Stable categories of graded Cohen-Macaulay modules over Gorenstein algebras of different parameters

In memoriam Ragnar-Olaf Buchweitz

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Setup

- Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a noetherian *n-bimodule Calabi-Yau algebra of Gorenstein parameter* ℓ, that is

$$\mathrm{RHom}_{A^{\mathrm{env}}}(A, A^{\mathrm{env}})[n](-\ell) \cong A \quad \text{in} \quad \mathcal{D}(\mathrm{Gr}\ A^{\mathrm{env}}).$$

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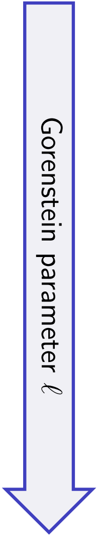
Example

$A = S\#G$, where $S = k[x_1, \ldots, x_n]$ and $G < SL(n, k)$ finite such that S^G is an isolated singularity. Take $e = \frac{1}{|G|}\sum_{g \in G} g$, in which case $eAe \cong S^G$.

Motivation



Theorem (Amiot-Iyama-Reiten)

$\ell = 1$. *There is a triangle equivalence*

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(eAe) \cong \mathcal{D}^{\mathrm{b}}(A_0/\langle e \rangle).$$

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Question: If A has Gorenstein parameter $\ell \geq 2$, when does $\underline{\mathrm{CM}}^{\mathbb{Z}}(eAe)$ have a tilting object?

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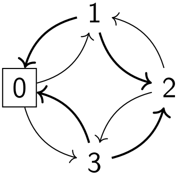
$\langle \frac{1}{4}(1,3) \rangle$		

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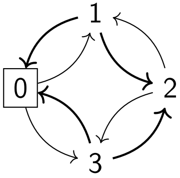
2-CY Gor. par 1: $\langle \frac{1}{4}(1,3) \rangle$		

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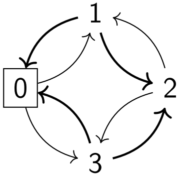
2-CY Gor. par 1:	$\langle \frac{1}{4}(1,3) \rangle$		$1 \longleftarrow 2 \longrightarrow 3$

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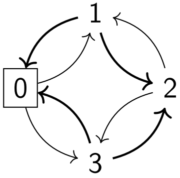
2-CY Gor. par 1:	$\langle \frac{1}{4}(1,3) \rangle$		$1 \longleftarrow 2 \longrightarrow 3$

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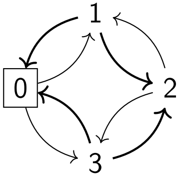
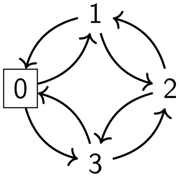
2-CY Gor. par 1: $\langle \frac{1}{4}(1,3) \rangle$		$1 \longleftarrow 2 \longrightarrow 3$
$\langle \frac{1}{4}(1,3) \rangle$		

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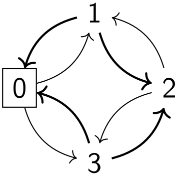
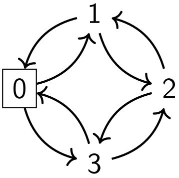
2-CY Gor. par 1:	$\langle \frac{1}{4}(1,3) \rangle$		$1 \longleftarrow 2 \longrightarrow 3$
2-CY Gor. par 2:	$\langle \frac{1}{4}(1,3) \rangle$		

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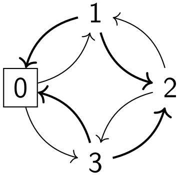
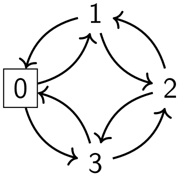
2-CY Gor. par 1:	$\langle \frac{1}{4}(1,3) \rangle$		$1 \longleftarrow 2 \longrightarrow 3$
2-CY Gor. par 2:	$\langle \frac{1}{4}(1,3) \rangle$		$1' \longleftarrow 2 \longrightarrow 3'$ $1 \longrightarrow 2' \longleftarrow 3$

