Dissertation

Etale (φ, Γ)-Modules with Values in Linear Algebraic Groups

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To Alexandra

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Contents

Abstract

We define a theory of etale (φ, Γ)-modules with values in a linear algebraic group $\mathbb{G}$ over the ring of integers of a p-adic field and show that under certain conditions on $\mathbb{G}$, there is an abstraction of Fontaine's method for GL_n, which gives rise to a correspondence to the theory of p-adic Galois representations with values in $\mathbb{G}$.

Introduction & Summary

In the p-adic Langlands programm, we are interested in representations of the absolute Galois group of $\mathbb{Q}_p$, which we denote by $G_{\mathbb{Q}_p}$, with values in a finite field extension $L|\mathbb{Q}_p$. These are finite-dimensional L-vector spaces V, such that $G_{\mathbb{Q}_p}$ acts linearly and continuously on V with respect to the p-adic topology. By choosing an L-basis of V, we can view such a representation as a continuous morphism of groups $G_{\mathbb{Q}_p} \to \mathrm{GL}_{\dim_L(V)}(L)$. Let $\mathcal{O}_L$ denote the ring of integers of L. Then it is well known that V contains an $\mathcal{O}_L$-lattice, which is invariant under the $G_{\mathbb{Q}_p}$-action. This means that the morphism $G_{\mathbb{Q}_p} \to \mathrm{GL}_{\dim_L(V)}(L)$ is conjugate to a continuous morphism $G_{\mathbb{Q}_p} \to \mathrm{GL}_{\dim_L(V)}(\mathcal{O}_L)$.

For $L = \mathbb{Q}_p$ Fontaine constructed a period ring $\mathcal{A}_{\mathbb{Q}_p}$, such that the multiplicative monoid $\mathbb{Z}_p^\bullet := \mathbb{Z}_p \backslash \{0\}$ acts on $\mathcal{A}_{\mathbb{Q}_p}$ with respect to the ring structure and showed that the (abelian) category of continuous $G_{\mathbb{Q}_p}$-representations finitely generated over $\mathbb{Z}_p$ is equivalent to the category of etale (φ, Γ)-modules over $\mathcal{A}_{\mathbb{Q}_p}$ (See Fon90, Theorem 3.4.3). This is the category of finitely generated $\mathcal{A}_{\mathbb{Q}_p}$-modules M equipped with a semilinear $\mathbb{Z}_p^\bullet$-action, which satisfies that the image of the action of $p \in \mathbb{Z}_p^\bullet$ generates M as an $\mathcal{A}_{\mathbb{Q}_p}$-module.

If such an M is free of rank n, choosing an $\mathcal{A}_{\mathbb{Q}_p}$-basis of M and the etaleness property allow us to view the $\mathbb{Z}_p^\bullet$-action on M as a 1-cocycle $\mathbb{Z}_p^\bullet \to \mathrm{GL}_n(\mathcal{A}_{\mathbb{Q}_p})$. Since the equivalence of Fontaine preserves freeness and the rank, the bijection of the isomorphism classes induced by the equivalence gives the following statement.

Theorem. *There are inverse bijections*

$$\mathbb{D} : (\mathrm{mor}^{cont}(G_{\mathbb{Q}_p}, \mathrm{GL}_n(\mathbb{Z}_p))/\sim) \leftrightarrow H^1(\mathbb{Z}_p^\bullet, \mathrm{GL}_n(\mathcal{A}_{\mathbb{Q}_p})) : \mathbb{V},$$

where $(\mathrm{mor}^{cont}(G_{\mathbb{Q}_p}, \mathrm{GL}_n(\mathbb{Z}_p))/\sim)$ *is the set of conjugacy classes of continuous morphisms and* $H^1(\mathbb{Z}_p^\bullet, \mathrm{GL}_n(\mathcal{A}_{\mathbb{Q}_p}))$ *is the set of of cohomology classes of the* 1*-cocycles denoted by* $C^1(\mathbb{Z}_p^\bullet, \mathrm{GL}_n(\mathcal{A}_{\mathbb{Q}_p}))$.

Now let $\mathbb{G}$ be a linear algebraic group over $\mathbb{Z}_p$, i.e. an affine group scheme of finite type over $\mathbb{Z}_p$. Then there is no canonical way to view a map $f \in \mathrm{mor}^{cont}(G_{\mathbb{Q}_p}, \mathbb{G}(\mathbb{Z}_p))$ as an object of an abelian category. Here $\mathbb{G}(\mathbb{Z}_p) \cong \varprojlim \mathbb{G}(\mathbb{Z}_p/p^n\mathbb{Z}_p)$ carries the profinite topology. But we will see how these maps can be seen as functors between certain tannakian categories, such that the conjugacy classes of $\mathrm{mor}^{cont}(G_{\mathbb{Q}_p}, \mathbb{G}(\mathbb{Z}_p))$ correspond to the isomorphism classes on these functors induced by tensorproduct-preserving natural isomorphisms.

We will give an abstraction of the methods of Fontaine, such that we can give a direct correspondence for certain $\mathbb{G}$ as in the theorem above. We will start by observing the case of "p-torsion" maps, so for now $\mathbb{G}$ is a linear algebraic group over $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. For this we recall, that $\mathcal{A}_{\mathbb{Q}_p}$ is a complete discrete valuation ring with uniformizer p and residue field $\mathbb{E} := \mathbb{F}_p((X))$, the field of Laurent series with coefficients in $\mathbb{F}_p$. Any 1-cocycle $c \in C^1(G_{\mathbb{E}}, \mathbb{G}(\mathbb{E}^{sep}))$ gives a pure inner form of the basechange $\mathbb{G} \underset{\mathbb{F}_p}{\otimes} \mathbb{E}$, which we denote by $\mathbb{G}^{(c)}$. Here $G_{\mathbb{E}}$ is the absolute Galois group of $\mathbb{E}$ and $\mathbb{E}^{sep}$ is the separable closure of $\mathbb{E}$ in an algebraic closure $\overline{\mathbb{E}}$ of $\mathbb{E}$. Considering that $G_{\mathbb{Q}_p}$ has a closed subgroup H, such that $H \cong G_{\mathbb{E}}$ via the p-power cyclotomic extension, we can define

$$j : \mathrm{mor}^{cont}(G_{\mathbb{Q}_p}, \mathbb{G}(\mathbb{F}_p)) \to C^1(G_{\mathbb{E}}, \mathbb{G}(\mathbb{E}^{sep})),$$

the map given by restriction $H \subset G_{\mathbb{Q}_p}$ and inclusion $\mathbb{F}_p \subset \mathbb{E}^{sep}$. The group $\mathbb{G}^{(c)}$ might not necessarily be a group over $\mathbb{F}_p$, but there is still a way to make $\mathbb{G}^{(c)}(\mathbb{E})$ into a $\mathbb{Z}_p^\bullet$-group, if c lies in the image of j and which is dependent on the choice of an inverse image of c under j, see part 2.2.2. Then we have the following correspondence.

Theorem A. *Let $\mathbb{G}$ be connected and $(f_i)_i \subset \mathrm{mor}^{cont}(G_{\mathbb{Q}_p}, \mathbb{G}(\mathbb{F}_p))$ be a family of elements, such that $(j(f_i))_i \subset \mathrm{im}(j)$ is a set of representatives in $\mathrm{im}(j)$ for the relation of cohomology. Then we have inverse bijections*

$$\mathbb{D} : (\mathrm{mor}^{cont}(G_{\mathbb{Q}_p}, \mathbb{G}(\mathbb{F}_p))/\sim) \leftrightarrow \coprod_i H^1(\mathbb{Z}_p^\bullet, \mathbb{G}^{(j(f_i))}(\mathbb{E})) : \mathbb{V},$$

such that $H^1(\mathbb{Z}_p^\bullet, \mathbb{G}^{(j(f_i))}(\mathbb{E}))$ corresponds to

$$\{[f]_\sim \in (\mathrm{mor}^{cont}(G_{\mathbb{Q}_p}, \mathbb{G}(\mathbb{F}_p))/\sim) \mid j(f) \textit{ is cohomological to } j(f_i)\}.$$

These bijections are natural in morphisms of linear algebraic groups up to some twisted conjugation dependent on the choice of the $(f_i)_i$, see Lemma 2.2.18.

We will calculate the map j for certain classes of linear algebraic groups.

Theorem B. *In "many cases" (see Theorem 2.3.26 for details) the map j is trivial up to cohomology for semisimple groups over $\mathbb{F}_p$, which are split over $\mathbb{F}_p$.*

In particular, if $H^1(G_{\mathbb{E}}, \mathbb{G}(\mathbb{E}^{sep})) = 1$ or more general, if the map j is trivial up to cohomology, we get a correspondence

$$\mathbb{D} : (\mathrm{mor}^{cont}(G_{\mathbb{Q}_p}, \mathbb{G}(\mathbb{F}_p))/\sim) \leftrightarrow H^1(\mathbb{Z}_p^{\bullet}, \mathbb{G}(\mathbb{E})) : \mathbb{V},$$

if $\mathbb{G}$ is connected. Now let $\mathbb{G}$ be a linear algebraic group over $\mathbb{Z}_p$. Then we get the following statement.

Theorem C. *If $\mathbb{G}$ is smooth over $\mathbb{Z}_p$, j is trivial up to cohomology and the base change $\mathbb{G} \underset{\mathbb{Z}_p}{\otimes} \mathbb{F}_p$ is connected, then we have inverse bijections*

$$\mathbb{D} : (\mathrm{mor}^{cont}(G_{\mathbb{Q}_p}, \mathbb{G}(\mathbb{Z}_p))/\sim) \leftrightarrow H^1(\mathbb{Z}_p^{\bullet}, \mathbb{G}(\mathcal{A}_{\mathbb{Q}_p})) : \mathbb{V}.$$

These bijections are natural in morphisms of such linear algebraic groups.

The condition on j is not necessary, although then we obtain a correspondence as in Theorem A, but the $\mathbb{Z}_p^{\bullet}$-groups on the right hand side might not necessarily be given by $\mathcal{A}_{\mathbb{Q}_p}$-valued points of forms of $\mathbb{G} \underset{\mathbb{Z}_p}{\otimes} \mathcal{A}_{\mathbb{Q}_p} = \mathrm{Spec}(A)$, but instead by $\mathcal{A}_{\mathbb{Q}_p}$-valued points of forms of the formal group $\mathrm{Spf}(\hat{A})$, where $\hat{A}$ is the p-adic completion of A and we view $\hat{A}$ with the p-adic topology.

We can also work with $\mathbb{F} := \mathbb{E}^{perf}$ instead of $\mathbb{E}$. Under certain conditions on $\mathbb{G}$, we will then show that the theory of etale (φ, Γ)-modules with values in $\mathbb{G}$ over $W(\mathbb{F})$ is the same as the theory of etale (φ, Γ)-modules with values in $\mathbb{G}$ over $\mathcal{A}_{\mathbb{Q}_p}$. Here $W(\mathbb{F})$ denotes the ring of Witt vectors of $\mathbb{F}$, which carries a $\mathbb{Z}_p^{\bullet}$-action of rings via functoriality in $\mathbb{F}$. Recall that there is a $\mathbb{Z}_p^{\bullet}$-equivariant embedding of rings $\mathcal{A}_{\mathbb{Q}_p} \subset W(\mathbb{F})$.

Theorem D. *Let $\mathbb{G}$ be smooth over $\mathbb{Z}_p$ and the basechange $\mathbb{G} \underset{\mathbb{Z}_p}{\otimes} \mathbb{F}_p$ be connected. The inclusion $\mathcal{A}_{\mathbb{Q}_p} \subset W(\mathbb{F})$ induces a bijection*

$$H^1(\mathbb{Z}_p^{\bullet}, \mathbb{G}(\mathcal{A}_{\mathbb{Q}_p})) \tilde{\to} H^1(\mathbb{Z}_p^{\bullet}, \mathbb{G}(W(\mathbb{F}))).$$

The proof of Theorem D in this work relies on Theorem C, but in the "p-torsion" case there exists a more direct proof, which doesn't rely on Theorem A and in this case one can drop the assumptions that $\mathbb{G}$ is smooth over $\mathbb{F}_p$ and connected.

In (Sch17) constructions are given for more general objects than the ones used by Fontaine. Those objects are associated to a finite extension $L|\mathbb{Q}_p$

instead of $\mathbb{Q}_p$. We will work in this setup to obtain the Theorems A to D for these objects associated to L.

This paper is organized in the following way. Segments of the work designated by a single number x are called "chapters", by two numbers x.y are called "sections" and by three numbers x.y.z are called "parts".

In chapter 1 we will give the constructions of the objects for which we want to define the theory of etale (φ, Γ)-modules with values in $\mathbb{G}$. Furthermore, we recall the correspondence given by Fontaine and rewrite it in a way to give a motivation for the generalisations made in the rest of the paper.

In chapter 2 we will follow the methods of Fontaine to construct the desired correspondence in the "p-torsion case", i.e. when $\mathbb{G}$ is a linear algebraic group over a finite field and we observe the objects with coefficients in certain field extensions of this finite field. We also show in this case, that the theory of perfect and non-perfect (φ, Γ)-modules with values in $\mathbb{G}$ is the same.

In chapter 3 we will work with $\mathbb{G}$ being a linear algebraic group over a ring of integers of a p-adic field. We will consider smoothness of $\mathbb{G}$, so that we can use the methods of Fontaine to succesively lift the desired correspondence from the correspondence in the "p-torsion case". Furthermore we also show, that the theory of perfect and non-perfect (φ, Γ)-modules with values in $\mathbb{G}$, if $\mathbb{G}$ is smooth and has connected base change to the residue field of the p-adic field.

Notation

We follow the usual convention in commutative algebra that, unless specifically stated otherwise, every ring is commutative with unit and every morphism of rings respects the unit.

If $(E, |\cdot|)$ is a non archimedean valued field, we denote its ring of integers by $\mathcal{O}_E := \{x \in E \mid |x| \leq 1\}$ and by $\mathfrak{m}_E := \{x \in E \mid |x| < 1\}$ the unique maximal ideal in $\mathcal{O}_E$. We define its residue field $k_E := \mathcal{O}_E/\mathfrak{m}_E$. Furthermore we denote the multiplicative monoid of its non zero numbers by $\mathcal{O}_E^\bullet := \mathcal{O}_E \backslash \{0\}$.

If A is a (non commutative) ring, then $A^\times$ denotes its group of units.

If E is any field, we denote by $G_E := \mathrm{Gal}(E^{sep}|E)$ its absolute Galois group with respect to a fixed separable algebraic closure $E^{sep}|E$.

When G is a group (or a monoid), which acts on a group M, then M^G denote the G-invariants of M. If $\alpha : M \to M$ is an endomorphism of groups, we denote by $M^{\alpha=1}$ the elements in M, which are fixed by α.

If A is a ring, we denote by $A - Alg$ the category of A-algebras and $A - Mod$ the category of A-modules and Grp denotes the category of groups.

If $X \to S$ and $S' \to S$ are (formal) schemes over a base S, then by $X_{S'}$ we denote the base change $X \underset{S}{\times} S' \to S'$. We further define $X(S') := \mathrm{mor}_S(S', X)$ to be the morphisms over S. If $S' = \mathrm{Spec}(B)$ ($S' = \mathrm{Spf}(B)$) is affine, we also write $X_B := X_{S'}$ and $X(B) := X(S')$. If $\phi : S_1 \to S_2$ is a morphism over S, we write $X(\phi) : X(S_2) \to X(S_1)$ for the induced morphism.

If $f : X \to Y$ and $g : X \to Z$ are morphisms in any category, we denote by $f \times g : X \to Y \times Z$ the morphism induced by the universal property of the product. If $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ are morphisms in any category, then $(f, g) : X_1 \times X_2 \to Y_1 \times Y_2$ denotes the morphism induced by functoriality of the product, i.e. $(f, g) = (f \circ \mathrm{pr}_{X_1}) \times (g \circ \mathrm{pr}_{X_2})$, where $\mathrm{pr}_{X_i} : X_1 \times X_2 \to X_i$ is the projection for $i = 1, 2$.

- Let $L|\mathbb{Q}_p$ be a finite extension with uniformizer π and residue field k with cardinality $q := \#k$ and let $\mathbb{C}_p$ be the completion of an algebraic closure $\overline{L}$ of L. We normalize the absolute value $|\cdot|$ on $\mathbb{C}_p$, so that $|\pi| = q^{-1}$.

- If B is an $\mathcal{O}_L$-Algebra, then $W(B)_L$ denotes the ring of over L ramified Witt vectors of B, see Proposition 1.1.4.

- Let $K|L$ be a finite extension in $\overline{L}$ with uniformizer π_K. Furthermore let $K_0|L$ with $K_0 \subset K$ denote an unramified extension with residue field κ. It is $W := W(\kappa)_L \cong \mathcal{O}_{K_0}$, see Corollary 1.1.14.

- Let $K_{0,\infty} \subset \overline{L}$ denote $K_{0,\infty} := K_0 L_\infty$ the Lubin-Tate extension of L corresponding to π adjoint to K_0, see Lemma 1.1.21. Furthermore let and $H_{K_0} \subset G_{K_0}$ be the absolute galoisgroup of $K_{0,\infty}$. We define $\Gamma_{K_0} := G_{K_0}/H_{K_0} = \mathrm{Gal}(K_{0,\infty}|K_0) \cong \mathrm{Gal}(L_\infty|L) \cong \mathcal{O}_L^\times$, see Proposition 1.1.23.

- We set $H_K := G_K \cap H_{K_0} \subset G_K \subset G_{K_0}$ and define $\Gamma_K := G_K/H_K \subset \Gamma_{K_0}$. Then we have $H_K = \mathrm{Gal}(\overline{L}|K_\infty)$ and $\Gamma_K = \mathrm{Gal}(K_\infty|K)$, where $K_\infty := KK_{0,\infty} = KL_\infty$. We define $\mathbb{O}_K \subset \mathcal{O}_L^\bullet$ to be the submonoid generated by Γ_K and π, see Definition 1.3.13.

- In chapter 2 $\mathbb{G}$ will denote a linear algebraic group over k, except in part 2.1.3, where it denotes a linear algebraic group over an arbitrary field E, see Definition 2.1.7.

  In chapter 3 $\mathbb{G}$ will denote a linear algebraic group over $\mathcal{O}_L$, except in part 3.1.2, where it denotes a linear algebraic group over an arbitrary complete discrete valuation ring R, see Definition 3.1.7.

- If $L \subset F \subset \mathbb{C}_p$ is a perfectoid field, see Definition 1.1.24, then $F^\flat$ denotes its tilt, see Proposition 1.1.28.

- Let $\kappa_E := \kappa((X))$ denote the field of Laurent series with coefficients in κ, see Definition 1.1.15, and let $\mathbb{E}_{K_0}$ denote the image of the embedding $\kappa_E \to \mathbb{C}_p^\flat$ as constructed before Definition 1.1.32. Let $\mathbb{E}_{K_0}^{sep}$ be the separable closure of $\mathbb{E}_{K_0}$ in $\mathbb{C}_p^\flat$. We set $\mathbb{E} := (\mathbb{E}_{K_0}^{sep})^{H_K}$ for the G_K-action on $\mathbb{C}_p^\flat$ defined before Lemma 1.1.30, see Definition 1.1.35. Furthermore let $\mathbb{F} := \mathbb{E}^{perf}$ be the perfect hull of $\mathbb{E}$, $\mathbb{E}^{sep}$ be the separable closure of $\mathbb{E}$ in $\mathbb{C}_p^\flat$ and $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$ in $\mathbb{C}_p^\flat$, see Proposition 1.1.29.i). Of course, it is $\mathbb{E}^{sep} = \mathbb{E}_{K_0}^{sep}$.

- Let $\mathbb{A}_K$ denote the complete discrete valuation ring with residue field $\mathbb{E}$ as defined in Definition 1.2.31 and $\mathbb{A} \subset W(\mathbb{E}^{sep})_L$ be the π-adic completion of the maximal unramified extension $\mathbb{A}_K$ as constructed in Definition 1.2.32.

- By $\varphi_L \in \mathrm{End}_{\mathcal{O}_L-Alg}(W(\overline{\mathbb{F}})_L)$ we denote the Frobenius on $W(\overline{\mathbb{F}})_L$, see Definition 1.1.5. By abuse of notation, we also denote $\varphi_L \in \mathrm{End}_{k-Alg}(\overline{\mathbb{F}})$ to be the induced map on the residue field.

- By $\rho : G_K \to \mathrm{Aut}_{\mathcal{O}_L-Alg}(W(\overline{\mathbb{F}})_L)$ we denote the action defined in Definition 1.2.16 and $\overline{\rho} : G_K \to \mathrm{Aut}_{k-Alg}(\overline{\mathbb{F}})$ denotes the induced action on the residue field, see Definition 1.1.38. Furthermore let $\tau : \Gamma_K \to \mathrm{Aut}_{\mathcal{O}_L-Alg}(W(\mathbb{F})_L)$ and $\overline{\tau} : \Gamma_K \to \mathrm{Aut}_{k-Alg}(\mathbb{F})$ denote the actions induced by ρ and $\overline{\rho}$, see Definition 1.2.16 and Definition 1.1.38. We also denote $\tau : \mathbb{O}_K \to \mathrm{End}_{\mathcal{O}_L-Alg}(W(\mathbb{F})_L)$ to be the extension of τ via $\pi \mapsto \varphi_L$, see Definition 1.3.13.

1 Period Rings and Motivation

In this chapter, we will follow (Sch17) to construct the objects over which we will define the theory of etale (φ, Γ)-modules with values in $\mathbb{G}$. For this, we will give a slightly generalized version of the constructions, which are defined in (Sch17).

Furthermore, we will give a version of the correspondence of Galois representations and (φ, Γ)-modules (Sch17, Theorem 3.3.10) in the language of the linear algebraic group GL_n, that will help us understand, where the constructions in the next chapters are motivated from.

In the same way, we will give a version of the comparasion of perfect and non-perfect (φ, Γ)-modules for GL_n via (Kle16, Theorem 3.2.15).

1.1 Preliminaries and Actions in the Torsion Case

In this section, we will follow (Sch17, chapter 1.1-1.4) to give an overview of the constructions necessary to obtain the rings, we want to work with. We will only give a shortened construction and drop most of the technicalities that arise, except in the places, where we want to make slight generalizations.

1.1.1 Ramified Witt Vectors

We choose L to be our base field and give an overview how to construct Witt vectors ramified over L. For this part B will always denote an $\mathcal{O}_L$-Algebra.

Definition 1.1.1. Let $n \geq 0$ be any integer and $\mathcal{O}_L[X_0, \dots, X_n]$ denote the ring of polynomials in $n+1$ indeterminants with coefficients in $\mathcal{O}_L$. We define

$$\Phi_n(X_0, \dots, X_n) := X_0^{q^n} + \pi X_1^{q^{n-1}} + \dots + \pi^n X_n,$$

which we call the *n-th Witt polynomial.*

By abuse of notation, we also define the following map

$$\Phi_n : B^{\mathbb{N}_0} \to B, (b_n)_n \mapsto \Phi_n(b_0, \dots, b_n)$$

and introduce the map

$$\Phi_B : B^{\mathbb{N}_0} \to B^{\mathbb{N}_0}, \mathbf{b} \mapsto (\Phi_n(\mathbf{b}))_n.$$

Lemma 1.1.2. *(Sch17, Lemma 1.1.3)*

i) If $\pi 1_B$ is not a zero divisor in B, then Φ_B is injective.

ii) If $\pi 1_B \in B^\times$, then Φ_B is bijective.

We furthermore introduce the maps

$$f_B : B^{\mathbb{N}_0} \to B^{\mathbb{N}_0}, (b_0, b_1, \dots) \mapsto (b_1, b_2, \dots),$$

which is a morphism of $\mathcal{O}_L$-algebras and

$$v_B : B^{\mathbb{N}_0} \to B^{\mathbb{N}_0}(b_0, b_1, \dots) \mapsto (0, \pi b_0, \pi b_1, \dots),$$

which is a morphism of $\mathcal{O}_L$-modules.

Proposition 1.1.3. *(Sch17, Proposition 1.1.5)*
If there exists an $\sigma \in \mathrm{End}_{\mathcal{O}_L-Alg}(B)$, such that

$$\sigma(b) \equiv b^q \mod \pi B \ \forall b \in B,$$

then we have the following.
It is $B' := \mathrm{im}(\Phi_B) \subset B^{\mathbb{N}_0}$ an $\mathcal{O}_L$-subalgebra, such that

$$B' = \{(b_n)_n \in B^{\mathbb{N}_0} \mid \sigma(b_n) \equiv b_{n+1} \mod \pi^{n+1} B \ \forall n \geq 0\}$$

and

$$f_B(B') \subset B', v_B(B') \subset B'.$$

We introduce the polynomial $\mathcal{O}_L$-algebra

$$A := \mathcal{O}_L[X_0, X_1, \dots, Y_0, Y_1, \dots]$$

in two infinite and countable sets of indeterminants. We introduce $\theta \in \mathrm{End}_{\mathcal{O}_L-Alg}(A)$ by setting

$$\theta(T_n) := T_n^q \ \forall n \geq 0 \text{ and } T_n \in \{X_n, Y_n\}.$$

Then θ satisfies the condition in Proposition 1.1.3 (See Sch17, Remark 1.1.6).

