g^{\mathcal{S}(\mathcal{A})}(A,e) = (A,g)$ in $\mathcal{S}(\mathcal{A})$ which is defined by the identity morphism $\mathrm{id}\colon R_g^{\mathcal{A}}(A) \to R_g^{\mathcal{A}}(A)$ in $\mathcal{A}$ and the identity $F(R_g^{\mathcal{S}(\mathcal{A})}(A,e)) = R_g^{\mathcal{B}}(F(A,e))$ which comes from the assumption that F is a functor of strict additive G-categories. One easily checks that $\alpha(F)$ satisfies condition (2.2) since it is satisfied for F.

Given a functor of additive categories with weak G-action $(F,T)\colon \mathcal{A} \to \mathrm{forget}(\mathcal{B})$, the functor of additive categories with strict G-action $\beta(F,T)\colon \mathcal{S}(\mathcal{A}) \to \mathcal{B}$ is defined as follows. It sends an object (A,h) to $R_h^{\mathcal{B}}(F(A))$. A morphism $\phi\colon (A,h) \to (B,k)$ in $\mathcal{S}(A)$ which is given by a morphism $\phi\colon R_h^{\mathcal{A}}A \to R_k^{\mathcal{A}}B$ in $\mathcal{A}$ is sent to morphism in $\mathcal{B}$ given by the composite

$$R_h^{\mathcal{B}}(F(A)) \xrightarrow{T_h(A)^{-1}} F(R_h^{\mathcal{A}}(A)) \xrightarrow{F(\phi)} F(R_k^{\mathcal{A}}(B)) \xrightarrow{T_k(B)} R_k^{\mathcal{B}}(F(B)).$$

The following calculation shows that $\beta(F,T)$ is indeed a functor of additive categories with strict G-action. Given an element $g \in G$ the morphism $R_g^{\mathcal{S}(\mathcal{A})}(\phi)\colon (A,hg) \to (B,kg)$ in $\mathcal{S}(\mathcal{A})$ is given by the morphism in $\mathcal{A}$

$$R_{hg}^{\mathcal{A}}(A) \xrightarrow{L_{hg}^{\mathcal{A}}(A)} R_g^{\mathcal{A}}(R_h^{\mathcal{A}}(A)) \xrightarrow{R_g^{\mathcal{A}}(\phi)} R_g^{\mathcal{A}}(R_k^{\mathcal{A}}(B)) \xrightarrow{L_{k,g}^{\mathcal{A}}(B)^{-1}} R_{kg}^{\mathcal{A}}(B).$$

Hence $\beta(F,T) \circ R_g^{\mathcal{S}(\mathcal{A})}(\phi)$ is the morphism in $\mathcal{B}$ given by the composite

$$\begin{aligned} R_{hg}^{\mathcal{B}}(F(A)) &\xrightarrow{T_{hg}(A)^{-1}} F(R_{hg}^{\mathcal{A}}(A)) \xrightarrow{F(L_{hg}^{\mathcal{A}}(A))} F(R_g^{\mathcal{A}}(R_h^{\mathcal{A}}(A))) \\ &\xrightarrow{F(R_g^{\mathcal{A}}(\phi))} F(R_g^{\mathcal{A}}(R_k^{\mathcal{A}}(B))) \xrightarrow{F(L_{k,g}^{\mathcal{A}}(B)^{-1})} F(R_{kg}^{\mathcal{A}}(B))) \\ &\xrightarrow{T_{kg}(B)} R_{kg}^{\mathcal{A}}(F(B))). \end{aligned}$$

The morphism $R_g^{\mathcal{B}}(B) \circ \beta(F,T)(\phi)$ in $\mathcal{B}$ is given by the composite

$$\begin{aligned} R_g^{\mathcal{B}}(R_h^{\mathcal{B}}(F(A))) &\xrightarrow{R_g^{\mathcal{B}}(T_h(A)^{-1})} R_g^{\mathcal{B}}(F(R_h^{\mathcal{A}}(A))) \xrightarrow{R_g^{\mathcal{B}}(F(\phi))} R_g^{\mathcal{B}}(F(R_k^{\mathcal{A}}(B))) \\ &\xrightarrow{R_g^{\mathcal{B}}(T_k(B))} R_g^{\mathcal{B}}(R_k^{\mathcal{B}}(F(B))). \end{aligned}$$

Since $\mathcal{B}$ is a additive category with strict G-action by assumption, we have the equalities $R_g^{\mathcal{B}}(R_h^{\mathcal{B}}(F(A))) = R_{hg}^{\mathcal{B}}(F(A))$ and $R_g^{\mathcal{B}}(R_k^{\mathcal{B}}(B)) = R_{kg}^{\mathcal{A}}(F(B)))$. We must show that under these identifications the two morphisms in $\mathcal{B}$ above agree. Since T_g is a natural transformation $F \circ R_g^{\mathcal{A}} \to R_g^{\mathcal{B}} \circ F$, the following diagram commutes

$$\begin{array}{ccc} F(R_g^{\mathcal{A}}(R_h^{\mathcal{A}}(A))) & \xrightarrow{F(R_g^{\mathcal{A}}(\phi))} & F(R_g^{\mathcal{A}}(R_k^{\mathcal{A}}(B))) \\ \downarrow{\scriptstyle T_g(R_h^{\mathcal{A}}(A))} & & \downarrow{\scriptstyle T_g(R_k^{\mathcal{A}}(B))} \\ R_g^{\mathcal{B}}(F(R_h^{\mathcal{A}}(A))) & \xrightarrow{R_g^{\mathcal{B}}(F(\phi))} & R_g^{\mathcal{B}}(F(R_k^{\mathcal{A}}(B))) \end{array}$$

Hence it suffices to show that the composites

$$R_g^{\mathcal{B}}(R_h^{\mathcal{B}}(F(A))) = R_{hg}^{\mathcal{B}}(F(A)) \xrightarrow{T_{hg}(A)^{-1}} F(R_{hg}^{\mathcal{A}}(A)) \xrightarrow{F(L_{hg}^{\mathcal{A}}(A))} F(R_g^{\mathcal{A}}(R_h^{\mathcal{A}}(A)))$$

and

$$R_g^{\mathcal{B}}(R_h^{\mathcal{B}}(F(A))) \xrightarrow{R_g^{\mathcal{B}}(T_h(A)^{-1})} R_g^{\mathcal{B}}(F(R_h^{\mathcal{A}}(A))) \xrightarrow{T_g(R_h^{\mathcal{A}}(A))^{-1}} F(R_g^{\mathcal{A}}(R_h^{\mathcal{A}}(A)))$$

agree and that the composites

$$F(R_g^{\mathcal{A}}(R_k^{\mathcal{A}}(B))) \xrightarrow{F(L_{k,g}^{\mathcal{A}}(B)^{-1})} F(R_{kg}^{\mathcal{A}}(B))) \xrightarrow{T_{kg}(B)} R_{kg}^{\mathcal{A}}(F(B))) = R_g^{\mathcal{B}}(R_k^{\mathcal{B}}(F(B)))$$

and

$$F(R_g^{\mathcal{A}}(R_k^{\mathcal{A}}(B))) \xrightarrow{T_g(R_k^{\mathcal{A}}(B))} R_g^{\mathcal{B}}(F(R_k^{\mathcal{A}}(B))) \xrightarrow{R_g^{\mathcal{B}}(T_k(B))} R_g^{\mathcal{B}}(R_k^{\mathcal{B}}(F(B))$$

agree. This follows in both cases from the commutativity of the diagram (2.2). This finishes the proof that $\beta(F)$ is a functor of additive categories with strict G-action. We leave it to the reader to check that both composites $\beta \circ \alpha$ and $\alpha \circ \beta$ are the identity.

(ii) The in $\mathcal{A}$ natural functor of additive categories with weak (G, v)-action

$$P_{\mathcal{A}} \colon \mathcal{A} \to \operatorname{forget}(\mathcal{S}(\mathcal{A}))$$

is defined to be the adjoint of the identity functor $\operatorname{id} \colon \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{A})$. Explicitly it sends an object A to the object (A, e) and a morphism $\phi \colon A \to B$ to the morphism $(A, e) \to (B, e)$ given by ϕ. Obviously $P_{\mathcal{A}}$ induces a bijection $\operatorname{mor}_{\mathcal{A}}(A, B) \to \operatorname{mor}_{\mathcal{S}(\mathcal{A})}(P_{\mathcal{A}}(A), P_{\mathcal{A}}(B))$ and for every object (A, g) in $\mathcal{S}(\mathcal{A})$ there is an object in the image of $P_{\mathcal{A}}$ which is isomorphic to (A, g), namely, $P_{\mathcal{A}}(R_g^{\mathcal{A}}(A)) = (R_g^{\mathcal{A}}(A), e)$. Hence the underlying functor $R_{\mathcal{A}}$ is an equivalence of additive categories. □

4. Crossed product rings and involutions

In this subsection we will introduce the concept of a crossed product ring. Let R be a ring and let G be a group. Let $e \in G$ be the unit in G and denote by 1 the multiplicative unit in R. Suppose that we are given maps of sets

$$c \colon G \to \operatorname{aut}(R), \quad g \mapsto c_g; \tag{4.1}$$

$$\tau \colon G \times G \to R^{\times}. \tag{4.2}$$

We require

$$c_{\tau(g,g')} \circ c_{gg'} = c_g \circ c_{g'}; \tag{4.3}$$

$$\tau(g, g') \cdot \tau(gg', g'') = c_g(\tau(g', g'')) \cdot \tau(g, g'g''); \tag{4.4}$$

$$c_e = \operatorname{id}_R; \tag{4.5}$$

$$\tau(e, g) = 1; \tag{4.6}$$

$$\tau(g, e) = 1, \tag{4.7}$$

for $g, g', g'' \in G$, where $c_{\tau(g,g')} \colon R \to R$ is conjugation with $\tau(g, g')$, i.e., it sends r to $\tau(g, g')r\tau(g, g')^{-1}$. Let $R * G = R *_{c,\tau} G$ be the free R-module with the set G as basis. It becomes a ring with the following multiplication

$$\left(\sum_{g \in G} \lambda_g g\right) \cdot \left(\sum_{h \in G} \mu_h h\right) = \sum_{g \in G} \left(\sum_{\substack{g', g'' \in G, \\ g'g'' = g}} \lambda_{g'} c_{g'}(\mu_{g''}) \tau(g', g'')\right) g.$$

This multiplication is uniquely determined by the properties $g \cdot r = c_g(r) \cdot g$ and $g \cdot g' = \tau(g, g') \cdot (gg')$. The conditions (4.3) and (4.4) relating c and τ are equivalent to the condition that this multiplication is associative. The other conditions (4.5), (4.6) and (4.7) are equivalent to the condition that the element $1 \cdot e$ is a multiplicative unit in $R * G$. We call

$$R * G = R *_{c,\tau} G \tag{4.8}$$

the *crossed product* of R and G with respect to c and τ.

Example 4.9. Let $1 \to H \xrightarrow{i} G \xrightarrow{p} Q \to 1$ be an extension of groups. Let $s\colon Q \to G$ be a map satisfying $p \circ s = \mathrm{id}$ and $s(e) = e$. We do not require s to be a group homomorphism. Define $c\colon Q \to \mathrm{aut}(RH)$ by $c_q(\sum_{h \in H} \lambda_h h) = \sum_{h \in H} \lambda_h s(q) h s(q)^{-1}$. Define $\tau\colon Q \times Q \to (RH)^{\times}$ by $\tau(q,q') = s(q)s(q')s(qq')^{-1}$. Then we obtain a ring isomorphism $RH * Q \to RG$ by sending $\sum_{q \in Q} \lambda_q q$ to $\sum_{q \in Q} i(\lambda_q)s(q)$, where $i\colon RH \to RG$ is the ring homomorphism induced by $i\colon H \to G$. Notice that s is a group homomorphism if and only if τ is constant with value $1 \in R$.

Next we consider the additive category with involution R-FGP of finitely generated projective R-modules. For $g \in G$ we obtain a functor $\mathrm{res}_{c_g}\colon R\text{-}\mathsf{FGP} \to R\text{-}\mathsf{FGP}$ by restriction with the ring automorphism $c_g\colon R \to R$. Define natural transformation of functors $R\text{-}\mathsf{FGP} \to R\text{-}\mathsf{FGP}$

$$L_{\tau(g,h)}\colon \mathrm{res}_{c_{gh}} \to \mathrm{res}_{c_h} \circ \mathrm{res}_{c_g}$$

by assigning to a finitely generated projective R-module the R-homomorphism

$$\mathrm{res}_{c_{gh}} P \to \mathrm{res}_{c_h} \mathrm{res}_{c_g} P, \quad p \mapsto \tau(g,h)p.$$

This is indeed a R-linear map because of the following computation for $r \in R$ and $p \in P$

$$\begin{aligned}\tau(g,h)c_{gh}(r) = \tau(g,h)c_{gh}(r)\tau(g,h)^{-1}\tau(g,h) &= c_{\tau(g,h)} \circ c_{gh}(r)\tau(g,h)\\ &= c_g \circ c_h(r)\tau(g,h).\end{aligned}$$

Lemma 4.10. *We get from the collections $\{\mathrm{res}_{c_g} \mid g \in G\}$ and $\{L_{\tau(g,h)} \mid g,h \in G\}$ the structure of an additive category with weak G-action on R-FGP.*

Proof. Condition (4.4) implies that for every finitely generated projective R-module the composites

$$\mathrm{res}_{c_{gg'g''}} P \xrightarrow{L_{\tau(g,g'g'')}} \mathrm{res}_{c_{g'g''}} \mathrm{res}_{c_g} P \xrightarrow{L_{c_g(\tau(g',g''))}} \mathrm{res}_{c_{g''}} \mathrm{res}_{C_{g'}} \mathrm{res}_{c_g} P$$

and

$$\mathrm{res}_{c_{gg'g''}} P \xrightarrow{L_{\tau(gg',g'')}} \mathrm{res}_{c_{g''}} \mathrm{res}_{c_{gg'}} P \xrightarrow{L_{\tau(g,g')}} \mathrm{res}_{c_{g''}} \mathrm{res}_{C_{g'}} \mathrm{res}_{c_g} P$$

agree. This takes care of condition (iii) in Definition 2.1. We conclude $(\mathrm{res}_{c(e)} = \mathrm{id}$, $L_{\tau(g,e)} = \mathrm{id}$ and $L_{\tau(e,g)} = \mathrm{id}$ for all $g \in G$ from (4.5), (4.6) and (4.7). □

Because of Lemma 4.10 we obtain two additive categories with strict G-action from the constructions of Section 3

(4.11) $$R\text{-}\mathsf{FGP}_{c,\tau} \quad := \quad \mathcal{S}(R\text{-}\mathsf{FGP});$$

¿From now on assume that R comes with an involution of rings $r \mapsto \overline{r}$. We want to consider extensions of it to an involution on $R * G$. Suppose that additionally we are given a map

(4.12) $$w\colon G \to R.$$

We require the following conditions for $g, h \in G$ and $r \in R$

(4.13) $$w(e) \quad = \quad 1;$$

(4.14) $$w(gh) \quad = \quad w(h)c_{h^{-1}}(w(g))\tau(h^{-1},g^{-1})c_{(gh)^{-1}}\left(\overline{\tau(g,h)}\right)^{-1};$$

(4.15) $$\overline{w(g)} \quad = \quad w(g)c_g^{-1}\left(\tau(g,g^{-1})\overline{\tau(g,g^{-1})}^{-1}\right);$$

(4.16) $$\overline{c_g(r)} \quad = \quad c_g\left(\left(w(g)\tau(g^{-1},g)\right)^{-1}\overline{r}\left(w(g)\tau(g^{-1},g)\right)\right).$$

We claim that there is precisely one involution on $R * G$ with the properties that it extends the involution on R and sends g to $w(g) \cdot g^{-1}$. The candidate for the involution is

$$\overline{\sum_{g \in G} r_g \cdot g} \quad := \quad \sum_{g \in G} w(g) c_{g^{-1}}(\overline{r_g}) \cdot g^{-1}. \tag{4.17}$$

One easily concludes from the requirements and the axioms of an involution that this is the only possible formula for such an involution. Namely,

$$\begin{aligned}\overline{\sum_{g \in G} r_g \cdot g} = \sum_{g \in G} \overline{r_g \cdot g} = \sum_{g \in G} \overline{g} \cdot \overline{r_g} = \sum_{g \in G} w(g) \cdot g^{-1} \cdot \overline{r_g} \\ = \sum_{g \in G} w(g) \cdot \left(g^{-1} \cdot \overline{r_g} \cdot g\right) \cdot g^{-1} = \sum_{g \in G} (w(g) c_{g^{-1}}(\overline{r_g}) \cdot g^{-1}.\end{aligned}$$

Before we explain that this definition indeed satisfies the axioms for an involution, we show that the conditions about w above are necessary for this map to be an involution on $R * G$. So assume that we have an involution on $R * G$ that extends the involution on R and sends g to $w(g) \cdot g^{-1}$ for a given map $w \colon G \to R$. Denote by 1 the multiplicative unit in both R and $R * G$. From

$$1 \cdot e = 1 = \overline{1} = \overline{1 \cdot e} = w(e) \cdot e$$

we conclude (4.13). The equality

$$\begin{aligned} w(gh) c_{(gh)^{-1}} \left(\overline{\tau(g,h)}\right) \cdot (gh)^{-1} = \overline{\tau(g,h) \cdot gh} = \overline{g \cdot h} \\ = \overline{h} \cdot \overline{g} = w(h) \cdot h^{-1} \cdot w(g) \cdot g^{-1} = w(h) \left(h^{-1} \cdot w(g) \cdot h\right) \cdot h^{-1} \cdot g^{-1} \\ = w(h) c_{h^{-1}}(w(g)) \tau(h^{-1}, g^{-1}) \cdot (gh)^{-1}\end{aligned}$$

implies (4.14). If we take $h = g^{-1}$ in (4.14) and use (4.13), we get

$$1 = w(e) = w(gg^{-1}) = w(g^{-1}) c_g(w(g)) \tau(g, g^{-1}) \overline{\tau(g, g^{-1})}^{-1}. \tag{4.18}$$

This implies that for all $g \in G$ the element $w(g)$ is a unit in R with inverse

$$w(g)^{-1} = c_{g^{-1}}(w(g^{-1})) \tau(g^{-1}, g) \overline{\tau(g^{-1}, g)}^{-1}.$$

The equality

$$\begin{aligned} g = \overline{\overline{g}} = \overline{w(g) \cdot g^{-1}} = \overline{g^{-1}} \cdot \overline{w(g)} = w(g^{-1}) \cdot g \cdot \overline{w(g)} \\ = w(g^{-1}) \cdot \left(g \cdot \overline{w(g)} \cdot g^{-1}\right) \cdot g = w(g^{-1}) c_g \left(\overline{w(g)}\right) \cdot g\end{aligned}$$

together with (4.18) implies

$$w(g^{-1}) c_g \left(\overline{w(g)}\right) = 1 = w(g^{-1}) c_g(w(g)) \tau(g, g^{-1}) \overline{\tau(g, g^{-1})}^{-1}.$$

If we multiply this equation with $w(g^{-1})^{-1}$ and apply the inverse c_g^{-1} of c_g, we derive condition (4.15). The equality

$$\begin{aligned} \overline{r} \cdot w(g) \cdot g^{-1} = \overline{r} \cdot \overline{g} = \overline{g \cdot r} = \overline{(g \cdot r \cdot g^{-1}) \cdot g} = \overline{c_g(r) \cdot g} = \overline{g} \cdot \overline{c_g(r)} \\ = w(g) \cdot g^{-1} \cdot \overline{c_g(r)} = w(g) \cdot \left(g^{-1} \cdot \overline{c_g(r)} \cdot g\right) \cdot g^{-1} = w(g) \cdot c_{g^{-1}} \left(\overline{c_g(r)}\right) \cdot g^{-1}\end{aligned}$$

implies that for all $g \in G$ and $r \in R$ we have $\overline{r} \cdot w(g) = w(g) \cdot c_{g^{-1}} \left(\overline{c_g(r)}\right)$ and hence

$$\overline{c_g(r)} = c_{g^{-1}}^{-1} \left(w(g)^{-1} \overline{r} w(g)\right).$$

¿From the relation (4.3) we conclude $c_{\tau(g^{-1},g)} = c_{g^{-1}} \circ c_g$ and hence $c_{g^{-1}}^{-1} = c_g \circ c_{\tau(g^{-1},g)}^{-1}$. Now condition (4.16) follows.