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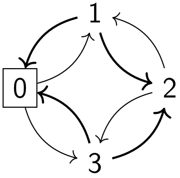
2-CY Gor. par 1:	$\langle \frac{1}{4}(1,3) \rangle$		$1 \longleftarrow 2 \longrightarrow 3$
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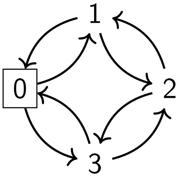
2-CY Gor. par 2:	$\langle \frac{1}{4}(1,3) \rangle$		$1' \longleftarrow 2 \longrightarrow 3'$ $1 \longrightarrow 2' \longleftarrow 3$

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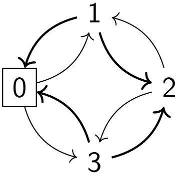
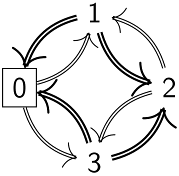
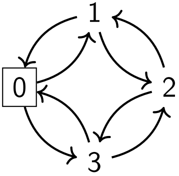
2-CY Gor. par 1:	$\langle \frac{1}{4}(1,3) \rangle$		$1 \longleftarrow 2 \longrightarrow 3$
	$\langle \frac{1}{4}(1,3,1,3) \rangle$		
2-CY Gor. par 2:	$\langle \frac{1}{4}(1,3) \rangle$		$1' \longleftarrow 2 \longrightarrow 3'$ $1 \longrightarrow 2' \longleftarrow 3$

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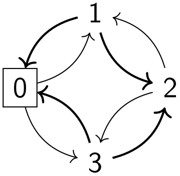
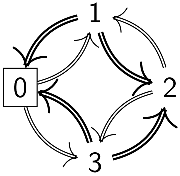
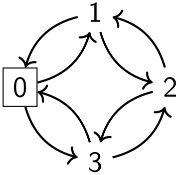
2-CY Gor. par 1:	$\langle \frac{1}{4}(1,3) \rangle$		$1 \longleftarrow 2 \longrightarrow 3$
4-CY Gor. par 2:	$\langle \frac{1}{4}(1,3,1,3) \rangle$		
2-CY Gor. par 2:	$\langle \frac{1}{4}(1,3) \rangle$		$1' \longleftarrow 2 \longrightarrow 3'$ $1 \longrightarrow 2' \longleftarrow 3$

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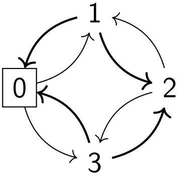

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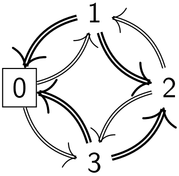

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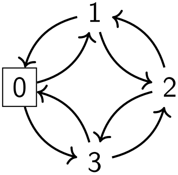

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Some bimodule CY algebras of higher Gorenstein parameter

Theorem (T.)

- *Let* Π_1 *and* Π_2 *be Koszul higher preprojective algebras. Then* $\Pi_1 \otimes \Pi_2$ *cannot be endowed with a grading structure of preprojective algebras.*
- *Suppose* $G < G_1 \times G_2$, $G_i < SL(n_i, k)$, *then* $S\#G$ *cannot be endowed with a grading structure of preprojective algebras.*

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However, they admit natural gradings of Calabi-Yau algebras of higher Gorenstein parameter.

Beilinson algebra [Minamoto-Mori]

Definition

The **Beilinson algebra**, ∇A, is defined as

$$\nabla A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-1} \\ 0 & A_0 & \cdots & A_{\ell-2} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}$$

The Beilinson algebra gives a "reduction" of the Gorenstein parameter. In fact, it is $(n-1)$-representation-infinite, so the preprojective algebra $B = \Pi(\nabla A)$ has Gorenstein parameter 1. Moreover, B is graded Morita equivalent to A and

$$\mathcal{D}^{\mathrm{b}}(\mathrm{qgr}\ eAe) \cong \mathcal{D}^{\mathrm{b}}(\mathrm{qgr}\ A) \cong \mathcal{D}^{\mathrm{b}}(\mathrm{qgr}\ B) \cong \mathcal{D}^{\mathrm{b}}(\nabla A).$$

Orlov embedding

We can use Orlov's fully faithful functors

$$\Phi_i : \underline{\mathsf{CM}}^{\mathbb{Z}}(eAe) \to \mathcal{D}^{\mathrm{b}}(\mathrm{qgr}\ eAe)$$

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Even if $\mathcal{D}^{\mathrm{b}}(\mathrm{qgr}\ eAe) \cong \mathcal{D}^{\mathrm{b}}(\mathrm{qgr}\ B)$, we cannot reduce our problem to the Gorenstein parameter 1 case. In fact, the induced idempotent $\tilde{e}$ on B rarely has the nice properties needed to restrict to this case.