We define $\mathbf{X} := (X_0, X_1, \dots) \in A^{\mathbb{N}_0}$ and analoguesly $\mathbf{Y}$. Then by Lemma 1.1.2 and Proposition 1.1.3 there exist unique elements $\mathbf{S}, \mathbf{P}, \mathbf{I}, \mathbf{F} \in A^{\mathbb{N}_0}$, satisfying the following conditions.

$$\begin{aligned}
\Phi_A(\mathbf{S}) &= \Phi_A(\mathbf{X}) + \Phi_A(\mathbf{Y}) \\
\Phi_A(\mathbf{P}) &= \Phi_A(\mathbf{X})\Phi_A(\mathbf{Y}) \\
\Phi_A(\mathbf{I}) &= -\Phi_A(X) \\
\Phi_A(\mathbf{F}) &= f_A(\Phi_A(X)).
\end{aligned}$$

We write $\mathbf{S} = (S_0, S_1, \dots)$ and analoguesly for $\mathbf{P}, \mathbf{I}, \mathbf{F}$. Then we have

$$\begin{aligned} S_n, P_n &\in \mathcal{O}_L[X_0, \dots, X_n, Y_0, \dots, Y_n], \\ I_n, F_{n-1} &\in \mathcal{O}_L[X_0, \dots, X_n] \text{ (see Sch17, discussion before Lemma 1.1.7).} \end{aligned}$$

We define $W(B)_L := B^{\mathbb{N}_0}$ with a new structure of an $\mathcal{O}_L$-algebra in the following way.

$$\begin{aligned} (a_n)_n +_{\mathbf{S}} (b_n)_n &:= (S_n(a_0, \dots, a_n, b_0, \dots, b_n))_n \\ (a_n)_n \cdot_{\mathbf{P}} (b_n)_n &:= (P_n(a_0, \dots, a_n, b_0, \dots, b_n))_n \\ \mathbf{0} &:= (0, 0, \dots) \\ \mathbf{1} &:= (1, 0, 0, \dots). \end{aligned}$$

For every morphism of $\mathcal{O}_L$-algebras $\phi : B_1 \to B_2$, we define

$$W(\phi)_L : W(B_1)_L \to W(B_2)_L, (b_n)_n \mapsto (\phi(b_n))_n.$$

Let $\phi_B : B \to \mathcal{O}_L$ be the map, which makes B into an $\mathcal{O}_L$-Algebra. The identity on $\mathcal{O}_L$ satisfies the condition in Proposition 1.1.3. It follows by Lemma 1.1.2 and Proposition 1.1.3 that we can introduce the map $\Omega : \mathcal{O}_L \to W(\mathcal{O}_L)_L$, which is given in the following way. For every $\lambda \in \mathcal{O}_L$ there exists a unique element $\Omega(\lambda)$, which satisfies $\Phi_{\mathcal{O}_L}(\Omega(\lambda)) = (\lambda, \lambda, \dots)$. We define

$$\Omega_B := W(\phi_B)_L \circ \Omega : \mathcal{O}_L \to W(B)_L.$$

Proposition 1.1.4. *(Sch17, Proposition 1.1.8)*

i) It is $(W(B)_L, +_{\mathbf{S}}, \cdot_{\mathbf{P}}, \Omega_B)$ *an* $\mathcal{O}_L$*-algebra, the* ring of ramified Witt vectors over B.

ii) The map $\Phi_B : W(B)_L \to B^{\mathbb{N}_0}$ *is a morphism of* $\mathcal{O}_L$*-algebras.*

iii) The construction $W(\cdot)_L$ *is an endofunctor of* $\mathcal{O}_L - Alg$*, i.e. for every morphism of* $\mathcal{O}_L$*-algebras* $\phi : B_1 \to B_2$*, the map* $W(\phi)_L$ *is also a morphism of* $\mathcal{O}_L$*-algebras satisfying the usual functorial identities.*

From now on, we will use the usual notation of addition and multiplication on $W(B)_L$ as well as the usual notation of scalar multiplication of $\mathcal{O}_L$ on $W(B)_L$.

Remark. The construction of $W(B)_L$ is independent of the choice of the uniformizer π.

Proof. We define the polynomial $\mathcal{O}_L$-algebra $B_0 := \mathcal{O}_L[\{X_b\}_{b\in B}]$ with set of indeterminants indexed by B. Let $\mathrm{pr} : B_0 \to B$ (resp $\theta : B_0 \to B_0$) be the $\mathcal{O}_L$-algebra morphism given by $X_b \mapsto b$ (resp. $X_b \mapsto X_b^q$). As before θ satisfies the condition of Proposition 1.1.3. It follows by Lemma 1.1.2.i) and Proposition 1.1.4 that $\Phi_{B_0} : W(B_0)_L \to \mathrm{im}(\Phi_{B_0})$ is an isomorphism of $\mathcal{O}_L$-algebras. By Proposition 1.1.3 $\mathrm{im}(\Phi_{B_0})$ is only dependent on the ideals $\pi^n B_0$ and not on the elements π^n for every $n \geq 1$, so $W(B_0)_L$ is independent on the choice of π. It follows that $W(B)_L \cong W(B_0)_L/\ker(W(\mathrm{pr})_L)$ is independent on the choice of π. $\square$

We consider the following maps on $W(B)_L$.

$$F : W(B)_L \to W(B)_L, (b_n)_n \mapsto (F_n(b_0, \dots, b_{n+1}))_n$$

and

$$V : W(B)_L \to W(B)_L, (b_n)_n \mapsto (0, b_0, b_1, \dots).$$

Definition 1.1.5. We call F the *Frobenius* and V the *Verschiebung* on $W(B)_L$.

Proposition 1.1.6. *(Sch17, Proposition 1.1.10)*
Let $\mathbf{a}, \mathbf{b} \in W(B)_L$. The maps F and V satisfy the following properties.

i) It is $f_B \circ \Phi_B = \Phi_B \circ F$ and $v_B \circ \Phi_B = \Phi_B \circ V$.

ii) It is $F \in \mathrm{End}_{\mathcal{O}_L-Alg}(W(B)_L)$.

iii) It is $V \in \mathrm{End}_{\mathcal{O}_L-Mod}(W(B)_L)$.

iv) It is $F(V(\mathbf{b})) = \pi\mathbf{b}$.

v) It is $V(\mathbf{a}\mathbf{b}) = V(\mathbf{a})\mathbf{b}$.

vi) It is $F(\mathbf{b}) \equiv \mathbf{b}^q \mod \pi W(B)_L$.

We define $V_m(B)_L := \mathrm{im}(V^m) \subset W(B)_L$ for every $m \geq 0$. Those are ideals by Proposition 1.1.6.v).

Definition 1.1.7. We call $W_m(B)_L := W(B)_L/V_m(B)_L$ the *ring of ramified Wittvectors of length m over B.*

Lemma 1.1.8. *(Sch17, Lemma 1.1.15)*
The map

$$\tau : B \to W(B)_L, b \mapsto (b, 0, 0, \dots)$$

is multiplicative. We call $\tau(b)$ the Teichmüller representative *or* Teichmüller lift *of $b \in B$.*

Definition 1.1.9. If B is a k-Algebra, then the q-Frobenius $B \to B, b \mapsto b^q$ is an endomorphism of $\mathcal{O}_L$-algebras. If this map is bijective, we call B *perfect.*

Proposition 1.1.10. *(Sch17, Proposition 1.1.18)*
If B is a k-Algebra, we have the following.

i) Any $\mathbf{b} = (b_n)_n \in W(B)_L$ satisfies

$$F(\mathbf{b}) = (b_n^q)_n \text{ and } \pi\mathbf{b} = F(V(\mathbf{b})) = V(F(\mathbf{b})) = (0, b_0^q, \dots).$$

ii) For every $m, n \geq 0$ we have $V_m(B)_L V_n(B)_L \subset V_{m+n}(B)_L$.

iii) For every $m \geq 1$ it is

$$\pi^m W(B)_L \subset V_1(B)_L^m = \pi^{m-1} V_1(B)_L \subset \pi^{m-1} W(B)_L.$$

iv) The canonical maps

$$W(B)_L \to \varprojlim W(B)_L / \pi^m W(B)_L$$

and

$$W(B)_L \to \varprojlim W(B)_L / V_1(B)_L^m$$

are bijective.

Proposition 1.1.11. *(Sch17, Proposition 1.1.19)*
If B is a perfect k-algebra we have the following.

i) The element $\pi 1_{W(B)_L} \neq 0$ is not a zero divisor in $W(B)_L$.

ii) For any $m \geq 0$, we have

$$V_m(B)_L = \pi^m W(B)_L = V_1(B)_L^m.$$

Proposition 1.1.12. *(Sch17, Proposition 1.1.21 & Remark 1.2.22)*
Let $B|k$ be a field extension.

i) The ring $W(B)_L$ is an integral domain and a local ring with maximal ideal $V_1(B)_L$ and residue field B. Furthermore its quotient field $\mathrm{Quot}(W(B)_L)$ *is of characteristic* 0.

ii) If B is perfect, then $W(B)_L$ is a complete discrete valuation ring with maximal ideal $\pi W(B)_L$ and residue field B. Furthermore any $\mathbf{b} = (b_n)_n \in W(B)_L$ has the convergent expansion

$$\mathbf{b} = \sum_{n \geq 0} \pi^n \tau(b_n^{q^{-n}}).$$

Lemma 1.1.13. *(Sch17, Proposition 1.1.23 & Corollary 1.1.24)*
Suppose that we have the following information.

i) The element $\pi 1_B$ is not a zero divisor in B.

ii) The k-algebra $B/\pi B$ is perfect.

iii) The natural map $B \to \varprojlim B/\pi^m B$ is an isomorphism.

iv) There exists an endomorphism of $\mathcal{O}_L$-algebras $\sigma : B \to B$, such that $\sigma(x) \equiv x^q \mod \pi B$ for all $x \in B$.

Then there exists a unique morphism of $\mathcal{O}_L$-algebras

$$s_B : B \to W(B)_L, \text{ such that } \Phi_B \circ s_B = (\sigma^n)_n.$$

Furthermore the morphism of $\mathcal{O}_L$-algebras

$$W(\mathrm{pr})_L \circ s_B : B \to W(B/\pi B)_L$$

is an isomorphism.

Corollary 1.1.14. *There exists an endomorphism of $\mathcal{O}_L$-algebras $\sigma : \mathcal{O}_{K_0} \to \mathcal{O}_{K_0}$, such that $\sigma(x) \equiv x^q \mod \pi\mathcal{O}_{K_0}$ for all $x \in \mathcal{O}_{K_0}$. So we have the isomorphism of $\mathcal{O}_L$-algebras*

$$W(\mathrm{pr})_L \circ s_{\mathcal{O}_{K_0}} : \mathcal{O}_{K_0} \to W(\kappa)_L$$

In particular, for any κ-algebra B, we can view $W(B)_L$ as an $\mathcal{O}_{K_0}$-algebra.

Proof. Since $K_0|L$ is unramified, the canonical map

$$f : \mathrm{End}_{\mathcal{O}_L}(\mathcal{O}_{K_0}) \to \mathrm{Gal}(\kappa|k), \phi \mapsto [x \mod \pi\mathcal{O}_{K_0} \to \phi(x) \mod \pi\mathcal{O}_{K_0}]$$

is an isomorphism. So there exists a (unique) $\sigma \in \mathrm{End}_{\mathcal{O}_L}(\mathcal{O}_{K_0})$, such that $f(\sigma) = (\cdot)^q$. Furthermore, since $K_0|L$ is unramified $\mathcal{O}_{K_0}/\pi\mathcal{O}_{K_0}$ is perfect and $\mathcal{O}_{K_0} \cong \varprojlim \mathcal{O}_{K_0}/\pi^n\mathcal{O}_{K_0}$. Obviously, $\pi \in \mathcal{O}_{K_0}$ is not a zero divisor, since $\mathcal{O}_{K_0}$ is an integral domain. □

From here on out, we will always identify $W := W(\kappa)_L = \mathcal{O}_{K_0}$.

1.1.2 Lubin-Tate Extensions

In this part, we will construct formal group laws over L and use those to construct field extensions over L, K_0 and K.

Definition 1.1.15. Let A be a ring.

i) We define the ring of *formal power series over A (in one indeterminant X)* to be the following.

$$A[[X]] := \{\sum_{n=0}^{\infty} a_n X^n \mid a_n \in A \ \forall n\}.$$

We successively define $A[[X_1, \ldots, X_m]] := A[[X_1, \ldots, X_{m-1}]][[X_m]]$.

ii) We define the ring of *Laurent series over A (in one indeterminant X)* to be the following.

$$A((X)) := \{\sum_{n=m}^{\infty} a_n X^n \mid m \in \mathbb{Z}, a_n \in A \ \forall n\}.$$

Remark. If A is a field, then $A[[X]]$ is a complete discrete valuation ring with uniformizer X, residue field A and quotient field $A((X))$.

Definition 1.1.16. A *commutative formal group law over $\mathcal{O}_L$* is a formal power series $F(X, Y) \in \mathcal{O}_L[[X, Y]]$ in two variables with coefficients in $\mathcal{O}_L$, which satisfies the following conditions.

- It is $F(X, 0) = X$ and $F(0, Y) = Y$.
- It is $F(X, F(Y, Z)) = F(F(X, Y), Z)$.
- It is $F(X, Y) = F(Y, X)$.

A morphism $h : F \to G$ between such formal group laws F and G is a formal power series $h(X) \in \mathcal{O}_L[[X]]$, such that $h(0) = 0$ and $h(F(X, Y)) = G(h(X), h(Y))$. By $\mathrm{mor}_{\mathcal{O}_L}(F, G)$ we denote the set of all those morphisms between F and G.

From this definition we see that the set of endomorphisms of a commutative formal group law F over $\mathcal{O}_L$, which we denote by $\mathrm{End}_{\mathcal{O}_L}(F)$, is a (possibly non-commutative) ring with the following structure.

$$(h_1 + h_2)(X) := F(h_1(X), h_2(X))$$
$$(h_1 \cdot h_2)(X) := h_1(h_2(X))$$

Lemma 1.1.17. *(Haz78, (A.4.7))*

Let F be a commutative formal group over $\mathcal{O}_L$ and let $E|L$ be a complete non archimedean field extension. By $\mathfrak{m}_E$ we denote the maximal ideal of the ring of integers of E.

i) There exists a unique formal power series $\iota_F(X) \in \mathcal{O}_L[[X]]$, such that

$$\iota_F(X) = -X + \text{ higher Terms and } F(X, \iota_F(X)) = 0.$$

ii) For $x, y \in \mathfrak{m}_E$ it is $x +_F y := F(x, y)$ a well defined element in $\mathfrak{m}_E$. Furthermore, $(\mathfrak{m}_E, +_F)$ is an abelian group and any $h \in \mathrm{End}_{\mathcal{O}_L}(F)$ induces an endomorphism of groups of $(\mathfrak{m}_E, +_F)$ by $x \mapsto h(x)$.

iii) The statement of ii) is also true for $\overline{L}$ instead of E.

Definition 1.1.18. A *Frobenius power series for π* is a formal power series $\phi(X) \in \mathcal{O}_L[[X]]$, which satisfies the following conditions.

$$\phi(X) = \pi X + \text{ higher terms } ; \phi(X) \equiv X^q \mod \pi\mathcal{O}_L[[X]].$$

Proposition 1.1.19. *(Sch17, Proposition 1.3.4)*

For any Frobenius power series $\phi(X)$ for π, there exists a unique commutative formal group law $F_\phi(X, Y)$ over $\mathcal{O}_L$, such that $\phi(X) \in \mathrm{End}_{\mathcal{O}_L}(F_\phi)$. We call F_ϕ the Lubin-Tate group law of ϕ.

Example. For $\phi = \pi X + X^q$, we call F_ϕ the *special Lubin-Tate group law of π*.

Proposition 1.1.20. *(Sch17, Proposition 1.3.6)*

For any Frobenius power series $\phi(X)$ for π, there exists a unique morphism of rings

$$\mathcal{O}_L \to \mathrm{End}_{\mathcal{O}_L}(F_\phi), \ a \mapsto [a]_\phi(X),$$

such that $[a]_\phi(X) = aX+$ higher terms and $[\pi]_\phi = \phi$. Furthermore, this morphism is injective.

For the rest of this part, we fix a Frobenius power series $\phi(X)$ for π and write $F := F_\phi$ for its Lubin-Tate group law. By Lemma 1.1.17.iii) and Proposition 1.1.20 we have an action

$$\mathcal{O}_L \times (\mathfrak{m}_{\overline{L}}, +_F) \to (\mathfrak{m}_{\overline{L}}, +_F), (a, z) \mapsto [a]_\phi(z).$$

So for any $n \geq 1$ we obtain the $\mathcal{O}_L$-submodule

$$\mathfrak{F}_n := \ker([\pi^n]_\phi) = \{z \in \mathfrak{m}_{\overline{L}} \mid [\pi^n]_\phi(z) = 0\},$$

which obviously is an $\mathcal{O}_K/\pi^n\mathcal{O}_K$-module and

$$\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \dots.$$

By adjoining $\mathfrak{F}_n$ to L we get the tower of field extensions

$$L \subset L_1 := K(\mathfrak{F}_1) \subset \dots \subset L_n := K(\mathfrak{F}_n) \subset L_\infty := \bigcup_n L_n \subset \overline{L}.$$

Lemma 1.1.21. *(Sch17, Remark 1.3.8)*
The extensions L_n and L_∞ depend only on the choice of π and not on the choice of ϕ. We call L_n the n-th Lubin-Tate extension for π *and L_∞ the* Lubin-Tate extension for π

Proposition 1.1.22. *(Sch17, Proposition 1.3.10)*
For any $n \geq 1$, it is $\mathfrak{F}_n$ a free $\mathcal{O}_L/\pi^n\mathcal{O}_L$-module of rank 1, such that there exist generators $z_n \in \mathfrak{F}_n$ for every $n \geq 1$ satisfying

$$[\pi]_\phi(z_{n+1}) = z_n \ \forall n \geq 1.$$

By (Ser79, II.§2 Corollary 3) every element in G_L respects the absolute value on $\overline{L}$. It follows that

$$\sigma[a]_\phi(z) = [a]_\phi(\sigma(z)) \text{ and } \sigma(F(z_1, z_2)) = F(\sigma(z_1), \sigma(z_2)) \ \forall \sigma \in G_K, z, z_1, z_2 \in \mathfrak{m}_{\overline{L}}, a \in \mathcal{O}_L.$$

So for every $n \geq 1$ we have a $\mathcal{O}_L/\pi^n\mathcal{O}_L$-linear action of G_L on $\mathfrak{F}_n$ given by

$$G_L \times \mathfrak{F}_n \to \mathfrak{F}_n, (\sigma, x) \mapsto \sigma(x).$$

Using Proposition 1.1.22, we see that for every $\sigma \in G_L$ there exists a unique $\chi_{L,n}(\sigma) \in (\mathcal{O}_L/\pi^n\mathcal{O}_L)^\times$, such that

$$\sigma(z) = [\chi_{L,n}(\sigma)](z) \ \forall z \in \mathfrak{F}_n.$$

It is $\chi_{L,n}(\sigma)$ independent on the choice of ϕ and only dependent on the choice of π (See Sch17, the discussion before Proposition 1.3.12).

Proposition 1.1.23. *(Sch17, Proposition 1.3.12)*
For any $n \geq 1$ the extension $L_n|L$ is finite Galois, and

$$\chi_{L,n} : Gal(L_n|L) \to (\mathcal{O}_L/\pi^n\mathcal{O}_L)^\times$$

is an isomorphism of groups. Furthermore the following holds.

i) The extension $L_n|L$ is totally ramified of degree $(q-1)q^{n-1}$.

ii) If $z_n \in \mathfrak{F}_n$ is a generator of $\mathfrak{F}_n$ as an $\mathcal{O}_L/\pi^n\mathcal{O}_L$-module, then we have the following statements.

a) It is $L_n = L(z_n)$.

b) The element z_n generates $\mathcal{O}_{L_n}$ as an $\mathcal{O}_L$-algebra.

c) The element z_n is a prime element of $\mathcal{O}_{L_n}$.

By passing to the projective limit with respect to n, we obtain an isomorphism

$$\chi_L : \mathrm{Gal}(L_\infty|L) \to \mathcal{O}_L^\times.$$

We define

$$K_{0,n} := K_0 L_n \text{ and } K_n := K L_n \ \forall n \in \mathbb{N} \cup \{\infty\}.$$

Since K_0 over L is unramified $K_{0,n}|K_0$ satisfies the properties of Proposition 1.1.23.i) & ii) and $K_n|K$ at least satisfies Proposition 1.1.23.ii) a)& b). Furthermore $K_{0,\infty}|K_0$ and $K_\infty|K$ are Galois extensions. We define

$$H_{K_0} := \mathrm{Gal}(\overline{L}|K_{0,\infty}) \subset G_{K_0}$$

and

$$\Gamma_{K_0} := G_{K_0}/H_{K_0} = \mathrm{Gal}(K_{0,\infty}|K_0) \cong \mathrm{Gal}(L_\infty|L) \cong \mathcal{O}_L^\times.$$

Furthermore we define

$$H_K := G_K \cap H_{K_0} = \mathrm{Gal}(\overline{L}|K_\infty) \text{ and } \Gamma_K := G_K/H_K \cong \mathrm{Gal}(K_{0,\infty}|K \cap K_{0,\infty}).$$

Since $G_K \subset G_{K_0}$ is an open subgroup, so is $\Gamma_K \subset \Gamma_{K_0} \cong \mathcal{O}_L^\times$.

1.1.3 Perfectoid Fields and Tilting

We give the Tilting construction for a perfectoid field $L \subset F \subset \mathbb{C}_p$ due to Scholze and give a small overview over the facts in this theory, that we need.

Definition 1.1.24. Let $L \subset F \subset \mathbb{C}_p$ be an intermediate field equiped with the non archimedean value $|\cdot|$ on $\mathbb{C}_p$. We say that F is *perfectoid*, if it satisfies the following conditions.

- The valued field $(F, |\cdot|)$ is complete.
- The subgroup $|F^\times| \subset \mathbb{R}_{>0}^\times$ is dense.
- It is $(\mathcal{O}_F/p\mathcal{O}_F)^p = (\mathcal{O}_F/p\mathcal{O}_F)$.

Example. Since $\mathbb{C}_p$ is algebraically closed (See Sch17, Remark 1.4.1), it is $(\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p})^p = (\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p})$. Furthermore we have $L_\infty \subset \mathbb{C}_p$, but $|L_\infty|^\times \subset \mathbb{R}^\times_{>0}$ is dense by Proposition 1.1.23.i), so $|\mathbb{C}_p^\times| \subset \mathbb{R}^\times_{>0}$ is dense. So $\mathbb{C}_p$ is perfectoid, since it is complete by definition.

We fix a perfectoid $K_0 \subset F \subset \mathbb{C}_p$ for the rest of this part.

Remark. (Sch17, Remark 1.4.3)
Every element of the value group $|F^\times|$ is a power of p.

We fix an element $\varpi \in \mathfrak{m}_F$, such that $|\varpi| \geq |\pi|$. We construct the k-algebra

$$\mathcal{O}_{F^\flat} := \varprojlim(\ldots \stackrel{(\cdot)^q}{\to} \mathcal{O}_F/\varpi\mathcal{O}_F \stackrel{(\cdot)^q}{\to} \mathcal{O}_F/\varpi\mathcal{O}_F \stackrel{(\cdot)^q}{\to} \ldots \stackrel{(\cdot)^q}{\to} \mathcal{O}_F/\varpi\mathcal{O}_F).$$

Remark 1.1.25. (Sch17, Remark 1.4.4)
The k-algebra $\mathcal{O}_{F^\flat}$ is perfect.

Lemma 1.1.26. *(Sch17, Discussion after Remark 1.4.4)*
Let $\alpha = (\ldots, \alpha_n, \ldots, \alpha_0) \in \mathcal{O}_{F^\flat}$ be an arbitrary element. Choose for any n an element $a_n \in \mathcal{O}_F$, such that $a_n \mod \varpi = \alpha_n$. Then

$$\alpha^\sharp := \lim_n a_n^{q^n} \in \mathcal{O}_F$$

is well defined and independent on the choice of the a_n.

Remark. (Sch17, Lemma 1.4.5)
The map

$$\varprojlim_{(\cdot)^q} \mathcal{O}_F \to \mathcal{O}_{F^\flat}, (\ldots, a_n, \ldots, a_0) \mapsto (\ldots, a_n \mod \varpi\mathcal{O}_F, \ldots, a_0 \mod \varpi\mathcal{O}_F)$$

is a multiplicative bijection. In particular, the k-algebra $\mathcal{O}_{F^\flat}$ is independent on the choice of ϖ.

Recall that we have the Teichmüller map $\tau : \kappa \to W = \mathcal{O}_{K_0}$. Let $\phi_F : \mathcal{O}_{K_0} \to \mathcal{O}_F$ be the map that makes $\mathcal{O}_F$ into an $\mathcal{O}_{K_0}$-algebra. We define $\tau_F : \kappa \stackrel{\tau}{\to} \mathcal{O}_{K_0} \stackrel{\phi_F}{\to} \mathcal{O}_F$.

Proposition 1.1.27. *(Compare to Sch09, Satz 2.2.2.iii))*
The map

$$\kappa \to \mathcal{O}_{F^\flat}, a \mapsto (\tau_F(a^{q^{-n}}) \mod \varpi\mathcal{O}_F)_n$$

is a morphism of k-algebras. In particular, we have that $\mathcal{O}_{F^\flat}$ is a κ-algebra.

Proposition 1.1.28. *(Sch17, Lemma 1.4.6 & Proposition 1.4.7)*

The map

$$|\cdot|_\flat : \mathcal{O}_{F^\flat} \to \mathbb{R}_{\geq 0}, \alpha \mapsto |\alpha^\flat|$$

is a nonarchimedean absolute value. Furthermore, it satisfies the following properties.

i) It is $|\mathcal{O}_{F^\flat}|_\flat = |\mathcal{O}_F|$.

ii) For any $\alpha, \beta \in \mathcal{O}_{F^\flat}$, it is

$$\alpha\mathcal{O}_{F^\flat} \subset \beta\mathcal{O}_{F^\flat} \text{ if and only if } |\alpha|_\flat \leq |\beta|_\flat.$$

iii) It is $\mathfrak{m}_{F^\flat} := \{\alpha \in \mathcal{O}_{F^\flat} \mid |\alpha|_\flat < 1\}$ the unique maximal ideal in $\mathcal{O}_{F^\flat}$.

iv) Let $\varpi^\flat \in \mathcal{O}_{F^\flat}$ be any element, such that $|\varpi^\flat|_\flat = |\varpi|$. Then the projection map sending $(\ldots, \alpha_0) \mapsto \alpha_0$ induces an isomorphism of rings

$$\mathcal{O}_{F^\flat}/\varpi^\flat\mathcal{O}_{F^\flat} \to \mathcal{O}_F/\varpi\mathcal{O}_F.$$

In particular, we have $\mathcal{O}_{F^\flat}/\mathfrak{m}_{F^\flat} \cong \mathcal{O}_F/\mathfrak{m}_F$.

In particular $\mathcal{O}_{F^\flat}$ is an integral domain. Its quotient field $F^\flat :=$ Quot$(\mathcal{O}_{F^\flat})$ has a unique multiplicative continuation of $|\cdot|_\flat$. With it, $F^\flat$ is a perfect and complete non archimedean field extension of κ, such that $\mathcal{O}_{F^\flat}$ is its ring of integers and $|F^\flat|_\flat = |F|$. We call $F^\flat$ the tilt *of F.*

Proposition 1.1.29. *i) The field $\mathbb{C}_p^\flat$ is algebraically closed.*

ii) The completions $\hat{L}_\infty$ of L_∞, $\hat{K}_\infty$ of K_∞ and $\hat{K}_{0,\infty}$ of $K_{0,\infty}$ are perfectoid and if $F_1 \subset F_2$ are two perfectoid fields over L, then we have the inclusion of valued fields

$$(F_1^\flat, |\cdot|_\flat) \subset (F_2^\flat, |\cdot|_\flat).$$

Proof. For i) and $\hat{L}_\infty$ being perfectoid, see (Sch17, Lemma 1.4.10 & Proposition 1.4.12). Since $K\hat{L}_\infty|\hat{L}_\infty$ is finite, it is complete and so $\hat{K}_\infty = K\hat{L}_\infty$. So by (Sch17, Proposition 1.6.8.i)) $\hat{K}_\infty$ is perfectoid. The same is analoguesly true for $\hat{K}_{0,\infty}$. The statement about the inclusions follows from the definition of $\mathcal{O}_{F^\flat}$ for perfectoid F and since $\mathcal{O}_{F_1} \cap \varpi\mathcal{O}_{F_2} = \varpi\mathcal{O}_{F_1}$. □

Since every $\sigma \in G_L$ acts continuous on the valued field $\overline{L}$, we get an action of G_L on $\mathbb{C}_p$. This action is continuous for the absolute value on $\mathbb{C}_p$ (See Sch17, Lemma 1.4.2).