Finally we show that the conditions (4.13), (4.14), (4.15) and (4.16) on w do imply that we get an involution of rings on $R*G$ by the formula (4.17). Obviously this formula is compatible with the additive structure on $R*G$ and sends 1 to 1. In order to show that it is an involution and compatible with the multiplicative structure we have to show $\overline{g\cdot h}=\overline{h}\cdot\overline{g}$, $\overline{rs}=\overline{s}\cdot\overline{r}$, $\overline{r\cdot g}=\overline{g}\cdot\overline{r}$, $\overline{g\cdot r}=\overline{r}\cdot\overline{g}$, $\overline{\overline{r}}=r$ and $\overline{\overline{g}}=g$ for $r,s\in R$ and $g,h\in G$. We get $\overline{rs}=\overline{s}\cdot\overline{r}$ and $\overline{\overline{r}}=r$ from the fact that we start with an involution on R. The other equations follow from the proofs above that (4.17) is the only possible candidate and that the conditions about w are necessary for the existence of the desired involution on $R*G$, just read the various equations and implications backwards. We will denote the resulting ring with involution by

$$R *_{c,\tau,w} G. \tag{4.19}$$

Example 4.20. Suppose that we are in the situation of Example 4.9. Suppose that we are additionally given a group homomorphism $w_1\colon G\to\operatorname{cent}(R)^\times$ to the abelian group of invertible central elements in R satisfying $\overline{w_1(g)}=w_1(g)$ for all $g\in G$. The w_1-twisted involution on RG is defined by $\overline{\sum_{g\in G} r_g\cdot g}=\sum_{g\in G}\overline{r_g}w_1(g)\cdot g^{-1}$. It extends the $w_1|_H$-involution on RH. We obtain an involution on $RH*Q$ if we conjugate the w_1-twisted involution with the isomorphism $RH*Q\xrightarrow{\cong}RG$ which we have introduced in Example 4.9. This involution on $RH*Q$ sends $q\in Q$ to the element $w_1(s(q))\tau(q^{-1},q)^{-1}\cdot q^{-1}$ because of the following calculation in RG for $q\in Q$

$$\begin{aligned}\overline{s(q)}&=w_1(s(q))\cdot s(q)^{-1}=w_1(s(q))\cdot s(q)^{-1}\cdot s(q^{-1})^{-1}\cdot s(q^{-1})\\&=w_1(s(q))\cdot\big(s(q^{-1})\cdot s(q)\big)^{-1}\cdot s(q^{-1})=w_1(s(q))\cdot\big(\tau(q^{-1},q)s(q^{-1}q)\big)^{-1}\cdot s(q^{-1})\\&=w_1(s(q))\tau(q^{-1},q)^{-1}\cdot s(q^{-1}).\end{aligned}$$

Define

$$w\colon Q\to RH,\quad q\mapsto w_1(s(q))\tau(q^{-1},q)^{-1}.$$

Then w satisfies the conditions (4.13), (4.14), (4.15) and (4.16) and the involution on $RH*Q$ determined by w corresponds under the isomorphism $RH*Q\xrightarrow{\cong}RG$ to the w_1-twisted involution on RG.

Let

$$t_g\colon \operatorname{res}_{c_g}\circ I_{R\text{-}\mathsf{FGP}}\to I_{R\text{-}\mathsf{FGP}}\circ\operatorname{res}_{c_g} \tag{4.21}$$

be the natural transformation which assigns to a finitely generated projective R-module P the R-isomorphism $t_g(P)\colon \operatorname{res}_{c_g}P^*\to(\operatorname{res}_{c_g}P)^*$ which sends the R-linear map $f\colon P\to R$ to the R-linear map

$$t_g(P)(f)\colon \operatorname{res}_{c_g}P\to R,\quad p\mapsto c_g^{-1}(f(p))\big(w(g)\tau(g^{-1},g)\big)^{-1}.$$

We firstly check that $t_g(P)(f)\colon \operatorname{res}_{c_g}P\to R$ is R-linear by the following computation

$$\begin{aligned}t_g(P)(f)(c_g(r)p)&=c_g^{-1}\left(f(c_g(r)p)\right)\big(w(g)\tau(g^{-1},g)\big)^{-1}\\&=c_g^{-1}\left(c_g(r)f(p)\right)\big(w(g)\tau(g^{-1},g)\big)^{-1}\\&=c_g^{-1}\left(c_g(r)\right)c_g^{-1}(f(p))\big(w(g)\tau(g^{-1},g)\big)^{-1}\\&=rc_g^{-1}(f(p))\big(w(g)\tau(g^{-1},g)\big)^{-1}\\&=rt_g(P)(f)(p).\end{aligned}$$

Finally we check that $t_g(P)\colon \operatorname{res}_{c_g} P^* \to (\operatorname{res}_{c_g} P)^*$ is R-linear by the following calculation for $f \in P^*$ and $p \in P$

$$\begin{aligned}
&t_g(P)\left((c_g(r)f)\right)(p)\\
&\quad= c_g^{-1}\left((c_g(r)f)(p)\right)\left(w(g)\tau(g^{-1},g)\right)^{-1}\\
&\quad= c_g^{-1}\left(f(p)\overline{c_g(r)}\right)\left(w(g)\tau(g^{-1},g)\right)^{-1}\\
&\quad= c_g^{-1}(f(p))c_g^{-1}(\overline{c_g(r)})\left(w(g)\tau(g^{-1},g)\right)^{-1}\\
&\quad= c_g^{-1}(f(p))c_g^{-1}\left(c_g\left(\left(w(g)\tau(g^{-1},g)\right)^{-1}\overline{r}\left(w(g)\tau(g^{-1},g)\right)\right)\right)\left(w(g)\tau(g^{-1},g)\right)^{-1}\\
&\quad= c_g^{-1}(f(p)\left(w(g)\tau(g^{-1},g)\right)^{-1}\overline{r}\left(w(g)\tau(g^{-1},g)\right)\left(w(g)\tau(g^{-1},g)\right)^{-1}\\
&\quad= c_g^{-1}(f(p)\left(w(g)\tau(g^{-1},g)\right)^{-1}\overline{r}\\
&\quad= t_g(P)(f)(p)\overline{r}\\
&\quad= \left(rt_g(P)\right)(f)(p).
\end{aligned}$$

Definition 4.22. An *additive G-category with involution $\mathcal{A}$* is an additive G-category, which is the same as an additive category with strict G-action (see Definition 2.1), together with an involution (I, E) of additive categories (see (1.1) and (1.2)) with the following properties: $I\colon \mathcal{A} \to \mathcal{A}$ is a contravariant functor of additive G-categories, i.e., $R_g \circ I = I \circ R_g$ for all $g \in G$, and $E\colon \mathrm{id}_{\mathcal{A}} \to I \circ I$ is a natural transformation of functors of additive G-categories, i.e., for every $g \in G$ and every object A in $\mathcal{A}$ the morphisms $E(R_g(A))$ and $R_g(E(A))$ from $R_g(A)$ to $I^2 \circ (R_g(A) = R_g \circ I^2(A)$ agree.

Lemma 4.23. *The additive category with strict G-action $R\text{-}\mathsf{FGP}_{c,\tau}$ of* (4.11) *inherits the structure of an additive G-category with involution in the sense of Definition 4.22.*

Proof. We firstly show that

$$I_{R\text{-}\mathsf{FGP}}\colon R\text{-}\mathsf{FGP} \to R\text{-}\mathsf{FGP}$$

together with the collection of the $\{t_g^{-1}\colon I_{R\text{-}\mathsf{FGP}} \circ \operatorname{res}_{c_g} \to \operatorname{res}_{c_g} \circ I_{R\text{-}\mathsf{FGP}} \mid g \in G\}$ (see (4.21)) is a contravariant functor of additive categories with weak G-action. We have to verify that the diagram (2.2) commutes. This is equivalent to show for every finitely generated projective R-module P and $g, h \in G$ that the following diagram commutes

$$\begin{tikzcd}
\operatorname{res}_{c_{gh}} P^* \arrow[rr, "t_{gh}(P)"] \arrow[d, "L_{\tau(g,h)}(P^*)"] & & (\operatorname{res}_{c_{gh}} P)^* \\
\operatorname{res}_{c_h}\operatorname{res}_{c_g} P^* \arrow[r, "\operatorname{res}_{c_h} t_g(P)"] & \operatorname{res}_{c_h}(\operatorname{res}_{c_g} P)^* \arrow[r, "t_h(\operatorname{res}_g P)"] & (\operatorname{res}_{c_h}\operatorname{res}_{c_g} P)^* \arrow[u, "L_{\tau(g,h)}(P)^*"']
\end{tikzcd}$$

We start with an element $f\colon P \to R$ in the left upper corner. Its image under the upper horizontal arrow is $p \mapsto c_{gh}^{-1}(f(p))\left(w(gh)\tau((gh)^{-1},gh)\right)^{-1}$. Next we list successively how its image looks like if we go in the anticlockwise direction from the left upper corner to the right upper corner. We first get $p \mapsto f(p)\overline{\tau(g,h)}$. After the second map we get $p \mapsto c_g^{-1}\left(f(p)\overline{\tau(g,h)}\right)\left(w(g)\tau(g^{-1},g)\right)^{-1}$. After applying the third map we obtain $p \mapsto c_h^{-1}\left(c_g^{-1}\left(f(p)\overline{\tau(g,h)}\right)\left(w(g)\tau(g^{-1},g)\right)^{-1}\right)\left(w(h)\tau(h^{-1},h)\right)^{-1}$. Finally we get $p \mapsto c_h^{-1}\left(c_g^{-1}\left(f(\tau(g,h)p)\overline{\tau(g,h)}\right)\left(w(g)\tau(g^{-1},g)\right)^{-1}\right)\left(w(h)\tau(h^{-1},h)\right)^{-1}$. Since f lies in P^*, we have $f(\tau(g,h)p) = \tau(g,h)f(p)$. Hence it suffices to show for all $r \in R$

$$c_h^{-1}\left(c_g^{-1}\left(\tau(g,h)r\overline{\tau(g,h)}\right)\left(w(g)\tau(g^{-1},g)\right)^{-1}\right)\left(w(h)\tau(h^{-1},h)\right)^{-1} = c_{gh}^{-1}(r)\left(w(gh)\tau((gh)^{-1},gh)\right)^{-1}.$$

(Notice that now f has been eliminated.) By applying c_{gh} we see that this is equivalent to showing

$$c_{gh}\left(c_h^{-1}\left(c_g^{-1}\left(\tau(g,h)r\overline{\tau(g,h)}\right)\right)\right) = rc_{gh}\left(\left(w(gh)\tau((gh)^{-1},gh)\right)^{-1}\left(w(h)\tau(h^{-1},h)\right)c_h^{-1}\left(w(g)\tau(g^{-1},g)\right)\right).$$

¿From the relation (4.3) we conclude that $c_{gh}\circ c_{h^{-1}}\circ c_{g^{-1}}(s) = \tau(g,h)^{-1}s\tau(g,h)$ holds for all $s \in R$. Hence it remains to show

$$\tau(g,h)^{-1}\left(\tau(g,h)r\overline{\tau(g,h)}\right)\tau(g,h) = rc_{gh}\left(\left(w(gh)\tau((gh)^{-1},gh)\right)^{-1}w(h)\tau(h^{-1},h)c_h^{-1}\left(w(g)\tau(g^{-1},g)\right)\right).$$

This reduces to proving for $g,h \in G$

$$\overline{\tau(g,h)}\tau(g,h) = c_{gh}\left(\tau((gh)^{-1},gh)^{-1}w(gh)^{-1}w(h)\tau(h^{-1},h)c_h^{-1}\left(w(g)\tau(g^{-1},g)\right)\right).$$

(Notice that now r has been eliminated.) By inserting condition (4.14) and the conclusions $c_{\tau(h^{-1},h)}\circ c_h^{-1} = c_{h^{-1}}$ and $c_{\tau((gh)^{-1},gh)}\circ c_{gh}^{-1} = c_{(gh)^{-1}}$ from conditions (4.3) and (4.5) we get

$$\begin{aligned}
& w(gh)^{-1}w(h)\tau(h^{-1},h)c_h^{-1}\left(w(g)\tau(g^{-1},g)\right) \\
= \ & \left(w(h)c_{h^{-1}}(w(g))\tau(h^{-1},g^{-1})c_{(gh)^{-1}}\left(\overline{\tau(g,h)}\right)^{-1}\right)^{-1} w(h) \\
& \qquad \tau(h^{-1},h)c_h^{-1}\left(w(g)\tau(g^{-1},g)\right)\tau(h^{-1},h)^{-1}\tau(h^{-1},h) \\
= \ & c_{(gh)^{-1}}\left(\overline{\tau(g,h)}\right)\tau(h^{-1},g^{-1})^{-1}c_{h^{-1}}(w(g))^{-1}w(h)^{-1}w(h) \\
& \qquad c_{h^{-1}}\left(w(g)\tau(g^{-1},g)\right)\tau(h^{-1},h) \\
= \ & c_{(gh)^{-1}}\left(\overline{\tau(g,h)}\right)\tau(h^{-1},g^{-1})^{-1}c_{h^{-1}}(w(g))^{-1}c_{h^{-1}}(w(g)) \\
& \qquad c_{h^{-1}}\left(\tau(g^{-1},g)\right)\tau(h^{-1},h) \\
= \ & c_{(gh)^{-1}}\left(\overline{\tau(g,h)}\right)\tau(h^{-1},g^{-1})^{-1}c_{h^{-1}}\left(\tau(g^{-1},g)\right)\tau(h^{-1},h) \\
= \ & \tau((gh)^{-1},gh)c_{gh}^{-1}\left(\overline{\tau(g,h)}\right)\tau((gh)^{-1},gh)^{-1}\tau(h^{-1},g^{-1})^{-1} \\
& \qquad c_{h^{-1}}\left(\tau(g^{-1},g)\right)\tau(h^{-1},h).
\end{aligned}$$

This implies

$$\begin{aligned}
& c_{gh}\left(\tau((gh)^{-1},gh)^{-1}w(gh)^{-1}w(h)\tau(h^{-1},h)c_h^{-1}\left(w(g)\tau(g^{-1},g)\right)\right) \\
= \ & c_{gh}\left(\tau((gh)^{-1},gh)^{-1}\tau((gh)^{-1},gh)c_{gh}^{-1}\left(\overline{\tau(g,h)}\right)\tau((gh)^{-1},gh)^{-1}\right. \\
& \qquad \left.\tau(h^{-1},g^{-1})^{-1}c_{h^{-1}}\left(\tau(g^{-1},g)\right)\tau(h^{-1},h)\right) \\
= \ & \overline{\tau(g,h)}c_{gh}\left(\tau((gh)^{-1},gh)^{-1}\tau(h^{-1},g^{-1})^{-1}c_{h^{-1}}\left(\tau(g^{-1},g)\right)\tau(h^{-1},h)\right).
\end{aligned}$$

Hence it remains to show

$$\tau(g,h) = c_{gh}\left(\tau((gh)^{-1},gh)^{-1}\tau(h^{-1},g^{-1})^{-1}c_{h^{-1}}\left(\tau(g^{-1},g)\right)\tau(h^{-1},h)\right).$$

(Notice that we have eliminated any expression involving the involution.) ¿From condition (4.3), (4.4) and (4.5) we conclude

$$\begin{aligned}
\tau(h^{-1},g^{-1})\tau((gh)^{-1},g) &= c_{h^{-1}}(\tau(g^{-1},g));\\
\tau(gh)^{-1},g)\tau(h^{-1},h) &= c_{(gh)^{-1}}(\tau(g,h))\tau((gh)^{-1},gh);\\
c_{gh}^{-1} &= c_{\tau((gh)^{-1},gh)^{-1}}\circ c_{(gh)^{-1}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\tau((gh)^{-1},gh)^{-1}\tau(h^{-1},g^{-1})^{-1}c_{h^{-1}}\left(\tau(g^{-1},g)\right)\tau(h^{-1},h)\\
&= \tau((gh)^{-1},gh)^{-1}\tau(h^{-1},g^{-1})^{-1}\tau(h^{-1},g^{-1})\tau((gh)^{-1},g)\tau(h^{-1},h)\\
&= \tau((gh)^{-1},gh)^{-1}\tau((gh)^{-1},g)\tau(h^{-1},h)\\
&= \tau((gh)^{-1},gh)^{-1}c_{(gh)^{-1}}(\tau(g,h))\tau((gh)^{-1},gh)\\
&= c_{gh}^{-1}(\tau(g,h)).
\end{aligned}$$

This finishes the proof of the commutativity of the diagram (2.2).

Next we show that $E_{R\text{-}\mathsf{FGP}}\colon \mathrm{id}_{R\text{-}\mathsf{FGP}} \to I_{R\text{-}\mathsf{FGP}}\circ I_{R\text{-}\mathsf{FGP}}$ is a natural transformation of contravariant functors of additive categories with weak G-action. We have to show that the diagram (2.3) commutes. This is equivalent to show for every finitely generated projective R-module P the following diagram commutes

$$\begin{tikzcd}
\mathrm{res}_{c_g} P \arrow[r, "E_{R\text{-}\mathsf{FGP}}(\mathrm{res}_{c_g} P)"] \arrow[d, "\mathrm{res}_{c_g} E_{R\text{-}\mathsf{FGP}}(P)"] & (\mathrm{res}_{c_g} P)^{**} \arrow[d, "t_g(P)^*"] \\
\mathrm{res}_{c_g}(P^{**}) \arrow[r, "t_g(P^*)"] & (\mathrm{res}_{c_g} P^*)^*
\end{tikzcd}$$

We start with an element $p \in P$ in the left upper corner. It is sent under the left vertical arrow to the element given by $f \mapsto f(p)$. The image of this element under the lower horizontal is given by $f \mapsto c_g^{-1}(f(p))\left(w(g)\tau(g^{-1},g)\right)^{-1}$. The image of $p \in P$ under the upper horizontal arrow is $f \mapsto f(p)$. The image of this element under the right vertical arrow sends f to $f\circ t_g(P)(p) = c_g^{-1}(f(p))\left(w(g)\tau(g^{-1},g)\right)^{-1}$.

¿From the naturality of the construction of the additive category with strict G-action $R\text{-}\mathsf{FGP}_{c,\tau} := \mathcal{S}(R\text{-}\mathsf{FGP})$ (see Section 3) we conclude that $(I_{R\text{-}\mathsf{FGP}}, \{t_g \mid g \in G\})$ induces a functor of additive categories with strict G-action

$$I_{R\text{-}\mathsf{FGP}_{c,\tau}}\colon R\text{-}\mathsf{FGP}_{c,\tau} \to R\text{-}\mathsf{FGP}_{c,\tau}$$

and $E_{R\text{-}\mathsf{FGP}}$ induces a natural transformation of functors of additive categories with strict G-action

$$E_{R\text{-}\mathsf{FGP}_{c,\tau}}\colon \mathrm{id}_{R\text{-}\mathsf{FGP}} \to I_{R\text{-}\mathsf{FGP}_{c,\tau}}\circ I_{R\text{-}\mathsf{FGP}_{c,\tau}}.$$

It remains to prove that condition (1.3) holds for $(I_{R\text{-}\mathsf{FGP}_{c,\tau}}, E_{R\text{-}\mathsf{FGP}_{c,\tau}})$. But this follows easily from the fact that condition (1.3) holds for $(I_{R\text{-}\mathsf{FGP}}, E_{R\text{-}\mathsf{FGP}})$. □

The additive G-category with involution constructed in Lemma 4.23 will be denoted in the sequel by

$$R\text{-}\mathsf{FGP}_{c,\tau,w}. \tag{4.24}$$

5. Connected groupoids and additive categories

Groupoids are always to be understood to be small. A groupoid is called *connected* if for two objects x and y there exists a morphism $f\colon x \to y$. Let $\mathcal{G}$ be a connected groupoid. Let $\mathsf{Add\text{-}Cat}$ be the category of small additive categories.