Strategy: Mutations of exceptional collections

One strategy, also used by [Amiot] to recover the equivalence of [Amiot-Iyama-Reiten] ($\ell = 1$), is to compare two semi-orthogonal decompositions in $\mathcal{D}^{\mathrm{b}}(\mathrm{qgr}\ eAe)$:

$$\mathcal{D}^{\mathrm{b}}(\mathrm{qgr}\ eAe) = \langle \pi Ae, \ldots, \pi Ae(\ell - 1) \rangle \qquad \text{([Minamoto-Mori])}$$
$$\mathcal{D}^{\mathrm{b}}(\mathrm{qgr}\ eAe) = \langle \pi eAe, \ldots, \pi eAe(\ell - 1), \Phi(\underline{\mathrm{CM}}^{\mathbb{Z}}(eAe)) \rangle \quad \text{([Orlov])}$$

We can then hope to obtain a tilting object in $\underline{\mathrm{CM}}^{\mathbb{Z}}(eAe)$ by using mutations.

Levelled algebras

Definition

A quiver Q is **ordered** if $Q_0 = \{0, \ldots, m\}$ and there is no arrow $i \to j$ if $j \leq i$.

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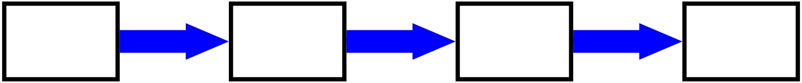

Levelled algebras behave well under mutations.

Derived equivalence in the case of levelled algebras

Theorem (T.)

If ∇A is a Koszul levelled algebra with respect to the radical grading, then there is a triangle equivalence

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Remark

The algebra A does not need to be Koszul itself.

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For levelled algebras, koszulity is closely related to being n-representation-infinite:

Lemma (Sandøy-T.)

If Λ is an n-representation-infinite n-levelled algebra, then Λ is Koszul.

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Lemma (Sandøy-T.)

If Λ is an n-representation-infinite n-levelled algebra, then Λ is Koszul.

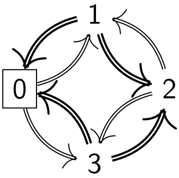
The theorem recovers the situation of the result by [Iyama-Takahashi, Mori-Ueyama] ($\ell = n$):

Proposition (T.)

If S is an AS-regular Koszul algebra, and $G < \mathrm{GrAut}\ S$ finite, then $\nabla(S\#G)$ is a Koszul levelled algebra.

Example

$S = k[x_1, x_2, x_3, x_4]$, $G = \langle \frac{1}{4}(1,3,1,3) \rangle$, A= $S\#G$, e is the idempotent corresponding to vertex 0, so that $eS\#Ge \cong S^G$.

$Q_{S\#G}$: 

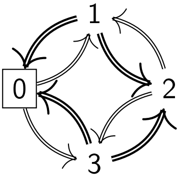
+ commutativity relations.

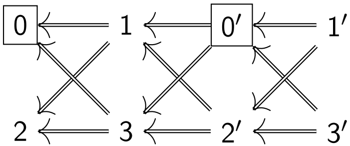
Grading: The **thick arrows** are in **degree** 1, the others in degree 0.

Then A is a (non-Koszul) 4-Calabi-Yau algebra of Gor. parameter 2.

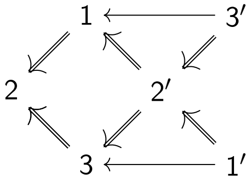
Remark: $S\#G$ cannot be endowed with a grading structure of Gor. parameter 1.

Example

$Q_{S\#G}$: 

, $Q_{(\nabla A)^!}$: 

The induced idempotent $\tilde{e}$ is the one corresponding to the vertices in the boxes.

$Q_{(1-\tilde{e})(\nabla A)^!(1-\tilde{e})}$: 

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$$\underline{\mathrm{CM}}^{\mathbb{Z}}(S^G) \cong \mathcal{D}^{\mathrm{b}}((1-\tilde{e})(\nabla A)^!(1-\tilde{e})).$$