From this point on we consider $\varpi := \pi$. Since every $\sigma \in G_L$ preserves $\pi\mathcal{O}_{\mathbb{C}_p}$, we obtain an action

$$G_L \times \mathcal{O}_{\mathbb{C}_p^\flat} \to \mathcal{O}_{\mathbb{C}_p^\flat}, (\sigma, (\dots, a_n \mod \pi\mathcal{O}_{\mathbb{C}_p}, \dots)) \mapsto (\dots, \sigma(a_n) \mod \pi\mathcal{O}_{\mathbb{C}_p}, \dots),$$

which acts by continuous endomorphisms of k-algebras. This action extends uniquely to an action on $\mathbb{C}_p^\flat$ by continuous endomorphisms of k-algebras and it preserves $|\cdot|_\flat$ by the definition of $\alpha^\sharp$ in Lemma 1.1.26. Furthermore $G_{K_0} \subset G_L$ even acts by continuous endomorphisms of κ-algebras.

Lemma 1.1.30. *(Sch17, Lemma 1.4.13) The G_L-action on the valued field $\mathbb{C}_p^\flat$ is continuous.*

The field extensions $K_\infty|K$ are normal and H_K fixes $\hat{K}_\infty$ and $\hat{K}_\infty^\flat$, since its acts by continuous morphisms. An analogues statement holds for $K_{0,\infty}|K_0$ and H_{K_0}. By (Bou66, III §2.4 Lemma 2), the projection $G_K \to \Gamma_K$ is an open map. It follows by an easy calculation (see for example (Kle16, Lemma 2.1.21)) that we get continuous actions

$$\Gamma_K \times \hat{K}_\infty^\flat \to \hat{K}_\infty^\flat$$

and

$$\Gamma_{K_0} \times \hat{K}_{0,\infty}^\flat \to \hat{K}_{0,\infty}^\flat$$

by Proposition 1.1.29.ii) and since the G_K-action on $\mathbb{C}_p^\flat$ is continuous by Lemma 1.1.30.

We fix a Frobenius power series ϕ for π and define the following $\mathcal{O}_L$-module.

$$T := \varprojlim(\dots \overset{[\pi]_\phi(\cdot)}{\to} \mathfrak{F}_n \overset{[\pi]_\phi(\cdot)}{\to} \mathfrak{F}_{n-1} \overset{[\pi]_\phi(\cdot)}{\to} \dots \overset{[\pi]_\phi(\cdot)}{\to} \mathfrak{F}_1).$$

Since $\phi(X) \equiv X^q \mod \pi\mathcal{O}_L[[X]]$, we have

$$y_{m+1}^q \equiv y_m \mod \pi\mathcal{O}_{\hat{K}_{0,\infty}} \ \forall m \geq 1, (y_n)_n \in T.$$

It follows that

$$\iota : T \to \mathcal{O}_{\hat{K}_{0,\infty}^\flat}, (y_n)_n \mapsto (\dots, y_n \mod \pi\mathcal{O}_{\hat{K}_{0,\infty}}, \dots, y_1 \mod \pi\mathcal{O}_{\hat{K}_{0,\infty}}, 0)$$

is a well defined map. By Proposition 1.1.22, we have that T is a free $\mathcal{O}_K$-module of rank one and that an element $(z_n)_n \in T$ is a generator of T as an $\mathcal{O}_L$-module if and only if z_n is a generator of $\mathfrak{F}_n$ as an $\mathcal{O}_L/\pi^n\mathcal{O}_L$-module for all n. We fix such a generator $t = (z_n)_n \in T$.

Lemma 1.1.31. *(Sch17, Lemma 1.4.14) It is $|\iota(t)|_\flat = |\pi|^{\frac{q}{q-1}} < |\pi|$.*

We set $\omega := \iota(t)$. By Lemma 1.1.31 we get a morphism of κ-algebras

$$\kappa[[X]] \to \mathcal{O}_{\hat{K}^\flat_{0,\infty}}, f(X) \mapsto f(\omega).$$

Since $\omega \in \hat{K}^\flat_{0,\infty}$ is invertible this morphism extends to an embedding of fields

$$\iota : \kappa((X)) \to \hat{K}^\flat_{0,\infty}.$$

Definition 1.1.32. We define the subfield $\mathbb{E}_{K_0} := \iota(\kappa((X))) \subset \hat{K}^\flat_{0,\infty}$.

Remark. The valued field $(\mathbb{E}_{K_0}, |\cdot|_\flat)$ is a complete non archimedean discretely valued field with residue field κ, uniformizer ω and its ring of integers satisfies $\mathcal{O}_{\mathbb{E}_{K_0}} \cong \kappa[[X]]$.

Lemma 1.1.33. *(Compare to Sch17, Lemma 1.4.15)*
For any $a \in \mathcal{O}_K$, we put $\overline{[a]}(X) := [a]_\phi(X) \mod \pi\mathcal{O}_L[[X]] \in \kappa[[X]]$.

i) For any $\gamma \in \Gamma_{K_0}$ we have $\gamma(\omega) = \overline{[\chi_L(\gamma)]}(\omega)$.

ii) The Γ_{K_0}-action on $\hat{K}^\flat_{0,\infty}$ preserves the subfield $\mathbb{E}_{K_0}$.

iii) The subfield $\mathbb{E}_{K_0}$ of $\hat{K}^\flat_{0,\infty}$ does not depend on the choice of the generator $t \in T$.

We define $\mathbb{E}^{sep}_{K_0} \subset \mathbb{C}_p^\flat$ to be the separable closure in $\mathbb{C}_p^\flat$. By Proposition 1.1.29.i) it is a separably algebraically closed extension of $\mathbb{E}_{K_0}$ and the G_{K_0}-action on $\mathbb{C}_p^\flat$ preserves $\mathbb{E}^{sep}_{K_0}$ by Lemma 1.1.33.ii).

Lemma 1.1.34. *Let $\mathbb{E}_L \subset \mathbb{E}_{K_0}$ be the image of $k((X)) \subset \kappa((X))$ under j. Then $\mathbb{E}^{sep}_L = \mathbb{E}^{sep}_{K_0}$. Furthermore by (Sch17, Lemma 1.4.15.ii)), the G_L-action on $\mathbb{C}_p^\flat$ preserves $\mathbb{E}^{sep}_L$.*

Proof. This follows from the fact that $\kappa((X))|k((X))$ is the unique unramified extension in $\overline{k}((X))$ with residue field κ and so, it is separable by Lemma 1.2.26.i). □

Definition 1.1.35. We define the κ-algebra

$$\mathbb{E} := (\mathbb{E}^{sep}_{K_0})^{H_K} = (\mathbb{E}^{sep}_L)^{H_K} \subset \hat{K}^\flat_\infty$$

The last inclusion is well defined by the remark after Lemma 1.1.26 and since $\hat{K}_\infty = \mathbb{C}_p^{H_K}$ by (Ax69, Theorem).

Remark. Since $H_K \subset G_K$ is normal, the continuous G_K-action on $\mathbb{C}_p^\flat$ preserves $\mathbb{E}$ and hence induces a continuous action of Γ_K on $\mathbb{E}$.

Definition 1.1.36. Let E be a field of characteristic p with a fixed algebraiclly closed field extension $C|E$. We define the *perfect hull of E in C* to be

$$E^{perf} := \{x \in C \mid \exists n \in \mathbb{N} : x^{p^n} \in E\}.$$

Remark. The subset $E^{perf} \subset C$ is a subfield, which is algebraic over E.

Lemma 1.1.37. *(Bou90, §§V.5.2 and V.7.7)*

Let E be a field of characteristic p with fixed algebraic closure $\overline{E}$. Then $E^{perf} \subset \overline{E}$ satisfies the following conditions.

i) The field extension $E^{perf}|E$ is the largest intermediate field of $\overline{E}|E$ which is purely inseparable over E.

ii) The field extension $E^{perf}|E$ is the smalles intermediate field of $\overline{E}|E$ which is perfect. In particular, the extension $\overline{E}|E^{perf}$ is Galois.

iii) If $E^{sep}|E$ denotes the separable closure of E in $\overline{E}$, then $E^{sep} \cap E^{perf} = E$ and $\overline{E} = (E^{perf})^{sep} = (E^{sep})^{perf} = E^{sep}E^{perf}$. In particular, restricting automorphisms to E^{sep} induces a topological isomorphism of groups $\mathrm{Gal}(\overline{E}|E^{perf}) \to \mathrm{Gal}(E^{sep}|E)$.

By Proposition 1.1.29.i), we can define the perfect hull of $\mathbb{E}$ in $\mathbb{C}_p^\flat$

$$\mathbb{F} := \mathbb{E}^{perf} \subset \mathbb{C}_p^\flat.$$

We furthermore define $\mathbb{E}^{sep}$ (resp. $\overline{\mathbb{F}}$) to be the separable closure of $\mathbb{E}$ (resp. $\mathbb{F}$) in $\mathbb{C}_p^\flat$. Again by Proposition 1.1.29.i), $\overline{\mathbb{F}}$ is an algebraically closed field and obviously $\mathbb{E}^{sep} = \mathbb{E}_L^{sep}$.

By Lemma 1.1.37.iii) we can identify the absolute Galois group $G_{\mathbb{E}}$ of $\mathbb{E}$ with the absolute Galois group $G_{\mathbb{F}}$ of $\mathbb{F}$.

Since the G_K-action on $\mathbb{C}_p^\flat$ preserves $\mathbb{E}_L^{sep}$ and $H_K \subset G_K$ is normal, it preserves $\mathbb{F}$ and $\mathbb{E}$.

Definition 1.1.38. The continuous G_K-action on $\overline{\mathbb{F}}$ gives us a map

$$\overline{\rho} : G_K \to \mathrm{Aut}_{\kappa-Alg}(\overline{\mathbb{F}}).$$

By abuse of notation we also denote $\overline{\rho} : G_K \to \mathrm{Aut}_{\kappa-Alg}(\mathbb{E}^{sep})$ to be the map given by restriction. We furthermore define $\overline{\tau} : \Gamma_K \to \mathrm{Aut}_{\kappa-Alg}(\mathbb{F})$ (and also $\overline{\tau} : \Gamma_K \to \mathrm{Aut}_{\kappa-Alg}(\mathbb{E})$) to be the map induced by $\overline{\rho}$.

Since $\overline{\rho}(H_K)$ fixes $\mathbb{E}$, the map $\overline{\rho}$ induces a map

$$\overline{\rho} : H_K \to G_{\mathbb{E}}$$

by restricting to H_K. Furthermore, since $\mathbb{E}_L \subset \hat{L}_\infty^\flat$, we have a restriction

$$\overline{\rho} : H_L := \mathrm{Gal}(\overline{L}|L_\infty) \to G_{\mathbb{E}_L}$$

Theorem 1.1.39. *(Sch17, Theorem 1.6.7)*
The map

$$\overline{\rho} : H_L \to G_{\mathbb{E}_L}$$

is a topological isomorphism of groups.

Corollary 1.1.40. *i) The map*

$$\overline{\rho} : H_K \to G_{\mathbb{E}}$$

is a topological isomorphism of groups.

ii) The extension $\mathbb{E}|\mathbb{E}_L$ is finite. In particular $\mathbb{E} \cong k_{\mathbb{E}}((Y))$ is a local field.

iii) It is $\mathbb{E}_{K_0} = (\mathbb{E}_L^{sep})^{H_{K_0}}$. In particular, if $K = K_0$ is unramified over L, we have $\mathbb{E} = \mathbb{E}_{K_0}$.

Proof. The first statement follows from Theorem 1.1.39 and the main theorem of Galois theory. Furthermore, since $K_\infty = KL_\infty|L_\infty$ is a finite extension, it is $H_K \subset H_L$ open and hence $\mathbb{E}|\mathbb{E}_L$ is finite by the main theory of Galois theory. For the third statement, we know that $\mathbb{E}_{K_0}|\mathbb{E}_L$ is a finite Galois extension, so by the main theorem of Galois theory there exists an open and normal $H \subset H_L$, such that $\mathbb{E}_{K_0} = (\mathbb{E}_L^{sep})^H$ and $H_L/H \cong \mathrm{Gal}(\mathbb{E}_{K_0}|\mathbb{E}_L)$. By (Ax69, Theorem) and the Remark after Lemma 1.1.26, it is $(\mathbb{C}_p^\flat)^{H_{K_0}} = \hat{K}_{0,\infty}^\flat$ and hence, $(\mathbb{E}_L^{sep})^H \subset (\mathbb{E}_L^{sep})^{H_{K_0}}$ and so $H_{K_0} \subset H \subset H_L$. Since $L_n|L$ is totally ramified for all $n \geq 1$ and $K_0|L$ is unramified and Galois, we have

$$H_L/H_{K_0} = \mathrm{Gal}(K_0L_\infty|L_\infty) \cong \mathrm{Gal}(K_0|L) \cong \mathrm{Gal}(\kappa|k).$$

On the other hand, since $\mathbb{E}_{K_0}|\mathbb{E}_L$ is unramified, we have

$$H_L/H \cong \mathrm{Gal}(\mathbb{E}_{K_0}|\mathbb{E}_L) \cong \mathrm{Gal}(\kappa|k)$$

and so $H_{K_0} \subset H$ is actually an equality. $\square$

We furthermore define

$$\varphi_L : \overline{\mathbb{F}} \to \overline{\mathbb{F}}, x \mapsto x^q.$$

Obviously, this map commutes with the automorphisms $\overline{\rho}(\sigma)$ for all $\sigma \in G_K$.

Definition 1.1.41. A topological field is a field E, which is a topological ring, such that $E^\times$ is a topological group.

We recall the following elementary fact.

Remark 1.1.42. Any field with an absolute value is a topological field.

Proof. Let $(E, |\cdot|)$ denote such a field. Let $\epsilon > 0$.

By the triangle inequality, the addition is continuous on such a field.

Let $x, y \in E$ and $x' \in B_{\delta_1}(x), y' \in B_{\delta_2}(y)$, then

$$|xy - x'y'| = |xy - x'y + x'y - x'y'| \leq |x - x'||y| + |x'||y - y'| < \delta_1|y| + \delta_2(|x| + \delta_1),$$

where the last inequality follows from

$$|x| - |x'| \leq |x - x'| < \delta_1$$

Choose $\delta_1 = \frac{\epsilon}{2|y|}$ and $\delta_2 = \frac{\frac{\epsilon}{2}}{|x|+\delta_1}$.

Let $x \in E^\times$ and $0 \neq y \in B_\delta(x)$. If $\delta \leq \frac{|x|}{2}$, then

$$|\frac{1}{x} - \frac{1}{y}| = \frac{|x - y|}{|x||y|} < \frac{\delta}{|x|(|x| - \delta)} \leq \frac{2\delta}{|x|^2}.$$

So choose $\delta := min\{\frac{|x|}{2}, \frac{\epsilon|x|^2}{2}\}$. □

In particular, all the fields we are observing in this part are topological fields with their non archimidean value.

Lastly, we will need to use the following map.

Lemma 1.1.43. *(Sch17, Lemma 1.4.18)*

The map

$$\Theta_F : W(\mathcal{O}_{F^\flat})_L \to \mathcal{O}_F, \sum_{n\geq 0} \tau(\alpha_n)\pi^n \to \sum_{n\geq 0} \alpha_n^\sharp \pi^n$$

is a well-defined surjective morphism of $\mathcal{O}_L$-algebras.

1.2 Topologies and the Period Rings

In this section, we want to construct the period rings in characteristic 0 as subrings of certain rings of ramified Witt vectors and furthermore lift the actions from the last section onto these rings. For this, we will need to endow the ramified Witt vectors with a topology, such that the lifted actions become continuous for this topology on the ring of Witt vectors over the fields we constructed in the last section.

We will follow (Sch17, chapter 1.5 & 1.7) for the general constructions of the topologies we want. Then we will embed the ring of integers of a two-dimensional local field with residue field $\kappa((X))$ into $W(\mathbb{E})_L$. Lastly, we will see that the maximal unramified extension of this two-dimensional local field can be embedded into $W(\mathbb{E}^{sep})_L$ and will define our ring of coefficients as a ring of H_K-invariants of this ring. These are slight generalizations of the constructions made in (Sch17, chapters 2.1 & 3.1).

1.2.1 Weak Topologies and Actions

We will begin this part by constructing a two-dimensional local field with residue field $\kappa((X))$. We will then construct certain topologies on this field and on rings of ramified Witt vectors.

Definition 1.2.1. We define the $\mathcal{O}_{K_0}$-algebra

$$\mathcal{A}_{K_0} := \varprojlim \mathcal{O}_{K_0}((X))/\pi^n \mathcal{O}_{K_0}((X)).$$

Remark. (Sch17, Discussion at the beginning of chapter 1.7)

We have the isomorphism of $\mathcal{O}_{K_0}$-algebras

$$\{\sum_{n\in\mathbb{Z}} a_n X^n \mid a_n \in \mathcal{O}_{K_0}, \lim_{n\to-\infty} a_n = 0\} \to \mathcal{A}_{K_0}, \sum_{n\in\mathbb{Z}} a_n X^n \mapsto (\sum_{n\in\mathbb{Z}} (a_n \mod \pi^m \mathcal{O}_{K_0}) X^n)_m.$$

From now on, we will often write $\sum_{n\in\mathbb{Z}} a_n X^n$ for elements in $\mathcal{A}_{K_0}$. So for $f(X) = \sum_{n\in\mathbb{Z}} a_n X^n \in \mathcal{A}_{K_0}$ we define

$$|f(X)| := \max_n |a_n|.$$

Note that this maximum exists, since K_0 is discretely valued.

Lemma 1.2.2. *(Sch17, Lemma 1.7.1, Remark 1.7.2 & the following discussion)* *i) The map $|\cdot|$ on $\mathcal{A}_{K_0}$ is a non archimedean absolute value, which makes $\mathcal{A}_{K_0}$ into a complete discrete valuation ring with uniformizer π and residue field $\kappa((X))$. The quotient field of $\mathcal{A}_{K_0}$ satisfies*

$$\mathcal{B}_{K_0} := \mathrm{Quot}(\mathcal{A}_{K_0}) = \{\sum_{n\in\mathbb{Z}} a_n X^n \mid a_n \in K_0, \sup_n |a_n| < \infty, \lim_{n\mapsto -\infty} |a_n| = 0\}$$

and its ring of integers is $\mathcal{A}_{K_0}$.

ii) If $g(X) \in X\mathcal{O}_{K_0}[[X]]$ and $g(X) \in \mathcal{A}_{K_0}^\times$, then g induces an endomorphism of $\mathcal{O}_{K_0}$-algebras

$$\mathcal{A}_{K_0} \to \mathcal{A}_{K_0}, f(X) \mapsto f(g(X)),$$

which extends to an endomorphism of the field of fractions

$$\mathcal{B}_{K_0} \to \mathcal{B}_{K_0}, f(X) \mapsto f(g(X)).$$

Let ϕ be a fixed Frobenius power series for π. Because of Lemma 1.2.2.ii) we get an action

$$\Gamma_{K_0} \times \mathcal{A}_{K_0} \to \mathcal{A}_{K_0}, (\gamma, f) \mapsto \ {}^{\gamma}f := f([\chi_L(\gamma)]_\phi(X))$$

and a injective Frobenius endomorphism of $\mathcal{O}_{K_0}$-algebras

$$\varphi_{K_0} : \mathcal{A}_{K_0} \to \mathcal{A}_{K_0}, f \mapsto f([\pi]_\phi(X)).$$

The induced map $\varphi_{K_0} : \kappa((X)) \to \kappa((X))$ is the κ-algebra morphism given by $X \mapsto X^q$, since $\phi(X) \equiv X^q \mod \pi\mathcal{O}_L[[X]]$.

Since $\mathcal{O}_L^\bullet$ is a commutative monoid, the map φ_{K_0} commutes with the action of every $\gamma \in \Gamma_{K_0}$.

We give $\mathcal{A}_{K_0}$ the topology of a topological $\mathcal{O}_K$-module by setting

$$U_m := X^m\mathcal{O}[[X]] + \pi^m\mathcal{A}_{K_0} \ \forall m \geq 1$$

as a fundamental system of open neighbourhoods of 0. This is possible, since $U_{\max(m,n)} \subset U_m \cap U_n$. Obviously, the following submodules also form a fundamental system of open neighbourhoods of 0 for this topology.

$$U_{l,m} := X^l\mathcal{O}[[X]] + \pi^m\mathcal{A}_{K_0} \ \forall l, m \geq 1.$$

Definition 1.2.3. We call this topology the *weak topology on $\mathcal{A}_{K_0}$*.

Proposition 1.2.4. *(Sch17, Lemma 1.7.6) The $\mathcal{O}_{K_0}$-algebra $\mathcal{A}_{K_0}$ is a complete Hausdorff topological $\mathcal{O}_{K_0}$-algebra with respect to the weak topology.*

Beware that we still need an endomorphism φ_L on $\mathcal{A}_K$ of $\mathcal{O}_L$-algebras, which is a lift of the map $(\cdot)^q$ and not of the κ-algebra endomorphism on $\kappa((X))$ given by $X \mapsto X^q$. We come to this later.

The ring $W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$ with the π-adic topology is not suitable for our purposes, as the following Lemma indicates.

Lemma 1.2.5. *(Kle16, Lemma 3.1.18)*

The action

$$G_L \times W(\mathcal{O}_{\mathbb{C}_p^\flat})_L \to W(\mathcal{O}_{\mathbb{C}_p^\flat})_L, (\sigma, (x_n)_n) \mapsto (\sigma(x_n))_n$$

is not *continuous for the π-adic topology.*

We introduce a new topology on ramified Witt vectors. Now let B be a perfect topological κ-algebra, such that there exists a fundamental system of open neighbourhoods of 0 given by ideals of B.

Definition 1.2.6. For any open ideal $\mathfrak{a} \subset B$ and $m \geq 1$ we define

$$\begin{aligned} V_{\mathfrak{a},m} &:= \ker(W(B)_L \overset{\mathrm{pr}}{\to} W_m(B)_L \overset{W_m(\mathrm{pr})_L}{\to} W_m(B/\mathfrak{a})_L) \\ &= \{(b_n)_n \in W(B)_L \mid b_i \in \mathfrak{a}\ \forall 0 \leq i < m\}. \end{aligned}$$

If $\mathfrak{b} \subset B$ is another open Ideal and $n \geq 1$, it is $V_{\mathfrak{a}\cap\mathfrak{b},\max(m,n)} \subset V_{\mathfrak{a},m} \cap V_{\mathfrak{b},n}$, so there is a structure of a topological $\mathcal{O}_{K_0}$-module on $W(B)_L$ for which the ideals $V_{\mathfrak{a},m}$ form a fundamental system of open neighbourhoods of 0. We call this topology the *weak topology on $W(B)_L$*.

Remark. (Sch17, Exercise 1.5.1) or (Kle16, Bemerkung after Definition 3.1.19)

The weak topology on $W(B)_L$ is the same as the product topology on $W(B)_L = B^{\mathbb{N}_0}$ induced by the topology on B. In particular, if $\rho : B_1 \to B_2$ is a continuous morphism between two perfect topological κ-algebras with a fundamental system of open neighbourhoods of 0 consisting of ideals, then $W(\rho)_L : W(B_1)_L \to W(B_2)_L$ is continuous for the weak topology.

Lemma 1.2.7. *(Sch17, Remark 1.5.2 & Lemma 1.5.3)*

Let G be a profinite group, which acts continuously on B by automorphisms of κ-algebras.

i) If the topology on B is Hausdorff (resp. complete), then the corresponding on $W(B)_L$ is Hausdorff (resp. complete).

ii) The action

$$G \times W(B)_L \to W(B)_L, (\sigma, (b_n)_n) \mapsto (\sigma(b_n))_n$$

is an action by automorphisms of $\mathcal{O}_{K_0}$-algebras which is continuous for the weak topology on $W(B)_L$.

Now let $F|\kappa$ be a perfect and complete non archimedean field extension with the absolute value denoted by $|\cdot|$. We consider the topological sub-κ-algebra $\mathcal{O}_F$. It has a fundamental system of open neighbourhoods of 0, consisting of the ideals $B_\varepsilon(0) := \{x \in \mathcal{O}_F \mid |x| < \epsilon\} \subset \mathcal{O}_F$ for any $\varepsilon > 0$.

Definition 1.2.8. For any open ideal $\mathfrak{a} \subset \mathcal{O}_F$ and $m \geq 1$, we define the $W(\mathcal{O}_F)_L$-submodules of $W(F)_L$

$$U_{\mathfrak{a},m} := V_{\mathfrak{a},m} + \pi^m W(F)_L = \{(b_n)_n \in W(F)_L \mid b_i \in \mathfrak{a} \ \forall 0 \leq i < m\}.$$

Then $W(F)_L$ carries the structure of a topological $\mathcal{O}_L$-module with respect to this fundamental system of open neighbourhoods of 0. We call it the *weak topology* on $W(F)_L$

Remark. (Sch17, Discussion before Lemma 1.5.4)

The weak topology on $W(F)_L$ is the same as the product topology $W(F)_L = F^{\mathbb{N}_0}$ induced by the topology on F given by its absolute value. In particular, if $F_1|F$ is a field extension of valued field, which satisfies the same conditions as F, then the topology induced by the inclusion $W(F)_L \subset W(F_1)_L$, where the right hand side carries the weak topology is the weak topology on the left hand side.

Lemma 1.2.9. *(Sch17, Lemma 1.5.4 & Lemma 1.5.5)*

The ring $W(F)_L$ is a complete and Hausdorff topological $\mathcal{O}_{K_0}$-algebra with respect to the weak topology.

Definition 1.2.10. Let X be a topological space and A be a topological Hausdorff group, which has an open neighbourhood of 0 consisting of subgroups. If $(f_n)_n : X \to Y$ is a sequence of continuous functions, which converges pointwise, i.e. there exists a function $f : X \to A$, such that

$$f(x) := \lim_n f_n(x)$$

exists for all $x \in X$. We say that the f_n *converge uniformly against* f, if for all open subgroups $H \subset A$ there exists $N \in \mathbb{N}$ such that

$$f_n(x)f(x)^{-1} \in H$$

for all $m \geq N$ and $x \in X$.