Given a contravariant functor $F\colon \mathcal{G} \to \mathsf{Add\text{-}Cat}$, we define a new small additive category, which we call its *homotopy colimit* (see for instance [15])

$$\int_{\mathcal{G}} F \tag{5.1}$$

as follows. An object is a pair (x, A) consisting of an object x in $\mathcal{G}$ and an object A in $F(x)$. A morphism in $\int_{\mathcal{G}} F$ from (x, A) to (y, B) is a formal sum

$$\sum_{f \in \operatorname{mor}_{\mathcal{G}}(x,y)} f \cdot \phi_f$$

where $\phi_f\colon A \to F(f)(B)$ is a morphism in $F(x)$ and only finitely many coefficients ϕ_f are different from zero. The composition of a morphism $\sum_{f \in \operatorname{mor}_{\mathcal{G}}(x,y)} f \cdot \phi_f\colon (x, A) \to (y, B)$ and a morphism $\sum_{g \in \operatorname{mor}_{\mathcal{G}}(y,z)} g \cdot \phi_g\colon (y, B) \to (z, C)$ is given by the formula

$$\sum_{h \in \operatorname{mor}_{\mathcal{G}}(x,z)} h \cdot \left(\sum_{\substack{f \in \operatorname{mor}_{\mathcal{G}}(x,y) \\ g \in \operatorname{mor}_{\mathcal{G}}(y,z) \\ h = g \circ f}} F(f)(\psi_g) \circ \phi_f \right).$$

The decisive special case is

$$(g \cdot \psi) \circ (f \cdot \phi) = (g \circ f) \cdot (F(f)(\psi) \circ \phi).$$

The $\mathbb{Z}$-module structure on $\operatorname{mor}_{\int_{\mathcal{G}} F}(x, y)$ is given by

$$\left(\sum_{f \in \operatorname{mor}_{\mathcal{G}}(x,y)} f \cdot \phi_f \right) + \left(\sum_{f \in \operatorname{mor}(\mathcal{G})} f \cdot \psi_f \right) \quad = \quad \sum_{f \in \operatorname{mor}_{\mathcal{G}}(x,y)} f \cdot (\phi_f + \psi_f).$$

A model for the sum of two objects (x, A) and (x, B) is $(x, A \oplus B)$ if $A \oplus B$ is a model for the sum of A and B in $F(x)$. Since $\mathcal{G}$ is by assumption connected, we can choose for any object (y, B) in $\int_{\mathcal{G}} F$ and any object x in $\mathcal{G}$ an isomorphism $f\colon x \to y$ and the objects $(x, F(f)(B))$ and (y, B) in $\int_{\mathcal{G}} F$ are isomorphic. Namely $f \cdot \operatorname{id}_{F(f)(B)}$ is an isomorphism $(x, F(f)(B) \xrightarrow{\cong} (y, B)$ whose inverse is $f^{-1} \cdot \operatorname{id}_B \cdot$ Hence the direct sum of two arbitrary objects (x, A) and (y, B) exists in $\int_{\mathcal{G}} F$.

Notice that we need the connectedness of $\mathcal{G}$ only to show the existence of a direct sum. This will become important later when we deal with non-connected groupoids.

This construction is functorial in F. Namely, if $S\colon F_0 \to F_1$ is a natural transformation of contravariant functors $\mathcal{G} \to \mathsf{Add\text{-}Cat}$, then it induces a functor

$$\int_{\mathcal{G}} S\colon \int_{\mathcal{G}} F_0 \to \int_{\mathcal{G}} F_1 \tag{5.2}$$

of additive categories as follows. It sends an object (x, A) in $\int_{\mathcal{G}} F_0$ to the object $(x, S(x)(A))$ in $\int_{\mathcal{G}} F_1$. A morphism $\sum_{f \in \operatorname{mor}_{\mathcal{G}}(x,y)} f \cdot \phi_f\colon (x, A) \to (y, B)$ is sent to the morphism

$$\sum_{f \in \operatorname{mor}_{\mathcal{G}}(x,y)} f \cdot S(x)(\phi_f)\colon (x, s(x)(A)) \to (y, s(y)(B)).$$

This makes sense since $S(x)(\phi_f)$ is a morphism in $F_1(x)$ from $S(x)(A)$ to $S(x)(F_0(f)(B)) = F_1(f)(S(y)(B))$. The decisive special case is that $\int_{\mathcal{G}} S$ sends $(f\colon x \to y) \cdot \phi$ to $(f\colon x \to y) \cdot S(x)(\phi)$. One easily checks that $\int_{\mathcal{G}} S$ is compatible with the structures of additive categories and we have

$$\left(\int_{\mathcal{G}} S_2\right) \circ \left(\int_{\mathcal{G}} S_1\right) = \int_{\mathcal{G}} (S_2 \circ S_1); \tag{5.3}$$

$$\int_{\mathcal{G}} \mathrm{id}_F = \mathrm{id}_{\int_{\mathcal{G}} F}. \tag{5.4}$$

The construction is also functorial in $\mathcal{G}$. Namely, let $W\colon \mathcal{G}_1 \to \mathcal{G}_2$ be a covariant functor of groupoids. Then we obtain a covariant functor

$$W_*\colon \textstyle\int_{\mathcal{G}_1} F \circ W \to \int_{\mathcal{G}_2} F \tag{5.5}$$

of additive categories as follows. An object (x_1, A) in $\int_{\mathcal{G}_1} F \circ W$ is sent to the object $(W(x_1), A)$ in $\int_{\mathcal{G}_2} F$. A morphism $\sum_{f \in \mathrm{mor}_{\mathcal{G}_1}(x_1,y_1)} f \cdot \phi_f \colon (x_1, A) \to (y_1, B)$ in $\int_{\mathcal{G}_1} F \circ W$ is sent to the morphism

$$\sum_{f \in \mathrm{mor}_{\mathcal{G}_2}(W(x_1),W(y_1))} f \cdot \left(\sum_{\substack{f_1 \in \mathrm{mor}_{\mathcal{G}_1}(x_1,y_1) \\ W(f_1)=f}} \phi_{f_1}\right) \colon (W(x_1), A) \to (W(y_1), B)$$

in $\int_{\mathcal{G}_2} F$. Here the decisive special case is that W_* sends the morphism $f \cdot \phi$ to $W(f) \cdot \phi$. One easily checks that W_* is compatible with the structures of additive categories and we have for covariant functors $W_1\colon \mathcal{G}_1 \to \mathcal{G}_2$, $W_2\colon \mathcal{G}_2 \to \mathcal{G}_3$ and a contravariant functor $F\colon \mathcal{G} \to \mathsf{Add\text{-}Cat}$

$$(W_2)_* \circ (W_1)_* = (W_2 \circ W_1)_*; \tag{5.6}$$

$$(\mathrm{id}_{\mathcal{G}})_* = \mathrm{id}_{\int_{\mathcal{G}} F}. \tag{5.7}$$

These two constructions are compatible. Namely, given a natural transformation $S_1\colon F_1 \to F_2$ of contravariant functors $\mathcal{G} \to \mathsf{Add\text{-}Cat}$ and a covariant functor $W\colon \mathcal{G}_1 \to \mathcal{G}$, we get

$$\left(\int_{\mathcal{G}} S\right) \circ W_* = W_* \circ \left(\int_{\mathcal{G}_1} (S \circ W)\right). \tag{5.8}$$

A functor $F\colon \mathcal{C}_0 \to \mathcal{C}_1$ of categories is called an *equivalence* if there exists a functor $F'\colon \mathcal{C}_1 \to \mathcal{C}_0$ with the property that $F' \circ F$ is naturally equivalent to the identity functor $\mathrm{id}_{\mathcal{C}_0}$ and $F \circ F'$ is naturally equivalent to the identity functor $\mathrm{id}_{\mathcal{C}_1}$. A functor F is a natural equivalence if and only if it is *full* and *faithful*, i.e., it induces a bijection on the isomorphism classes of objects and for any two objects c, d in $\mathcal{C}_0$ the induced map $\mathrm{mor}_{\mathcal{C}_0}(c, d) \to \mathrm{mor}_{\mathcal{C}_1}(F(c), F(d))$ is bijective. If $\mathcal{C}_0$ and $\mathcal{C}_1$ come with an additional structure such as of an additive category (with involution) and F is compatible with this structure, we require that F' and the two natural equivalences $F' \circ F \simeq \mathrm{id}_{\mathcal{C}_0}$ and $F \circ F' \simeq \mathrm{id}_{\mathcal{C}_1}$ are compatible with these. In this case it still true that F is an equivalence of categories with this additional structure if and only if F is full and faithful.

One easily checks

Lemma 5.9. *(i) Let $W\colon \mathcal{G}_1 \to \mathcal{G}$ be an equivalence of connected groupoids. Let $F\colon \mathcal{G} \to \mathsf{Add\text{-}Cat}$ be a contravariant functor. Then*

$$W_*\colon \int_{\mathcal{G}_1} F \circ W \to \int_{\mathcal{G}} F$$

is an equivalence of additive categories.

(ii) Let $\mathcal{G}$ be a connected groupoid. Let $S\colon F_1 \to F_2$ be a transformation of contravariant functors $\mathcal{G} \to \mathsf{Add\text{-}Cat}$ such that for every object x in $\mathcal{G}$

the functor $S(x)\colon F_0(x) \to F_1(x)$ is an equivalence of additive categories. Then

$$\int_{\mathcal{G}} S\colon \int_{\mathcal{G}} F_1 \to \int_{\mathcal{G}} F_2$$

is an equivalence of additive categories.

6. From crossed product rings to additive categories

Example 6.1. Here is our main example of a contravariant functor $\mathcal{G} \to \mathsf{Add\text{-}Cat}$. Notice that a group G is the same as a groupoid with one object and hence a contravariant functor from a group G to $\mathsf{Add\text{-}Cat}$ is the same as an additive G-category what is the same as an additive category with strict G-action (see Definition 2.1). Let R be a ring together with maps of sets

$$\begin{array}{rcll} c\colon G & \to & \operatorname{aut}(R), & g \mapsto c_g; \\ \tau\colon G \times G & \to & R^{\times}. & \end{array}$$

satisfying (4.3), (4.4), (4.5), (4.6) and (4.7). We have introduced the additive G-category $R\text{-}\mathsf{FGP}_{c,\tau}$ in (4.11). All the construction restrict to the subcategory $R\text{-}\mathsf{FGF} \subseteq R\text{-}\mathsf{FGP}$ of finitely generated free R-modules and lead to the additive G-category

$$R\text{-}\mathsf{FGF}_{c,\tau} \quad := \quad \mathcal{S}(R\text{-}\mathsf{FGP}); \tag{6.2}$$

Lemma 6.3. *Consider the data (R, c, τ) and the additive category $R\text{-}\mathsf{FGF}_{c,\tau}$ appearing in Example 6.1. Let $\int_G R\text{-}\mathsf{FGF}_{c,\tau}$ be the additive category defined in (5.1). Since G regarded as a groupoid has precisely one object, we can (and will) identify the set of objects in $\int_G R\text{-}\mathsf{FGF}_{c,\tau}$ with the set of objects in $R\text{-}\mathsf{FGF}_{c,\tau}$ which consists of pairs (M, g) for M a finitely generated free R-module and $g \in G$. Denote by $\left(\int_G R\text{-}\mathsf{FGF}_{c,\tau}\right)_e$ the full subcategory of $\int_G R\text{-}\mathsf{FGF}_{c,\tau}$ consisting of objects of the shape (M, e) for $e \in G$ the unit element. Denote by $R * G = R *_{c,\tau} G$ the crossed product ring (see (4.8)). Then*

(i) *There is an equivalence of additive categories*

$$\alpha\colon \left(\int_G R\text{-}\mathsf{FGF}_{c,\tau}\right)_e \to R *_{c,\tau} G\text{-}\mathsf{FGF};$$

(ii) *The inclusion*

$$\left(\int_G R\text{-}\mathsf{FGF}_{c,\tau}\right)_e \to \int_G R\text{-}\mathsf{FGF}_{c,\tau}$$

is an equivalence of additive categories.

Proof. (i) An object (M, e) in $\left(\int_G R\text{-}\mathsf{FGF}_{c,\tau}\right)_e$ is sent under α to the finitely generated free $R *_{c,\tau} G$-module $R *_{c,\tau} G \otimes_R M$. A morphism $\phi = \sum_{g\in G} g \cdot \bigl(\phi_g\colon M \to \operatorname{res}_{c_g}(N)\bigr)$ from (M, e) to (N, e) is sent to the $R *_{c,\tau} G$-homomorphism

$$\alpha(\phi)\colon R *_{c,\tau} G \otimes_R M \to R *_{c,\tau} G \otimes_R N, \quad u \otimes x \mapsto \sum_{g\in G} u \cdot \tau(g^{-1}, g)^{-1} \cdot g^{-1} \otimes \phi_g(x)$$

for $u \in R *_{c,\tau} G$ and $x \in M$. This is well-defined, i.e., compatible with the tensor relation, by the following calculation for $r \in R$ using (4.3) and (4.5).

$$\begin{aligned}
&u \cdot \tau(g^{-1},g)^{-1} \cdot g^{-1} \otimes \phi_g(rx)\\
&\quad = u \cdot \tau(g^{-1},g)^{-1} \cdot g^{-1} \otimes c_g(r)\phi_g(x)\\
&\quad = u \cdot \tau(g^{-1},g)^{-1} \cdot g^{-1} \cdot c_g(r) \otimes \phi_g(x)\\
&\quad = u \cdot \tau(g^{-1},g)^{-1} c_{g^{-1}}(c_g(r)) \cdot g^{-1} \otimes \phi_g(x)\\
&\quad = u \cdot \tau(g^{-1},g)^{-1} c_{g^{-1}}(c_g(r))\tau(g^{-1},g)\tau(g^{-1},g)^{-1} \cdot g^{-1} \otimes \phi_g(x)\\
&\quad = u \cdot c_{g^{-1}g}(r)\tau(g^{-1},g)^{-1} \cdot g^{-1} \otimes \phi_g(x)\\
&\quad = u \cdot c_e(r)\tau(g^{-1},g)^{-1} \cdot g^{-1} \otimes \phi_g(x)\\
&\quad = (u \cdot r)\tau(g^{-1},g)^{-1} \cdot g^{-1} \otimes \phi_g(x).
\end{aligned}$$

Next we show that α is a covariant functor. Obviously $\alpha(\mathrm{id}_{(M,e)}) = \mathrm{id}_{\alpha(M,e)}$. Consider morphisms $\phi = \sum_{g\in G} g \cdot \phi_g \colon (M,e) \to (N,e)$ and $\psi = \sum_{g\in G} g \cdot \psi_g \colon (N,e) \to (P,e)$ in $\left(\int_G R\text{-}\mathsf{FGF}_{c,\tau}\right)_e$. A direct computation shows for $u \in R *_{c,\tau} G$ and $x \in M$

$$\begin{aligned}
&\alpha(\psi)\left(\alpha(\phi)(u \otimes x)\right)\\
&\quad = \alpha(\psi)\left(\sum_{k\in G} u \cdot \tau(k^{-1},k)^{-1} \cdot k^{-1} \otimes \phi_k(x)\right)\\
&\quad = \sum_{h\in G}\sum_{k\in G} u \cdot \tau(k^{-1},k)^{-1} \cdot k^{-1} \cdot \tau(h^{-1},h)^{-1} \cdot h^{-1} \otimes \psi_h \circ \phi_k(x)\\
&\quad = \sum_{h,k\in G} u \cdot \tau(k^{-1},k)^{-1} c_{k^{-1}}(\tau(h^{-1},h)^{-1}) \cdot k^{-1} \cdot h^{-1} \otimes \psi_h \circ \phi_k(x)\\
&\quad = \sum_{h,k\in G} u \cdot \tau(k^{-1},k)^{-1} c_{k^{-1}}(\tau(h^{-1},h)^{-1})\tau(k^{-1},h^{-1}) \cdot (hk)^{-1} \otimes \psi_h \circ \phi_k(x)
\end{aligned}$$

and

$$\begin{aligned}
&\alpha(\psi \circ \phi)(u \otimes x)\\
&\quad = \sum_{g\in G} u \cdot \tau(g^{-1},g)^{-1} \cdot g^{-1} \otimes (\psi \circ \phi)_g(x)\\
&\quad = \sum_{g\in G} u \cdot \tau(g^{-1},g)^{-1} \cdot g^{-1} \otimes \left(\sum_{\substack{hk\in G,\\ hk=g}} r_k(\psi_h) \circ \phi_k(x)\right)\\
&\quad = \sum_{g\in G} u \cdot \tau(g^{-1},g)^{-1} \cdot g^{-1} \otimes \left(\sum_{\substack{hk\in G,\\ hk=g}} \tau(h,k)^{-1}\psi_h \circ \phi_k(x)\right)\\
&\quad = \sum_{g\in G}\sum_{\substack{hk\in G,\\ hk=g}} u \cdot \tau(g^{-1},g)^{-1} \cdot g^{-1} \cdot \tau(h,k)^{-1} \otimes \psi_h \circ \phi_k(x)\\
&\quad = \sum_{g\in G}\sum_{\substack{hk\in G,\\ hk=g}} u \cdot \tau(g^{-1},g)^{-1} c_{g^{-1}}(\tau(h,k)^{-1}) \cdot g^{-1} \otimes \psi_h \circ \phi_k(x)\\
&\quad = \sum_{hk\in G} u \cdot \tau((hk)^{-1},hk)^{-1} c_{(hk)^{-1}}(\tau(h,k)^{-1}) \cdot (hk)^{-1} \otimes \psi_h \circ \phi_k(x).
\end{aligned}$$

Hence it remains to show for $h,k \in G$

$$\tau(k^{-1},k)^{-1} c_{k^{-1}}(\tau(h^{-1},h)^{-1})\tau(k^{-1},h^{-1}) = \tau((hk)^{-1},hk)^{-1} c_{(hk)^{-1}}(\tau(h,k)^{-1}),$$

or, equivalently,

$$\tau(k^{-1},h^{-1})c_{(hk)^{-1}}(\tau(h,k))\tau((hk)^{-1},hk) \quad = \quad c_{k^{-1}}(\tau(h^{-1},h))\tau(k^{-1},k).$$

Since (4.4) yields

$$\tau((hk)^{-1},h)\tau(k^{-1},k) = c_{(hk)^{-1}}(\tau(h,k)\tau((hk)^{-1},hk),$$

it suffices to show

$$\tau(k^{-1},h^{-1})\tau((hk)^{-1},h) \quad = \quad c_{k^{-1}}(\tau(h^{-1},h)).$$

But this follows from (4.4) and (4.7). This finishes the proof that α is a covariant functor. Obviously it is compatible with the structures of an additive category. One easily checks that α induces a bijection between the isomorphism classes of objects. In order to show that α is a weak equivalence, we have to show for two objects (M,e) and (N,e) that α induces a bijection

$$\operatorname{mor}_{\left(\int_G R\text{-}\mathsf{FGF}_{c,\tau}\right)_e}((M,e),(N,e)) \xrightarrow{\cong} \hom_{R*_{c,\tau}G}(R*_{c,\tau}\otimes_R M, R*_{c,\tau}\otimes_R N).$$

Since α is compatible with the structures of an additive category, it suffices to check this in the special case $M=N=R$, where it is obvious.

(ii) An object of the shape (M,g) in $\int_G R\text{-}\mathsf{FGF}_{c,\tau}$ is isomorphic to the object (M,e), namely an isomorphism $(M,g) \xrightarrow{\cong} (M,e)$ in $\int_G R\text{-}\mathsf{FGF}_{c,\tau}$ is given by $g\cdot \mathrm{id}_{\mathrm{res}_{c_g}(M)}$. □

7. Connected groupoids and additive categories with involutions

Next we want to enrich the constructions of Section 5 to additive categories with involutions. Let $\mathsf{Add\text{-}Cat}_{\mathrm{inv}}$ be the category of additive categories with involution. Given a contravariant functor $(F,T)\colon \mathcal{G}\to \mathsf{Add\text{-}Cat}_{\mathrm{inv}}$, we want to define on the additive category $\int_{\mathcal{G}} F$ the structure of an additive category with involution. Here the pair (F,T) means that we assign to every object x in $\mathcal{G}$ an additive category with involution $F(x)$ and for every morphism $f\colon x\to y$ in $\mathcal{G}$ we have a functor of additive categories with involution $(F(f),T(f))\colon F(y)\to F(x)$.