Lemma 1.2.11. *(Uniform Limit Theorem) Let $X, A, (f_n)_n, f$ be as in the last Definition. Then f is continuous, if the f_n converge uniformly against f.*

Proof. Let $H \subset A$ be an open subgroup and $x \in X$. By the hypothesis and since the f_n are continuous and H is a group, there exists $N \in \mathbb{N}$ and $x \in U \subset X$ open such that

$$f(x)f(y)^{-1} = (f(x)f_N(x)^{-1})(f_N(x)f_N(y)^{-1})(f_N(y)f(y)^{-1}) \in H$$

for all $y \in U$. $\square$

Remark. Let $A = W(F)_L$. If a sequence

$$f_n : X \to W(F)_L$$

converges uniformly against an

$$f : X \to W(F)_L$$

in the π-adic topology, then it converges against f in the weak topology.

Proof. First of all, if the $\lim_n f_n(x)$ exist in the π-adic topology and converge against $f(x)$, then the same is true for the weak topology, since if

$$f(x) - f_n(x) \in \pi^n W(F)_L,$$

then

$$f(x) - f_n(x) \in \pi^n W(F)_L + V_{\mathfrak{a},n} = U_{\mathfrak{a},n}$$

for all open $\mathfrak{a} \in W(\mathcal{O}_F)_L$. The same argument shows that they converge uniformly in the weak topology, if they converge uniformly in the π-adic topology. $\square$

Proposition 1.2.12. *The group of units $W(F)_L^\times$ is a topological group.*

Proof. By Lemma 1.1.8 and Proposition 1.1.12.ii), it is

$$W(F)_L^\times \cong \tau(F^\times) \times (1 + \pi W(F)_L)$$

a isomorphism of groups, which is a homeomorphism for the subset topologies of weak topology, since $W(F)_L$ is a topological ring. Since $\tau : F^\times \to W(F)_L^\times$ is a homeomorphism onto its image and F is a topological field by Remark 1.1.42, it suffices to show that inverting on

$$U_W := 1 + \pi W(F)_L$$

is continuous. But for every $c \in \pi W(F)_L$, the geometric series

$$\sum_{n \geq 0} c^n$$

converges for the π-adic and hence for the weak topology. It follows that for $u \in U_W$ with

$$u = 1 - c \text{ for } c \in \pi W(F)_L$$

that

$$u^{-1} = \sum_{n \geq 0} c^n.$$

Since $W(F)_L$ is a topological ring for the weak topology, the map

$$U_W \to \pi W(F)_L, u \mapsto 1 - u$$

is a homeomorphism for the weak topology. So we need to show that

$$f : \pi W(F)_L \to W(F)_L, c \mapsto \sum_{n \geq 0} c^n$$

is continuous. Consider for every $m \in \mathbb{N}$ the map

$$f_m : \pi W(F)_L \to W(F)_L, c \mapsto \sum_{n=0}^{m} c^n.$$

This is the composition of the maps

$$f_m : \pi W(F)_L \xrightarrow{c \mapsto (1, c, \ldots, c^m)} \prod_{n=0}^{m} W(F)_L \xrightarrow{\Sigma} W(F)_L.$$

These are continuous, since $W(F)_L$ is a topological ring. Since $W(F)_L$ is Hausdorff and the f_m clearly converge pointwise against f, it suffices to show that they converge uniformly. But for every $c \in \pi W(F)_L$, it is

$$\sum_{n \geq 0} c^n - \sum_{n=0}^{m} c^n \in \pi^{m+1} W(F)_L.$$

for all $m \geq 0$. $\square$

Now let $E|\kappa$ be a complete non archimedean but not necessarily perfect field extension. By (Neu99, II Theorem 4.8) the non archimedean value on E extends uniquely to an algebraic closure $\overline{E}$ of E and especially to $F := E^{perf}$ and a separable closure E^{sep}. We furthermore set $\overline{F} := \overline{E}$. Since the map $(\cdot)^p$ is continuous on F and $\overline{F}$ for the topology induced by the absolute value, the completion $\hat{F}$ and $\hat{\overline{F}}$ are still perfect.

Definition 1.2.13. We define the *weak topology* on $W(E)_L$ (resp. on $W(E^{sep})_L$) to be the topology induced by the inclusion

$$W(E)_L \subset W(\hat{F})_L \text{ (resp. } W(E^{sep})_L \subset W(\hat{\overline{F}})_L),$$

where the right hand side is equipped with the weak topology as defined in Definition 1.2.8.

Analoguesly, we define the *weak topology* on $W(F)_L$ (resp. on $W(\overline{F})_L$) as the topology induced by the inclusion

$$W(F)_L \subset W(\hat{F})_L \text{ (resp. } W(\overline{F})_L \subset W(\hat{\overline{F}})_L).$$

By this definition and the remark after Definition 1.2.8 we get the following remark.

Remark. The weak topology on $W(E)_L$ (resp. on $W(E^{sep})_L$) is the same as the product topology on $W(E)_L = E^{\mathbb{N}_0}$ (resp. on $W(E^{sep}) = (E^{sep})^{\mathbb{N}_0}$), induced by the topology on E (resp. on E^{sep}) given by its absolute value. In particular, the topology induced by the inclusion $W(E)_L \subset W(E^{sep})_L$, where the right hand side carries the weak topology is the weak topology on the left hand side.

The same statements hold for $W(F)_L$ (resp. for $W(\overline{F})_L$).

Proposition 1.2.14. *(Sch17, Proposition 1.4.27)*

i) It is $\hat{\mathbb{F}} \subset \hat{K}_\infty^\flat$.

ii) It is $\widehat{\mathbb{E}^{sep}} = \hat{\overline{\mathbb{F}}} = \mathbb{C}_p^\flat$.

Proposition 1.2.15. *The actions*

$$G_K \times W(\overline{\mathbb{F}})_L \to W(\overline{\mathbb{F}})_L, (\sigma, (x_n)_n) \mapsto (\overline{\rho}(\sigma)(x_n))_n$$

and

$$\Gamma_K \times W(\mathbb{F})_L \to W(\mathbb{F})_L, (\gamma, (x_n)_n) \mapsto (\overline{\tau}(\gamma)(x_n))_n$$

define actions of $\mathcal{O}_{K_0}$-algebras and are continuous for the weak topologies.

The same statements hold for $\mathbb{E}$ instead of $\mathbb{F}$ (resp. for $\mathbb{E}^{sep}$ instead of $\overline{\mathbb{F}}$.)

Proof. (Inspired by Sch17, Remark 2.1.14)

By Definition 1.2.13 the weak topology on $W(E)_L$ for any field extension $E|L$ with $E \subset \mathbb{C}_p$ is given by the weak topology on $\mathbb{C}_p$. By (Bou66, III §2.4

Lemma 2) the projection $G_K \to \Gamma_K$ is open. So by an easy calculation (see again (Kle16, Lemma 2.1.21)) it suffices to show, that

$$G_K \times W(\mathbb{C}_p^\flat)_L \to W(\mathbb{C}_p^\flat)_L, (\sigma, (x_n)_n) \mapsto (\overline{\rho}(\sigma)(x_n))_n$$

is continuous for the weak topology. By Lemma 1.1.30 and Lemma 1.2.7.ii)

$$G_K \times W(\mathcal{O}_{\mathbb{C}_p^\flat})_L \to W(\mathcal{O}_{\mathbb{C}_p^\flat})_L, (\sigma, (x_n)_n) \mapsto (\overline{\rho}(\sigma)(x_n))_n$$

is continuous for the weak topology, but since $G_K(\pi^m W(\mathbb{C}_p^\flat)_L) = \pi^m W(\mathbb{C}_p^\flat)_L$ this remains true for the G_K-action on $W(\mathbb{C}_p^\flat)_L$. □

Definition 1.2.16. The continuous G_K-action on $W(\overline{\mathbb{F}})_L$ gives us a map

$$\rho : G_K \to \mathrm{Aut}_{\mathcal{O}_{K_0}-Alg}(W(\overline{\mathbb{F}})_L).$$

By abuse of Notation we also denote $\rho : G_K \to \mathrm{Aut}_{\mathcal{O}_{K_0}-Alg}(W(\mathbb{E}^{sep})_L)$ to be the map given by restriction. We furthermore define $\tau : \Gamma_K \to \mathrm{Aut}_{\mathcal{O}_{K_0}-Alg}(W(\mathbb{F})_L)$ (and also $\tau : \Gamma_K \to \mathrm{Aut}_{\mathcal{O}_{K_0}-Alg}(W(\mathbb{E})_L)$) to be the map induced by ρ.

Furthermore, we define

$$\varphi_L : W(\overline{\mathbb{F}})_L \to W(\overline{\mathbb{F}})_L, x \mapsto F_{\overline{\mathbb{F}}}(x),$$

where $F_{\overline{\mathbb{F}}}$ denotes the Frobenius on $W(\overline{\mathbb{F}})_L$ (see Definition 1.1.5). This is an abuse of notation, since it is a lift of the map $\varphi_L = (\cdot)^q$ defined after Theorem 1.1.39. Since it is $\varphi_L = W((\cdot)^q)_L$ by Proposition 1.1.10.i) φ_L is continuous for the weak topology and commutes with the automorphisms $\rho(\sigma)$ for all $\sigma \in G_K$.

Lastly, we will need to use that $\Theta_{\mathbb{C}_p}$ is compatible with the actions and topologies defined in this part.

Lemma 1.2.17. *(Sch17, Lemma 1.6.1)*
The map

$$\Theta_{\mathbb{C}_p} : W(\mathcal{O}_{\mathbb{C}_p^\flat})_L \to \mathcal{O}_{\mathbb{C}_p}$$

from Lemma 1.1.43 satisfies the following properties. For this, we also denote $\rho : G_K \to \mathrm{Aut}_{\mathcal{O}_{K_0}-Alg}(W(\mathcal{O}_{\mathbb{C}_p^\flat})_L)$ *for the map induced by the action.*

i) It is $\sigma(\Theta_{\mathbb{C}_p}(a)) = \Theta_{\mathbb{C}_p}(\rho(\sigma)(a))$ *for all* $\sigma \in G_K$ *and* $a \in W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$.

ii) The map $\Theta_{\mathbb{C}_p}$ *is continuous and open with respect to the weak topology on* $W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$.

1.2.2 The Period Ring

In this part, we will construct a lift of the isomorphism $\iota : \kappa((X)) \to \mathbb{E}_{K_0}$ to an embedding $\mathfrak{j} : \mathcal{A}_{K_0} \to W(\mathbb{E}_{K_0})_L$, which is topological for the weak topologies and such that the Γ_{K_0}-action and φ_L on the right hand side preserve the image of $\mathfrak{j}$, which we will denote by $\mathbb{A}_{K_0}$.

Definition 1.2.18. We define $\mathbb{M}_{\mathbb{E}_{K_0}} \subset W(\mathbb{E}_{K_0})_L$ to be the maximal ideal

$$\mathbb{M}_{\mathbb{E}_{K_0}} := \Phi_0^{-1}(\mathfrak{m}_{\mathbb{E}_{K_0}}).$$

Remark. (Sch17, Remark 2.1.2)

With respect to its weak topology, the ring $W(\mathcal{O}_{\mathbb{E}_{K_0}})_L$ is a topological $\mathcal{O}_{K_0}$-algebra, which is Hausdorff and complete. Furthermore $\mathbb{M}_{\mathbb{E}_{K_0}} \subset W(\mathcal{O}_{\mathbb{E}_{K_0}})_L$ is closed and hence complete.

Lemma 1.2.19. *(Sch17, Lemma 2.1.4 & Lemma 2.1.6)*

i) The ideals $(\Theta_{\mathbb{C}_p}^{-1}(\pi\mathcal{O}_{\mathbb{C}_p})^m)_m$ form a fundamental system of open neighbourhoods of 0 for the weak topology on $W(\mathcal{O}_{\mathbb{C}_p^\flat})$.

ii) It is $\mathbb{M}_{\mathbb{E}_{K_0}} \subset \Theta_{\mathbb{C}_p}^{-1}(\pi\mathcal{O}_{\mathbb{C}_p})$.

iii) With respect to the weak topology any element $\alpha \in \mathbb{M}_{\mathbb{E}_{K_0}}$ is topologically nilpotent, i.e. $\lim_n \alpha^n = 0$.

Proof. We only need to prove *ii*). Let $\alpha = (\alpha_n)_n \in \mathbb{M}_{\mathbb{E}_{K_0}}$. Then $|\alpha_0^\sharp| = |\alpha_0|_\flat \le |\omega| < |\pi|$. On the other hand we have $\Theta_{\mathbb{C}_p}(\alpha) \equiv \alpha_0^\sharp \mod \pi\mathcal{O}_{\mathbb{C}_p}$ by definition, so we obtain $|\Theta_{\mathbb{C}_p}(\alpha)| \le |\pi|$. $\square$

Corollary 1.2.20. *Let ϕ be a Frobenius power series for π and $F := F_\phi$ denote the corresponding Lubin-Tate formal group law. Then $(\mathbb{M}_{\mathbb{E}_{K_0}}, +_F)$ is a $\mathcal{O}_L$-module via*

$$\mathcal{O}_L \times \mathbb{M}_{\mathbb{E}_{K_0}} \to \mathbb{M}_{\mathbb{E}_{K_0}}, (b, z) \mapsto [b]_\phi(z).$$

Furthermore, any formal power series $X\mathcal{O}_{K_0}[[X]]$ converges on $\mathbb{M}_{\mathbb{E}_{K_0}}$ and so for any $\alpha \in \mathbb{M}_{\mathbb{E}_{K_0}}$ and $f \in \mathcal{O}_{K_0}[[X]]$, it is $f(\alpha) \in W(\mathcal{O}_{\mathbb{E}_{K_0}})_L$.

We consider $\mathcal{A}_L := \varprojlim \mathcal{O}_L((X))/\pi^n\mathcal{O}_L((X)) \subset \mathcal{A}_{K_0}$. This ring has the same properties we established for $\mathcal{A}_{K_0}$ and in particular has a *weak topology*. By going through the definitions one easily sees, that the weak topology on $\mathcal{A}_L$ is the same topology as the topology induced by the inclusion $\mathcal{A}_L \subset \mathcal{A}_{K_0}$.

Furthermore the residue field of $\mathcal{A}_L$ is $k((X)) \cong \mathbb{E}_L$. We define the maximal ideal

$$\mathbb{M}_{\mathbb{E}_L} := \Phi_0^{-1}(\mathfrak{m}_{\mathbb{E}_L}) \subset W(\mathcal{O}_{\mathbb{E}_L})_L.$$

Just like $\mathbb{M}_{\mathbb{E}_{K_0}}$ this maximal ideal $\mathbb{M}_{\mathbb{E}_L}$ satisfies the properties of Corollary 1.2.20.

Lemma 1.2.21. *(Sch17, Lemma 2.1.11)*
There exists a unique endomorphism of $\mathcal{O}_L$-algebras

$$\{\} : \mathbb{M}_{\mathbb{E}_L} \to \mathbb{M}_{\mathbb{E}_L},$$

which satisfies

$$\Phi_0 \circ \{\} = \Phi_0 \text{ and } [\pi]_\phi \circ \{\} = F_{\mathcal{O}_{\mathbb{E}_L}} \circ \{\},$$

where $F_{\mathcal{O}_{\mathbb{E}_L}}$ denotes the Frobenius on $W(\mathcal{O}_{\mathbb{E}_L})_L$.

We define

$$\tau_\phi : \mathfrak{m}_{\mathbb{E}_L} \xrightarrow{\tau} \mathbb{M}_{\mathbb{E}_L} \xrightarrow{\{\}} \mathbb{M}_{\mathbb{E}_L} \subset \mathbb{M}_{\mathbb{E}_{K_0}} \text{ and } \iota_\phi := \tau_\phi \circ \iota : T \to \mathbb{M}_{\mathbb{E}_{K_0}}.$$

Let $t \in T$ be a generator as an $\mathcal{O}_L$-module and $\omega := \iota(t) \in \mathbb{E}_{K_0}$ be the corresponding uniformizer. We furthermore define $\omega_\phi := \iota_\phi(t) = \tau_\phi(\omega)$. By Lemma 1.2.21, it is $\Phi_0(\omega_\phi) = \omega$. By Corollary 1.2.20, we obtain a map

$$\mathcal{O}_{K_0}[[X]] \to W(\mathcal{O}_{\mathbb{E}_{K_0}})_L, f(X) \mapsto f(\omega_\phi).$$

Since $\Phi_0(\omega_\phi) = \omega \neq 0$, the element $\omega_\phi \in W(\mathbb{E}_{K_0})_L^\times$ is a unit by Proposition 1.1.12.i). It follows that we have a map

$$\mathcal{O}_{K_0}((X)) \to W(\mathbb{E}_{K_0})_L$$

and by passing to the π-adic completion we get a map of $\mathcal{O}_{K_0}$-algebras

$$\mathrm{j} : \mathcal{A}_{K_0} = \varprojlim \mathcal{O}_{K_0}((X))/\pi^n\mathcal{O}_{K_0}((X)) \to \varprojlim W(\mathbb{E}_{K_0})_L/\pi^n W(\mathbb{E}_{K_0})_L \cong W(\mathbb{E}_{K_0})_L,$$

see Proposition 1.1.10.iv). This map is an embedding, since we can further lift them to their quotient fields. It obviously satisfies, that the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{A}_{K_0} \arrow[r, "\mathrm{j}"] \arrow[d, "\mathrm{pr}"'] & W(\mathbb{E}_{K_0})_L \arrow[d, "\Phi_0"] \\
\kappa((X)) \arrow[r, "\iota"] & \mathbb{E}_{K_0}
\end{tikzcd}$$

Definition 1.2.22. We define

$$\mathbb{A}_{K_0} := \mathrm{im}(\mathfrak{j}) \text{ and } \mathbb{A}_L := \mathfrak{j}(\mathcal{A}_L).$$

Lemma 1.2.23. *(Compare to Sch17, Proposition 2.1.16.i)) The map $\mathfrak{j}$ is a topological embedding for the weak topologies.*

Proof. We have $|\Phi_0(\omega_\phi)|_\flat = |\pi|^{\frac{q}{q-1}} < 1$, so the statement follows from (Sch17, Remark 2.1.5.ii)). □

The ring $\mathbb{A}_L$ is invariant under the Frobenius and the G_{K_0}-action.

Proposition 1.2.24. *(Sch17, Proposition 2.1.16)*

For any $f \in \mathcal{A}_L$ and $\gamma \in \Gamma_{K_0}$, we have

i) $F_{\mathbb{E}_{K_0}}(\mathfrak{j}(f)) = \mathfrak{j}(\varphi_{K_0}(f))$,

ii) $\gamma(\mathfrak{j}(f)) = \mathfrak{j}({}^{\gamma}f)$.

The second identity extends by continuity and $\mathcal{O}_{K_0}$-linearity of the Γ_{K_0}-action for every $f \in \mathcal{A}_{K_0}$. The first identity cannot extend for $\mathcal{A}_{K_0}$, if $K_0 \neq L$, since $F_{\mathbb{E}_{K_0}}$ induces the map $(\cdot)^q$ on $\mathbb{E}_{K_0}$, but φ_{K_0} induces the endomorphism of κ-algebras $X \mapsto X^q$ on $\kappa((X))$. In the next section we will see that $\mathbb{A}_{K_0}$ is still invariant under the Frobenius.

Remark. (Compare to Sch17, Remark 2.1.17)

The ring $\mathbb{A}_{K_0}$ does not depend on the choice of generator $t \in T$.

1.2.3 Unramified Extensions

In this part we will give a brief reminder of the theory of unramified extensions and then construct our Period ring $\mathbb{A}_K \subset W(\mathbb{E})_L$ and furthermore the completion of the maximal unramified extension $\mathbb{A} \subset W(\mathbb{E}^{sep})_L$.

Definition 1.2.25. Let E be a complete, discretely valued non archimedean field with uniformizer π_E and residue field k_E. A finite extension $E_0|E$ is called *unramified*, if E_0 has uniformizer π_E and the extension of residue fields $k_{E_0}|k_E$ is separable. In this case we also call the extension $\mathcal{O}_{E_0}|\mathcal{O}_E$ *unramified.*

Lemma 1.2.26. *(Sch17, Lemma 1.2.4)*

For any unramified extension $E_0|E$, we have the following.

i) The extension $E_0|E$ is separable.

ii) If $a \in \mathcal{O}_{E_0}$ is such that $k_{E_0} = k_E(\alpha)$ for the image $\alpha \in k_{E_0}$ of a, then $E_0 = E[a]$ and $1, a, \dots, a^{[E_0:E]-1}$ is an $\mathcal{O}_E$-basis of $\mathcal{O}_{E_0}$.

By (Sch17, Lemma 1.7.1.ii)) $\mathcal{A}_L$ is a complete discrete valuation ring with uniformizer π.

Example 1.2.27. The extension $\mathcal{A}_{K_0}|\mathcal{A}_L$ is unramified, because an element $a \in \mathcal{O}_{K_0}$, such that $k_{K_0} = k_L[\alpha]$ as in Lemma 1.2.26.ii) satisfies $\mathcal{A}_{K_0} = \mathcal{A}_L[a]$, since we can write any element in $\mathcal{A}_{K_0}$ as

$$\sum_{n\in\mathbb{Z}}(\sum_{m} b_m^{(n)} a^m)X^n \text{ with } b_m \in \mathcal{O}_L$$

by the remark after Definition 1.2.1 and Lemma 1.2.26.ii). Since the π-adic value is non archmidean, we can change the order of the summands in the "powerseries" to obtain the equality

$$\sum_{n\in\mathbb{Z}}(\sum_{m} b_m^{(n)} a^m)X^n = \sum_{m}(\sum_{n\in\mathbb{Z}} b_m^{(n)} X^n)a^m.$$

So $\mathcal{A}_{K_0}|\mathcal{A}_L$ is finite. Furthermore, both rings have uniformizer π and $\kappa((X))|k((X))$ is separable by Lemma 1.2.26.i).

Lemma 1.2.28. *(See Sch17, Lemma 1.2.5)*

For finite extensions $E \subset E_0 \subset E_1$ and $E \subset E_0' \subset E_1$, we have the following.

i) The extension $E_1|E$ is unramified if and only if $E_1|E_0$ and $E_0|E$ are unramified.

ii) If $E_0|E$ is unramified, then $E_0E_0'|E_0'$ is unramified.

iii) If $E_0|E$ and $E_0'|E$ are unramified, then $E_0E_0'|E$ is unramified.

Fix a separable closure E^{sep} of E. By Lemma 1.2.28, the union E^{nr} of all unramified extensions of E in E^{sep} is a Galois extension $E^{nr}|E$, which we call the *maximal unramified extension.*

Proposition 1.2.29. *(See Sch17, Proposition 1.2.6 & Exercise 1.2.7)*

i) The residue field of E^{nr} is a separable closure k_E^{sep} of k_E in an algebraically closed field containing k_E.

ii) The natural maps

$$\mathrm{Gal}(E^{nr}|E) \xrightarrow{f_1} \mathrm{Aut}_{\mathcal{O}_E}(\mathcal{O}_{E^{nr}}) \xrightarrow{f_2} G_{k_E},$$

with

$$f_1(\sigma) = \sigma_{|\mathcal{O}_{E^{nr}}}$$

and

$$f_2(\phi) = [x \mod \mathfrak{m}_{E^{nr}} \mapsto \phi(x) \mod \mathfrak{m}_{E^{nr}}$$

are isomorphisms and $f_2 \circ f_1$ is a topological isomorphism for the Krull topologies.

Lemma 1.2.30. *(See Sch17, Lemma 3.1.3)*

Let $E|\mathbb{E}_L$ be any finite extension contained in $\mathbb{E}_L^{sep}$. There exists a unique finite ring extension $\mathbb{A}_L \subset \mathbb{A}_L(E) \subset W(\mathbb{E}_L^{sep})$, which satisfies the following properties.

a) The ring $\mathbb{A}_L(E)$ is a complete discrete valuation ring with prime element π.

b) The map $\Phi_0 : W(\mathbb{E}_L^{sep})_L \to \mathbb{E}_L^{sep}$ induces an isomorphism

$$\mathbb{A}_L(E)/\pi\mathbb{A}_L(E) \to E.$$

Furthermore, we have the following.

c) It is $\mathbb{A}_L(E) \subset W(E)_L$ and the quotient field $\mathrm{Quot}(\mathbb{A}_L(E))$ is a finite unramified extension of $\mathbb{B}_L$.

d) The Frobenius F on $W(\mathbb{E}_L^{sep})$ preserves $\mathbb{A}_L(E)$.

By the uniqueness, we have that $\mathbb{A}_{K_0} = \mathbb{A}_L(\mathbb{E}_{K_0})$ and so by Lemma 1.2.30.d) the Frobenius preserves $\mathbb{A}_{K_0}$.

Definition 1.2.31. We set

$$\mathbb{A}_K := \mathbb{A}_L(\mathbb{E}), \mathbb{B}_K := \mathrm{Quot}(\mathbb{A}_K).$$

We view $\mathbb{A}_K \subset W(\mathbb{E})_L$ with the subset topology of the weak topology.

We furthermore set

$$\mathbb{A}_L^{nr} := \bigcup_E \mathbb{A}_L(E), \mathbb{B}_L^{nr} := \mathrm{Quot}(\mathbb{A}_L^{nr}).$$

By Lemma 1.2.30, Proposition 1.2.29 and Corollary 1.1.40.i), we see that

- the map $\Phi_0 : \mathbb{A}_L^{nr}/\pi\mathbb{A}_L^{nr} \to \mathbb{E}_L^{sep}$ is an isomorphism,
- the Frobenius φ_L and G_L-action ρ on $W(\mathbb{E}_L^{sep})_L$ preserve $\mathbb{A}_L^{nr}$ and
- the G_K-action on $W(\mathbb{E}_L^{sep})_L$ induces isomorphisms

$$H_K \tilde{\to} \mathrm{Gal}(\mathbb{B}_L^{nr}|\mathbb{B}_K) \tilde{\to} G_{\mathbb{E}}.$$

Definition 1.2.32. We set $\mathbb{A}$ to be the π-adic completion of $\mathbb{A}_L^{nr}$ in $W(\mathbb{E}_L^{sep})_L$.

Remark 1.2.33. (See Sch17, Remark 3.1.4)

i) The π-adic topology on $W(\mathbb{E}_L^{sep})_L$ induces the π-adic topology on $\mathbb{A}_L^{nr}$.

ii) The map $\varprojlim \mathbb{A}_L^{nr}/\pi^m\mathbb{A}_L^{nr} \to \mathbb{A}$ is an isomorphism.

Since any $\sigma \in G_L$ and the Frobenius on $W(\mathbb{E}_L^{sep})_L$ act continuously for the π-adic topology, the list of properties above yields that

- the map $\Phi_0 : \mathbb{A}/\pi\mathbb{A} \to \mathbb{E}_L^{sep}$ is an isomorphism,
- the Frobenius φ_L and the G_L-action ρ on $W(\mathbb{E}_L^{sep})_L$ preserve $\mathbb{A}$ and H_K fixes $\mathbb{A}_K$.

Remark 1.2.34. (Compare to Sch17, Remark 3.1.5)

i) The G_L-action on $W(\mathbb{C}_p^\flat)_L$ commutes with the Frobenius F.

ii) It is $(W(\mathbb{C}_p^\flat)_L)^{F=1} = W(k)_L = \mathcal{O}_L$.

Lemma 1.2.35. *It is $\mathbb{A}^{H_K} = \mathbb{A}_K$. In particular, the G_K-action on $\mathbb{A}$ induces a continuous Γ_K-action on $\mathbb{A}_K$, which we also denote by τ.*

Proof. (Inspired by Sch17, Lemma 3.1.6)

Since $H_K \cong G_{\mathbb{E}}$, we have that $(\mathbb{A}/\pi\mathbb{A})^{H_K} = (\mathbb{E}_L^{sep})^{H_K} = \mathbb{E} = \mathbb{A}_K/\pi\mathbb{A}_K$. Considering the commutative and exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{A}_K/\pi^m\mathbb{A}_K \arrow[r] \arrow[d] & \mathbb{A}_K/\pi^{m+1}\mathbb{A}_K \arrow[r] \arrow[d] & \mathbb{A}_K/\pi\mathbb{A}_K \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & (\mathbb{A}/\pi^m\mathbb{A})^{H_K} \arrow[r] & (\mathbb{A}/\pi^{m+1}\mathbb{A})^{H_K} \arrow[r] & (\mathbb{A}/\pi\mathbb{A})^{H_K} &
\end{tikzcd}$$

we deduce from the snake lemma and induction that $(\mathbb{A}/\pi^m\mathbb{A})^{H_K} = \mathbb{A}_K/\pi^m\mathbb{A}_K$ for all $m \geq 1$. By Remark 1.2.33.ii), we see that

$$\mathbb{A}^{H_K} = (\varprojlim \mathbb{A}/\pi^m\mathbb{A})^{H_K} = \varprojlim (\mathbb{A}/\pi^m\mathbb{A})^{H_K} = \varprojlim \mathbb{A}_K/\pi^m\mathbb{A}_K = \mathbb{A}_K.$$

$\square$

1.3 Modules and Motivation

In this section, we want to give the definition of (φ, Γ)-modules and cite the classical correspondence to Galois representations. We will then proceed how this correspondence translates to a correspondence in the context of linear algebraic groups, which will serve as the motivation for us to obtain a more general statement.

1.3.1 Etale (φ_L, Γ_K)-Modules

We begin with the definition of the weak topologies for finitely generated modules over our period rings. Let $\mathcal{R} \in \{\mathbb{A}_K, W(\mathbb{F})_L\}$.

Definition 1.3.1. Let M be a finitely generated $\mathcal{R}$-module with projection $\mathcal{R}^n \to M$. We give $\mathcal{R}^n$ the product topology of the weak topology on $\mathcal{R}$ and M the quotient topology of the projection.

Remark. (Compare to Sch17, Exercise 2.2.3)
The topology on M from the last definition is independent on the choice of projection. We call this topology the *weak topology* on M. With its weak topology M is a topological $\mathcal{R}$-module.

Lemma 1.3.2. *(Compare to Sch17, Remark 2.2.5)*
Let $\alpha : \mathcal{R} \to \mathcal{R}$ be a continuous ring homomorphism, and let $\beta : M \to N$ be any α-semilinear homomorphism between finitely generated $\mathcal{R}$-modules M and N. Then β is continuous for the weak topologies on M and N.

Definition 1.3.3. Let Γ be a topological group, which acts continuously on $\mathcal{R}$ via automorphisms of $\mathcal{O}_L$-algebras.

i) Let V be a finitely generated $\mathcal{O}_L$-module. If $\sigma : G_K \to \mathrm{Aut}_{\mathcal{O}_L}(V)$ is an action, which is continuous for the π-adic topology on V, then we call (V, σ) a *continuous G_K-representation over $\mathcal{O}_L$*. A homomorphism between two representations (V, σ_V) and (W, σ_W) is a linear map $V \to W$, which commutes with the actions. By $Rep_{\mathcal{O}_L}(G_K)$ we denote the category of continuous G_K-representations over $\mathcal{O}_L$.

ii) Let M be a finitely generated $\mathcal{R}$-module. If $\alpha : \Gamma \to \mathrm{Aut}_{\mathcal{O}_L}(M)$ is an action, which is semilinear for the action on $\mathcal{R}$ and the map $\Gamma \times M \to M$ induced by α is continuous for the weak topology on M and $\varphi_M : M \to M$ is an φ_L-semilinear endomorphism, which commutes with every $\alpha(\gamma)$, $\gamma \in \Gamma$, then we call (M, φ_M, α) a *(φ_L, Γ)-module over $\mathcal{R}$*. A homomorphism between (φ_L, Γ)-modules is an $\mathcal{R}$-linear map, which commutes with the Γ-actions and the φ_L-semilinear maps. By $\Gamma\Phi_{\mathcal{R}}$, we denote the category of (φ_L, Γ)-modules over $\mathcal{R}$. If $\Gamma = \{1\}$ is the trivial group, we just write $\Phi_{\mathcal{R}}$ for $\{1\}\Phi_{\mathcal{R}}$.

iii) By $\Gamma\Phi_{\mathcal{R}}^{et} \subset \Gamma\Phi_{\mathcal{R}}$ we denote the full subcategory of (φ_L, Γ)-modules (M, φ_M, α), such that the linearisation of φ_M

$$\varphi_M^{lin} : \mathcal{R} \underset{\varphi_L, \mathcal{R}}{\otimes} M \to M, x \otimes m \mapsto x\varphi_M(m)$$

is an isomorphism. Here $\mathcal{R} \underset{\varphi_L, \mathcal{R}}{\otimes} M$ denotes the basechange of M via $\varphi_L : \mathcal{R} \to \mathcal{R}$. We call an object in $\Gamma\Phi_{\mathcal{R}}^{et}$ an *etale* (φ_L, Γ)*-module over* $\mathcal{R}$.

We will sometimes implicitly use the following fact.

Remark. (Compare to Sch17, Exercise after Proposition 2.2.7) or (See Kle16, Proposition 2.1.18)

An object $(M, \varphi_M) \in \Phi_{\mathcal{R}}$ is etale, if and only if $\varphi_M(M) \subset M$ generates M as an $\mathcal{R}$-module.

Theorem 1.3.4. *(See Sch17, Theorem 3.3.10)*

If $K = L$, then we have quasi-inverse equivalences of categories

$$\begin{aligned} \mathbb{D} : Rep_{\mathcal{O}_L}(G_L) &\leftrightarrow \Gamma_L \Phi_{\mathbb{A}_L}^{et} : \mathbb{V} \\ (V, \sigma) &\mapsto (\mathbb{A} \underset{\mathcal{O}_L}{\otimes} V)^{H_L} \\ (\mathbb{A} \underset{\mathbb{A}_L}{\otimes} M)^{\varphi=1} &\leftarrow\!\shortmid (M, \varphi_M, \alpha), \end{aligned}$$

where $\mathbb{A} \underset{\mathcal{O}_L}{\otimes} V$ carries the diagonal G_L-action $\rho \otimes \sigma$ and the Frobenius is given by $\varphi_L \otimes \mathrm{id}$ for the Frobenius φ_L on $\mathbb{A}$. On the other side the Frobenius φ on $\mathbb{A} \underset{\mathbb{A}_L}{\otimes} M$ is given by $\varphi_L \otimes \varphi_M$ and the G_L-action on $\mathbb{A} \underset{\mathbb{A}_L}{\otimes} M$ is given by the diagonal action $\rho \otimes (\alpha \circ \mathrm{pr}_{H_L})$ for the projection $\mathrm{pr}_{H_L} : G_L \to \Gamma_L$. Furthermore these functors preserve elementary divisors and the rank of a module.

Remark. The author thinks that one should be able to drop the assumption that $K = L$, but since this work will be going into a different direction, we will not prove this here. To prove this, it is advised to work through (Sch17, chapter 2 & 3) with the setup of rings and modules we have established here, but one has to be careful that $\mathbb{A}_K$ might not necessarily have a description as in Definition 1.2.1, if $K|L$ is ramified. So some statements and proofs in (Sch17, chapter 2 & 3) might have to be changed.

Theorem 1.3.5. *(See Kle16, Lemma 2.2.16, Proposition 2.2.25 & Proposition 3.2.16)*

i) Let $E|k$ be any field extension and $\mathcal{O}_{\mathcal{E}}$ be a complete discrete valuation ring, which is an $\mathcal{O}_L$-algebra with uniformizer π and residue field E. Let $\varphi \in \mathrm{End}_{\mathcal{O}_L-Alg}(\mathcal{O}_{\mathcal{E}})$ be a local endomorphism, which lifts the q-Frobenius $(\cdot)^q : E \to E$. Let $F := E^{perf}$ be the perfect hull of E. Then there exists an embedding $\mathcal{O}_{\mathcal{E}} \subset W(F)_L$, such that the Frobenius on $W(F)_L$ induces φ on $\mathcal{O}_{\mathcal{E}}$. We also denote φ for the Frobenius on $W(F)_L$. As in ii) and iii) of the last definition, we define Φ_R^{et} to be the category of etale φ-modules over R, where $R \in \{\mathcal{O}_{\mathcal{E}}, W(F)_L\}$. Then for every $(M, \varphi_M) \in \Phi^{et}_{W(F)_L}$, there exists a unique $\mathcal{O}_{\mathcal{E}}$-submodule $M_E \subset M$, such that $(M_E, \varphi_{M|M_E}) \in \Phi^{et}_{\mathcal{O}_{\mathcal{E}}}$ and for every $N \subset M$ with $(N, \varphi_{M|N}) \in \Phi^{et}_{\mathcal{O}_{\mathcal{E}}}$, we have $N \subset M_E$. This construction is functorial and induces a quasi-inverse for the quasi-equivalence

$$W(F)_L \underset{\mathcal{O}_{\mathcal{E}}}{\otimes} \cdot : \Phi^{et}_{\mathcal{O}_{\mathcal{E}}} \to \Phi^{et}_{W(F)_L}.$$

ii) If $K = L$, then the functors in i) induce quasi-equivalences

$$W(\mathbb{F}_L)_L \underset{\mathbb{A}_L}{\otimes} \cdot : \Gamma_L \Phi^{et}_{\mathbb{A}_L} \leftrightarrow \Gamma_L \Phi^{et}_{W(\mathbb{F}_L)_L} : (\cdot)_{\mathbb{E}_L}.$$

We will generalize the following statement.

Theorem 1.3.6. *Let $(M, \varphi_M) \in \Phi^{et}_{\mathbb{A}_L}$. If there is a Γ_K-action on M, which is semilinear for the action of $\mathbb{A}_L$ denoted by $\alpha : \Gamma_K \to \mathrm{Aut}_{\mathcal{O}_L}(M)$, such that every $\alpha(\gamma)$, $\gamma \in \Gamma_K$ commutes with φ_M, then $(M, \varphi_M, \alpha) \in \Gamma_K \Phi^{et}_{\mathbb{A}_L}$, i.e. the map $\Gamma_K \times M \to M$ induced by α is automatically continuous for the weak topology on M.*

Proof. For $\Gamma_K = \Gamma_L$ this is (See Sch17, Theorem 2.2.8). But the proof there works just as well for open subgroups $\Gamma \subset \Gamma_L$. □

To generalize the last theorem for $\mathcal{R}$ instead of $\mathbb{A}_L$, we will deduce the general case from the special case above. To do this, we need some technical Lemmas.

Lemma 1.3.7. *Let $(M, \varphi_M, \alpha) \in \Gamma_K \Phi^{et}_{\mathbb{A}_K}$. Then*

$$(W(\mathbb{F})_L \underset{\mathbb{A}_K}{\otimes} M, \varphi_L \otimes \varphi_M, \tau \otimes \alpha) \in \Gamma_K \Phi^{et}_{W(\mathbb{F})_L},$$

where $\tau \otimes \alpha$ denotes the diagonal Γ_K-action. The same is true for $\mathbb{A}_L$ and $W(\mathbb{F}_L)_L$ instead of $\mathbb{A}_K$ and $W(\mathbb{F})_L$.

Proof. This can be proven in the same way as (Sch17, Lemma 3.1.11). $\square$

Lemma 1.3.8. *(based on Kle16, Lemma 3.2.3)*
Let $\pi^\infty = 0$. Let $n \in \mathbb{N} \cup \{\infty\}$ We endow $W(\mathbb{F})_L/\pi^n W(\mathbb{F})_L$ with the weak topology as a $W(\mathbb{F})_L$-module. Then the subset topology on

$$\mathbb{A}_K/\pi^n \mathbb{A}_K \subset W(\mathbb{F})_L/\pi^n W(\mathbb{F})_L$$

is the same as the weak topology of $\mathbb{A}_K/\pi^n\mathbb{A}_K$ as a $\mathbb{A}_K/\pi^n\mathbb{A}_K$.

Proof. For $n = \infty$ this is just by definition. So let $n \in \mathbb{N}$. We set $\mathcal{R} := \mathbb{A}_K$ and $\mathcal{S} := W(\mathbb{F})_L$. Since

$$\mathrm{pr}_{\pi^n \mathcal{R}} = \mathrm{pr}_{\pi^n \mathcal{S}|\mathcal{R}}$$

is open for the weak topology and $\mathcal{R}/\pi^n\mathcal{R}$ resp. $\mathcal{S}/\pi^n\mathcal{S}$ is a topological $\mathcal{R}$- resp. $\mathcal{S}$-module, it suffices to show that

$$\mathrm{pr}_{\pi^n \mathcal{S}}(U_{\mathfrak{a},m} \cap \mathcal{R}) = \mathrm{pr}_{\pi^n \mathcal{S}}(U_{\mathfrak{a},m}) \cap \mathrm{pr}_{\pi^n \mathcal{S}}(\mathcal{R})$$

where $m \geq n$ and $U_{\mathfrak{a},m}$ is the $W(\mathcal{O}_{\mathbb{F}})_L$-module corresponding to an open ideal $\mathfrak{a} \subset \mathcal{O}_{\mathbb{F}}$, which defines the weak topology on $\mathcal{S}$. By definition, it is

$$U_{\mathfrak{a},m} = \{(a_0, a_1, \dots) \in \mathcal{S} \mid a_i \in \mathfrak{a} \text{ for all } i \in \{0, \dots, m-1\}\}.$$

By (Sch17, Lemma 1.1.13.i)), we have

$$(a_n + b_n)_n = (a_n)_n + (b_n)_n \text{ for all } (a_n)_n, (b_n)_n \in \mathcal{S} \text{ with } a_n b_n = 0 \text{ for all } n. \tag{$*$}$$

Since $m \geq n$ and $\overline{\mathbb{F}_L}$ is perfect, we calculate for $a := (a_0, \dots)$ and the corresponding $a^{(m)} := (a_0, \dots, a_{m-1}, 0, \dots)$

$$\begin{aligned}
\mathrm{pr}_{\pi^n \mathcal{S}}(U_{\mathfrak{a},m} \cap \mathcal{R}) &= \{\mathrm{pr}_{\pi^n \mathcal{S}}(a) \mid a_i \in \mathfrak{a} \text{ for all } i \in \{0, \dots, m-1\},\ a \in \mathcal{R}\} \\
&\overset{(*)}{=} \{a^{(m)} + \pi^n \mathcal{S} \mid a_i \in \mathfrak{a}, \exists b \in \pi^m \mathcal{S} : a^{(m)} + b \in \mathcal{R}\} \\
&= \{a^{(m)} + \pi^n \mathcal{S} \mid a_i \in \mathfrak{a}, a^{(m)} \in \mathcal{R} + \pi^m \mathcal{S}\} \\
&= \mathrm{pr}_{\pi^n \mathcal{S}}(U_{\mathfrak{a},m}) \cap \mathrm{pr}_{\pi^n \mathcal{S}}(\mathcal{R} + \pi^m \mathcal{S}) = \mathrm{pr}_{\pi^n \mathcal{S}}(U_{\mathfrak{a},m}) \cap \mathrm{pr}_{\pi^n \mathcal{S}}(\mathcal{R}).
\end{aligned}$$

$\square$

Proposition 1.3.9. *Let $(M, \varphi_M) \in \Phi^{et}_{\mathbb{A}_K}$ (resp. $(M, \varphi_M) \in \Phi^{et}_{W(\mathbb{F})_L}$) together with a Γ_K-semilinear action M, which commutes with φ_M. As always, we denote this action by α. Then $(M, \varphi_M, \alpha) \in \Gamma_K \Phi^{et}_{\mathbb{A}_K}$ (resp. $(M, \varphi_M, \alpha) \in \Gamma_K \Phi^{et}_{W(\mathbb{F})_L}$), if and only if $(W(\mathbb{F})_L \underset{\mathbb{A}_K}{\otimes} M, \varphi_L \otimes \varphi_M, \tau \otimes \alpha) \in \Gamma_K \Phi^{et}_{W(\mathbb{F})_L}$ (resp. $(M_{\mathbb{E}}, \varphi_{M|M_{\mathbb{E}}}, \alpha_{|M_{\mathbb{E}}}) \in \Gamma_K \Phi^{et}_{\mathbb{A}_K}$). The same is true for $\mathbb{A}_L$ and $W(\mathbb{F}_L)_L$ instead of $\mathbb{A}_K$ and $W(\mathbb{F})_L$.*

Proof. By Theorem 1.3.5.i), the two cases here are linked by natural isomorphisms. These are topological isomorphisms for the weak topology by Lemma 1.3.2, so the two cases are equivalent. The direction "'nonperfect to perfect" is Lemma 1.3.7. For the other direction, we will prove that, if $(M, \varphi_M, \alpha) \in \Gamma_K \Phi^{et}_{W(\mathbb{F})_L}$, then $(M_{\mathbb{E}}, \varphi_{M|M_{\mathbb{E}}}, \alpha_{|\mathbb{M}_{\mathbb{E}}}) \in \Gamma_K \Phi^{et}_{\mathbb{A}_K}$. By Theorem 1.3.5.i) it suffices to show that α induces an action on $M_{\mathbb{E}}$ and the map $\Gamma \times M_{\mathbb{E}} \to M_{\mathbb{E}}$ induced by α restricted to $M_{\mathbb{E}}$ is continuous. For the first statement, let $\gamma \in \Gamma_K$. Then $(\alpha(\gamma)(M_{\mathbb{E}}), \varphi_M)$ is a finitely generated φ_L-module over $\mathbb{A}_K$, since φ_M and $\alpha(\gamma)$ commute by assumption. Since $\tau(\gamma) : \mathbb{A}_K \to \mathbb{A}_K$ is bijective and $M_{\mathbb{E}}$ is etale, we calculate

$$\mathbb{A}_K \cdot \varphi_M((\alpha(\gamma)(M_{\mathbb{E}})) = \tau(\gamma)(\mathbb{A}_K) \cdot \alpha(\gamma)(\varphi_M(M_{\mathbb{E}})) = \alpha(\gamma)(\mathbb{A}_K \cdot \varphi_M(M_{\mathbb{E}})) = \alpha(\gamma)(M_{\mathbb{E}}),$$

so $\alpha(\gamma)(M_{\mathbb{E}})$ is etale. By the uniqueness of Theorem 1.3.5.i), it follows that $\alpha(\gamma)(M_{\mathbb{E}}) \subset M_{\mathbb{E}}$.

For the second statement, we have by the equivalence of Theorem 1.3.5.i) and the elementary divisor theorem that there are topological isomorphisms, such that the following diagram is commutative.

$$\begin{tikzcd}
M \arrow[r, "\cong"] & W(\mathbb{F})_L^n \oplus W(\mathbb{F})_L/(\pi^{n_1}) \oplus \cdots \oplus W(\mathbb{F})_L/(\pi^{n_r}) \\
M_{\mathbb{E}_L^{sep}} \arrow[u, "\subset"] \arrow[r, "\cong"] & \mathbb{A}_K^n \oplus \mathbb{A}_K/(\pi^{n_1}) \oplus \cdots \oplus \mathbb{A}_K/(\pi^{n_r}) \arrow[u, "\subset"]
\end{tikzcd}$$

Since the weak topology is compatible with direct sums by (Sch17, Exercise 2.2.3.(3)), it suffices to prove the statement for $M = W(\mathbb{F})_L/(\pi^n)$ and $M_{\mathbb{E}} \cong \mathbb{A}_K/(\pi^n)$ for $n \in \mathbb{N} \cup \{\infty\}$. But then the weak topology on $M_{\mathbb{E}}$ is the subspace topology $M_{\mathbb{E}} \subset M$ of the weak topology on M by Lemma 1.3.8, so since α is continuous, so is α restricted to $M_{\mathbb{E}}$. The argumentation is the same for $\mathbb{A}_L$ and $W(\mathbb{F}_L)_L$. □

We also the need the following property in the perfect case.

Lemma 1.3.10. *It is $(W(\mathbb{F})_L, \varphi_L, \tau) \in \Gamma_K \Phi^{et}_{W(\mathbb{F}_L)_L}$, i.e. $W(\mathbb{F})_L$ is a finite $W(\mathbb{F}_L)_L$-module and the Frobenius on $W(\mathbb{F})_L$ has bijective linearisation as an $W(\mathbb{F}_L)_L$-module.*

Proof. First, we show that $W(\mathbb{F})_L$ is a finite unramified extension of $W(\mathbb{F}_L)_L$. By Proposition 1.2.29.ii), there exists a finite unramified extension C of the quotient field $\mathrm{Quot}(W(\mathbb{F}_L)_L)$ with residue field $\mathbb{F}$. By the universal property of the maximal unramified extension (See Kle16, Satz 2.1.10.ii)) or by a variant of (Sch17, Lemma 3.1.2) there exists a lift of the q-Frobenius $(\cdot)^q : \mathbb{F} \to \mathbb{F}$ on $\mathcal{O}_C$, which we denote by $\varphi_C : \mathcal{O}_C \to \mathcal{O}_C$. It follows that we can deduce from Lemma 1.1.13 that

$$\mathcal{O}_C \cong W(\mathbb{F})_L,$$

in particular $W(\mathbb{F})_L \subset W(\overline{\mathbb{F}})_L$ is the finite unramified extension of $W(\mathbb{F})_L$ with residue field $\mathbb{F}$, which is unique by Proposition 1.2.29.ii). Since $\mathbb{F}$ is perfect, the Frobenius on $W(\mathbb{F})_L$ is surjective. So the $W(\mathbb{F}_L)_L$-linearisation of the Frobenius is bijective, since then the image trivially is a subset generating $W(\mathbb{F})_L$ as a $W(\mathbb{F}_L)_L$-module. □

Corollary 1.3.11. *If $M \in \Phi^{et}_{W(\mathbb{F})_L}$, then $M \in \Phi^{et}_{W(\mathbb{F}_L)_L}$.*

Proof. In Lemma 1.3.10, we have seen that $W(\mathbb{F})_L$ is a finite $W(\mathbb{F}_L)_L$-module, so M is a finite $W(\mathbb{F}_L)_L$-module. We only need to show that the Frobenius φ_M on M is etale as an $W(\mathbb{F}_L)_L$-module, but

$$\begin{aligned} W(\mathbb{F}_L)_L \cdot \varphi_M(M) &= W(\mathbb{F}_L)_L \cdot \varphi_L(W(\mathbb{F})_L) \cdot \varphi_M(M) \\ &= W(\mathbb{F})_L \cdot \varphi_M(M) \\ &= M, \end{aligned}$$

where the first equality comes from the semilinearity of φ_M, the second equality is Lemma 1.3.10 and the last equality is due to the hypothesis. So $\varphi_M(M)$ generates M as an $W(\mathbb{F}_L)_L$ and so M is etale over this ring. □

As before, let

$$\mathcal{R} \in \{\mathbb{A}_K, W(\mathbb{F})_L\}.$$

Now we can prove the generalized Theorem.

Theorem 1.3.12. *Let $(M, \varphi_M) \in \Phi^{et}_{\mathcal{R}}$. If there is a Γ_K-action on M, which is semilinear for the action of $\mathcal{R}$ denoted by $\alpha : \Gamma_K \to \mathrm{Aut}_{\mathcal{O}_L}(M)$, such that every $\alpha(\gamma)$, $\gamma \in \Gamma_K$ commutes with φ_M, then $(M, \varphi_M, \alpha) \in \Gamma_K\Phi^{et}_{\mathcal{R}}$, i.e. the map $\Gamma_K \times M \to M$ induced by α is automatically continuous for the weak topology on M.*

Proof. Let M be as in the hypothesis. By Proposition 1.3.9, we can restrict ourselves to the perfect case $\mathcal{R} = W(\mathbb{F})_L$ as it is equivalent to the nonperfect one. Now by Corollary 1.3.11, we can view M as an etale $W(\mathbb{F}_L)_L$-module. Again by Proposition 1.3.9, we can instead show that $M_{\mathbb{E}_L} \in \Gamma_K\Phi^{et}_{\mathbb{A}_L}$. But this is exactly Theorem 1.3.6. □

1.3.2 Motivation

In this part, we will reformulate the statements of Theorem 1.3.4 and Theorem 1.3.5.ii) to statements for the linear algebraic group GL_n over $\mathcal{O}_L$. Recall that Γ_K has an open embedding in $\mathcal{O}_L^\times$ by Proposition 1.1.23. Let again $\mathcal{R} \in \{\mathbb{A}_K, W(\mathbb{F})_L\}$.

Definition 1.3.13. Let $\mathbb{O}_K \subset \mathcal{O}_L^\bullet$ be the submonoid generated by Γ_K and π. Then $\mathbb{O}_K \cong \Gamma_K \times \pi^{\mathbb{N}_0}$. We view $\mathbb{O}_K \subset \mathcal{O}_L$ with the subset topology of the π-adic topology. Furthermore, we extend $\tau : \mathbb{O}_K \to \mathrm{End}_{\mathcal{O}_K-Alg}(\mathcal{R})$ via $\pi^n \mapsto \varphi_L^n$ for $n \in \mathbb{N}$.

Remark. An object in $M \in \Gamma_K\Phi_{\mathcal{R}}$ is the same as a morphism of monoids $\mathbb{O}_K \to \mathrm{End}_{\mathcal{O}_L}(M)$, such that the induced $\mathbb{O}_K$-action on M is semilinear for the $\mathbb{O}_K$-action on $\mathcal{R}$ and continuous on Γ_K for the weak topology on M. The object is etale, if and only if the action of π is etale.