Next we construct for a functor $\mathcal{G}\to \mathsf{Add\text{-}Cat}_{\mathrm{inv}}$ an involution of additive categories

$$(I_{\int_{\mathcal{G}} F}, E_{\int_{\mathcal{G}} F}) \tag{7.1}$$

on the additive category $\int_{\mathcal{G}} F$ which we have introduced in (5.1). On objects we put

$$I_{\int_{\mathcal{G}} F}(x,A) := (x, I_{\mathcal{G}}(A)) = (x,A^*).$$

Let $\phi = \sum_{f\in \operatorname{mor}_{\mathcal{G}}(x,y)} f\cdot \phi_f\colon (x,A)\to (y,B)$ be a morphism in $\int_{\mathcal{G}} F$. Define $I_{\int_{\mathcal{G}} F}(\phi)\colon B^*\to A^*$ to be the morphism $\phi^* = \sum_{f\in \operatorname{mor}_{\mathcal{G}}(y,x)} f\cdot (\phi^*)_f\colon (y,B^*)\to (x,A^*)$ in $\int_{\mathcal{G}} F$ whose component for $f\in \operatorname{mor}_{\mathcal{G}}(y,x)$ is given by the composite

$$\begin{aligned}(\phi^*)_f\colon B^* = F(f)\left(F(f^{-1})(B^*)\right) &\xrightarrow{F(f)(T(f^{-1})(B))} F(f)\left(F(f^{-1})(B)^*\right)\\ &\xrightarrow{F(f)\left((\phi_{f^{-1}})^*\right)} F(f)(A^*).\end{aligned}$$

Next we show that $I_{\int_{\mathcal{G}} F}$ is a contravariant functor. Obviously $I_{\int_{\mathcal{G}} F}$ sends the identity id_A to $\mathrm{id}_{I_{\int_{\mathcal{G}} F}(A)}$. We have to show

$$I_{\int_{\mathcal{G}} F}(\psi\circ\phi) = I_{\int_{\mathcal{G}} F}(\phi)\circ I_{\int_{\mathcal{G}} F}(\psi)$$

for morphisms $\phi = \sum_{h\in \operatorname{mor}_{\mathcal{G}}(x,y)} h\cdot\phi_h\colon (x,A)\to (y,B)$ and $\psi\colon \sum_{k\in \operatorname{mor}_{\mathcal{G}}(y,z)} k\cdot \psi_k\colon (y,B\to (z,C)$, or in short notation $(\psi\circ\phi)^* = \phi^*\circ\psi^*$.

By definition $(\phi^* \circ \psi^*) = \sum_{g \in \operatorname{mor}_{\mathcal{G}}(z,x)} g \cdot (\phi^* \circ \psi^*)_g$ for

$$(\phi^* \circ \psi^*)_g := \sum_{\substack{k \in \operatorname{mor}_{\mathcal{G}}(z,y),\\ h \in \operatorname{mor}_{\mathcal{G}}(y,x),\\ hk=g}} F(k)((\phi^*)_h) \circ (\psi^*)_k.$$

By definition

$$(\psi^*)_k \colon C^* = F(k)(F(k^{-1})(C^*)) \xrightarrow{F(k)(T(k^{-1})(C))} F(k)(F(k^{-1})(C)^*) \xrightarrow{F(k)((\psi_{k^{-1}})^*)} F(k)(B^*)$$

and

$$(\phi^*)_h \colon B^* = F(h)(F(h^{-1})(B^*)) \xrightarrow{F(h)(T(h^{-1})(B))} F(h)(F(h^{-1})(B)^*) \xrightarrow{F(h)((\phi_{h^{-1}})^*)} F(h)(A^*).$$

Hence the component $(\phi^* \circ \psi^*)_g$ of $(\phi^* \circ \psi^*)$ at $g \colon z \to x$ is given by the sum of morphisms from C^* to $F(g)(A^*)$

$$\sum_{\substack{k \in \operatorname{mor}_{\mathcal{G}}(z,y),\\ h \in \operatorname{mor}_{\mathcal{G}}(y,x),\\ hk=g}} F(k)\left(F(h)((\phi_{h^{-1}})^*)\right) \circ F(k)\left(F(h)(T(h^{-1})(B))\right) \circ F(k)((\psi_{k^{-1}})^*) \circ F(k)(T(k^{-1})(C)).$$

The component of $(\psi \circ \phi)^*_g$ of $(\psi \circ \phi)^*$ at $g \colon z \to x$ is given by

$$C^* = F(g)\left(F(g^{-1})(C^*)\right) \xrightarrow{F(g)(T(g^{-1})(C))} F(g)\left(F(g^{-1})(C)^*\right) \xrightarrow{F(g)\left(((\psi\circ\phi)_{g^{-1}})^*\right)} F(g)(A^*).$$

Since for $g \colon z \to x$ we have

$$(\psi \circ \phi)_{g^{-1}} = \sum_{\substack{h \in \operatorname{mor}_{\mathcal{G}}(y,z),\\ k \in \operatorname{mor}_{\mathcal{G}}(x,y),\\ hk=g^{-1}}} F(k)(\psi_h) \circ \phi_k,$$

the component of $(\psi \circ \phi)^*_g$ of $(\psi \circ \phi)^*$ at $g \colon z \to x$ is given by the sum of morphisms C^* to $F(g)(A^*)$

$$\sum_{\substack{h \in \operatorname{mor}_{\mathcal{G}}(y,z),\\ k \in \operatorname{mor}_{\mathcal{G}}(x,y),\\ hk=g^{-1}}} F(g)\left((\phi_k)^*\right) \circ F(g)\left(F(k)(\psi_h)^*\right) \circ F(g)(T(g^{-1})(C)).$$

By changing the indexing by replacing h with k^{-1} and k by h^{-1}, this transforms to

$$\sum_{\substack{k \in \operatorname{mor}_{\mathcal{G}}(z,y),\\ h \in \operatorname{mor}_{\mathcal{G}}(y,x),\\ hk=g}} F(g)\left((\phi_{h^{-1}})^*\right) \circ F(g)\left(F(h^{-1})(\psi_{k^{-1}})^*\right) \circ F(g)(T(g^{-1})(C)).$$

Hence we have to show for every $k \colon z \to y$ and $h \colon y \to x$ with $hk = g$ that the two composites

$$F(k)\left(F(h)((\phi_{h^{-1}})^*)\right) \circ F(k)\left(F(h)(T(h^{-1})(B))\right) \circ F(k)((\psi_{k^{-1}})^*) \circ F(k)(T(k^{-1})(C))$$

and

$$(F(g)((\phi_{h^{-1}})^*) \circ F(g)\left(F(h^{-1})(\psi_{k^{-1}})^*\right) \circ F(g)(T(g^{-1})(C))$$

agree. We compute for the first one

$$\begin{aligned} & F(k)\left(F(h)((\phi_{h^{-1}})^*)\right) \circ F(k)\left(F(h)(T(h^{-1})(B))\right) \circ F(k)((\psi_{k^{-1}})^*) \circ F(k)(T(k^{-1})(C)) \\ = \quad & \left(F(g)((\phi_{h^{-1}})^*\right) \circ F(g)(T(h^{-1})(B)) \circ F(g)\left(F(h^{-1})((\psi_{k^{-1}})^*)\right) \\ & \qquad \circ F(g)\left(F(h^{-1})(T(k^{-1})(C))\right). \end{aligned}$$

Hence it remains to show that the composites

$$F(g^{-1})(C^*) \xrightarrow{T(g^{-1})(C)} F(g^{-1})(C)^* \xrightarrow{F(h^{-1})(\psi_{k^{-1}})^*} F(h^{-1})(B)^*$$

and

$$\begin{aligned} F(g^{-1})(C^*) = F(h^{-1})\left(F(k^{-1})(C^*)\right) & \xrightarrow{F(h^{-1})(T(k^{-1})(C))} F(h^{-1})(F(k^{-1})(C)^*) \\ & \xrightarrow{F(h^{-1})((\psi_{k^{-1}})^*)} F(h^{-1})(B^*) \xrightarrow{T(h^{-1})(B)} F(h^{-1})(B)^* \end{aligned}$$

agree. The second one agrees with the composite

$$\begin{aligned} F(g^{-1})(C^*) = F(h^{-1})\left(F(k^{-1})(C^*)\right) & \xrightarrow{F(h^{-1})(T(k^{-1})(C))} F(h^{-1})(F(k^{-1})(C)^*) \\ & \xrightarrow{T(h^{-1})(F(k^{-1})(C))} F(h^{-1})(F(k^{-1})(C))^* \xrightarrow{F(h^{-1})(\psi_{k^{-1}})^*} F(h^{-1})(B)^* \end{aligned}$$

since $T(h^{-1})$ is a natural transformation $F(h^{-1}) \circ I_{F(y)} \to I_{F(x)} \circ F(h^{-1})$. Since $(F(h^{-1}), T(h^{-1})) \circ (F(k^{-1}), T(k^{-1})) = (F(k^{-1}h^{-1}), T(k^{-1}h^{-1})) = (F(g^{-1}), T(g^{-1}))$ the map $T(g^{-1})(C)$ can be written as the composite

$$\begin{aligned} T(g^{-1})(C) \colon F(g^{-1})(C^*) & = F(h^{-1})\left(F(k^{-1})(C^*)\right) \\ & \xrightarrow{F(h^{-1})\left(T(k^{-1})(C)\right)} F(h^{-1})\left(F(k^{-1})(C)^*\right) \\ & \xrightarrow{T(h^{-1})\left(F(k^{-1})(C)\right)} F(h^{-1})\left(F(k^{-1})(C)\right)^* = F(g)(C)^*. \end{aligned}$$

This finishes the proof that $I_{\int_{\mathcal{G}} F}$ is a contravariant functor.

The natural equivalence

$$E_{\int_{\mathcal{G}} F} \colon \mathrm{id}_{\int_{\mathcal{G}} F} \to I_{\int_{\mathcal{G}} F} \circ I_{\int_{\mathcal{G}} F}$$

assigns to an object (x, A) in $\int_{\mathcal{G}} F$ the isomorphism

$$\mathrm{id}_x \cdot \left(E_{\mathcal{G}}(A) \colon A \xrightarrow{\cong} A^{**}\right) \colon (x, A) \to (x, A^{**}).$$

We have to check that $E_{\int_{\mathcal{G}} F}$ is a natural equivalence. Consider a morphism $\phi = \sum_{f \in \mathrm{mor}_{\mathcal{G}}(x,y)} f \cdot (\phi_f \colon (x, A) \to (y, B))$ in $\int_{\mathcal{G}} F$. Then $\left(I_{\int_{\mathcal{G}} F} \circ I_{\int_{\mathcal{G}} F}\right)(\phi)$ has as the component for $f \colon x \to y$ the composite

$$\begin{aligned} A^{**} = F(f)\left(F(f^{-1})(A^{**})\right) & \xrightarrow{F(f)\left((T(f^{-1})(A^*)\right)} F(f)\left(F(f^{-1})(A^*)^*\right) \\ & \xrightarrow{F(f)\left(F(f^{-1})((\phi_f)^*)^*\right)} F(f)\left(F(f^{-1})(F(f)(B)^*)^*\right) \\ & \xrightarrow{F(f)\left(F(f^{-1})(T(f)(B))^*\right)} F(f)\left(F(f^{-1})(F(f)(B^*))^*\right) = F(f)(B^{**}). \end{aligned}$$

Hence $\left(I_{\int_{\mathcal{G}} F} \circ I_{\int_{\mathcal{G}} F}\right)(\phi) \circ E_{\int_{\mathcal{G}} F}(x, A)$ has as component for $f \colon x \to y$ the composite

$$\begin{aligned} A \xrightarrow{E_{\mathcal{A}}(A)} A^{**} = F(f)\left(F(f^{-1})(A^{**})\right) &\xrightarrow{F(f)\left((T(f^{-1})(A^*)\right)} F(f)\left(F(f^{-1})(A^*)^*\right) \\ &\xrightarrow{F(f)\left(F(f^{-1})(\phi_f^*)^*\right)} F(f)\left(F(f^{-1})(F(f)(B)^*)^*\right) \\ &\xrightarrow{F(f)\left(F(f^{-1})(T(f)(B))^*\right)} F(f)\left(F(f^{-1})(F(f)(B^*))^*\right) = F(f)(B^{**}). \end{aligned}$$

The component of $E_{\int_{\mathcal{G}} F}(y, B) \circ \phi$ at $f \colon x \to y$ is the composite

$$A \xrightarrow{\phi_f} F(f)(B) \xrightarrow{F(f)(E_{\mathcal{A}}(B))} F(f)(B^{**}).$$

It remains to show that these two morphisms $A \to F(f)(B^{**})$ agree. The following two diagrams commute since $E_{\mathcal{A}}$ and $T(f^{-1})$ are natural transformations

$$\begin{tikzcd} A \arrow[r, "E_{\mathcal{A}}(A)"] \arrow[d, "\phi_f"] & A^{**} = F(f)\left(F(f^{-1})(A^{**})\right) \arrow[d, "\phi_f^{**} = F(f)\left(F(f^{-1})(\phi_f^{**})\right)"] \\ F(f)(B) \arrow[r, "E_{\mathcal{A}}(F(f)(B))"] & F(f)(B)^{**} = F(f)\left(F(f^{-1})(F(f)(B)^{**})\right) \end{tikzcd}$$

and

$$\begin{tikzcd} F(f)\left(F(f^{-1})(A^{**})\right) \arrow[r, "F(f)(T(f^{-1})(A^*))"] \arrow[d, "F(f)\left(F(f^{-1})(\phi_f^{**})\right)"] & F(f)\left(F(f^{-1})(A^*)^*\right) \arrow[d, "F(f)\left(F(f^{-1})(\phi_f^*)^*\right)"] \\ F(f)\left(F(f^{-1})(F(f)(B)^{**})\right) \arrow[r, "F(f)\left(T(f^{-1})(F(f)(B)^*)\right)"] & F(f)\left(F(f^{-1})(F(f)(B)^*)^*\right). \end{tikzcd}$$

Hence we have to show that

$$F(f)(B) \xrightarrow{F(f)(E_{\mathcal{A}}(B))} F(f)(B^{**})$$

agrees with the composite

$$\begin{aligned} F(f)(B) \xrightarrow{E_{\mathcal{A}}(F(f)(B))} F(f)(B)^{**} &= F(f)\left(F(f^{-1})(F(f)(B)^{**})\right) \\ &\xrightarrow{F(f)\left(T(f^{-1})(F(f)(B)^*)\right)} F(f)\left(F(f^{-1})(F(f)(B)^*)^*\right) \\ &\xrightarrow{F(f)\left(F(f^{-1})(T(f)(B))^*\right)} F(f)\left(F(f^{-1})(F(f)(B^*))^*\right) = F(f)(B^{**}). \end{aligned}$$

(Notice that ϕ is not involved anymore.) The following diagram commutes by the axioms (see (1.6))

$$\begin{tikzcd} F(f)(B) \arrow[r, "E_{\mathcal{A}}(F(f)(B))"] \arrow[d, "F(f)(E_{\mathcal{A}}(B))"] & F(f)(B)^{**} \arrow[d, "T(f)(B)^*"] \\ F(f)(B^{**}) \arrow[r, "T(f)(B^*)"] & F(f)(B^*)^* \end{tikzcd}$$

Hence it remains to show the commutativity of the following diagram (which does not involve ϕ and $E_{\mathcal{A}}$ anymore).

$$\begin{tikzcd} F(f)(B)^{**} = F(f)\left(F(f^{-1})(F(f)(B)^*)\right)^* \arrow[r, "F(f)\left(T(f^{-1})(F(f)(B)^*)\right)"] \arrow[d, "T(f)(B)^*"] & F(f)\left(F(f^{-1})(F(f)(B)^*)^*\right) \arrow[d, "F(f)\left(F(f^{-1})(T(f)(B))^*\right)"'] \\ F(f)(B^*)^* & F(f)(B^{**}) = F(f)\left(F(f^{-1})(F(f)(B^*))^*\right) \arrow[l, "T(f)(B^*)"] \end{tikzcd}$$

Since $(F(f),T(f))\circ(F(f^{-1}),T(f^{-1})) = \mathrm{id}$, we have

$$T(f)\left(F(f^{-1})(F(f)(B)^*)\right) \ \circ\ F(f)\left(T(f^{-1})(F(f)(B)^*)\right) = \mathrm{id}\,.$$

Hence it suffices to prove the commutativity of the following diagram

$$\begin{tikzcd} F(f)(B)^{**} = F(f)\left(F(f^{-1})(F(f)(B)^*)\right)^* \arrow[d, "T(f)(B)^*"] & F(f)\left(F(f^{-1})(F(f)(B)^*)^*\right) \arrow[l, "T(f)\left(F(f^{-1})(F(f)(B)^*)\right)"'] \arrow[d, "F(f)\left(F(f^{-1})(T(f)(B))^*\right)"'] \\ F(f)(B^*)^* & F(f)(B^{**}) = F(f)\left(F(f^{-1})(F(f)(B^*))^*\right) . \arrow[l, "T(f)(B^*)"] \end{tikzcd}$$

This follows because this diagram is obtained by applying the natural transformation $T(f)$ to the morphism

$$F(f^{-1})(F(f)(B^*)) \xrightarrow{F(f^{-1})(T(f)(B))} F(f^{-1})(F(f)(B)^*).$$

The condition (1.3) is satisfied for $(I_{\int_{\mathcal{G}} F}, E_{\int_{\mathcal{G}} F})$ since it holds for $(I_{\mathcal{A}}, E_{\mathcal{A}})$.

We will denote the resulting additive category $\int_{\mathcal{A}} F$ with involution $(I_{\int_{\mathcal{G}} F}, E_{\int_{\mathcal{G}} F})$ by

$$\int_{\mathcal{G}}(F,T). \tag{7.2}$$

Let (F_0,T_0) and (F_1,T_1) be two contravariant functors $\mathcal{G} \to \mathtt{Add\text{-}Cat}_{\mathrm{inv}}$. Let $(S,U)\colon (F_0,T_0) \to (F_1,T_1)$ be a natural transformation of such functors. This means that we for each object x in $\mathcal{G}$ we have an equivalence $(S(x),U(x))\colon F_0(x) \to F_1(y)$ of additive categories with involution such that for all $f\colon x \to y$ in $\mathcal{G}$ the following diagram of functors of additive categories with involution commutes

$$\begin{tikzcd} F_0(y) \arrow[r, "(S(y),U(y))"] \arrow[d, "(F_0(f),T_0(f))"] & F_1(y) \arrow[d, "(F_1(f),T_1(f))"] \\ F_0(x) \arrow[r, "(S(x),U(x))"] & F_1(x) \end{tikzcd} \tag{7.3}$$

Then both $\int_{\mathcal{G}}(F_0,T_0)$ and $\int_{\mathcal{G}}(F_1,T_1)$ are additive categories with involutions. The functor of additive categories $\int_{\mathcal{G}} S\colon \int_{\mathcal{G}} F_0 \to \int_{\mathcal{G}} F_1$ defined in (5.2) extends to a functor of additive categories with involution

$$\int_{\mathcal{G}}(S,U)\colon \int_{\mathcal{G}}(F_0,T_0) \to \int_{\mathcal{G}}(F_1,T_1) \tag{7.4}$$

as follows. We have to specify a natural equivalence

$$\widehat{U}\colon \left(\int_{\mathcal{G}} S\right) \circ I_{\int_{\mathcal{G}}(F_0,T_0)} \to I_{\int_{\mathcal{G}}(F_1,T_1)} \circ \int_{\mathcal{G}} S.$$

For an object (x,A) in $\int_{\mathcal{G}} F_0$ the isomorphism

$$\widehat{U}(x,A)\colon \left(\int_{\mathcal{G}} S\right) \circ I_{\int_{\mathcal{G}}(F_0,T_0)}(x,A) \to I_{\int_{\mathcal{G}}(F_1,T_1)} \circ \int_{\mathcal{G}} S(x,A)$$

is given by the isomorphism

$$\mathrm{id}_x \cdot U(x)(A) \colon (x, S(x)(A^*)) \to (x, S(x)(A)^*)$$

in $\int_{\mathcal{G}} F_1$. Next we check that $\widehat{U}$ is a natural equivalence.