By $\Gamma_K\Phi_{\mathcal{R}}^{(n)} \subset \Gamma_K\Phi_{\mathcal{R}}$, we denote the full subcategory of those modules, which are free of rank n. Analoguesly, we define $\Gamma_K\Phi_{\mathcal{R}}^{et,(n)}$ and $Rep_{\mathcal{O}_L}(G_K)^{(n)}$. Let M be a free $\mathcal{R}$-module of rank n with an $\mathbb{O}_K$-semilinear action. We consider an $\mathcal{R}$-basis $\underline{x} := (x_i)_{1\leq i\leq n}$. Let $\gamma \in \mathbb{O}_K$. By $\gamma * m$, we denote the action of γ on $m \in M$. We define $A := A_{\gamma,\underline{x}} \in Mat_{n\times n}(\mathcal{R})$ to be the Matrix, which satisfies

$$\gamma * x_i = \sum_{j\leq n} A_{ji} x_j.$$

Lemma 1.3.14. *A module $(M, \varphi_M) \in \Phi_{\mathcal{R}}^{(n)}$ is etale if and only if $A_{\pi,\underline{x}} \in \mathrm{GL}_n(\mathcal{R})$ for some $\mathcal{R}$-basis $\underline{x} = (x_i)_i$ of M, where $A_{\pi,\underline{x}}$ is defined as above for the semilinear map φ_M considered as an action of π.*

Proof. Let $1 \otimes \underline{x} := (1 \otimes x_i)_i$ be the corresponding $\mathcal{R}$-basis of $\mathcal{R} \underset{\varphi_L,\mathcal{R}}{\otimes} M$. Let $\varphi_M^{lin} : \mathcal{R} \underset{\varphi_L,\mathcal{R}}{\otimes} M \to M$ be the linearisation of φ_M. Then $A_{\pi,\underline{x}} = {}_{\underline{x}}[\varphi_M^{lin}]_{1\otimes\underline{x}}$ is the Matrix that describes φ_M^{lin} for the $\mathcal{R}$-bases $1 \otimes \underline{x}$ on the left hand side and $\underline{x}$ on the right hand side. So φ_M^{lin} is an isomorphism if and only if $A_{\pi,\underline{x}} \in \mathrm{GL}_n(\mathcal{R})$. □

Let $\gamma * B$ denote the canonical action of $\gamma \in \mathcal{O}_K$ on $B \in Mat_{n\times n}(\mathcal{R})$ given by the action τ on the entries of B. Then

$$(\gamma\delta)*x_i = \gamma*(\delta*x_i) = \gamma*\sum_{j\leq n}(A_{\delta,\underline{x},ji}x_j) = \sum_{j\leq n}\sum_{k\leq n}(\gamma*A_{\delta,\underline{x},ji}\cdot A_{kj}x_k)\ \forall \gamma,\delta \in \mathbb{O}_K.$$

It follows that

$$A_{\gamma\delta,\underline{x}} = A_{\gamma,\underline{x}} \cdot \gamma * A_{\delta,\underline{x}}.$$

Since any morphism of rings sends units into units and the determinant is polynomial over $\mathbb{Z}$, the canonical action of $\mathbb{O}_K$ on $Mat_{n\times n}(\mathcal{R})$ restricts to an $\mathbb{O}_K$-action on $\mathrm{GL}_n(\mathcal{R}) = \{A \in Mat_{n\times n} \mid \det(A) \in \mathcal{R}^\times\}$, which makes it into an $\mathbb{O}_K$-group. It is continuous on Γ_K for the topology on $\mathrm{GL}_n(\mathcal{R})$, which is induced by the weak topology on $\mathcal{R}$. We denote the action also by $\mathrm{GL}_n(\tau(\gamma))(B)$ instead of $\gamma * B$, if $B \in \mathrm{GL}_n(\mathcal{R})$.

Definition 1.3.15. Set

$$\begin{aligned}&C^1(\mathbb{O}_K, \mathrm{GL}_n(\mathcal{R}))\\ :=&\{\alpha : \mathbb{O}_K \to \mathrm{GL}_n(\mathcal{R}) \mid \alpha(\gamma\delta) = \alpha(\gamma)\cdot\gamma * \alpha(\delta)\ \forall \gamma, \delta \in \mathbb{O}_K, \alpha_{|\Gamma_K} \text{ is continuous.}\}.\end{aligned}$$

Lemma 1.3.16. *If M is a free $\mathcal{R}$-module of rank n with a semilinear $\mathbb{O}_K$-action, then $M \in {}_{\Gamma_K}\Phi_{\mathcal{R}}^{et,(n)}$, if and only if*

$$c_{\underline{x}} := [\gamma \mapsto A_{\gamma,\underline{x}}] \in C^1(\mathbb{O}_K, \mathrm{GL}_n(\mathcal{R}))$$

for some $\mathcal{R}$-basis $\underline{x} = (x_i)_i$ of M.

Proof. By Lemma 1.3.14 and the calculations above, it suffices to show that $c_{\underline{x}}$ is continuous on Γ_K if and only if the Γ_K-action on M is continuous. Let the Γ_K-action on M be continuous. By Lemma 1.3.2 the isomorphism

$$f : \mathcal{R}^n \to M, e_i \mapsto x_i$$

is a topological isomorphism for the weak topologies and we have

$$f((A_{\gamma,\underline{x},ji})_j) = \gamma * x_i.$$

So $\gamma \mapsto A_{\gamma,\underline{x}}$ is continuous if and only if $\gamma \mapsto \gamma * x_i$ is continuous for all $1 \le i \le n$. But this holds true, since Γ_K acts continuously on M. On the other hand, let $c_{\underline{x}}$ be continuous on Γ_K. We have already seen that

$$\alpha_i : \Gamma \to M, \gamma \mapsto \gamma * x_i$$

is continuous for all x_i in $\underline{x}$. Let f be as above, $\mu : M \times \mathcal{R} \to M$ be the scalar multiplication and

$$\tau_n : \Gamma_K \times \mathcal{R}^n \to \mathcal{R}^n, (\gamma, (a_i)_i) \mapsto (\tau(\gamma)(a_i))_i.$$

By semilinearity the action $\Gamma_K \times M \to M$ is given by the continuous maps

$$\Gamma_K \times M \overset{(\mathrm{id}_{\Gamma_K}, f^{-1})}{\to} \Gamma_K \times \mathcal{R}^n \overset{\mathrm{pr}_{\Gamma_K} \times \tau_n}{\to} \Gamma_K \times \mathcal{R}^n \overset{(\prod_i \alpha_i, \mathrm{id}_{\mathcal{R}^n})}{\to} M^n \times \mathcal{R}^n \tilde{\to} (M \times \mathcal{R})^n \overset{(\mu)^n}{\to} M^n \overset{\sum}{\to} M.$$

$\square$

Because of Lemma 1.3.16, we call elements $c \in C^1(\mathbb{O}_K, \mathrm{GL}_n(\mathcal{R}))$ *etale (φ_L, Γ_K)-modules over $\mathcal{R}$ with values in* GL_n.

Proposition 1.3.17. *Let $M \in \Gamma_K\Phi_{\mathcal{R}}^{et,(n)}$ and $\underline{x} = (x_i)_i$ be as in the previous Lemma. Then the construction $c_{\underline{x}}$ as in the previous Lemma induces a bijection*

$$(\Gamma_K\Phi_{\mathcal{R}}^{et,(n)}/\cong) \to H^1(\mathbb{O}_K, \mathrm{GL}_n(\mathcal{R})), [M] \mapsto [c_{\underline{x}}],$$

which is independent on the choice of $\underline{x}$.

Proof. Let $\underline{y} = (y_i)_i$ be another $\mathcal{R}$-basis and $X := {}_{\underline{y}}[\mathrm{id}_M]_{\underline{x}} \in \mathrm{GL}_n(\mathcal{R})$ be the Matrix, which describes the basechange from $\underline{x}$ to $\underline{y}$. Then

$$\begin{aligned}\sum_{j\leq n}\sum_{k\leq n} c_{\underline{y}}(\gamma)_{ji} X_{kj} x_k &= \gamma * y_i \\ &= \gamma * \sum_{j\leq n} X_{ji} x_j \\ &= \sum_{j\leq m} \tau(\gamma)(X_{ji})\gamma * x_j \\ &= \sum_{j\leq n}\sum_{k\leq n} \tau(\gamma)(X_{ji}) c_{\underline{x}}(\gamma)_{kj} x_k.\end{aligned}$$

Since the $(x_k)_k$ are linearly independent over $\mathcal{R}$ it follows that

$$c_{\underline{x}}(\gamma) \cdot \mathrm{GL}_n(\tau(\gamma))(X) = X \cdot c_{\underline{y}}(\gamma) \ \forall \gamma \in \mathbb{O}_K.$$

Let $f : M \tilde{\to} N$ be an isomorphism of etale (φ_L, Γ_K)-modules over $\mathcal{R}$ and let $\underline{f(x)} := (f(x_i))_i$. Then $c_{\underline{x}} = c_{\underline{f(x)}}$, since $\gamma * f(x_i) = f(\gamma * x_i)$ and f is linear. Let $c \in C^1(\mathbb{O}_K, \mathrm{GL}_n(\mathcal{R}))$, then we define for every $\gamma \in \mathbb{O}_K$ a γ-semilinear map on $\mathcal{R}^n$ via $\gamma * e_i := c(\gamma) \cdot e_i$ for the standard $\mathcal{R}$-basis $(e_i)_i$ of $\mathcal{R}^n$. This makes $\mathcal{R}^n$ into an object of $\Gamma_K\Phi_{\mathcal{R}}^{et,(n)}$, since it is an action by the cocycle condition, so by Lemma 1.3.16 it is in $\Gamma_K\Phi_{\mathcal{R}}^{et,(n)}$. If $c_1, c_2 \in C^1(\mathbb{O}_K, \mathrm{GL}_n(\mathcal{R}))$ are cohomological, i.e. there exists $B \in \mathrm{GL}_n(\mathcal{R})$, such that

$$c_1(\gamma) \cdot \mathrm{GL}_n(\tau(\gamma))(B) = B \cdot c_2(\gamma) \ \forall \gamma \in \mathbb{O}_K,$$

then B induces an isomorphism $\mathcal{R}^n \to \mathcal{R}^n$, where the left hand side carries the action induced by c_2 and the right hand side carries the action induced by c_1, which can be shown by a calculation as in the beginning of the proof. This induces an inverse map, since if we start with $c \in C^1(\mathbb{O}_K, \mathrm{GL}_n(\mathcal{R}))$ and we take the standard $\mathcal{R}$-basis $(e_i)_i$ of $\mathcal{R}^n$, then

$$\gamma * e_i = \sum_{j\leq n} c(\gamma)_{ji} e_j.$$

On the other hand, if we start with $M \in \Gamma_K\Phi_{\mathcal{R}}^{et,(n)}$ and $\mathcal{R}$-basis $\underline{x} = (x_i)_i$ of M and we make $\mathcal{R}^n$ into an object in $\Gamma_K\Phi_{\mathcal{R}}^{et,(n)}$ via $c_{\underline{x}}$, then the isomorphism $f : \mathcal{R}^n \to M, e_i \mapsto x_i$ satisfies

$$f(\gamma * e_i) = \sum_{j \leq n} f(c_{\underline{x},ji}(\gamma)e_j) = \sum_{j \leq n} c_{\underline{x},ji}(\gamma)x_j = \gamma * x_i = \gamma * f(e_i)\ \forall i.$$

Since the e_i are an $\mathcal{R}$-basis of $\mathcal{R}^n$ and the action is semilinear, this identity holds for all $x \in \mathcal{R}^n$. □

By doing an analogues construction and argument for $Rep_{\mathcal{O}_L}(G_K)^{(n)}$, we get a bijection

$$(Rep_{\mathcal{O}_L}(G_K)^{(n)}) \tilde{\to} \operatorname{mor}^{cont}(G_K, \mathrm{GL}_n(\mathcal{O}_L))/\sim .$$

Then Theorem 1.3.4 and Theorem 1.3.5.ii) give us the following statements.

Theorem 1.3.18. *If $K = L$, then we have the following.*

i) There exist inverse bijections

$$\mathbb{D}_n : (\operatorname{mor}^{cont}(G_L, \mathrm{GL}_n(\mathcal{O}_L))/\sim) \leftrightarrow H^1(\mathcal{O}_L^\bullet, \mathrm{GL}_n(\mathbb{A}_L)) : \mathbb{V}_n,$$

such that

$$\begin{CD}
(\operatorname{mor}^{cont}(G_L, \mathrm{GL}_n(\mathcal{O}_L))/\sim) @>{\mathbb{D}_n}>> H^1(\mathcal{O}_L^\bullet, \mathrm{GL}_n(\mathbb{A}_L)) \\
@V{(\sigma_{\mathcal{O}_L})_*}VV @VV{(\sigma_{\mathbb{A}_L})_*}V \\
(\operatorname{mor}^{cont}(G_L, \mathrm{GL}_m(\mathcal{O}_L))/\sim) @>>{\mathbb{D}_m}> H^1(\mathcal{O}_L^\bullet, \mathrm{GL}_m(\mathbb{A}_L))
\end{CD}$$

commutes for any morphism of groups $\sigma : \mathrm{GL}_n \to \mathrm{GL}_m$ over $\mathcal{O}_L$.

ii) The inclusion $\mathbb{A}_L \subset W(\mathbb{F}_L)_L$ induces a bijection

$$H^1(\mathcal{O}_L^\bullet, \mathrm{GL}_n(\mathbb{A}_L)) \to H^1(\mathcal{O}_L^\bullet, \mathrm{GL}_n(W(\mathbb{F}_L)_L)).$$

Remark. The part about the commutative diagram does not directly follow from the results we established here. We will later prove it in a more general setting and show that the map here induced by Fontaine's functor is the one, we will develop later on.

In the following chapters, we want to generalize this statement for smooth linear algebraic groups $\mathbb{G}$ over $\mathcal{O}_L$ instead of just GL_n.

2 The Case of Characteristic p Coefficients

In this chapter, we will prove the correspondence for general linear algebraic groups over k, i.e. in the "π-torsion" case. For this, we will start with recalling some general theories and theorems that will help us. Then we will show for a general linear algebraic group $\mathbb{G}$ over k that the continuous morphisms $\mathrm{mor}^{cont}(G_K, \mathbb{G}(k))$ can be viewed as a category of functors, such that the conjugacy classes of morphisms correspond to isomorphism classes of these functors. Afterward we will give a generalisation of Theorem 1.3.18 in the "π-torsion" case. In the end, we will calculate some examples.

2.1 General Theories

In this section, we will give an overview of the theories of tannakian categories and forms of linear algebraic groups. We will also recall a theorem of Steinberg, which deals with the surjectivity of a self-map of a linear algebraic group over an algebraically closed field of positive characteristic and generalize it for separably algebrically closed fields.

2.1.1 Tannakian Categories

We will follow Delignes and Milnes (Del12, chapter 1 and 2) to give an overview of those parts in the theory of Tannakian categories that we will need to give a "categorification" of the set of continuous morphisms from G_K to the k-valued points of a linear algebraic group over k.

Let $(\mathcal{C}, \otimes, \phi, \psi)$ be a tensorcategory as in (Del12, Definition 1.1), i.e. $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor together with an associativity constraint ϕ and a commutativity constraint ψ, which is compatible with ϕ (See Del12, (1.0.1) & (1.0.2)), such that there exists an object $\mathbb{1}$ in $\mathcal{C}$, called *unit object*, with the property that $\cdot \otimes \mathbb{1}$ is an auto-equivalence of $\mathcal{C}$.

Definition 2.1.1. Let $(\mathcal{C}', \otimes', \phi', \psi')$ be another tensorcategory. For convenience, we will also write $\otimes$ for $\otimes'$.

i) A *tensor functor from $\mathcal{C}$ to $\mathcal{C}'$* is a pair (F, c), where

$$F : \mathcal{C} \to \mathcal{C}'$$

is a functor and

$$c : \otimes \circ (F \times F) \to F \circ \otimes$$

is a natural isomorphism, which satisfies the following properties.

a) For all X, Y, Z in $\mathcal{C}$ the following diagram is commutative.

$$\begin{tikzcd}
F(X) \otimes (F(Y) \otimes F(Z)) \arrow[r, "\mathrm{id} \otimes c"] \arrow[d, "\phi'"'] & F(X) \otimes F(Y \otimes Z) \arrow[r, "c"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F(\phi)"] \\
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "c \otimes \mathrm{id}"'] & F(X \otimes Y) \otimes F(Z) \arrow[r, "c"'] & F((X \otimes Y) \otimes Z)
\end{tikzcd}$$

b) For all X, Y in $\mathcal{C}$ the following diagram is commutative.

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "c"] \arrow[d, "\psi'"'] & F(X \otimes Y) \arrow[d, "F(\psi)"] \\
F(Y) \otimes F(X) \arrow[r, "c"'] & F(Y \otimes X)
\end{tikzcd}$$

c) For any unit object $\mathbb{1}$ in $\mathcal{C}$, $F(\mathbb{1})$ is a unit object in $\mathcal{C}'$.

ii) Let $\mathcal{C}''$ be a third tensorcategory. If (F, c) is a tensor functor from $\mathcal{C}$ to $\mathcal{C}'$ and (G, d) be a tensor functor from $\mathcal{C}'$ to $\mathcal{C}''$. Then we define the concatenation

$$(G, d) \circ (F, c) := (G \circ F, G(c) \circ d),$$

where $(G(c) \circ d)_{X,Y}$ is the map

$$(G \circ F)(X) \otimes (G \circ F)(Y) \overset{d_{F(X),F(Y)}}{\to} G(F(X) \otimes F(Y)) \overset{G(c_{F(X),F(Y)})}{\to} (G \circ F)(X \otimes Y)$$

for every X, Y in $\mathcal{C}$.

Remark. If $(F, c), (G, d)$ are as in ii) in the Definition above, then $(G, d) \circ (F, c)$ is a tensor functor.

Lemma 2.1.2. *(See Del12, Proposition 1.3.(b))*
Let U, V in $\mathcal{C}$ be two unit objects. Let $u : U \to U \otimes U$ and $v : V \to V \otimes V$ be isomorphisms. Then there exists a unique isomorphism $\alpha = \alpha_{u,v} : U \to V$, such that $(\alpha \otimes \alpha) \circ u = v \circ \alpha$.

By abuse of notation, we also refer to (U, u) as a *unit object in* $\mathcal{C}$, where U and u are as in Lemma 2.1.2.

Definition 2.1.3. Let $(F, c), (G, d)$ be two tensor functors from $\mathcal{C}$ to $\mathcal{C}'$. A *tensor natural morphism between* F *and* G is a natural transformation $\lambda : F \to G$, which satisifies the following properties.

i) For all X, Y in $\mathcal{C}$, the following diagram commutes.

$$\begin{tikzcd} F(X) \otimes F(Y) \arrow[r, "c"] \arrow[d, "\lambda_X \otimes \lambda_Y"'] & F(X \otimes Y) \arrow[d, "\lambda_{X \otimes Y}"] \\ G(X) \otimes G(Y) \arrow[r, "d"'] & G(X \otimes Y) \end{tikzcd}$$

ii) For all unit objects (U, u) in $\mathcal{C}$ and all unit objects (U', u') in $\mathcal{C}'$, the following diagram commutes.

$$\begin{tikzcd} F(U) \arrow[r, "\alpha_{c^{-1} \circ F(u), u'}"] \arrow[d, "\lambda_U"'] & U' \arrow[d, "\mathrm{id}"] \\ G(U) \arrow[r, "\alpha_{d^{-1} \circ G(u), u'}"'] & U' \end{tikzcd}$$

Here, the α are as in Lemma 2.1.2.

We fix a unit object $\mathbb{1}$ in $\mathcal{C}$. Let X, Y be in $\mathcal{C}$, then an *inner hom of* X *and* Y is a representing object for the contravariant functor

$$\mathrm{mor}_{\mathcal{C}}(\cdot \otimes X, Y) : \mathcal{C} \to Set,$$

if it is a representable functor. We denote such an object by $\underline{Hom}(X, Y)$ and by $ev_{X,Y} : \underline{Hom}(X, Y) \otimes X \to Y$, we denote the morphism, which corresponds to $\mathrm{id}_{\underline{Hom}(X,Y)}$. We set $X^\vee := \underline{Hom}(X, \mathbb{1})$ and $ev_X := ev_{X,\mathbb{1}}$. Then there is a morphism $\iota_X : X \to (X^\vee)^\vee$ corresponding to $ev_X \circ \psi : X \otimes X^\vee \to \mathbb{1}$, if $X^\vee$ and $(X^\vee)^\vee$ exist.

Definition 2.1.4. We call X in $\mathcal{C}$ *reflexive*, if $X^\vee$ and $(X^\vee)^\vee$ exist and ι_X is an isomorphism.

Furthermore there exists a morphism

$$\Phi : \underline{Hom}(X, Y) \otimes \underline{Hom}(\tilde{X}, \tilde{Y}) \to \underline{Hom}(X \otimes \tilde{X}, Y \otimes \tilde{Y})$$

corresponding to

$$(\underline{Hom}(X, Y) \otimes \underline{Hom}(\tilde{X}, \tilde{Y})) \otimes (X \otimes \tilde{X}) \overset{\cong}{\to} (\underline{Hom}(X, Y) \otimes X) \otimes (\underline{Hom}(\tilde{X}, \tilde{Y}) \otimes \tilde{X}) \overset{ev \otimes ev}{\to} Y \otimes \tilde{Y},$$

where the isomorphism is given by the associativity and commutativity constraint, if all those objects exist.

Definition 2.1.5. The tensorcategory $(\mathcal{C}, \otimes)$ is called *rigid*, if it satisfies the following properties.

i) For all X, Y in $\mathcal{C}$, the inner hom $\underline{Hom}(X, Y)$ exists.

ii) For all $X, Y, \tilde{X}, \tilde{Y}$ in $\mathcal{C}$, the morphism Φ defined above is an isomorphism.

iii) Every X in $\mathcal{C}$ is reflexive.

Definition 2.1.6. If $\mathcal{C}$ is additive (resp. abelian), we say that the tensorcategory $(\mathcal{C}, \otimes)$ is an *additive (resp. abelian) tensorcategory*, if $\otimes$ is a bi-additive functor.

We fix a field E. From now on, we consider that $(\mathcal{C}, \otimes)$ is an abelian tensorcategory, such that $\mathcal{C}$ is E-linear.

Definition 2.1.7. A linear algebraic group over E is an affine group scheme of finite type over E.

Example. i) The abelian category vec_E of finite dimensional vector spaces over E is a rigid abelian tensorcategory with the usual tensorproduct over E, since $\underline{Hom}(X, Y) = \mathrm{mor}_E(X, Y)$ is an inner hom by the adjointness of tensorproduct and morphisms. Furthermore, the morphisms $X \to (X^\vee)^\vee$ and $\mathrm{mor}_E(X, Y) \otimes \mathrm{mor}_E(\tilde{X}, \tilde{Y}) \to \mathrm{mor}_E(X \otimes \tilde{X}, Y \otimes \tilde{Y})$ for the definition of rigidity are the obvious ones. It is $\mathrm{End}_E(\mathbb{1}) = E$.

ii) Let G be a topological group and $Rep_E(G)$ be category of continuous representations on finite dimensional vector spaces over E, i.e. (V, ρ_V) is in $Rep_E(G)$, if V is in vec_E and $\rho_V : G \to \mathrm{Aut}_E(V)$ is a morphism of groups, such that the induced action $G \times V \to V$ is continuous for the discrete topology on V. The morphisms in $Rep_E(G)$ are those E-linear morphisms, which respect the G-actions. This is a rigid abelian tensor category with $(V, \rho_V) \otimes (W, \rho_W) := (V \underset{E}{\otimes} W, \rho_V \otimes \rho_W)$, since we can equip $\mathrm{mor}_E(V, W)$ with a G-action by conjugation of the G-actions on V and W. Then the forgetful functor $\omega_G : Rep_E(G) \to vec_E$ is faithful and exact. It is $\mathrm{End}_{Rep_E(G)}(\mathbb{1}) = E$.

iii) Let $\mathbb{G}$ be a linear algebraic group over E. By $Rep_E(\mathbb{G})$, we denote the category of E-linear representations over $\mathbb{G}$, i.e. (V, σ_V) is in $Rep_E(\mathbb{G})$, if V is in vec_E and σ_V is a collection of R-linear actions

$$\sigma_{V,R} : \mathbb{G}(R) \times (V \underset{E}{\otimes} R) \to (V \underset{E}{\otimes} R)$$

for every E-algebra R, which is functorial in R. The morphisms of $Rep_E(\mathbb{G})$ are those E-linear morphisms $f : V \to W$, such that $f \otimes \mathrm{id} :$

$V \underset{E}{\otimes} R \to W \underset{E}{\otimes} R$ respects the actions $\sigma_{V,R}$ and $\sigma_{W,R}$ for every E-algebra R. This is an abelian rigid tensorcategory with tensorproduct similiarly defined as in ii). The forgetful functor $Rep_E(\mathbb{G}) \to vec_E$ is exact and faithful. Furthermore, it is $\mathrm{End}_{Rep_E(\mathbb{G})}(\mathbb{1}) = E$.

Definition 2.1.8. Let $\omega : \mathcal{C} \to vec_E$ be a faithful, exact and E-linear functor, such that (ω, c) is a tensor functor for some c as in the definiton of a tensor functor.

i) We say that $(\mathcal{C}, (\omega, c))$ is a *(neutral) tannakian category over E*, if $\mathcal{C}$ it is rigid and $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = E$. We call (ω, c) a *fibre functor of $\mathcal{C}$*.

ii) If $(\mathcal{C}, (\omega_{\mathcal{C}}, c_{\mathcal{C}})), (\mathcal{D}, (\omega_{\mathcal{D}}, c_{\mathcal{D}}))$ are two tannakian categories over E and (F, c) is a tensor functor between $\mathcal{C}$ and $\mathcal{D}$, we say that (F, c) is a *tannakian functor between $\mathcal{C}$ and $\mathcal{D}$*, if F is E-linear and $(\omega_{\mathcal{D}}, c_{\mathcal{D}}) \circ (F, c) = (\omega_{\mathcal{C}}, c_{\mathcal{C}})$. We set

$$Fun^{tan}(\mathcal{C}, \mathcal{D})$$

to be the collection of all tannakian functors between $\mathcal{C}$ and $\mathcal{D}$.

Remark. Our definition of neutral tannakian category differs from the one in (Del12, Definition 2.19) in the way that the fibre functor is part of the datum for us, where in (Del12, Definition 2.19), such a fibre functor is only required to exist, but not part of the datum. We do this here for the definition of a tannakian functor.