Let $\sum_{f \in \mathrm{mor}_{\mathcal{G}}(x,y)} f \cdot \phi_f \colon (x, A) \to (y, B)$ be a morphism in $\int_{\mathcal{G}} F_0$, where by definition $\phi_f \colon A \to F(f)(B)$ is a morphism in the additive category $F_0(x)$. We have to show the commutativity of the following diagram in the additive category $\int_{\mathcal{G}} F_1$

$$\begin{tikzcd}
\left(\int_{\mathcal{G}} S\right) \circ I_{\int_{\mathcal{G}}(F_0,T_0)}(y,B) \arrow[r, "(\int_{\mathcal{G}} S) \circ I_{\int_{\mathcal{G}}(F_0,T_0)}(\phi)"] \arrow[d, "\widehat{U}(y,B)"] & \left(\int_{\mathcal{G}} S\right) \circ I_{\int_{\mathcal{G}}(F_0,T_0)}(x,A) \arrow[d, "\widehat{U}(x,A)"] \\
I_{\int_{\mathcal{G}}(F_1,T_1)} \circ \int_{\mathcal{G}} S(y,B) \arrow[r, "I_{\int_{\mathcal{G}}(F_1,T_1)} \circ \int_{\mathcal{G}} S(\phi)"] & I_{\int_{\mathcal{G}}(F_1,T_1)} \circ \int_{\mathcal{G}} S(x,A)
\end{tikzcd}$$

The morphism $I_{\int_{\mathcal{G}}(F_0,T_0)}(\phi)$ in $\int_{\mathcal{G}} F_0$ is given by

$$\phi^* = \sum_{f \in \mathrm{mor}_{\mathcal{G}}(x,y)} f \cdot (\phi^*)_f \colon (y, B^*) \to (x, A^*),$$

where the component $(\phi^*)_f$ is the composite

$$(\phi^*)_f \colon B^* = F_0(f)\left(F_0(f^{-1})(B^*)\right) \xrightarrow{F_0(f)\left(T_0(f^{-1})(B)\right)} F_0(f)\left(F_0(f^{-1})(B)^*\right) \xrightarrow{F_0(f)((\phi_{f^{-1}})^*)} F_0(f)\,(A^*)\,.$$

The morphism $\left(\int_{\mathcal{G}} S\right) \circ I_{\int_{\mathcal{G}}(F_0,T_0)}(\phi) \colon (y, B^*) \to (x, A^*)$ in $\int_{\mathcal{G}} F_1$ is given by $\sum_{f \in \mathrm{mor}_{\mathcal{G}}(x,y)} f \cdot \psi_f \colon (y, B^*) \to (x, A^*)$, where ψ_f is the composite

$$\begin{aligned}
\psi_f \colon S(y)(B^*) &= S(y)\left(F_0(f)\left(F_0(f^{-1})(B^*)\right)\right) \\
&\xrightarrow{S(y)\left(F_0(f)\left(T_0(f^{-1})(B)\right)\right)} S(y)\left(F_0(f)\left(F_0(f^{-1})(B)^*\right)\right) \\
&\xrightarrow{S(y)(F_0(f)((\phi_{f^{-1}})^*))} S(y)\left(F_0(f)\,(A^*)\right) = F_1(f)\left(S(x)\,(A^*)\right).
\end{aligned}$$

Hence the morphism $\widehat{U}(x, A) \circ \left(\int_{\mathcal{G}} S\right) \circ I_{\int_{\mathcal{G}}(F_0,T_0)}(\phi) \colon (y, B^*) \to (x, A^*)$ in $\int_{\mathcal{G}} F_1$ is given by $\sum_{f \in \mathrm{mor}_{\mathcal{G}}(x,y)} f \cdot \mu_f \colon (y, B^*) \to (x, A^*)$, where μ_f is the composite in $F_1(y)$

$$\begin{aligned}
\mu_f \colon S(y)(B^*) &= S(y)\left(F_0(f)\left(F_0(f^{-1})(B^*)\right)\right) \\
&\xrightarrow{S(y)\left(F_0(f)\left(T_0(f^{-1})(B)\right)\right)} S(y)\left(F_0(f)\left(F_0(f^{-1})(B)^*\right)\right) \\
&\xrightarrow{S(y)(F_0(f)((\phi_{f^{-1}})^*))} S(y)\left(F_0(f)\,(A^*)\right) = F_1(f)\left(S(x)\,(A^*)\right) \\
&\xrightarrow{F_1(f)(U(x)(A))} F_1(f)\left(S(x)(A)^*\right).
\end{aligned}$$

The morphism $\int_{\mathcal{G}} S(\phi) \colon (x, S(x)(A)) \to (y, S(y)(B))$ in $\int_{\mathcal{G}} F_1$ is given by

$$\sum_{f \in \mathrm{mor}_{\mathcal{G}}(x,y)} f \cdot (S(x)(\phi_f) \colon S(x)(A) \to S(x)(F_0(f)(B) = F_1(f)(S(y)(B))\,.$$

The morphism $I_{\int_{\mathcal{G}}(F_1,T_1)} \circ \int_{\mathcal{G}} S(\phi)\colon (y, S(y)(B)^*) \to (x, S(x)(A)^*)$ in $\int_{\mathcal{G}} F_1$ is given by $\sum_{f \in \operatorname{mor}_{\mathcal{G}}(y,x)} f \cdot \nu_f$, where ν_f is the composite in $F_1(y)$.

$$\nu_f\colon S(y)(B)^* = F_1(f)\left(F_1(f^{-1})\left(S(y)(B)^*\right)\right)$$
$$\xrightarrow{F_1(f)\left(T_1(f^{-1})(S(y)(B))\right)} F_1(f)\left(F_1(f^{-1})\left(S(y)(B)\right)^*\right) = F_1(f)\left(S(x)\left(F_0(f^{-1})(B)\right)^*\right)$$
$$\xrightarrow{F_1(f)\left((S(x)(\phi_{f^{-1}}))^*\right)} F_1(f)\left(S(x)(A)^*\right).$$

The morphism $I_{\int_{\mathcal{G}}(F_1,T_1)} \circ \int_{\mathcal{G}} S(\phi) \circ \widehat{U}(y,B)\colon (y, S(y)(B^*)) \to (x, S(x)(A)^*)$ in $\int_{\mathcal{G}} F_1$ is given by $\sum_{f \in \operatorname{mor}_{\mathcal{G}}(y,x)} f \cdot \omega_f$, where ω_f is the composite in $F_1(y)$.

$$\omega_f\colon S(y)(B^*) \xrightarrow{U(y)(B)} S(y)(B)^* = F_1(f)\left(F_1(f^{-1})\left(S(y)(B)^*\right)\right)$$
$$\xrightarrow{F_1(f)\left(T_1(f^{-1})(S(y)(B))\right)} F_1(f)\left(F_1(f^{-1})\left(S(y)(B)\right)^*\right) = F_1(f)\left(S(x)\left(F_0(f^{-1})(B)\right)^*\right)$$
$$\xrightarrow{F_1(f)\left((S(x)(\phi_{f^{-1}}))^*\right)} F_1(f)\left(S(x)(A)^*\right).$$

Hence we have to show for all $f\colon y \to x$ in $\operatorname{mor}_{\mathcal{G}}(y,x)$ that the two composites in $F_1(y)$

$$S(y)(B^*) = S(y)\left(F_0(f)\left(F_0(f^{-1})(B^*)\right)\right)$$
$$\xrightarrow{S(y)\left(F_0(f)\left(T_0(f^{-1})(B)\right)\right)} S(y)\left(F_0(f)\left(F_0(f^{-1})(B)^*\right)\right)$$
$$\xrightarrow{S(y)(F_0(f)((\phi_{f^{-1}})^*))} S(y)\left(F_0(f)\left(A^*\right)\right) = F_1(f)\left(S(x)\left(A^*\right)\right)$$
$$\xrightarrow{F_1(f)(U(x)(A))} F_1(f)\left(S(x)(A)^*\right)$$

and

$$S(y)(B^*) \xrightarrow{U(y)(B)} S(y)(B)^* = F_1(f)\left(F_1(f^{-1})\left(S(y)(B)^*\right)\right)$$
$$\xrightarrow{F_1(f)\left(T_1(f^{-1})(S(y)(B))\right)} F_1(f)\left(F_1(f^{-1})\left(S(y)(B)\right)^*\right) = F_1(f)\left(S(x)\left(F_0(f^{-1})(B)\right)^*\right)$$
$$\xrightarrow{F_1(f)\left((S(x)(\phi_{f^{-1}}))^*\right)} F_1(f)\left(S(x)(A)^*\right).$$

agree. Since S is a natural transformation from $F_0 \to F_1$, the first composite can be rewritten as the composite

$$S(y)(B^*) = F_1(f)\left(S(x)\left(F_0(f^{-1})(B^*)\right)\right)$$
$$\xrightarrow{F_1(f)\left(S(x)\left(T_0(f^{-1})(B)\right)\right)} F_1(f)\left(S(x)\left(F_0(f^{-1})(B)^*\right)\right)$$
$$\xrightarrow{F_1(f)(S(x)((\phi_{f^{-1}})^*))} F_1(f)\left(S(x)\left(A^*\right)\right)$$
$$\xrightarrow{F_1(f)(U(x)(A))} F_1(f)\left(S(x)(A)^*\right).$$

Since $U(x)$ is a natural transformation from $S(x) \circ I_{F_0(x)}$ to $I_{F_1(x)} \circ S(x)$, this agrees with the composite

$$\begin{aligned} S(y)(B^*) &= F_1(f)\left(S(x)\left(F_0(f^{-1})(B^*)\right)\right) \\ &\xrightarrow{F_1(f)\left(S(x)\left(T_0(f^{-1})(B)\right)\right)} F_1(f)\left(S(x)\left(F_0(f^{-1})(B)^*\right)\right) \\ &\xrightarrow{F_1(f)\left(U(x)\left(F_0(f^{-1})(B)\right)\right)} F_1(f)\left(S(x)\left(F_0(f^{-1})(B)\right)^*\right) \\ &\xrightarrow{F_1(f)((S(x)(\phi_{f^{-1}}))^*)} F_1(f)\left(S(x)(A)^*\right). \end{aligned}$$

Hence it suffices to show that the following two composites agree

$$\begin{aligned} S(y)(B^*) &= F_1(f)\left(S(x)\left(F_0(f^{-1})(B^*)\right)\right) \\ &\xrightarrow{F_1(f)\left(S(x)\left(T_0(f^{-1})(B)\right)\right)} F_1(f)\left(S(x)\left(F_0(f^{-1})(B)^*\right)\right) \\ &\xrightarrow{F_1(f)\left(U(x)\left(F_0(f^{-1})(B)\right)\right)} F_1(f)\left(S(x)\left(F_0(f^{-1})(B)\right)^*\right) \end{aligned}$$

and

$$\begin{aligned} &S(y)(B^*) \xrightarrow{U(y)(B)} S(y)(B)^* = F_1(f)\left(F_1(f^{-1})\left(S(y)(B)^*\right)\right) \\ &\xrightarrow{F_1(f)\left(T_1(f^{-1})(S(y)(B))\right)} F_1(f)\left(F_1(f^{-1})\left(S(y)(B)\right)^*\right) = F_1(f)\left(S(x)\left(F_0(f^{-1})(B)\right)^*\right) \end{aligned}$$

(Notice that $\phi_{f^{-1}}$ has been eliminated.) This will follow by applying $F_1(f)$ to the following diagram, provided we can show that it does commute.

$$\begin{tikzcd} S(x)\left(F_0(f^{-1})(B^*)\right) = F_1(f^{-1})\left(S(y)(B^*)\right) \arrow[r, "S(x)\left(T_0(f^{-1})(B)\right)"] \arrow[d, "F_1(f^{-1})(U(y)(B))"'] & S(x)\left(F_0(f^{-1})(B)^*\right) \arrow[d, "U(x)\left(F_0(f^{-1})(B)\right)"'] \\ F_1(f^{-1})\left(S(y)(B)^*\right) \arrow[r, "T_1(f^{-1})(S(y)(B))"] & S(x)\left(F_0(f^{-1})(B)\right)^* \end{tikzcd}$$

But the latter diagram commutes because we require the following equality of functors of additive categories with involution for $f^{-1}\colon x \to y$ (see (7.3))

$$(F_0(f^{-1}), T_0(f^{-1})) \circ (S(x), U(x)) = (S(y), U(y)) \circ (F_1(f^{-1}), T_1(f^{-1})).$$

This finishes the proof that $\widehat{U}$ is a natural equivalence. One easily checks that condition (1.6) is satisfied by $\widehat{U}$ since it holds for $U(x)$ for all objects x in $\mathcal{G}$. This finishes the construction of the functor of additive categories with involution (S, U) (see (7.4)).

One easily checks

$$\left(\int_{\mathcal{G}} (S_2, U_2)\right) \circ \left(\int_{\mathcal{G}} (S_1, U_1)\right) = \int_{\mathcal{G}} (S_2, U_2) \circ (S_1, U_1) \tag{7.5}$$

$$\int_{\mathcal{G}} \mathrm{id}_F = \mathrm{id}_{\int_{\mathcal{G}} F}. \tag{7.6}$$

Given a functor of groupoids $W\colon \mathcal{G}_1 \to \mathcal{G}$ and a functor $(F, T)\colon \mathcal{G} \to \mathsf{Add\text{-}Cat}_{\mathrm{inv}}$, the composition with W a yields a functor $(F \circ W, T \circ W)$. Hence both $\int_{\mathcal{G}_1} (F, T) \circ W$ and $\int_{\mathcal{G}} (F, T)$ are additive categories with involutions. One easily checks that $I_{\int_{\mathcal{G}} F} \circ W_* = W_* \circ I_{\int_{\mathcal{G}_1} F \circ W}$ holds for the functor W_* defined in (5.5). Hence

$$(W_*, \mathrm{id})\colon \int_{\mathcal{G}_1} (F, T) \circ W \to \int_{\mathcal{G}} (F, T). \tag{7.7}$$

is a functor of additive categories with involution. One easily checks

$$((W_2)_*, \mathrm{id}) \circ ((W_1)_*, \mathrm{id}) = ((W_2 \circ W_1)_*, \mathrm{id}); \tag{7.8}$$

$$(\mathrm{id}_{\mathcal{G}})_* = \mathrm{id}_{\int_{\mathcal{G}} F}. \tag{7.9}$$

These two constructions are compatible. Namely, we get

$$\left(\int_{\mathcal{G}} (S, U)\right) \circ (W_*, \mathrm{id}) = (W_*, \mathrm{id}) \circ \left(\int_{\mathcal{G}_1} (S \circ W, U \circ W)\right). \tag{7.10}$$

One easily checks

Lemma 7.11. *(i) Let $W: \mathcal{G}_1 \to \mathcal{G}$ be an equivalence of connected groupoids. Let $(F, T): \mathcal{G} \to \mathtt{Add\text{-}Cat}_{inv}$ be a contravariant functor. Then*

$$W_*: \int_{\mathcal{G}_1} (F, T) \circ W \to \int_{\mathcal{G}} (F, T)$$

is an equivalence of additive categories with involution.

(ii) Let $\mathcal{G}$ be a connected groupoid. Let $S: (F_1, T_1) \to (F_2, T_2)$ be a transformation of contravariant functors $\mathcal{G} \to \mathtt{Add\text{-}Cat}_{inv}$ such that for every object x in $\mathcal{G}$ the functor $S(x): F_0(x) \to F_1(x)$ is an equivalence of additive categories. Then

$$\int_{\mathcal{G}} S: \int_{\mathcal{G}} (F_1, T_1) \to \int_{\mathcal{G}} (F_2, T_2)$$

is an equivalence of additive categories with involution.

8. From crossed product rings with involution to additive categories with involution

Next we want to extend Example 6.1 and Lemma 6.3 to rings and additive categories with involutions. Let R be a ring and let G be a group. Suppose that we are given maps of sets

$$\begin{aligned} c: G &\to \mathrm{aut}(R), \quad g \mapsto c_g; \\ \tau: G \times G &\to R^\times; \\ w: G &\to R, \end{aligned}$$

satisfying conditions (4.3), (4.4), (4.5), (4.6), (4.7), (4.13), (4.14), (4.15), and (4.16). We have constructed in Section 4 an involution on the crossed product $R * G = R *_{c,\tau} G$. We have denoted this ring with involution by $R*G = R*_{c,\tau,w}G$ (see (4.19)). The additive category $R*G\text{-}\mathsf{FGF}$ inherits the structure of an additive category with involution (see Example 1.5).

We have introduced notion of an additive G-category with involution in Definition 4.22 and constructed an explicit example $R\text{-}\mathsf{FGP}_{c,\tau,w}$ in (4.24). All these constructions restrict to the subcategory $R\text{-}\mathsf{FGF} \subseteq R\text{-}\mathsf{FGP}$ of finitely generated free R-modules. Thus we obtain the additive G-category with involution

$$R\text{-}\mathsf{FGP}_{c,\tau,w} \tag{8.1}$$

Lemma 8.2. *Consider the data (R, c, τ, w) and the additive G-category with involution $R\text{-}\mathsf{FGF}_{c,\tau,w}$ of (8.1). Let $\int_G R\text{-}\mathsf{FGF}_{c,\tau,w}$ be the additive category with involution defined in (7.1). Since G regarded as a groupoid has precisely one object, we can (and will) identify the set of objects in $\int_G R\text{-}\mathsf{FGF}_{c,\tau,w}$ with the set of objects in $R\text{-}\mathsf{FGF}_{c,\tau,w}$ which consists of pairs (M, g) for M a finitely generated free R-module and $g \in G$. Denote by $\left(\int_G R\text{-}\mathsf{FGF}_{c,\tau,w}\right)_e$ the full subcategory of $\int_G FGFR_{c,\tau,w}$ consisting of objects of the shape (M, e) for $e \in G$ the unit element. Denote by $R * G = R *_{c,\tau,w} G$ the ring with involution given by the crossed product ring (see 4.19). Then* *(i) There is an equivalence of additive categories with involution*

$$(\alpha, \beta)\colon \left(\int_G R\text{-}\mathsf{FGF}_{c,\tau,w}\right)_e \to R *_{c,\tau,w} G\text{-}\mathsf{FGF};$$

(ii) The inclusion

$$\left(\int_G R\text{-}\mathsf{FGF}_{c,\tau,w}\right)_e \to \int_G R\text{-}\mathsf{FGF}_{c,\tau,w}$$

is an equivalence of additive categories with involution.

Proof. (i) We have already constructed an equivalence of categories

$$\alpha\colon \left(\int_G R\text{-}\mathsf{FGF}_{c,\tau}\right)_e \to R *_{c,\tau} G\text{-}\mathsf{FGF};$$

in Lemma 6.3 (i). We want to show that α is compatible with the involution, i.e., there is a functor of categories with involutions

$$(\alpha, \beta)\colon \left(\int_G R\text{-}\mathsf{FGF}_{c,\tau,w}\right)_e \to R *_{c,\tau,w} G\text{-}\mathsf{FGF}.$$

The natural equivalence $\beta\colon \alpha \circ I_{\left(\int_G R\text{-}\mathsf{FGF}_{c,\tau,w}\right)_e} \to I_{R *_{c,\tau,w} G\text{-}\mathsf{FGF}} \circ \alpha$ assigns to an object (M, e) in $\left(\int_G R\text{-}\mathsf{FGF}_{c,\tau,w}\right)_e$ the $R *_{c,\tau} G$-isomorphism

$$\beta(M, e)\colon R *_{c,\tau,w} G \otimes_R M^* \xrightarrow{\cong} (R *_{c,\tau,w} G \otimes_R M)^*$$

given by $\beta(M, e)(u \otimes f)(v \otimes m) = vf(m)\overline{u}$ for $f \in M^*$, $u \in R *_{c,\tau} G$ and $m \in M$. Obviously β is compatible with the structures of additive categories.

Next we check that β is a natural transformation. We have to show for a morphism $\phi\colon (M, e) \to (N, e)$ in $\left(\int_G R\text{-}\mathsf{FGF}_{c,\tau,w}\right)_e$ that the following diagram commutes

$$\begin{tikzcd}
R * G \otimes_R N^* \arrow[r, "\alpha(\phi^*)"] \arrow[d, "\beta(N,e)"] & R * G \otimes_R M^* \arrow[d, "\beta(M,e)"] \\
(R * G \otimes_R N)^* \arrow[r, "\alpha(\phi)^*"] & (R * G \otimes_R M)^*
\end{tikzcd}$$

Recall that a morphism

$$\phi = \sum_{g \in G} g \cdot \phi_g\colon (M, e) \to (N, e)$$

in $\left(\int_G R\text{-}\mathsf{FGF}_{c,\tau,w}\right)_e$ is given by a collection of morphisms $\phi_g\colon (M, e) \to R_g(N, e) = (N, g)$ in $R\text{-}\mathsf{FGF}_{c,\tau,w}$ for $g \in G$, where ϕ_g is a R-homomorphism $M \to \operatorname{res}_{c_g} N$. We want to unravel what the dual morphism

$$\phi^* = \sum_{g \in G} g \cdot (\phi^*)_g\colon (N, e)^* = (N^*, e) \to (M, e)^* = (M^*, e)$$

in $\left(\int_G R\text{-}\mathsf{FGF}_{c,\tau,w}\right)_e$ is. It is given by a collection of morphisms $\{(\phi^*)_g\colon (N^*, e) \to R_g(M^*, e) = (M^*, g) \mid g \in G\}$ in $R\text{-}\mathsf{FGF}_{c,\tau,w}$, where $(\phi^*)_g$ is a R-homomorphism $N^* \to \operatorname{res}_{c_g} M^*$. In $R\text{-}\mathsf{FGF}_{c,\tau,w}$ the morphism $(\phi^*)_g$ is given by the composite

$$(N^*, e) = (N^*, g^{-1}) \cdot g = R_g(N^*, g^{-1}) \xrightarrow{R_g((\phi_{g^{-1}})^*)} R_g(M^*, e) = (M^*, g).$$

The morphism $(\phi_{g^{-1}})^*$ is given by the composite

$$\operatorname{res}_{c_{g^{-1}}} N^* \xrightarrow{t_{g^{-1}}(N)} \left(\operatorname{res}_{c_{g^{-1}}} N\right)^* \xrightarrow{(\phi_{g^{-1}})^*} M^*.$$

Explicitly this is the map

$$N^* \to M^*, \quad f(x) \mapsto c_{g^{-1}}^{-1}\left(f \circ \phi_{g^{-1}}(x)\right)\left(w(g^{-1})\tau(g, g^{-1})\right)^{-1}.$$

The morphism $R_g((\phi_{g^{-1}})^*)$ is the composite

$$N^* = \operatorname{res}_{c_{g^{-1}g}} N^* \xrightarrow{L_{\tau(g^{-1},g)}} \operatorname{res}_g \operatorname{res}_{c_{g^{-1}}} N^* \xrightarrow{\operatorname{res}_{c_g}(\phi_{g^{-1}})^*} \operatorname{res}_{c_g} M^*.$$

Hence the R-linear map $(\phi^*)_g \colon N^* \to \operatorname{res}_{c_g} M^*$ sends $f \in N^*$ to the element in M^* given by

$$x \mapsto c_{g^{-1}}^{-1}\left(f \circ \phi_{g^{-1}}(x)\overline{\tau(g^{-1},g)}\right)\left(w(g^{-1})\tau(g,g^{-1})\right)^{-1}.$$

This implies that the $R*G$-homomorphism

$$\alpha(\phi^*) \colon R*G \otimes_R N^* \to R*G \otimes_R M^*$$

sends $u \otimes f$ for $u \in R*G$ and $f \in N^*$ to the R-linear map $M \to R$ given by

$$\sum_{g\in G} u \cdot \tau(g^{-1},g)^{-1} \cdot g^{-1} \otimes \left(c_{g^{-1}}^{-1} \circ f \circ \phi_{g^{-1}}\right) c_{g^{-1}}^{-1}\left(\overline{\tau(g^{-1},g)}\right)\left(w(g^{-1})\tau(g,g^{-1})\right)^{-1}.$$

We conclude that the composite $\beta(M,e)\circ\alpha(\phi^*)$ sends $u \otimes f$ for $u \in R*G$ and $f \in N^*$ to the R-linear map $R*G \otimes_R M \to R*G$ which maps $v \otimes x$ for $v \in R*G$ and $x \in M$ to the element in $R*G$

$$\sum_{g\in G} v \cdot \left(c_{g^{-1}}^{-1} \circ f \circ \phi_{g^{-1}}\right)(x) c_{g^{-1}}^{-1}\left(\overline{\tau(g^{-1},g)}\right)\left(w(g^{-1})\tau(g,g^{-1})\right)^{-1} \cdot \overline{u \cdot \tau(g^{-1},g)^{-1} \cdot g^{-1}}.$$

We compute that the composite $\alpha(\phi)^* \circ \beta(N,e)$ sends $u \otimes f$ for $u \in R*G$ and $f \in N^*$ to the R-linear map $R*G \otimes_R M \to R*G$ which maps $v \otimes x$ for $v \in R*G$ and $x \in M$ to the element in $R*G$

$$\begin{aligned}
&\beta(N,e)(u\otimes f)\left(\alpha(\phi)(v\otimes x)\right)\\
&= \beta(N,e)(u\otimes f)\left(\sum_{g\in G} v \cdot \tau(g^{-1},g)^{-1}\cdot g^{-1} \otimes \phi_g(x)\right)\\
&= \sum_{g\in G} \beta(N,e)(u\otimes f)\left(v \cdot \tau(g^{-1},g)^{-1}\cdot g^{-1} \otimes \phi_g(x)\right)\\
&= \sum_{g\in G} v \cdot \tau(g^{-1},g)^{-1}\cdot g^{-1} \cdot f\left(\phi_g(x)\right)\cdot \overline{u}.
\end{aligned}$$

Hence it suffices to show for each $g \in G$, $u, v \in R*G$ and $x \in M$

$$\begin{aligned}
v\left(c_{g^{-1}}^{-1} \circ f \circ \phi_{g^{-1}}\right)(x) c_{g^{-1}}^{-1}\left(\overline{\tau(g^{-1},g)}\right)\left(w(g^{-1})\tau(g,g^{-1})\right)^{-1}&\\
\cdot \overline{u \cdot \tau(g^{-1},g)^{-1} \cdot g^{-1}}&\\
= v \cdot \tau(g,g^{-1})^{-1} \cdot g \cdot f\left(\phi_{g^{-1}}(x)\right)\cdot \overline{u}.&
\end{aligned}$$

Since

$$\begin{aligned}
\overline{u \cdot \tau(g^{-1},g)^{-1} \cdot g^{-1}} &= w(g^{-1})c_g(\overline{\tau(g^{-1},g)^{-1}}) \cdot g \cdot \overline{u};\\
g \cdot f\left(\phi_{g^{-1}}(x)\right)\cdot \overline{u} &= c_g\left(f\left(\phi_{g^{-1}}(x)\right)\right)\cdot g \cdot \overline{u},
\end{aligned}$$

it remains to show for all $g \in G$ and $x \in M$

$$\begin{aligned}
\left(c_{g^{-1}}^{-1} \circ f \circ \phi_{g^{-1}}\right)(x) c_{g^{-1}}^{-1}\left(\overline{\tau(g^{-1},g)}\right)\left(w(g^{-1})\tau(g,g^{-1})\right)^{-1}&\\
\cdot w(g^{-1})c_g(\overline{\tau(g^{-1},g)^{-1}})&\\
= \tau(g,g^{-1})^{-1} \cdot c_g\left(f\left(\phi_{g^{-1}}(x)\right)\right).&
\end{aligned}$$

If we put $r = f \circ \phi_{g^{-1}}(x)$, this becomes equivalent to showing for all $g \in G$ and $r \in R$

$$c_{g^{-1}}^{-1}(r)c_{g^{-1}}^{-1}\left(\overline{\tau(g^{-1},g)}\right)\left(w(g^{-1})\tau(g,g^{-1})\right)^{-1} \cdot w(g^{-1})c_g(\overline{\tau(g^{-1},g)^{-1}}) \\ = \tau(g,g^{-1})^{-1} \cdot c_g(r).$$

This is equivalent to showing

$$\tau(g,g^{-1})c_{g^{-1}}^{-1}\left(r\overline{\tau(g^{-1},g)}\right)\tau(g,g^{-1})^{-1} = c_g(r\overline{\tau(g^{-1},g)}).$$

From (4.3) and (4.5) we conclude for $x \in R$

$$\tau(g,g^{-1})c_{g^{-1}}^{-1}(x)\tau(g,g^{-1})^{-1} = c_g(x),$$

and the claim follows, i.e., β is a natural equivalence.

It remains to check that the following diagram (see (1.6)) commutes for every object (M,e) in $R\text{-}\mathsf{FGF}_{c,\tau}[G]^e$.

$$\begin{tikzcd}
R * G \otimes_R M \arrow[r, "E_{R*_{c,\tau,w}G\text{-}\mathsf{FGF}}(R*G\otimes_R M)"] \arrow[d, "\alpha(E_{R\text{-}\mathsf{FGF}_{c,\tau,w}}(M,e))"] & (R * G \otimes_R M)^{**} \arrow[d, "(\beta(M,e))^*"] \\
R * G \otimes_R M^{**} \arrow[r, "\beta((M,e)^*)"] & (R * G \otimes_R M^*)^*
\end{tikzcd}$$

We consider an element $u \otimes x$ in the left upper corner for $u \in R*G$ and $x \in M$. It is sent by the upper horizontal arrow to the element in $(R * G \otimes_R M)^{**}$ which maps $h \in (R * G \otimes_R M)^*$ to $\overline{h(u \otimes x)}$. This element is mapped by the right vertical arrow to the element in $(R * G \otimes_R M^*)^*$ which sends $v \otimes f$ for $v \in R * G$ and $f \in M^*$ to

$$\overline{\beta(M,e)(v \otimes f)(u \otimes x)} = \overline{uf(x)\overline{v}} = v\overline{f(x)}\overline{u}.$$

The left vertical arrow sends $u \otimes x$ to $u \otimes I_{R\text{-}\mathsf{FGF}}(x)$, where $I_{R\text{-}\mathsf{FGF}}(x)$ sends $f \in M^*$ to $f(x)$. This element is mapped by the lower horizontal arrow to the element in $(R * G \otimes_R M^*)^*$ which sends $v \otimes f$ for $v \in R * G$ and $f \in M^*$ to

$$vI_{R\text{-}\mathsf{FGF}}(x)(f)\overline{u} \quad = \quad v\overline{f(x)}\overline{u}.$$

This finishes the proof that

$$(\alpha,\beta)\colon \left(\int_G R\text{-}\mathsf{FGF}_{c,\tau,w}\right)_e \to R *_{c,\tau,w} G\text{-}\mathsf{FGF}$$

is an equivalence of additive category with involutions.

(ii) This has already been proved in Lemma 6.3 (ii). □

9. G-homology theories

In this section we construct G-homology theories and discuss induction.

Definition 9.1 (Transport groupoid)**.** Let G be a group and let ξ be a G-set. Define the *transport groupoid* $\mathcal{G}^G(\xi)$ to be the following groupoid. The set of objects is ξ itself. For $x_1, x_2 \in \xi$ the set of morphisms from x_1 to x_2 consists of those elements g in G for which $gx_1 = x_2$ holds. Composition of morphisms comes from the group multiplication in G.

A G-map $\alpha\colon \xi \to \eta$ of G-sets induces a covariant functor $\mathcal{G}^G(\alpha)\colon \mathcal{G}^G(\xi) \to \mathcal{G}^G(\eta)$ by sending an object $x \in \xi$ to the object $\alpha(x) \in \eta$. A morphism $g\colon x_1 \to x_2$ is sent to the morphism $g\colon \alpha(x_1) \to \alpha(x_2)$.

Fix a functor

$$\mathbf{E}\colon \mathsf{Add\text{-}Cat}_{\mathrm{inv}} \to \mathsf{Spectra}$$

which sends weak equivalences of additive categories with involutions to weak homotopy equivalences of spectra.

Let G be a group. Let $\mathsf{Groupoids} \downarrow G$ be the category of connected groupoids over G considered as a groupoid with one object, i.e., objects a covariant functors $F\colon \mathcal{G} \to G$ with a connected groupoid as source and G as target and a morphism from $F_0\colon \mathcal{G}_0 \to G$ to $F_1\colon \mathcal{G}_1 \to G$ is a covariant functor $W\colon \mathcal{G}_0 \to \mathcal{G}_1$ satisfying $F_1 \circ W = F_0$. For a G-set S let

$$\mathrm{pr}_G\colon \mathcal{G}^G(S) \to \mathcal{G}(G/G) = G$$

be the functor induced by the projection $S \to G/G$. The transport category yields a functor

$$\mathcal{G}^G\colon \mathsf{Or}G \to \mathsf{Groupoids} \downarrow G$$

by sending G/H to $\mathrm{pr}_G\colon \mathcal{G}^G(G/H) \to \mathcal{G}^G(G/G) = G$.

Let $\mathcal{A}$ be an additive G-category with involution in the sense of Definition 4.22. We obtain a functor

$$\mathbf{E}_{\mathcal{A}}\colon \mathsf{Or}G \to \mathsf{Spectra}, \quad G/H \mapsto \mathbf{E}\left(\textstyle\int_{\mathcal{G}(G/H)} \mathcal{A} \circ \mathrm{pr}_G\right). \tag{9.2}$$

Associated to it there is a G-homology theory in the sense of [9, Section 1]

$$H_*^G(-;\mathbf{E}_{\mathcal{A}}) \tag{9.3}$$

such that $H_*^G(G/H;\mathbf{E}_{\mathcal{A}}) \cong \pi_n\left(\mathbf{E}_{\mathcal{A}}(G/H)\right)$ holds for every $n \in \mathbb{Z}$ and every subgroup $H \subseteq G$. Namely, define for a G-CW-complex X

$$H_n^G(X;\mathbf{E}_{\mathcal{A}}) = \pi_n\left(\mathrm{map}_G(G/?, X)_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}_{\mathcal{A}}(G/?)\right).$$

For more details about spectra and spaces over a category and associated homology theories we refer to [5]. (Notice that there $\wedge_{\mathsf{Or}(G)}$ is denoted by $\otimes_{\mathsf{Or}(G)}$.)

Lemma 9.4. *Let $f\colon \mathcal{A} \to \mathcal{B}$ be a weak equivalence of additive G-categories with involution. Then the induced map*

$$H_n^G(X;\mathbf{E}_f)\colon H_n^G(X;\mathbf{E}_{\mathcal{A}}) \xrightarrow{\cong} H_n^G(X;\mathbf{E}_{\mathcal{A}})$$

is a bijection for all $n \in \mathbb{Z}$.

Proof. This follows from Lemma 7.11 and [5, Lemma 4.6]. □

Let $\phi\colon K \to G$ be a group homomorphism. Given a K-CW-complex X, let $G \times_\phi X$ be the G-CW-complex obtained from X by induction with ϕ. If $\mathcal{H}_*^G(-)$ is a G-homology theory, then $\mathcal{H}^G(\phi_*(-))$ is a K-homology theory. The next result is essentially the same as the proof of the existence of an induction structure in [1, Lemma 6.1].

Lemma 9.5. *Let $\phi\colon K \to G$ be a group homomorphism. Let $\mathcal{A}$ be an additive G-category with involution in the sense of Definition 4.22. Let $\mathrm{res}_\phi \mathcal{A}$ be the additive K-category with involution obtained from $\mathcal{A}$ by restriction with ϕ.*

Then there is a transformation of K-homology theories

$$\sigma_*\colon H_*^K(-;\mathbf{E}_{\mathrm{res}_\phi \mathcal{A}}) \to H_*^G(\phi_*(-);\mathbf{E}_{\mathcal{A}})$$

If X is a K-CW-complex on which $\ker(\phi)$ acts trivially, then

$$\sigma_n\colon H_n^K(X;\mathbf{E}_{\mathrm{res}_\phi \mathcal{A}}) \xrightarrow{\cong} H_n^G(\phi_* X;\mathbf{E}_{\mathcal{A}})$$

is bijective for all $n \in \mathbb{Z}$.

Proof. We have to construct for every K-CW-complex X a natural transformation

$$(9.6)\quad \operatorname{map}_K(K/?,X)_+ \wedge_{\mathsf{Or}(K)} \mathbf{E}\left(\int_{\mathcal{G}^K(K/?)} \operatorname{res}_\phi \mathcal{A}\circ \operatorname{pr}_K\right) \\ \to \operatorname{map}_G(G/?,\phi_* X)_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}\left(\int_{\mathcal{G}^G(G/?)} \mathcal{A}\circ \operatorname{pr}_G\right).$$

The group homomorphism ϕ induces for every transitive K-set ξ a functor, natural in ξ,

$$\mathcal{G}^\phi(\xi)\colon \mathcal{G}^K(\xi)\to \mathcal{G}^G(\phi_*\xi)$$

which sends an object $x\in\xi$ to the object (e,x) in $G\times_\phi \xi$ and sends a morphism given by $k\in K$ to the morphism given by $\phi(k)$. We obtain for every transitive K-set ξ a functor of additive categories with involutions, natural in ξ (see (7.7))

$$\mathcal{G}^\phi(\xi)_*\colon \int_{\mathcal{G}^K(\xi)} \operatorname{res}_\phi \mathcal{A}\circ \operatorname{pr}_K = \int_{\mathcal{G}^K(\xi)} \mathcal{A}\circ \operatorname{pr}_G \circ \mathcal{G}^\phi(\xi) \to \int_{\mathcal{G}^G(\phi_*\xi)} \mathcal{A}\circ \operatorname{pr}_G.$$

Thus we obtain a map of spectra

$$\operatorname{map}_K(K/?,X)_+ \wedge_{\mathsf{Or}(K)} \mathbf{E}\left(\int_{\mathcal{G}^K(K/?)} \operatorname{res}_\phi \mathcal{A}\circ \operatorname{pr}_K\right) \\ \to \operatorname{map}_K(K/?,X)_+ \wedge_{\mathsf{Or}(K)} \mathbf{E}\left(\int_{\mathcal{G}^G(\phi_*(K/?))} \mathcal{A}\circ \operatorname{pr}_G\right).$$

¿From the adjunction of induction and restriction with the functor

$$\mathsf{Or}(\phi)\colon \mathsf{Or}(K)\to \mathsf{Or}(G),\quad K/H\mapsto \phi_* K/H,$$

and the canonical map of contravariant $\mathsf{Or}(G)$-spaces

$$\mathsf{Or}(\phi)_*\left(\operatorname{map}_K(K/?,X)\right)\to \operatorname{map}_G(G/?,\phi_*X),$$

which is an isomorphism for a K-CW-complexes X, we obtain maps of spectra

$$\operatorname{map}_K(K/?,X)_+ \wedge_{\mathsf{Or}(K)} \mathbf{E}\left(\int_{\mathcal{G}^G(\phi_*(K/?))} \mathcal{A}\circ \operatorname{pr}_G\right) \\ \cong \operatorname{map}_K(K/?,X)_+ \wedge_{\mathsf{Or}(K)} \mathsf{Or}(\phi)^*\left(\mathbf{E}\left(\int_{\mathcal{G}^G(G/?)} \mathcal{A}\circ \operatorname{pr}_G\right)\right) \\ \cong \mathsf{Or}(\phi)_*\left(\operatorname{map}_K(K/?,X)\right)_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}\left(\int_{\mathcal{G}^G(G/?)} \mathcal{A}\circ \operatorname{pr}_G\right) \\ \cong \operatorname{map}_G(G/?,\phi_*X)_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}\left(\int_{\mathcal{G}^G(G/?)} \mathcal{A}\circ \operatorname{pr}_G\right).$$

Now the desired map of spectra (9.6) is the composite of the two maps above.