Furthermore by using tensor natural transformations as morphisms, we can make $Fun^{tan}(\mathcal{C}, \mathcal{D})$ into a category.

Let $\mathbb{G}$ be a linear algebraic group and $(Rep_E(\mathbb{G}), (\omega_{\mathbb{G}}, \mathrm{id}))$ be the neutral tannakian category of E-linear $\mathbb{G}$-representations with the forgetful fibre functor $\omega := (\omega_{\mathbb{G}}, \mathrm{id})$. By $\mathrm{Aut}^{\otimes}(\omega)$, we denote the group of all tensor natural automorphisms of ω.

Lemma 2.1.9. *(See Del12, Proposition 2.8)*

If (V, σ_V) is in $Rep_E(\mathbb{G})$ and $g \in \mathbb{G}(E)$, we write $\sigma_V^*(g)$ for the automorphism of V induced by the $\mathbb{G}(E)$-action $\sigma_{V,E}$ under g. The map

$$\mathbb{G}(E) \to \mathrm{Aut}^{\otimes}(\omega), g \mapsto (\sigma_V^*(g))_{(V,\sigma_V)}$$

is a well defined isomorphism of groups.

Proposition 2.1.10. *(See Wat79, 3.4 Theorem)*

A group scheme $\mathbb{G}$ over E is a linear algebraic group over E if and only if there exists a closed immersion $\iota : \mathbb{G} \to \mathrm{GL}_n$ of groups over E for some $n \in \mathbb{N}$.

2.1.2 Theorem of Lang-Steinberg

In this part, we recall the classical theorem of Steinberg for the so called *Lang map* and give a slight generalization, so that we can also use it in the non perfect setting for the next section. So let $\mathbb{G}$ be a linear algebraic group over k. Let R be a k-algebra. By $\varphi_R : R \to R, x \mapsto x^q$, we denote the q-Frobenius on R.

Definition 2.1.11. We define the *Lang map on* $\mathbb{G}(R)$ to be

$$\Psi_R : \mathbb{G}(R) \to \mathbb{G}(R), A \mapsto A^{-1} \cdot \mathbb{G}(\varphi_R)(A).$$

Theorem 2.1.12. *(Theorem of Lang-Steinberg)(See Ste68, Theorem 10.1)*
If $E|k$ is an algebraically closed field extension and $\mathbb{G}$ is connected, then Ψ_E is surjective.

Remark. Since Steinberg works with classical group varieties over an algebraically closed field, the correct assumption on $\mathbb{G}$ for this theorem is that the group $\mathbb{G}(E)$ should be connected for the Zariski topology. But $\mathbb{G}(E)$ is connected for the Zariski topology, if and only if the base change $\mathbb{G}_E$ of $\mathbb{G}$ to E is connected by (Gö10, Corollary 3.36), which says that $\mathbb{G}(E)$ corresponds to the closed points of $\mathbb{G}_E$ and that this is a very dense subset in $\mathbb{G}_E$, i.e. its intersection with any closed subset $X \subset \mathbb{G}_E$ is dense in X for the subset topology of $X \subset \mathbb{G}_E$. Here, we used that for $\mathbb{G}_E = \mathrm{Spec}(A)$ the embedding

$$\mathbb{G}(E) = \mathrm{mor}_{E-alg}(A, E) \to \mathbb{G}_E, f \mapsto \ker(f)$$

onto the closed points is a topological embedding for the Zariski topologies, but this is immediate by the definition of these topologies (Compare to Gö10, Example 2.15). Now, every connected group over a field is automatically geometrically connected (See Var19, Proposition 38.7.11), so our assumption for this Theorem is the correct one.

We will generalize this for *separably algebraically closed field* extensions $E|k$, which means that every algebraic and separable element over E already lies in E.

Theorem 2.1.13. *(See Sch07, Satz 2.1)*
Let $E|k$ be a separably algebraically closed field extension. Let V be a finite dimensional E-vector space together with a φ_E-semilinear endomorphism $f : V \to V$, which is etale. Then there exists an E-basis $(v_i)_i$ of V, such that $f(v_i) = v_i$ for all i.

Lemma 2.1.14. *If $E|k$ is a separably algebraically closed field extension, then for $\mathbb{G} = \mathrm{GL}_n$, we have that*

$$\Psi_E^{(n)} : \mathrm{GL}_n(E) \to \mathrm{GL}_n(E), A \mapsto A^{-1} \cdot \mathrm{GL}_n(\varphi_E)(A)$$

is surjective for all $n \in \mathbb{N}$.

Proof. Let $B \in \mathrm{GL}_n(E)$ be arbitrary. Let (e_i) denote the standard E-basis of E^n. Define a φ_E-semilinear endomorphism $\varphi_B : E^n \to E^n$ by extending $(e_i \mapsto B \cdot e_i)_i$ φ_E-semilinearly. Since B is invertible, φ_B is etale (compare to Lemma 1.3.14). So there exists a φ_B-invariant E-basis $(x_i)_i$ of E^n by Theorem 2.1.13. By $x_j^{(i)}$, we denote the i-th entry of x_j. Define $X := (x_j^{(i)})_{ij}$. Since the $(x_i)_i$ form an E-basis of E^n, it is $X \in \mathrm{GL}_n(E)$. We calculate for any j that

$$x_j = \varphi_B(x_j) = \sum_i \varphi_E(x_j^{(i)})(B \cdot e_i) = (\sum_i \varphi_E(X_{ij}) B_{ki})_k.$$

It follows that $X = B \cdot \mathrm{GL}_n(\varphi_E)(X)$, so $B = \Psi_E^{(n)}(X^{-1})$. □

To show the surjectivity for general connected groups, we need the following technical Lemma.

Lemma 2.1.15. *Let $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ be two schemes over a basering C, such that there exists a closed immersion $\iota : X \to Y$, with corresponding projection $\iota^* : A \to B$. Let $R \subset S$ be two C-algebras. Then*

$$X(R) = \iota_S(X(S)) \cap Y(R)$$

via the embedding $\iota_S : X(S) \to Y(S)$ induced by ι.

Proof. The inclusion $X(R) \subset X(S) \cap Y(R)$ follows from the inclusions $R \subset S$ and $X \subset Y$. Let $f \in X(S) \cap Y(R)$. This is a morphism $f : B \to S$ over C, such that $f \circ \iota^* = g$ for a morphism $g : B \to R$ over C. So $\mathrm{im}(f) \subset R$ and so $f \in X(R)$. □

We will mostly use the following special case, which is why we write it down redundantly as its own Lemma.

Lemma 2.1.16. *Let $\mathbb{H} \subset \mathrm{GL}_n$ be a closed subgroup over a ring C and let R, S be two C-algebras with $R \subset S$. Then*

$$\mathbb{H}(R) = \mathbb{H}(S) \cap \mathrm{GL}_n(R).$$

Proof. It is $\mathbb{H} = \mathrm{Spec}(C[\{X_{ij}\}_{1\leq i,j\leq n}]_{det(X_{ij})})/I)$ for some ideal $I \subset C[\{X_{ij}\}_{1\leq i,j\leq n}]$. Then for any C-algebra T, we have

$$\mathbb{H}(T) = \{A \in \mathrm{GL}_n(T) \mid P(A) = 0 \ \forall P \in I\},$$

so $\mathbb{H}(R) = \mathbb{H}(S) \cap \mathrm{GL}_n(R)$. □

Proposition 2.1.17. *Let R be a k-algebra, which is an integral domain. Then $\Psi_R : \mathbb{G}(R) \to \mathbb{G}(R)$ is surjective for all connected linear algebraic groups $\mathbb{G}$ over k, if and only if $\Psi_R^{(n)} : \mathrm{GL}_n(R) \to \mathrm{GL}_n(R)$ is surjective for all $n \in \mathbb{N}$.*

Proof. All the GL_n are connected, so the 'only if' part is immediate. Let $E := \mathrm{Quot}(R)$ be the quotient field of R and $\overline{E}|E$ be an algebraic closure of E. Let $\mathbb{G}$ be connected. We fix an embedding $\mathbb{G} \subset \mathrm{GL}_n$. Let $A \in \mathbb{G}(R)$ be arbitary. By the Theorem of Lang-Steinberg there exists an $B \in \mathbb{G}(\overline{E})$, such that $\Psi_{\overline{E}}(B) = A$ and by hypothesis, there exists an $\tilde{B} \in \mathrm{GL}_n(R)$, such that $\Psi_R^{(n)}(\tilde{B}) = A$. By functoriality of $\mathbb{G}$, it is also $\Psi_{\overline{E}}^{(n)}(\tilde{B}) = A$. We calculate

$$(\Psi_{\overline{E}}^{(n)})^{-1}\{A\} = \mathrm{GL}_n(k) \cdot \tilde{B},$$

since $B_0\tilde{B} \in (\Psi_{\overline{E}}^{(n)})^{-1}\{A\}$, if and only if

$$\tilde{B}^{-1}B_0^{-1} \cdot \mathrm{GL}_n(\varphi_{\overline{E}})(B_0)\,\mathrm{GL}_n(\varphi_{\overline{E}})(\tilde{B}) = \tilde{B}^{-1} \cdot \mathrm{GL}_n(\varphi_{\overline{E}})(\tilde{B}),$$

if and only if $B_0 = \mathrm{GL}_n(\varphi_{\overline{E}})(B_0)$, if and only if

$$B_0 \in \mathrm{GL}_n(\overline{E}) \cap Mat_{n\times n}(k) = \mathrm{GL}_n(k).$$

Since $\mathbb{G} \subset \mathrm{GL}_n$ is a natural transformation, we conclude

$$B \in (\Psi_{\overline{E}})^{-1}\{A\} \subset (\Psi_{\overline{E}}^{(n)})^{-1}\{A\} = \mathrm{GL}_n(k) \cdot \tilde{B} \subset \mathrm{GL}_n(R).$$

So $B \in \mathbb{G}(\overline{E}) \cap \mathrm{GL}_n(R) = \mathbb{G}(R)$ by Lemma 2.1.16. So $A = \Psi_{\overline{E}}(B) = \Psi_R(B)$ by functoriality of $\mathbb{G}$. □

Corollary 2.1.18. *If $\mathbb{G}$ is connected and $E|k$ is a separably algebraically closed field extension, then $\Psi_E : \mathbb{G}(E) \to \mathbb{G}(E)$ is surjective.*

Proof. Lemma 2.1.14 and Proposition 2.1.17. □

2.1.3 Pure Inner Forms

In this section, we recall the theory of forms of a linear algebraic group $\mathbb{G}$ over a field E.

Definition 2.1.19. Let $E_0|E$ be an extension of fields. Then a linear algebraic group $\mathbb{H}$ over E is called an $E_0|E$*-Form of* $\mathbb{G}$, if

$$\mathbb{G}_{E_0} \cong \mathbb{H}_{E_0}$$

for the basechange to E_0. If $E_0 = E^{sep}$ is the separable closure in an algebraic closure of E, we call an $E^{sep}|E$-Form just an E-Form.

Let $E_0|E$ be Galois with galoisgroup $G_{E_0|E}$. Let $\mathbb{H}$ be a $E_0|E$-Form of $\mathbb{G}$. Choose an E_0-isomorphism $\alpha : \mathbb{G}_{E_0} \tilde{\to} \mathbb{H}_{E_0}$. For any $s \in G_{E_0|E}$ we define an E_0-automorphism of $\mathbb{G}_{E_0}$ denoted by

$$\alpha_s := [\alpha, \mathrm{id} \otimes s] := \alpha^{-1} \circ (\mathrm{id}_{\mathbb{H}}, \mathrm{Spec}(s^{-1})) \circ \alpha \circ (\mathrm{id}_{\mathbb{G}}, \mathrm{Spec}(s)).$$

Beware that Spec is contravariant, which is why we have to conjugate α with $\mathrm{Spec}(s^{-1})$ and not with $\mathrm{Spec}(s)$ for the following proposition. We say that two Forms of $\mathbb{G}$ are isomorphic, if they are isomorphic as groups over E. Let $F(\mathbb{G}, E_0|E)$ be the set of isomorphism classes of $E_0|E$-Forms of $\mathbb{G}$ and $A_{E_0} := \mathrm{Aut}_{E_0}(\mathbb{G}_{E_0})$. We make the latter into a discrete $G_{E_0|E}$-group by the formula

$${}^s f := (\mathrm{id}_{\mathbb{G}}, \mathrm{Spec}(s^{-1})) \circ f \circ (\mathrm{id}_{\mathbb{G}}, \mathrm{Spec}(s)),\ \forall f \in A_{E_0}, s \in G_{E_0|E}.$$

Proposition 2.1.20. *(Compare to Ser97, III.§1 Proposition 5)*
The construction above induces a well defined bijection

$$\theta_{E_0|E} : F(\mathbb{G}, E_0|E) \tilde{\to} H^1(G_{E_0|E}, A_{E_0}), \mathbb{H} \mapsto [s \mapsto \alpha_s].$$

Remark 2.1.21. (Based on Spr79, Discussion on p. 11)
Let $(c_s)_s \in C^1(G_{E_0|E}, A_{E_0})$ be a cocycle. Then $\theta^{-1}_{E_0|E}$ is induced by a construction of a form $\mathbb{G}^{(c)}$, for which there exists an identification of the points $\mathbb{G}^{(c)}(E_0) = \mathbb{G}(E_0)$, such that

$$\mathbb{G}^{(c)}(s)(A) = c_{s,E_0} \circ \mathbb{G}(s)(A)\ \forall s \in G_{E_0|E}, A \in \mathbb{G}(E_0), \tag{1}$$

where $c_{s,E_0} \in \mathrm{Aut}(\mathbb{G}(E_0))$ is the automorphism induced by c_s via the canonical isomorphism $\mathbb{G}_{E_0}(E_0) \cong \mathbb{G}(E_0)$. Furthermore, let $F_0 := E_0^{perf}$ be the perfect hull. Recall that $F_0|E^{perf}$ is galois with the same galoisgroup as $E_0|E$. Thus, it makes sense to say that we also have

$$\mathbb{G}^{(c)}(s)(A) = (c_{s,E_0}, \mathrm{id}) \circ \mathbb{G}(s)(A)\ \forall s \in G_{E_0|E}, A \in \mathbb{G}(F_0), \tag{2}$$

where $(c_{s,E_0}, \mathrm{id}) \in \mathrm{Aut}(\mathbb{G}(F_0))$ is the automorphism induced by $(c_s, \mathrm{id}_{F_0}) \in \mathrm{Aut}_{F_0}(\mathbb{G}_{F_0})$ via the canonical isomorphism $(\mathbb{G}_{E_0})_{F_0}(F_0) \cong \mathbb{G}(F_0)$. The construction is as follows. We define an action of $G_{E_0|E}$ on $\mathbb{G}_{E_0}$ via

$$s \mapsto (\mathrm{id}_{\mathbb{G}}, \mathrm{Spec}(s)) \circ c_s^{-1}.$$

If $\mathbb{G} = \mathrm{Spec}(A)$, then this induces a $G_{E_0|E}$-semilinar action of Hopfalgebras on $A \underset{E}{\otimes} E_0$. We define

$$\mathbb{G}^{(c)} := \mathrm{Spec}((A \underset{E}{\otimes} E_0)^{G_{E_0|E}})$$

for the invariants under this action. We define the E-algebra

$$A^{(c)} := (A \underset{E}{\otimes} E_0)^{G_{E_0|E}}.$$

By classical Galois descent (Compare to Sil09, II Lemma 5.8.1), we have that the scalar multiplication induces an isomorphism of Hopfalgebras

$$\mu : A^{(c)} \underset{E}{\otimes} E_0 \to A \underset{E}{\otimes} E_0.$$

So $\mathbb{G}^{(c)}$ is an $E_0|E$-Form of $\mathbb{G}$. Let

$$c_s^* : A \underset{E}{\otimes} E_0 \to A \underset{E}{\otimes} E_0$$

be the E_0-hopfalgebra morphism, which is induced by c_s for $s \in E_{E_0|E}$. Then μ induces the following diagrams to be commutative for every $s \in G_{E_0|E}$.

$$\begin{tikzcd}
A^{(c)} \underset{E}{\otimes} E_0 \arrow[r, "\mathrm{id}_{A^{(c)}} \otimes s"] \arrow[d, "\mu"'] & A^{(c)} \underset{E}{\otimes} E_0 \arrow[d, "\mu"] \\
A \underset{E}{\otimes} E_0 \arrow[r, "(c_s^*)^{-1} \circ (\mathrm{id}_A \otimes s)"'] & A \underset{E}{\otimes} E_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
A^{(c)} \underset{E}{\otimes} F_0 \arrow[r, "\mathrm{id}_{A^{(c)}} \otimes s"] \arrow[d, "\mathrm{Spec}(\mu \otimes \mathrm{id}_{F_0})"'] & A^{(c)} \underset{E}{\otimes} F_0 \arrow[d, "\mathrm{Spec}(\mu \otimes \mathrm{id}_{F_0})"] \\
A \underset{E}{\otimes} F_0 \arrow[r, "(c_s^* \otimes \mathrm{id}_{F_0})^{-1} \circ (\mathrm{id}_A \otimes s)"'] & A \underset{E}{\otimes} F_0
\end{tikzcd}$$

We will only calculate (1). The calculation for (2) goes analogously.

Let $f \in \mathrm{mor}_{E-Alg}(A, E_0) = \mathbb{G}(E_0)$,

$$\iota : A \to A \underset{E}{\otimes} E_0,\ a \mapsto a \otimes 1$$

and

$$\iota^{(c)} : A^{(c)} \to A^{(c)} \underset{E}{\otimes} E_0,\ x \mapsto x \otimes 1.$$

Then we have the following chain of corresponding elements

$$\begin{aligned}
&(f \in \mathrm{mor}_{E-Alg}(A, E_0)) \\
\hat{=}&((f \otimes \mathrm{id}_{E_0}) \in \mathrm{mor}_{E_0-Alg}(A \underset{E}{\otimes} E_0, E_0)) \\
\hat{=}&((f \otimes \mathrm{id}_{E_0}) \circ \mu \in \mathrm{mor}_{E_0-Alg}(A^{(c)} \underset{E}{\otimes} E_0, E_0)) \\
\hat{=}&((f \otimes \mathrm{id}_{E_0}) \circ \mu \circ \iota^{(c)} \in \mathrm{mor}_{E-Alg}(A^{(c)}, E_0)) = ((f \otimes \mathrm{id}_{E_0})_{|A^{(c)}} \in \mathrm{mor}_{E-Alg}(A^{(c)}, E_0)).
\end{aligned}$$

Applying $\mathbb{G}^{(c)}(s)$ gives us

$$(s \circ f \otimes s)_{|A^{(c)}} \in \mathrm{mor}_{E-Alg}(A^{(c)}, E_0).$$

We then have the following chain of corresponding elements.

$$\begin{aligned}
&((s \circ f \otimes s)_{|A^{(c)}} \in \mathrm{mor}_{E-Alg}(A^{(c)}, E_0)) \\
\hat{=}&((s \circ f \otimes s)_{|A^{(c)}} \otimes \mathrm{id}_{E_0}) \in \mathrm{mor}_{E_0-Alg}(A^{(c)} \underset{E}{\otimes} E_0, E_0)) \\
\hat{=}&((s \circ f \otimes s)_{|A^{(c)}} \otimes \mathrm{id}_{E_0}) \circ \mu^{-1} \in \mathrm{mor}_{E_0-Alg}(A \underset{E}{\otimes} E_0, E_0)) \\
\hat{=}&((s \circ f \otimes s)_{|A^{(c)}} \otimes \mathrm{id}_{E_0}) \circ \mu^{-1} \circ \iota \in \mathrm{mor}_{E-Alg}(A, E_0)).
\end{aligned}$$

It follows that to calculate (1), we have to show the equality

$$((s \circ f \otimes s)_{|A^{(c)}} \otimes \mathrm{id}_{E_0}) \circ \mu^{-1} \circ \iota = ((s \circ f) \otimes \mathrm{id}_{E_0}) \circ c_s^* \circ \iota.$$

For this we note that by the definition of the $G_{E_0|E}$-action and $A^{(c)}$ it is

$$(c_s^*)_{|A^{(c)}} = (\mathrm{id}_A \otimes s)_{|A^{(c)}}\ \forall s \in G_{E_0|E}. \tag{C}$$

Let $a \in A$. Since μ is bijective, it is

$$a \otimes 1 = \sum_{i,j} a_i^{(j)} \otimes x_i^{(j)} y_j = \mu(\sum_j (\sum_i a_i^{(j)} \otimes x_i^{(j)}) \otimes y_j)$$

for some $\sum_i a_i^{(j)} \otimes x_i^{(j)} \in A^{(c)}$ and $y_j \in E_0$ for every j. It follows by (C) and the fact that c_s^* is E_0-linear that

$$c_s^*(a \otimes 1) = \sum_j c_s^*(\sum_i a_i^{(j)} \otimes x_i^{(j)}) y_j = \sum_{i,j} a_i^{(j)} \otimes s(x_i^{(j)}) y_j.$$

So by the multiplicativity of $s \in G_{E_0|E}$ and since

$$\mu^{-1}(a \otimes 1) = \sum_j (\sum_i a_i^{(j)} \otimes x_i^{(j)}) \otimes y_j$$

it follows that

$$\begin{aligned}((s \circ f) \otimes \mathrm{id}_{E_0}) \circ c_s^*(a \otimes 1) &= \sum_{i,j} s(f(a_i^{(j)}) x_i^{(j)}) y_j \\ &= ((s \circ f \otimes s)_{|A^{(c)}} \otimes \mathrm{id}_{E_0}) \circ \mu^{-1}(a \otimes 1).\end{aligned}$$

This is the desired equality.

Any $g \in \mathbb{G}(E_0)$ can be made into an inner automorphism of $\mathbb{G}(R)$, where R is an E_0-algebra with the same formula $[\mathbb{G}(R) \ni y \mapsto gyg^{-1}]$. Via this construction and the Yoneda Lemma, Φ extends to a morphism of groups

$$\Phi : \mathbb{G}(E_0) \to A_{E_0}.$$

Remark 2.1.22. Let $\mu : \mathbb{G}_{E_0} \times \mathbb{G}_{E_0} \times \mathbb{G}_{E_0} \to \mathbb{G}_{E_0}$ be the map induced by the multiplication. Then for every $g \in \mathbb{G}(E_0)$, we have

$$\Phi(g) =$$

$$\mathbb{G}_{E_0} \overset{\mathrm{pr}_{E_0} \times \mathrm{id} \times \mathrm{pr}_{E_0}}{\to} \mathrm{Spec}(E_0) \times \mathbb{G}_{E_0} \times \mathrm{Spec}(E_0) \overset{(g \times \mathrm{id}_{E_0}, \mathrm{id}, g^{-1} \times \mathrm{id}_{E_0})}{\to} \mathbb{G}_{E_0} \times \mathbb{G}_{E_0} \times \mathbb{G}_{E_0} \overset{\mu}{\to} \mathbb{G}_{E_0}.$$

Proof. Let R be a E_0-algebra and $can : \mathrm{Spec}(R) \to \mathrm{Spec}(E_0)$ be the canonical morphism. Let $f \in \mathbb{G}_{E_0}(R)$. It is $f = f_1 \times can$ for some $f_1 \in \mathbb{G}(R)$. We have to show that

$$((g \times \mathrm{id}_{E_0}) \circ can) \cdot f \cdot ((g^{-1} \times \mathrm{id}_{E_0}) \circ can) = \mu \circ (g \times \mathrm{id}_{E_0}, \mathrm{id}, g^{-1} \times \mathrm{id}_{E_0}) \circ (\mathrm{pr}_{E_0} \times \mathrm{id} \times \mathrm{pr}_{E_0}) \circ f.$$

By the universal property of the product and $f = f_1 \times can$, it is

$$(\mathrm{pr}_{E_0} \times \mathrm{id} \times \mathrm{pr}_{E_0}) \circ f = can \times f \times can.$$

Again by the universtial property of the product, we have

$$(g \times \mathrm{id}_{E_0}, \mathrm{id}, g^{-1} \times \mathrm{id}_{E_0}) \circ (can \times f \times can) = ((g \times \mathrm{id}_{E_0}) \circ can) \times f \times ((g^{-1} \times \mathrm{id}_{E_0}) \circ can).$$

Since μ induces the multiplication, we have

$$\mu \circ ((g \times \mathrm{id}_{E_0}) \circ can) \times f \times ((g^{-1} \times \mathrm{id}_{E_0}) \circ can) = ((g \times \mathrm{id}_{E_0}) \circ can) \cdot f \cdot ((g^{-1} \times \mathrm{id}_{E_0}) \circ can).$$

□

Lemma 2.1.23. *The map $\Phi : \mathbb{G}(E_0) \to A_{E_0}$ is $G_{E_0|E}$-equivariant.*

Proof. We have to show that

$$\Phi(g \circ \operatorname{Spec}(s)) = (\mathrm{id}_{\mathbb{G}}, \operatorname{Spec}(s^{-1})) \circ \Phi(g) \circ (\mathrm{id}_{\mathbb{G}}, \operatorname{Spec}(s)).$$

We use the identity in Remark 2.1.22 to calculate this. So first we calculate by using the universal property of the product that

$$(\mathrm{pr}_{E_0} \times \mathrm{id} \times \mathrm{pr}_{E_0}) \circ (\mathrm{id}_{\mathbb{G}}, \operatorname{Spec}(s)) = (\operatorname{Spec}(s) \circ \mathrm{pr}_{E_0}) \times (\mathrm{id}_{\mathbb{G}}, \operatorname{Spec}(s)) \times (\operatorname{Spec}(s) \circ \mathrm{pr}_{E_0}).$$

Using the fact that $\cdot \circ \operatorname{Spec}(s)$ is an endomorphism of groups on $\mathbb{G}(E_0)$, we calculate

$$\begin{aligned}
&(g \times \mathrm{id}_{E_0}, \mathrm{id}, g^{-1} \times \mathrm{id}_{E_0}) \circ ((\operatorname{Spec}(s) \circ \mathrm{pr}_{E_0}) \times (\mathrm{id}_{\mathbb{G}}, \operatorname{Spec}(s)) \times (\operatorname{Spec}(s) \circ \mathrm{pr}_{E_0})) \\
=&(g \circ \operatorname{Spec}(s) \times \operatorname{Spec}(s)) \times (\mathrm{id}_{\mathbb{G}}, \operatorname{Spec}(s)) \times ((g \circ \operatorname{Spec}(s))^{-1} \times \operatorname{Spec}(s)).
\end{aligned}$$

Let $\mu_1 : \mathbb{G} \times \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ be the map induced by the multiplication on $\mathbb{G}$. Then

$$\mu = (\mu_1, \mathrm{id}_{E_0}) \circ (\mathrm{pr}_{\mathbb{G}}, \mathrm{pr}_{\mathbb{G}}, \mathrm{pr}_{\mathbb{G}}) \times \mathrm{pr}_{E_0}.$$

Via this, we deduce

$$(\mathrm{id}_{\mathbb{G}}, \operatorname{Spec}(s^{-1})) \circ \mu = \mu \circ (\mathrm{id}_{\mathbb{G}}, \operatorname{Spec}(s^{-1}))^3,$$

where

$$(\mathrm{id}, \operatorname{Spec}(s^{-1}))^3 := (\mathrm{id}_{\mathbb{G}}, \operatorname{Spec}(s^{-1}), \mathrm{id}_{\mathbb{G}}, \operatorname{Spec}(s^{-1}), \mathrm{id}_{\mathbb{G}}, \operatorname{Spec}(s^{-1})).$$

It follows that

$$\begin{aligned}
&(\mathrm{id}, \operatorname{Spec}(s^{-1}))^3 \circ (g \circ \operatorname{Spec}(s) \times \operatorname{Spec}(s)) \times (\mathrm{id}_{\mathbb{G}}, \operatorname{Spec}(s)) \times ((g \circ \operatorname{Spec}(s))^{-1} \times \operatorname{Spec}(s)) \\
=&(g \circ \operatorname{Spec}(s) \times \mathrm{id}_{E_0}) \times \mathrm{id} \times ((g \circ \operatorname{Spec}(s))^{-1} \times \mathrm{id}_{E_0}) \\
=&(g \circ \operatorname{Spec}(s) \times \mathrm{id}_{E_0}, \mathrm{id}, (g \circ \operatorname{Spec}(s))^{-1} \times \mathrm{id}_{E_0}) \circ (\mathrm{pr}_{E_0} \times \mathrm{id} \times \mathrm{pr}_{E_0}).
\end{aligned}$$

So by using the identity in Remark 2.1.22 again for $g \circ \operatorname{Spec}(s)$, we get the desired identity. $\square$

We obtain a map

$$\overline{\Phi}^{(p)} : H^1(G_{E_0|E}, \mathbb{G}(E_0)) \to H^1(G_{E_0|E}, A_{E_0}).$$

Beware that this map is in general neither surjective nor injective.