The proof that $\tau_n(X)$ is bijective if $\ker(\phi)$ acts freely on X is the same as the one of [1, Lemma 1.5]. ☐

10. $\mathbb{Z}$-categories and additive categories with involutions

For technical reason it will be useful that $\mathcal{A}$ comes with a (strictly associative) functorial direct sum. It will be used in the definition of the category $\operatorname{ind}_\phi \mathcal{A}$ in (11.5) and in functorial constructions about categories arising in controlled topology. (See for instance [2, Section 2.2], [3, Section 3].)

Definition 10.1 ($\mathbb{Z}$-category (with involution))**.** A *$\mathbb{Z}$-category* $\mathcal{A}$ is an additive category except that we drop the condition that finite direct sums do exists. More precisely, a $\mathbb{Z}$-category $\mathcal{A}$ is a small category such that for two objects A and B the morphism set $\operatorname{mor}_{\mathcal{A}}(A,B)$ has the structure of an abelian group and composition yields bilinear maps $\operatorname{mor}_{\mathcal{A}}(A,B) \times \operatorname{mor}_{\mathcal{A}}(B,C) \to \operatorname{mor}_{\mathcal{A}}(B,C)$.

The notion of a *$\mathbb{Z}$-category with involution* $\mathcal{A}$ is defined analogously. Namely, we require the existence of the pair $(I_{\mathcal{A}}, E_{\mathcal{A}})$ with the same axioms as in Section 1 except that we forget everything about finite direct sums.

Of course an additive category (with involution) is a $\mathbb{Z}$-category (with involution), just forget the existence of the direct sum of two objects.

Given a $\mathbb{Z}$-category $\mathcal{A}$, we can enlarge it to an additive category $\mathcal{A}_{\oplus}$ with a functorial direct sums as follows. The objects in $\mathcal{A}_{\oplus}$ are n-tuples $\underline{A} = (A_1, A_2, \ldots, A_n)$ consisting of objects A_i in $\mathcal{A}$ for $i = 1, 2, \ldots, n$ and $n = 0, 1, 2, \ldots$, where we think of the empty set as 0-tuple which we denote by 0. the $\mathbb{Z}$-module of morphisms from $\underline{A} = (A_1, \ldots, A_m)$ to $\underline{B} = (B_1, \ldots, B_n)$ is given by

$$\operatorname{mor}_{\mathcal{A}_{\oplus}}(\underline{A}, \underline{B}) := \bigoplus_{1 \le i \le m, 1 \le j \le n} \operatorname{mor}_{\mathcal{A}}(A_i, B_j).$$

Given a morphism $f\colon \underline{A} \to \underline{B}$, we denote by $f_{i,j}\colon A_i \to B_j$ the component which belongs to $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. If A or B is the empty tuple, then $\operatorname{mor}_{\mathcal{A}_{\oplus}}(A,B)$ is defined to be the trivial $\mathbb{Z}$-module. The composition of $f\colon \underline{A} \to \underline{B}$ and $g\colon \underline{B} \to \underline{C}$ for objects $\underline{A} = (A_1, \ldots, A_m)$, $\underline{B} = (B_1, \ldots, B_n)$ and $\underline{C} = (C_1, \ldots, C_p)$ is defined by

$$(g \circ f)_{i,k} := \sum_{j=1}^{n} g_{j,k} \circ f_{i,j}.$$

The sum on $\mathcal{A}_{\oplus}$ is defined on objects by sticking the tuples together, i.e., for $\underline{A} = (A_1, \ldots, A_m)$ and $\underline{B} = (B_1, \ldots, B_n)$ define

$$\underline{A} \oplus \underline{B} := (A_1, \ldots, A_m, B_1, \ldots, B_n).$$

The definition of the sum of two morphisms is now obvious. The zero object is given by the empty tuple 0. The construction is strictly associative. These data define the structure of an additive category with functorial direct sum on $\mathcal{A}_{\oplus}$. Notice that this is more than an additive category since for an additive category the existence of the direct sum of two objects is required but not a functorial model.

In the sequel functorial direct sum is always to be understood to be strictly associative, i.e., we have for three objects A_1, A_2 and A_3 the equality $(A_1 \oplus A_2) \oplus A_3 = A_1 \oplus (A_2 \oplus A_3)$ and we will and can omit the brackets from now on in the notion. We have constructed a functor from the category of $\mathbb{Z}$-categories to the category of additive categories with functorial direct sum

$$\oplus\colon \mathbb{Z}\text{-}\mathsf{Cat} \to \mathsf{Add}\text{-}\mathsf{Cat}_{\oplus}, \quad \mathcal{A} \mapsto \mathcal{A}_{\oplus}.$$

Let

$$\operatorname{forget}\colon \mathsf{Add}\text{-}\mathsf{Cat}_{\oplus} \to \mathbb{Z}\text{-}\mathsf{Cat}$$

be the forgetful functor.

Lemma 10.2. *(i) We obtain an adjoint pair of functors* $(\oplus, \operatorname{forget})$.

(ii) We get for every $\mathbb{Z}$-category $\mathcal{A}$ a functor of $\mathbb{Z}$-categories

$$Q_{\mathcal{A}}\colon \mathcal{A} \to \operatorname{forget}(\mathcal{A}_{\oplus})$$

which is natural in $\mathcal{A}$.

If $\mathcal{A}$ is already an additive category, $Q_{\mathcal{A}}$ is an equivalence of additive categories.

Proof. (i) We have to construct for every $\mathbb{Z}$-category $\mathcal{A}$ and every additive category $\mathcal{B}$ with functorial direct sum to one another inverse maps

$$\alpha\colon \operatorname{func}_{\mathtt{Add\text{-}Cat}_\oplus}(\mathcal{A}_\oplus, \mathcal{B}) \to \operatorname{func}_{\mathbb{Z}\text{-}\mathtt{Cat}}(\mathcal{A}, \operatorname{forget}(\mathcal{B}))$$

and

$$\beta\colon \operatorname{func}_{\mathbb{Z}\text{-}\mathtt{Cat}}(\mathcal{A}, \operatorname{forget}(\mathcal{B})) \to \operatorname{func}_{\mathtt{Add\text{-}Cat}_\oplus}(\mathcal{A}_\oplus, \mathcal{B}).$$

Given $F\colon \mathcal{A}_\oplus \to \mathcal{B}$, define $\alpha(F)\colon \mathcal{A} \to \mathcal{B}$ to be the composite of F with the obvious inclusion $Q_\mathcal{A}\colon \mathcal{A} \to \mathcal{A}_\oplus$ which sends A to (A). Given $F\colon \mathcal{A} \to \operatorname{forget}(\mathcal{B})$, define $\beta(F)\colon \mathcal{A}_\oplus \to \mathcal{B}$ by sending $(A_1, A_2, \ldots, A_n)$ to $F(A_1) \oplus F(A_2) \oplus \cdots \oplus F(A_n)$.

(ii) We have defined $Q_\mathcal{A}$ already above. It is the adjoint of the identity on $\mathcal{A}_\oplus$. Obviously $Q_\mathcal{A}$ induces a bijection $\operatorname{mor}_\mathcal{A}(A, B) \to \operatorname{mor}_{\mathcal{A}_\oplus}(Q_\mathcal{A}(A), Q_\mathcal{A}(B))$ for every objects $A, B \in \mathcal{A}$. Suppose that $\mathcal{A}$ is an additive category. Then every object $(A_1, A_2, \ldots, A_n)$ in $\mathcal{A}_\oplus$ is isomorphic to an object in the image of $P_\mathcal{A}$, namely to $P_\mathcal{A}(A_1 \oplus A_2 \oplus \cdots A_n) = (A_1 \oplus A_2 \oplus \cdots \oplus A_n)$. Hence $Q_\mathcal{A}$ is an equivalence of additive categories. □

Definition 10.3 (Additive category with functorial direct sum and involution)**.** An *additive category with functorial sum and involution* is an additive category with (strictly associative) functorial sum $\oplus$ and involution (I, E) which are strictly compatible with one another, i.e., if A_1 and A_2 are two objects in $\mathcal{A}$, then $I(A_1 \oplus A_2) = I(A_1) \oplus I(A_2)$ and $E(A_1 \oplus A_2) = E(A_1) \oplus E(A_2)$ hold.

One easily checks that if the $\mathbb{Z}$-category $\mathcal{A}$ comes with an involution $(I_\mathcal{A}, E_\mathcal{A})$, the additive category $\mathcal{A}_\oplus$ constructed above inherits the structure of an additive category with functorial direct sum and involution in the sense of Definition 10.3. Namely, define

$$\begin{aligned} I_{\mathcal{A}_\oplus}((A_1, A_2, \ldots, A_n)) &= (I_\mathcal{A}(A_1), I_\mathcal{A}(A_1), \ldots, I_\mathcal{A}(A_1))\,; \\ E_{\mathcal{A}_\oplus}((A_1, A_2, \ldots, A_n)) &= E_\mathcal{A}(A_1) \oplus E_\mathcal{A}(A_2) \oplus \cdots \oplus E_\mathcal{A}(A_1). \end{aligned}$$

We obtain a functor from the category of $\mathbb{Z}$-categories with involution to the category of additive categories with functorial direct sum and involution

$$\oplus\colon \mathbb{Z}\text{-}\mathtt{Cat}_{\mathrm{inv}} \to \mathtt{Add\text{-}Cat}_{\mathrm{inv}\oplus}, \quad \mathcal{A} \mapsto \mathcal{A}_\oplus.$$

Let

$$\operatorname{forget}\colon \mathtt{Add\text{-}Cat}_{\mathrm{inv}\oplus} \to \mathbb{Z}\text{-}\mathtt{Cat}_{\mathrm{inv}}$$

be the forgetful functor. One easily extends the proof of Lemma 10.2 to the case with involution.

Lemma 10.4. *(i) We obtain an adjoint pair of functors* $(\oplus, \operatorname{forget})$.

(ii) We get for every $\mathbb{Z}$-category with involution $\mathcal{A}$ a functor of $\mathbb{Z}$-categories with involution

$$Q_\mathcal{A}\colon \mathcal{A} \to \operatorname{forget}(\mathcal{A}_\oplus)$$

which is natural in $\mathcal{A}$.

If $\mathcal{A}$ is already an additive category with involution, then $Q_\mathcal{A}$ is an equivalence of additive categories with involution.

Definition 10.5. A $\mathbb{Z}$*-G-category with involution* $\mathcal{A}$ is the same as an additive G-category in the sense of Definition 4.22 except that one forgets about the direct sum.

Definition 10.6 (Additive G-category with functorial sum and (strict) involution)**.** An *additive G-category with functorial sum and involution* is an additive G-category with (strictly associative) functorial sum $\oplus$ and involution (I, E) which are strictly compatible with one another, i.e., we have:

(i) If A_1 and A_2 are two objects in $\mathcal{A}$, then $I(A_1 \oplus A_2) = I(A_1) \oplus I(A_2)$ and $E(A_1 \oplus A_2) = E(A_1) \oplus E(A_2)$ hold;
(ii) If A_1 and A_2 are two objects in $\mathcal{A}$ and $g \in G$, then $R_g(A_1) \oplus R_g(A_2) = R_g(A_1 \oplus A_2)$ holds;
(iii) If A is an object in $\mathcal{A}$, then $I(R_g(A)) = R_g(I(A))$ and $E(R_g(A)) = R_g(E(A))$ hold.

If the involution is strict in the sense of Section 1, i.e., $E = \mathrm{id}$ and $I \circ I = \mathrm{id}$, we call $\mathcal{A}$ an *additive G-category with functorial sum and strict involution.*

Define a $\mathbb{Z}$-G-category with (strict) involution analogously, just forget the direct sum.

We obtain a functor from the category of $\mathbb{Z}$-G-categories with involution to the category of additive categories with functorial direct sum and involution

$$\oplus \colon \mathbb{Z}\text{-}\mathsf{G}\text{-}\mathsf{Cat}_{\mathrm{inv}} \to \mathsf{Add}\text{-}\mathsf{G}\text{-}\mathsf{Cat}_{\mathrm{inv}\oplus}, \quad \mathcal{A} \mapsto \mathcal{A}_\oplus.$$

Let

$$\mathrm{forget} \colon \mathsf{Add}\text{-}\mathsf{G}\text{-}\mathsf{Cat}_{\mathrm{inv}\oplus} \to \mathbb{Z}\text{-}\mathsf{Cat}_{\mathrm{inv}}$$

be the forgetful functor. One easily extends the proof of Lemma 10.2 to the case with G-action and involution.

Lemma 10.7. *(i) We obtain an adjoint pair of functors $(\oplus, \mathrm{forget})$.*
(ii) We get for every $\mathbb{Z}$-G-category with involution $\mathcal{A}$ a functor of $\mathbb{Z}$-categories with involution

$$Q_{\mathcal{A}} \colon \mathcal{A} \to \mathrm{forget}(\mathcal{A}_\oplus)$$

which is natural in $\mathcal{A}$.
If $\mathcal{A}$ is already an additive G-category with involution, then $Q_{\mathcal{A}}$ is an equivalence of additive G-categories with involution;
(iii) The corresponding definitions and results carry over to the case of strict involutions.

Remark 10.8. Given an additive G-category $\mathcal{A}$ and a G-set T, we have constructed the additive G-category $\left(\int_{\mathcal{G}(T)} \mathcal{A} \circ \mathrm{pr}_G\right)_\oplus$. Let $\mathcal{A} *_G T$ be the additive G-category defined in [4, Definition 2.1]. We obtain a functor of $\mathbb{Z}$-categories

$$\rho(T) \colon \int_{\mathcal{G}(T)} \mathcal{A} \circ \mathrm{pr}_G \to \mathcal{A} *_G T$$

by sending an object (x, A) to the object $\{B_t \mid t \in T\}$ for which $B_x = A$ if $x = t$ and $B_x = 0$ if $x \neq t$. It induces a functor of additive categories with functorial direct sum

$$\rho(T)_\oplus \colon \left(\int_{\mathcal{G}(T)} \mathcal{A} \circ \mathrm{pr}_G\right)_\oplus \to (\mathcal{A} *_G T)_\oplus .$$

Recall that we have the functor of $\mathbb{Z}$-categories

$$Q_{\mathcal{A} *_G T} \colon \mathcal{A} *_G T \to (\mathcal{A} *_G T)_\oplus .$$

One easily checks that both $\rho(T)_\oplus$ and $Q_{\mathcal{A} *_G T}$ are equivalences of additive categories and natural in T.

If $\mathcal{A}$ is an additive G-category with strict involution, then we obtain on the source and the target of $\rho(T)_\oplus$ and of $Q_{\mathcal{A} *_G T}$ strict involutions such that both $\rho(T)_\oplus$ and $Q_{\mathcal{A} *_G T}$ are equivalences of additive categories with strict involution.

This implies that the G-homology theories constructed for K- and L-theory here and in [4, Definition 2.1] are naturally isomorphic and lead to isomorphic assembly maps.

11. G-homology theories and restriction

Fix a functor

$$\mathbf{E}\colon \mathsf{Add\text{-}Cat}_{\mathrm{inv}} \to \mathsf{Spectra} \tag{11.1}$$

which sends weak equivalences of additive categories with involutions to weak homotopy equivalences of spectra. We call it *compatible with direct sums* if for any family of additive categories with involutions $\{\mathcal{A}_i \mid i \in I\}$ the map induced by the canonical inclusions $\mathcal{A}_i \to \bigoplus_{i\in I} \mathcal{A}_i$ for $i \in I$

$$\bigvee_{i\in I} \mathbf{E}(\mathcal{A}_i) \to \mathbf{E}\left(\bigoplus_{i\in I} \mathcal{A}_i\right)$$

is a weak homotopy equivalence of spectra.

Example 11.2. The most important examples for $\mathbf{E}$ will be for us the functor which sends an additive category $\mathcal{A}$ to its non-connective algebraic K-theory spectrum $\mathbf{K}_{\mathcal{A}}$ in the sense of Pedersen-Weibel [11], and the functor which sends an additive category with involution $\mathcal{A}$ to its algebraic $L^{-\infty}$-spectrum $\mathbf{L}^{-\infty}_{\mathcal{A}}$ in the sense of Ranicki (see [12], [13] and [14]). Both functors send weak equivalences to weak homotopy equivalences and are compatible with direct sums. The latter follows from the fact that they are compatible with finite direct sums and compatible with directed colimits. This is proven for rings in [1, Lemma 5.2], the proof carries over to additive categories with involution.

Given a G-CW-complex X and a group homomorphism $\phi\colon K \to G$, let ϕ^*X be the K-CW-complex obtained from X by restriction with ϕ. Given a K-homology theory $\mathcal{H}^K_*$, we obtain a G-homology theory by sending a G-CW-complex X to $\mathcal{H}^K_*(\phi^*X)$. Recall that we have assigned to an additive G-category $\mathcal{A}$ with involution a G-homology theory $H^G_*(-;\mathbf{E}_{\mathcal{A}})$ in (9.3). The main result of this section is

Theorem 11.3. *Suppose that the functor* $\mathbf{E}$ *of* (11.1) *is compatible with direct sums. Let* $\phi\colon K \to G$ *be a group homomorphism. Let* $\mathcal{A}$ *be a* $\mathbb{Z}$-K*-category with involution in the sense of Definition 10.5. Let* $\operatorname{ind}_\phi \mathcal{A}$ *be the* G-$\mathbb{Z}$*-category with involution defined in* (11.5).

Then there is a natural equivalence of G*-homology theories*

$$\tau_*\colon \mathcal{H}^K\big(\phi^*(-);\mathbf{E}_{\mathcal{A}_\oplus}\big) \xrightarrow{\cong} \mathcal{H}^G\big(-;\mathbf{E}_{(\operatorname{ind}_\phi \mathcal{A})_\oplus}\big).$$

Its proof needs some preparation.

Given a contravariant functor $F\colon \mathcal{G} \to \mathsf{Add\text{-}Cat}_{\mathrm{inv}}$ from a groupoid into the category $\mathsf{Add\text{-}Cat}_{\mathrm{inv}}$ of additive categories with involution, we have defined an additive category with involution $\int_{\mathcal{G}} F$ in (7.2), provided that $\mathcal{G}$ is connected. We want to drop the assumption that $\mathcal{G}$ is connected. The connectedness of $\mathcal{G}$ was only used in the construction of the direct sum of two objects in $\int_{\mathcal{G}} F$. Hence everything goes through if we refine us to the construction of $\mathbb{Z}$-categories with involution. Namely, if we drop the connectivity assumption on $\mathcal{G}$, all constructions and all the functoriality properties explained in Section 7 remain true if we work within the category $\mathbb{Z}\text{-}\mathsf{Cat}_{\mathrm{inv}}$ instead of $\mathsf{Add\text{-}Cat}_{\mathrm{inv}}$.

Let G and K be groups. Consider a (left) K-set ξ and a K-G-biset η. Then G acts from the right on the transport groupoid $\mathcal{G}^K(\eta)$. Namely, for an element $g \in G$ the map $R_g\colon \eta \to \eta,\ x \mapsto xg$ is K-equivariant and induces a functor $\mathcal{G}^K(R_g)\colon \mathcal{G}^K(\eta) \to \mathcal{G}^K(\eta)$.

Consider a K-$\mathbb{Z}$-category with involution $\mathcal{A}$. Let $\mathrm{pr}_K\colon \mathcal{G}^K(\eta) \to \mathcal{G}^K(K/K) = K$ be the functor induced by the projection $\eta \to K/K$. Then $\mathcal{A}\circ\mathrm{pr}_K$ is a contravariant functor $\mathcal{G}^K(\eta) \to \mathbb{Z}\text{-}\mathfrak{Cat}_{\mathrm{inv}}$. We obtain a $\mathbb{Z}$-category with involution $\int_{\mathcal{G}^K(\eta)} \mathcal{A} \circ \mathrm{pr}_K$ (compare (7.2)). Given $g \in G$, the functor $\mathcal{G}^G(R_g) \colon \mathcal{G}^K(\eta) \to \mathcal{G}^K(\eta)$ induces a functor of $\mathbb{Z}$-categories with involution (compare (7.7))

$$\mathcal{G}^G(R_g) \colon \int_{\mathcal{G}^K(\eta)} \mathcal{A} \circ \mathrm{pr}_K = \int_{\mathcal{G}^K(\eta)} \mathcal{A} \circ \mathrm{pr}_K \circ \mathcal{G}^K(R_g) \to \int_{\mathcal{G}^K(\eta)} \mathcal{A} \circ \mathrm{pr}_K,$$

which strictly commutes with the involution. Thus $\int_{\mathcal{G}^K(\eta)} \mathcal{A} \circ \mathrm{pr}_K$ becomes a $\mathbb{Z}$-G-category with involution in the sense of Definition 10.5. We conclude that $\left(\int_{\mathcal{G}^K(\eta)} \mathcal{A} \circ \mathrm{pr}_K\right) \circ \mathrm{pr}_G$ is a contravariant functor $\mathcal{G}^G(\xi) \to \mathbb{Z}\text{-}\mathfrak{Cat}_{\mathrm{inv}}$. We obtain a $\mathbb{Z}$-category with involution (compare (7.2))

$$\int_{\mathcal{G}^G(\xi)} \left(\int_{\mathcal{G}^K(\eta)} \mathcal{A} \circ \mathrm{pr}_K\right) \circ \mathrm{pr}_G .$$

Consider $\eta \times \xi$ as a left $G \times K$ set by $(g,k) \cdot (y,x) = (kyg^{-1}, gx)$. Then $\mathcal{A} \circ \mathrm{pr}_{G\times K}$ is a contravariant functor $\mathcal{G}^{G\times K}(\eta \times \xi) \to \mathbb{Z}\text{-}\mathfrak{Cat}_{\mathrm{inv}}$. We obtain a $\mathbb{Z}$-category with involution (compare (7.2))

$$\int_{\mathcal{G}^{G\times K}(\eta\times\xi)} \mathcal{A} \circ \mathrm{pr}_{G\times K} .$$

Lemma 11.4. *There is an isomorphism of $\mathbb{Z}$-categories with involution*

$$\omega \colon \int_{\mathcal{G}^G(\xi)} \left(\int_{\mathcal{G}^K(\eta)} \mathcal{A} \circ \mathrm{pr}_K\right) \circ \mathrm{pr}_G \xrightarrow{\cong} \int_{\mathcal{G}^{G\times K}(\eta\times\xi)} \mathcal{A} \circ \mathrm{pr}_{G\times K}$$

which is natural in both ξ and η.

Proof. An object in $\int_{\mathcal{G}^G(\xi)} \left(\int_{\mathcal{G}^K(\eta)} \mathcal{A} \circ \mathrm{pr}_K\right) \circ \mathrm{pr}_G$ is given by $(x,(y,A))$, where $x \in \xi$ is an object in $\mathcal{G}^G(\xi)$ and (y,A) is an object in $\int_{\mathcal{G}^K(\eta)} \mathcal{A} \circ \mathrm{pr}_K$ which is given by an object $y \in \eta$ in $\mathcal{G}^K(\eta)$ and an object A in $\mathcal{A}$. The object $(x,(y,A))$ is sent under ω to the object $((y,x),A)$ given by the object (y,x) in $\mathcal{G}^{G\times K}(\eta\times\xi)$ and the object $A \in \mathcal{A}$.

A morphism ϕ in $\int_{\mathcal{G}^G(\xi)} \left(\int_{\mathcal{G}^K(\eta)} \mathcal{A} \circ \mathrm{pr}_K\right) \circ \mathrm{pr}_G$ from $(x_1,(y_1,A_1))$ to $(x_1,(y_2,A_2))$ is given by $g \cdot \psi$ for a morphism $g \colon x_1 \to x_2$ in $\mathcal{G}^G(\xi)$ and a morphism $\psi \colon (y_1,A_1) \to \mathcal{G}^K(R_g)_*(y_2,A_2)$. The morphism ψ itself is given by $k \cdot \nu$ for a morphism $k \colon y_1 \to y_2 g$ in $\mathcal{G}^K(\eta)$ and a morphism $\nu \colon A \to r_k(A)$ in $\mathcal{A}$. Define the image of ϕ under ω to be the morphism in $\int_{\mathcal{G}^{G\times K}(\eta\times\xi)} \mathcal{A} \circ \mathrm{pr}$ given by the morphism $(g^{-1},k) \colon (y_1,x_1) \to (y_2,x_2)$ in $\mathcal{G}^{G\times K}(\eta\times\xi)$ and the morphism $\phi \colon A \to r_k(A)$. This makes sense since $r_k(A)$ is the image of A under the functor $\mathcal{A} \circ \mathrm{pr}(g^{-1},k)$.

One easily checks that ω is an isomorphism of $\mathbb{Z}$-categories with involution and natural with respect to ξ and η. $\square$

Let $\phi \colon K \to G$ be a group homomorphism and ξ be a G-set. Let $\phi^*\xi$ be the K-set obtained from the G-set ξ by restriction with ϕ. Consider a K-$\mathbb{Z}$-category with involution $\mathcal{A}$ in the sense of Definition 10.5. Let ϕ^*G be the K-G-biset for which multiplication with $(k,g) \in K \times G$ sends $x \in G$ to $\phi(k)xg^{-1}$. We have explained above how $\int_{\mathcal{G}^K(\phi^*G)} \mathcal{A}$ can be considered as a G-$\mathbb{Z}$-category with involution. We will denote it by

$$\mathrm{ind}_\phi \mathcal{A} \quad := \quad \int_{\mathcal{G}^K(\phi^*G)} \mathcal{A}. \tag{11.5}$$

Lemma 11.6. *For every G-set ξ there is a natural equivalence of $\mathbb{Z}$-categories with involutions*

$$\tau\colon \int_{\mathcal{G}^G(\xi)} \operatorname{ind}_\phi \mathcal{A} \xrightarrow{\simeq} \int_{\mathcal{G}^K(\phi^*\xi)} \mathcal{A}\circ \operatorname{pr}_K .$$

It is natural in ξ.

Proof. Because of Lemma 11.4 it suffices to construct a natural equivalence

$$\tau\colon \int_{\mathcal{G}^{G\times K}(\phi^*G\times\xi)} \mathcal{A}\circ \operatorname{pr} \xrightarrow{\simeq} \int_{\mathcal{G}^K(\phi^*\xi)} \mathcal{A}\circ \operatorname{pr}_K .$$

Consider the functor

$$W\colon \mathcal{G}^{G\times K}(\phi^*G\times\xi)\to \mathcal{G}^K(\phi^*\xi)$$

sending an object $(x,y)\in G\times\xi$ in $\mathcal{G}^{G\times K}(\phi^*G\times\xi)$ to the object $xy\in\xi$ in $\mathcal{G}^K(\phi^*\xi)$ and a morphism $(g,k)\colon (x_1,y_1)\to(x_2,y_2)$ to the morphism $k\colon xy_1\to xy_2$. Now define τ to be

$$W_*\colon \int_{\mathcal{G}^{G\times K}(\phi^*G\times\xi)} \mathcal{A}\circ \operatorname{pr} = \int_{\mathcal{G}^{G\times K}(\phi^*G\times\xi)} \mathcal{A}\circ \operatorname{pr}_K\circ W \xrightarrow{\simeq} \int_{\mathcal{G}^K(\phi^*\xi)} \mathcal{A}$$

(see (7.7)). Since W is a weak equivalence of groupoids, τ is a weak equivalence of additive categories with involution by Lemma 5.9 (i). One easily checks that this construction is natural in ξ. $\square$

Now we can give the proof of Theorem 11.3.

Proof. In the sequel we write $E_\oplus\colon \mathbb{Z}\text{-}\mathsf{Cat}_{\mathrm{inv}}\to \mathsf{Spectra}$ for the composite of the functor $\mathbf{E}$ of (11.1) and the functor $\mathbb{Z}\text{-}\mathsf{Cat}_{\mathrm{inv}}\to \mathsf{Add}\text{-}\mathsf{Cat}_{\mathrm{inv}}$ sending $\mathcal{A}$ to $\mathcal{A}_\oplus$. Given a G-CW-complex X, we have to define a weak equivalence of spectra

$$\begin{aligned}&\operatorname{map}_K(K/?,\phi^*X)_+\wedge_{\mathsf{Or}(K)} \mathbf{E}_\oplus\left(\int_{\mathcal{G}^K(K/?)} \mathcal{A}\circ\operatorname{pr}_K\right)\\ &\qquad\to \operatorname{map}_G(G/?,X)_+\wedge_{\mathsf{Or}(G)} \mathbf{E}_\oplus\left(\int_{\mathcal{G}^G(G/?)} \operatorname{ind}_\phi\mathcal{A}\circ\operatorname{pr}_G\right).\end{aligned}$$

The left hand side can be rewritten as

$$\begin{aligned}&\operatorname{map}_K(K/?,\phi^*X)_+\wedge_{\mathsf{Or}(K)} \mathbf{E}_\oplus\left(\int_{\mathcal{G}^K(K/?)} \mathcal{A}\circ\operatorname{pr}_K\right)\\
&= \operatorname{map}_G(\phi_*(K/?),X)_+\wedge_{\mathsf{Or}(K)} \mathbf{E}_\oplus\left(\int_{\mathcal{G}^K(K/?)} \mathcal{A}\circ\operatorname{pr}_K\right)\\
&= \operatorname{map}_G(G/?,X)_+\wedge_{\mathsf{Or}(G)} \operatorname{map}_K(\phi_*(K/??),G/?)_+\wedge_{\mathsf{Or}(K)} \mathbf{E}_\oplus\left(\int_{\mathcal{G}^K(K/??)} \mathcal{A}\circ\operatorname{pr}_K\right)\\
&= \operatorname{map}_G(G/?,X)_+\wedge_{\mathsf{Or}(G)} \operatorname{map}_K(K/??,\phi^*G/?)_+\wedge_{\mathsf{Or}(K)} \mathbf{E}_\oplus\left(\int_{\mathcal{G}^K(K/??)} \mathcal{A}\circ\operatorname{pr}_K\right).\end{aligned}$$

Because of Lemma 11.6 the right hand side can be identified with

$$\begin{aligned}&\operatorname{map}_G(G/?,X)_+\wedge_{\mathsf{Or}(G)} \mathbf{E}_\oplus\left(\int_{\mathcal{G}^G(G/?)} \operatorname{ind}_\phi\mathcal{A}\circ\operatorname{pr}_G\right)\\
&\quad= \operatorname{map}_G(G/?,X)_+\wedge_{\mathsf{Or}(G)} \mathbf{E}_\oplus\left(\int_{\mathcal{G}^K(\phi^*G/?)} \mathcal{A}\circ\operatorname{pr}_K\right).\end{aligned}$$

Hence we need to construct for every K-set ξ a weak homotopy equivalence, natural in ξ

$$\rho(\xi)\colon \operatorname{map}_K(K/??,\xi)_+ \wedge_{\mathrm{Or}(K)} \mathbf{E}_\oplus\left(\int_{\mathcal{G}^K(K/??)} \mathcal{A}\circ \mathrm{pr}_K\right) \to \mathbf{E}_\oplus\left(\int_{\mathcal{G}^K(\xi)} \mathcal{A}\circ \mathrm{pr}_K\right).$$

The map $\rho(\xi)$ sends an element in the source given by (ϕ,z) for a K-map $\phi\colon K/??\to \xi$ and $z \in \mathbf{E}_\oplus\left(\int_{\mathcal{G}^K(K/??)} \mathcal{A}\circ \mathrm{pr}_K\right)$ to $\mathbf{E}_\oplus\left(\mathcal{G}^K(\phi)_*\right)(z)$, where

$$\mathcal{G}^K(\phi)_*\colon \int_{\mathcal{G}^K(K/??)} \mathcal{A}\circ \mathrm{pr}_K = \int_{\mathcal{G}^K(K/??)} \mathcal{A}\circ \mathrm{pr}_K \circ \mathcal{G}^K(\phi) \to \int_{\mathcal{G}^K(\xi)} \mathcal{A}\circ \mathrm{pr}_K$$

has been defined in (7.7). Obviously it is natural in ξ and is an isomorphism if ξ is a K-orbit. For a family of K-sets $\{\xi_i \mid i \in I\}$ there is a natural isomorphism of spectra

$$\begin{aligned}\bigvee_{i\in I}\left(\operatorname{map}_K(K/??,\xi_i)_+ \wedge_{\mathrm{Or}(K)} \mathbf{E}_\oplus\left(\int_{\mathcal{G}^K(K/??)} \mathcal{A}\circ \mathrm{pr}_K\right)\right)&\\ \xrightarrow{\cong} \operatorname{map}_K\left(K/??,\coprod_{i\in I}\xi_i\right)_+ \wedge_{\mathrm{Or}(K)} \mathbf{E}_\oplus\left(\int_{\mathcal{G}^K(K/??)} \mathcal{A}\circ \mathrm{pr}_K\right).&\end{aligned}$$

We have

$$\begin{aligned}\coprod_{i\in I}\mathcal{G}^K(\xi_i) &\cong \mathcal{G}^K\left(\coprod_{i\in I}\xi_i\right);\\ \coprod_{i\in I}\int_{\mathcal{G}^K(\xi_i)}\mathcal{A}\circ \mathrm{pr}_k &\cong \int_{\coprod_{i\in I}\mathcal{G}^K(\xi_i)}\mathcal{A}\circ \mathrm{pr}_k;\\ \bigoplus_{i\in I}\left(\int_{\mathcal{G}^K(\xi_i)}\mathcal{A}\circ \mathrm{pr}_k\right)_\oplus &\cong \left(\coprod_{i\in I}\int_{\mathcal{G}^K(\xi_i)}\mathcal{A}\circ \mathrm{pr}_k\right)_\oplus.\end{aligned}$$

By assumption $\mathbf{E}$ is compatible with direct sums. Hence we obtain a weak equivalence

$$\bigvee_{i\in I}\mathbf{E}_\oplus\left(\int_{\mathcal{G}^K(\xi_i)}\mathcal{A}\circ \mathrm{pr}_K\right) \xrightarrow{\simeq} \mathbf{E}_\oplus\left(\int_{\mathcal{G}^K(\coprod_{i\in I}\xi_i)}\mathcal{A}\circ \mathrm{pr}_K\right).$$

We conclude that $\rho\left(\coprod_{i\in I}\xi_i\right)$ is a weak homotopy equivalence if and only if $\bigvee_{i\in I}\rho(\xi_i)$ is a weak homotopy equivalence. Since a K-set is the disjoint union of its K-orbits and a wedge of weak homotopy equivalences of spectra is again a weak homotopy equivalence, $\rho(\xi)$ is a weak homotopy equivalence for every K-set ξ. This finishes the proof of Theorem 11.3. □

12. Proof of the main theorems

In this section we can finally give the proofs of Theorem 0.4, Theorem 0.7 and Theorem 0.12.

Proof of Theorem 0.4. This follows from Lemma 7.11 and Lemma 8.2. □

Proof of Theorem 0.7. Let $\phi\colon K \to G$ be a group homomorphism and let $\mathcal{B}$ be a additive K-category with involution. We have to show that the following assembly map is bijective

$$\operatorname{asmb}_n^{K,\mathcal{B}}\colon H_*^K(E_{\phi^*\mathcal{VC}\mathrm{yc}}(K);\mathbf{L}_\mathcal{B}) \to H_n^K(\mathrm{pt};\mathbf{L}_\mathcal{B}) = L_n\left(\int_K \mathcal{B}\right).$$

Since $\phi^* E_{\mathcal{VC}\text{yc}}(G)$ is a model for $E_{\phi^*\mathcal{VC}\text{yc}}(K)$, this follows from the commutative diagram

$$\begin{tikzcd}
H_n^K(E_{\phi^*\mathcal{VC}\text{yc}}(K);\mathbf{L}_{\mathcal{B}}) \arrow[r, "H_n^K(\mathrm{pr};\mathbf{L}_{\mathcal{B}})"] \arrow[d, "\cong"', "H_n^K(\mathrm{id};\mathbf{L}_{Q_{\mathcal{B}}})"] & H_n^K(\mathrm{pt};\mathbf{L}_{\mathcal{B}}) = L_n\left(\int_K \mathcal{B}\right) \arrow[d, "\cong"', "H_n^K(\mathrm{id};\mathbf{L}_{Q_{\mathcal{B}}})"] \\
H_n^K(E_{\phi^*\mathcal{VC}\text{yc}}(K);\mathbf{L}_{\mathcal{B}_\oplus}) \arrow[r, "H_n^K(\mathrm{pr};\mathbf{L}_{\mathcal{B}_\oplus})"] \arrow[d, "\cong"', "\tau_n^\phi(E_{\mathcal{VC}\text{yc}}(G))"] & H_n^K(\mathrm{pt};\mathbf{L}_{\mathcal{B}_\oplus}) = L_n\left(\int_K \mathcal{B}_\oplus\right) \arrow[d, "\cong"', "\tau_n^\phi(\mathrm{pt})"] \\
H_n^G\big(E_{\mathcal{VC}\text{yc}}(G);\mathbf{L}_{(\mathrm{ind}_\phi \mathcal{B})_\oplus}\big) \arrow[r, "H_n^G\left(\mathrm{pr};\mathbf{L}_{(\mathrm{ind}_\phi \mathcal{B})_\oplus}\right)"] & H_n^G\big(\mathrm{pt};\mathbf{L}_{(\mathrm{ind}_\phi \mathcal{B})_\oplus}\big) = L_n\left(\int_G (\mathrm{ind}_\phi \mathcal{B})_\oplus\right)
\end{tikzcd}$$

where pr denotes the projection onto the one-point-space pt and $Q_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{B}_\oplus$ is the natural equivalence coming from Lemma 10.7 and the vertical arrows are isomorphisms because of Lemma 9.4 and Theorem 11.3. □

Proof of Theorem 0.12. Given an additive category $\mathcal{A}$ with involution $\mathcal{A}$, we can consider it as an additive category with $(\mathbb{Z}/2, v)$-operation as explained in Example 2.4. If we apply Lemma 3.2, we obtain an additive category with strict involution $\mathcal{S}^{\mathbb{Z}/2}(\mathcal{A})$ together with a weak equivalence of additive categories with involutions

$$P_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{S}^{\mathbb{Z}/2}(\mathcal{A})$$

If $\mathcal{A}$ is an additive G-category with involution in the sense of Definition 4.22, then $P_{\mathcal{A}}$ is an equivalence of additive G-categories with involution.

If we apply Lemma 10.7, we obtain an additive G-category with functorial direct sum and strict involution $\mathcal{S}^{\mathbb{Z}/2}(\mathcal{A})_\oplus$ in the sense of Definition 10.6 and an equivalence of additive G-categories with strict involution

$$Q_{\mathcal{S}^{\mathbb{Z}/2}(\mathcal{A})}\colon \mathcal{S}^{\mathbb{Z}/2}(\mathcal{A}) \to \mathcal{S}^{\mathbb{Z}/2}(\mathcal{A})_\oplus$$

The composite

$$f := Q_{\mathcal{S}^{\mathbb{Z}/2}(\mathcal{A})} \circ P_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{S}^{\mathbb{Z}/2}(\mathcal{A})_\oplus$$

is a weak equivalence of additive G-category with involution. Now the claim follows from the following commutative diagram

$$\begin{tikzcd}
H_n^G(E_{\mathcal{VC}\text{yc}}(G);\mathbf{L}_{\mathcal{A}}) \arrow[r, "H_n^K(\mathrm{pr};\mathbf{L}_{\mathcal{A}})"] \arrow[d, "\cong"', "H_n^K(\mathrm{id};\mathbf{L}_f)"] & H_n^G(\mathrm{pt};\mathbf{L}_{\mathcal{A}}) = L_n\left(\int_G \mathcal{A}\right) \arrow[d, "\cong"', "H_n^K(\mathrm{id};\mathbf{L}_f)"] \\
H_n^G\big(E_{\mathcal{VC}\text{yc}}(G);\mathbf{L}_{\mathcal{S}^{\mathbb{Z}/2}(\mathcal{A})_\oplus}\big) \arrow[r, "H_n^K\left(\mathrm{pr};\mathbf{L}_{\mathcal{S}^{\mathbb{Z}/2}(\mathcal{A})_\oplus}\right)"] & H_n^G\big(\mathrm{pt};\mathbf{L}_{\mathcal{A}}\big) = L_n\left(\int_G \mathcal{S}^{\mathbb{Z}/2}(\mathcal{A})_\oplus\right)
\end{tikzcd}$$

whose vertical arrows are isomorphisms by Lemma 9.4. □

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