Definition 2.1.24. We call an $E_0|E$-Form $\mathbb{H}$ of $\mathbb{G}$ *a pure inner form*, if its isomorphism class is in $\theta^{-1}_{E_0|E}(\mathrm{im}(\overline{\Phi}^{(p)}))$.

We close this part with the following technical Lemma that will often be used for calculating points of schemes, in particular for pure inner forms of some group.

Lemma 2.1.25. *If $X = \mathrm{Spec}(A)$ is an affine scheme over an ring R, B is an R-algebra and $S \subset \mathrm{End}_{R-alg}(B)$ is any subset of endomorphisms, then*

$$X(B^S) = X(B)^S$$

for the "action" of S on $X(B)$ induced by $X(s)$ for any $s \in S$. In particular for X over E, we have

$$X(E) = X(E_0)^{G_{E_0|E}}.$$

Proof. It is $X(B) = \mathrm{mor}_{R-alg}(A, B)$ and $X(s)(f) = s \circ f$ for any $s \in S$ and $f \in X(B)$. It is $s \circ f = f$ for all s, if and only if $\mathrm{im}(f) \in B^S$. □

Appendix: Smooth Schemes over an separably algebraically closed field

The following result ties in neatly into the thematics of comparing perfect with non perfect setups, so the author chose to include it.

Definition 2.1.26.

i) Denote by Var_E the category of pairs $(X, A \tilde{\to} E[X_1, \ldots, X_n]/I)$, where $X = \mathrm{Spec}(A)$ is a smooth, affine schemes over E together with a fixed isomorphism of E-algebras $A \tilde{\to} E[X_1, \ldots, X_n]/I$. The morphisms $(X, \cong) \to (Y, \cong)$ are morphisms $X \to Y$ of schemes over E.

ii) Let Var_E^{cl} be the category of Zariski closed embeddings $\iota : X(E) \subset \mathbb{A}^n(E) := E^n$ for some $n \in \mathbb{N}$, where X is a smooth, affine scheme over E and ι is given by an closed embedding $\tilde{\iota} : X \to \mathbb{A}^n$ of schemes over E. A morphism $(\iota_1 : X(E) \to E^n) \to (\iota_2 : Y(E) \to E^m)$ is a polynomial map $f : \mathrm{im}(\iota_1) \to \mathrm{im}(\iota_2)$, i.e. there exist polynomials $P_1, \ldots, P_m \in E[X_1, \ldots, X_n]$, such that

$$f(x_1, \ldots, x_n) = (P_1(x_1, \ldots, x_n), \ldots, P_m(x_1, \ldots, x_m)).$$

Proposition 2.1.27. *(Nonperfect Nullstellensatz)*

Let $X = \mathrm{Spec}(A) \in Var_E$ with the fixed isomorphism $A \cong E[X_1, \ldots, X_n]/I$. View $X(E) = \{x \in E^n \mid f(x) = 0 \ \forall f \in I\}$. Then

$$I(X(E)) := \{P \in E[X_1, \ldots X_n] \mid P(x) = 0 \ \forall x \in X(E)\} = I.$$

Proof. Obviously $I \subset I(X(E))$. Let $g \in I(X(E))$ and $\bar{g} \in A$ be the corresponding element. Define $J := IE[X_1, \ldots, X_{n+1}] + \langle gX_{n+1} - 1) \rangle$. Then $E[X_1, \ldots, X_{n+1}]/J \cong A_{\bar{g}}$. Set $X_g := \mathrm{Spec}(A_{\bar{g}})$. Then we calculate

$$X_g(E) = \{(x_1, \ldots, x_{n+1}) \in E^{n+1} \mid f(x_1, \ldots x_n) = 0 \, \forall f \in I, g(x_1, \ldots, x_n)x_{n+1} = 1\} = \emptyset,$$

since $g \in I(X(E))$. But $X_g(E) \subset X_g$ is dense (See Gö10, B.74, Corollary 6.32, Proposition 6.21), since X_g is smooth over E as an open subscheme of X (See Gö10, Proposition 6.15.(5)). It follows that $A_{\bar{g}} = 0$, so there exists $m \in \mathbb{N}$, such that $\bar{g}^m = 0$ in A. It follows that $g^m \in I$, so $g \in Rad(I) = I$, since X is reduced as a smooth scheme (See Gö10, B.74, Corollary 6.32). □

We have the following functors

Definition 2.1.28.

i) Let $(X, \iota_1 : A \tilde{\rightarrow} E[X_1, \ldots, X_n]/I), (Y, \iota_2 : B \tilde{\rightarrow} E[X_1, \ldots, X_m]/J) \in Var_E$ with a morphism $\tilde{f} : X \to Y$. This induces a morphism $f : B \to A$, which gives a unique morphism $g : E[X_1, \ldots, X_m]/J \to E[X_1, \ldots, X_n]/I$, such that $g \circ \iota_2 = \iota_1 \circ f$. This induces a well defined polynomial map

$$\begin{aligned} g_{poly} : \mathrm{im}(\mathrm{Spec}(\iota_1^{-1})_E) &\to \mathrm{im}(\mathrm{Spec}(\iota_2^{-1})_E) \\ (x_1, \ldots, x_n) &\mapsto (g(X_1)(x_1, \ldots, x_n), \ldots, g(X_m)(x_1, \ldots, x_n)), \end{aligned}$$

We obtain a functor

$$ev_E : Var_E \to Var_E^{cl}, (X, \iota : A \tilde{\rightarrow} E[X_1, \ldots, X_n]/I) \mapsto (\mathrm{Spec}(\mathrm{pr}_I \circ \iota^{-1})_E : X(E) \to E^n),$$

where $\mathrm{pr}_I : E[X_1, \ldots, X_n] \to E[X_1, \ldots, X_n]/I$ is the projection.

ii) We have a contravariant functor

$$F_I : Var_E^{cl} \to E - alg, (\iota : X(E) \to \mathbb{A}^n(E)) \mapsto E[X_1, \ldots, X_n]/I(X(E)),$$

since any polynomial map $f : \mathrm{im}(\iota_1) \to \mathrm{im}(\iota_2)$ with $\mathrm{im}(\iota_i) \subset E^{n_i}$ for $i = 1, 2$ defines a well defined morphism

$$E[X_1, \ldots, X_{n_2}]/I(\mathrm{im}(\iota_2)) \to E[Y_1, \ldots, Y_{n_1}]/I(\mathrm{im}(\iota_1))$$

by sending X_i to P_i, if f is defined by polynomials $P_1, \ldots, P_{n_2}$.

Lemma 2.1.29. *The functors ev_E and F_I are fully faithful.*

Proof. By the nonperfect Nullstellensatz, it is $\Gamma \cong F_I \circ ev_E$, where $\Gamma : Var_E \to E - alg$ is taking global sections. Beware that this isomorphism is natural and not only pointwise, since the isomorphism $\Gamma(X) \to F_I \circ ev_E(X)$ is part of the datum in Var_E and the morphism of $F_I \circ ev_E(Y) \to F_I \circ ev_E(X)$ induced by $X \to Y$ is by construction the morphism, which is induced by the isomorphisms of the data $(X, \cong), (Y, \cong)$. But Γ is fully faithful, so ev_E is faithful and F_I is full. To show that they are fully faithful, we only have to show that F_I is faithful. But if $F_I(f) = F_I(g)$ for polynomial maps $f, g : \mathrm{im}(\iota_1) \to \mathrm{im}(\iota_2)$ with $\mathrm{im}(\iota_1) \subset E^n$, then f and g are equal on

$$V(I(\mathrm{im}(\iota_1))) := \{x \in E^n \mid P(x) = 0 \ \forall P \in I(\mathrm{im}(\iota_2))\}$$

by construction of $F_I(f)$ and $F_I(g)$. Let $x \in \mathrm{im}(\iota_1)$, then $P(x) = 0 \ \forall P \in I(\mathrm{im}(\iota_1))$ by definition of $I(\mathrm{im}(\iota_1))$, so $\mathrm{im}(\iota_1) \subset V(I(\iota_1))$, which means that $f = g$. $\square$

2.2 The Correspondences

In this section, we will finally construct the bijections to generalize Theorem 1.3.18 in the "π-torsion" case. In the first part, we will give a categorification of Galois representations with values in a linear algebraic group as a category of Tannakian Functors for certain Tannakian categories.

In the second part, we will give a correspondence for those representations to (φ_L, Γ_K)-modules with values in Forms of the linear algebraic group we start with.

In the last part, we will give a correspondence of the nonperfect and the perfect setting.

2.2.1 Galois Representations as Tannakian Functors

Let $\mathbb{G}$ be a linear algebraic group over k. We view $\mathbb{G}(k)$ with the discrete topology.

Definition 2.2.1. A *continuous G_K-representation over k with values in* $\mathbb{G}$ is an element

$$f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(k)) := \{a \in \mathrm{mor}_{Grp}(G_K, \mathbb{G}(k)) \mid a \text{ is continuous}\}.$$

Recall the Tannakian categories $(Rep_k(\mathbb{G}), \omega)$ and $(Rep_k(G_K), \omega_K)$ over k, where the fibre functors are the forgetful ones.

Remark. Let $(\mathcal{C}, \omega_{\mathcal{C}})$ and $(\mathcal{D}, \omega_{\mathcal{D}})$ be two neutral Tannakian categories over k. A Tannakian functor (F, c) from $(\mathcal{C}, \omega_{\mathcal{C}})$ to $(\mathcal{D}, \omega_{\mathcal{D}})$ is a pair, where $F : \mathcal{C} \to \mathcal{D}$ is a k-linear functor, such that (F, c) is a tensor functor, which satisfies $\omega_{\mathcal{D}} \circ (F, c) = \omega_{\mathcal{C}}$. Since $\omega_{\mathcal{D}}$ is faithful, c is uniquely determined by $\omega_{\mathcal{C}}$ and $\omega_{\mathcal{D}}$. This is why we will only write F for (F, c) in the following. For example, if $\mathcal{C} = Rep_k(\mathbb{G}), \mathcal{D} = Rep_k(G_K)$, then $(F, c) = (F, \mathrm{id})$. Furthermore, since $\omega_{\mathcal{D}}$ is faithful, F is already uniquely determined on the morphisms by $\omega_{\mathcal{C}}$ and $\omega_{\mathcal{D}}$, which is why in the following we will only write what F does on the objects. For example, if $\mathcal{C} = Rep_k(\mathbb{G}), \mathcal{D} = Rep_k(G_K)$, then $F((V, \sigma_V))$ is of the form (V, ρ_V) for any (V, σ_V) in $Rep_k(\mathbb{G})$ and $F(\phi) = \phi$ for all morphisms $\phi \in Rep_k(\mathbb{G})$. In particular, in this case any such functor is automatically k-linear.

Proposition 2.2.2. *We have a well defined bijection*

$$\begin{aligned} \mathrm{mor}^{cont}(G_K, \mathbb{G}(k)) &\to ob(Fun^{tan}(Rep_k(\mathbb{G}), Rep_k(G_K))) \\ f &\mapsto [(V, \sigma_V) \mapsto (V, [g \mapsto \sigma_V^*(f(g))])] \end{aligned}$$

which induces a bijection from the set of conjugacy classes to the set of isomorphism classes.

Proof. Via the isomorphism $\mathbb{G}(k) \to \mathrm{Aut}^{\otimes}(\omega)$ of Lemma 2.1.9, the statement becomes showing that

$$\begin{aligned}
\mathrm{mor}^{cont}(G_K, \mathrm{Aut}^{\otimes}(\omega)) &\leftrightarrow ob(Fun^{tan}(Rep_k(\mathbb{G}), Rep_k(G_K))) \\
f &\mapsto [F_f : (V, \sigma_V) \mapsto (V, [\rho_{f,\sigma_V} : g \mapsto f(g)_{\sigma_V}])] \\
[f_F : g \mapsto (\rho^*_{F(V,\sigma_V)}(g))_{(V,\sigma_V)}] &\leftarrow\!\shortmid F
\end{aligned}$$

are welldefined maps, which are inverse to each other. Here $\rho^*_{F(V,\sigma_V)}$ is the map $G_K \to \mathrm{Aut}_k(V)$ induced by the G_K-action on V of $F(V, \sigma_V)$. We only show that the maps are welldefined, because then it is easy to see that they are inverse to each other.

First we show that F_f is a Tannakian functor for any $f \in \mathrm{mor}^{cont}(G_K, \mathrm{Aut}^{\otimes}(\omega))$. We need to show that ρ_{f,σ_V} is an automorphism of groups, which induces an continuous action on V for every (V, σ_V) in $Rep_k(\mathbb{G})$. Since f is a morphism of groups, so is ρ_{f,σ_V}. To show that ρ_{f,σ_V} induces a continuous action on V, we need to show that the stabilizer

$$G_v := \{g \in G_K \mid \rho_{f,\sigma_V}(g)(v) = v\}$$

is open for any $v \in V$. But we see that

$$\ker(f) \subset \{g \in G \mid f(g)_{\sigma_V} = \mathrm{id}_V\} \subset G_v$$

and since $\ker(f)$ is open, so is G_v as a subgroup of G_K. Now let $\phi : (V, \sigma_V) \to (W, \sigma_W)$ be a morphism in $Rep_k(\mathbb{G})$. Since $f(g)$ is a natural transformation of the forgetful functor $\omega : Rep_k(\mathbb{G}) \to vec_k$ for every $g \in G_K$, the morphism $F_f(\phi) = \phi$ commutes with $\rho_{f,\sigma_V}(g)$ and $\rho_{f,\sigma_W}(g)$ for every $g \in G_K$. So F_f is a functor. By definition it commutes with the forgetful functors ω and ω_K and it is a Tensor functor sind $f(g)$ is a tensor autormorphism for every $g \in G_K$.

Now, we show that f_F is a continuous morphism of groups for any Tensor functor F. First of all, $F(\phi) = \phi$ for ϕ as above and so ϕ commutes with $f_F(g)_{\sigma_V}$ and $f_F(g)_{\sigma_W}$, so $f_F(g)$ is a natural automorphism of ω for any $g \in G_K$. Furthermore, since F is a tensor functor, it is $f_F(g) \in \mathrm{Aut}^{\otimes}(\omega)$. The map f_F is a morphism of groups, since $F(\sigma_V)$ is a morphism of groups for any (V, σ_V). Since G_K is a profinite group and $\mathrm{Aut}^{\otimes}(\omega)$ is finite by Lemma 2.1.9 and Proposition 2.1.10, it suffices to show that $\ker(f_F) \subset G_K$ is closed for f_F to be continuous. Let $G_v^{(\sigma_V)}$ be the stabilizer of $v \in V$ for the action induced by $F(\sigma_V)$. This is open and hence closed in G_K, since $F(\sigma_V)$ is an object in $Rep_k(G_K)$. We calculate

$$\ker(f_F) = \{g \in G \mid f_F(g)_{\sigma_V} = \mathrm{id}_V \;\; \forall \sigma_V\} = \bigcap_{\sigma_V} \Big(\bigcap_{v \in V} G_v^{(\sigma_V)} \Big).$$

Beware that this intersection makes sense in a set theoretical way, since any $\bigcap_{v \in V} G_v^{\sigma_V}$ is an element in the powerset of G_K for any (V, σ_V). So f_F is a continuous morphism of groups.

Lastly, we want to show that these bijections respect the equivalences on the respective set. For this take note that any Isomorphism between Tannakian functors in $ob(Fun^{tan}(Rep_k(\mathbb{G}), Rep_k(G_K)))$ is an element in $\mathrm{Aut}^{\otimes}(\omega)$, by an argument as for showing that $f_F(g) \in \mathrm{Aut}^{\otimes}(\omega)$ for any $g \in G_K$, since such functors commute with the forgetful functors. So let $\tau : F \to G$ be such an isomorphism between two Tannakian functors F and G. Then we calculate for any (V, σ) in $Rep_k(\mathbb{G})$ that

$$\tau_\sigma \circ f_G(g)_\sigma = \tau_\sigma \circ G(\sigma)(g) = F(\sigma)(g) \circ \tau_\sigma = f_F(g)_\sigma \circ \tau_\sigma.$$

So f_F and f_G are conjugated via τ. A similar calculation gives that $\tau : F_f \to F_{\tau^{-1} \circ f \circ \tau}$ is an isomorphism. □

Remark. Following through the proof of this Proposition, we see that one can exchange G_K by any profinite group $\mathcal{G}$.

2.2.2 Galois Representations and Etale (φ_L, Γ_K)-Modules

In this part, $\mathbb{G}$ is a linear algebraic group over k. We fix an embedding $\mathbb{G} \subset \mathrm{GL}_n$ over k. Let

$$\mathbb{K} \in \{\mathbb{E}, \mathbb{F}\} \text{ and } f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(k)).$$

Let

$$j_{\mathbb{K}} : \mathrm{mor}^{cont}(G_K, \mathbb{G}(k)) \to C^1(H_K, \mathbb{G}(\mathbb{K}^{sep}))$$

and

$$\bar{j}_{\mathbb{K}} : \mathrm{mor}^{cont}(G_K, \mathbb{G}(k)) \to H^1(H_K, \mathbb{G}(\mathbb{K}^{sep}))$$

be the maps induced by restricting to H_K and the inclusion $k \subset \mathbb{K}^{sep}$.

Remark. Let C be a ring and R, S be C-algebras. Let $\phi : R \to S$ be a morphism of C-Algebras and X be a scheme over C. We view $A \in X(C)$ as an element in $X(R)$ via the map that makes R into a C-algebra. Then we have $X(\phi)(A) = A$. We will use this fact for $C = k, R = S = \mathbb{K}^{sep}, X = \mathbb{G}$ and the G_K-action $\bar{\rho}(g)$ for $g \in G_K$ and the Frobenius φ_L several times in this part.

Remark 2.2.3. The map

$$\mathrm{im}(\bar{j}_{\mathbb{E}}) \to \mathrm{im}(\bar{j}_{\mathbb{F}})$$

induced by the inclusion $\mathbb{E}^{sep} \subset \overline{\mathbb{F}}$ is bijective.

Proof. Let $f_1, f_2 \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(k))$. Let $j_{\mathbb{E}}(f_1), j_{\mathbb{E}}(f_2) \in \mathrm{im}(j)$, such that

$$f_1(h) = B^{-1} f_2(h) \mathbb{G}(\bar{\rho}(h))(B)$$

for all $h \in H_L$ and some $B \in \mathbb{G}(\overline{\mathbb{F}})$. Since $f_1(h), f_2(h) \in \mathbb{G}(k)$ and φ_L commutes with the $\bar{\rho}(h)$, we have

$$f_1(h) = \mathbb{G}(\varphi_L^m)(B)^{-1} \cdot f_2(h) \cdot \mathbb{G}(\bar{\rho}(h))(\mathbb{G}(\varphi_L^m)(B))$$

for all $m \in \mathbb{N}$. It is $\mathbb{G}(\mathbb{E}_L^{sep}) = \mathrm{GL}_n(\mathbb{E}_L^{sep}) \cap \mathbb{G}(\overline{\mathbb{F}}_L)$ by Lemma 2.1.16 and so there exists an $N > 0$, such that

$$\mathbb{G}(\varphi_L^N)(B) \in \mathbb{G}(\mathbb{E}_L^{sep}) \text{ and so } \bar{j}_{\mathbb{E}}(f_1) = \bar{j}_{\mathbb{E}}(f_2), .$$

since $\overline{\mathbb{F}} = (\mathbb{E}^{sep})^{perf}$ and $\varphi_L = (\cdot)^q$. $\square$

By Lemma 2.1.23 there are maps

$$\overline{\Phi}^{(p)} : H^1(H_K, \mathbb{G}(\mathbb{E}^{sep})) \to H^1(H_K, A_{\mathbb{E}^{sep}}) \text{ and } \overline{\Phi}^{(p)} : H^1(H_K, \mathbb{G}(\overline{\mathbb{F}})) \to H^1(H_K, A_{\overline{\mathbb{F}}})$$

and

$$\overline{\Phi}_c^{(p)} : C^1(H_K, \mathbb{G}(\mathbb{E}^{sep})) \to C^1(H_K, A_{\mathbb{E}^{sep}}) \text{ and } \overline{\Phi}_c^{(p)} : C^1(H_K, \mathbb{G}(\overline{\mathbb{F}})) \to C^1(H_K, A_{\overline{\mathbb{F}}}),$$

where $A_{\mathbb{K}^{sep}} := \mathrm{Aut}_{\mathbb{K}^{sep}}(\mathbb{G}_{\mathbb{K}^{sep}})$ are the automorphisms of groupschemes over $\mathbb{K}^{sep}$. The map

$$\mathrm{Aut}_{\mathbb{E}^{sep}}(\mathbb{G}_{\mathbb{E}^{sep}}) \to \mathrm{Aut}_{\overline{\mathbb{F}}}(\mathbb{G}_{\overline{\mathbb{F}}}), f \mapsto (f, \mathrm{id}_{\overline{\mathbb{F}}})$$

is H_K-equivariant, since conjugating $(f, \mathrm{id}_{\overline{\mathbb{F}}})$ with $(\mathrm{id}_{\mathbb{G}}, \mathrm{Spec}(s^{-1}))$ for $s \in H_K$ cancels itself out on $\mathrm{Spec}(\overline{\mathbb{F}})$, so

$$(\mathrm{id}_{\mathbb{G}}, \mathrm{Spec}(s^{-1})) \circ (f, \mathrm{id}_{\overline{\mathbb{F}}}) \circ (\mathrm{id}_{\mathbb{G}}, \mathrm{Spec}(s)) = ((\mathrm{id}_{\mathbb{G}}, \mathrm{Spec}(s^{-1})) \circ f \circ (\mathrm{id}_{\mathbb{G}}, \mathrm{Spec}(s)), \mathrm{id}_{\overline{\mathbb{F}}}).$$

Lemma 2.2.4. *The following diagram is commutative.*

$$\begin{tikzcd}
C^1(H_K, \mathbb{G}(\mathbb{E}^{sep})) \arrow[r, "(\subset)_*"] \arrow[d, "\overline{\Phi}_c^{(p)}"'] & C^1(H_K, \mathbb{G}(\overline{\mathbb{F}})) \arrow[d, "\overline{\Phi}_c^{(p)}"] \\
C^1(H_K, A_{\mathbb{E}^{sep}}) \arrow[r, "(c_s)_s \mapsto (c_s, \mathrm{id}_{\overline{\mathbb{F}}})_s"'] & C^1(H_K, A_{\overline{\mathbb{F}}})
\end{tikzcd}$$

Proof. Let $(c_s)_s \in C^1(H_K, \mathbb{G}(\mathbb{E}^{sep}))$. Then we need to show that $(\Phi(c_s), \mathrm{id}_{\overline{\mathbb{F}}})_s$ induces conjugating with c_s on $\mathbb{G}(R)$ for every $\overline{\mathbb{F}}$-algebra R and $s \in H_K$. So let $f \in \mathbb{G}(R)$ and $f \times can \in \mathbb{G}_{\overline{\mathbb{F}}}(R)$ be the corresponding element, where $can : \mathrm{Spec}(R) \to \mathrm{Spec}(\overline{\mathbb{F}})$ is the canonical morphism. Then by definition of Φ and the universal property of the fiber product, we calculate

$$(\Phi(c_s), \mathrm{id}_{\overline{\mathbb{F}}}) \circ (f \times can) = (\Phi(c_s) \circ f) \times can = (c_s \cdot f \cdot c_s^{-1}) \times can,$$

which corresponds to $c_s \cdot f \cdot c_s^{-1} \in \mathbb{G}(R)$. $\square$

For this section, we shorten $j := j_{\mathbb{E}}$ and $\bar{j} := \bar{j}_{\mathbb{E}}$. For $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(k))$ we define

$$\mathbb{G}^{(f)} := (\mathbb{G}_{\mathbb{E}})^{(\overline{\Phi}_c^{(p)}(j(f))}.$$

This is a pure inner form of $\mathbb{G}_{\mathbb{E}}$ (over $\mathbb{E}$!) and we have an identification

$$\mathbb{G}^{(f)}(\mathbb{K}^{sep}) = \mathbb{G}(\mathbb{K}^{sep}), \tag{F1}$$

$$\mathbb{G}^{(f)}(\bar{\rho}(h))(A) = f(h) \cdot \mathbb{G}(\bar{\rho}(h))(A) \cdot f(h)^{-1} \; \forall A \in \mathbb{G}^{(f)}(\mathbb{K}^{sep}), h \in H_K \tag{F2}$$

$$\mathbb{G}^{(f)}(\mathbb{K}) = \mathbb{G}^{(f)}(\mathbb{K}^{sep})^{H_K} \tag{F3}$$

by Remark 2.1.21.(1) & (2), Lemma 2.2.4 and Lemma 2.1.25. For $\mathbb{K} = \mathbb{F}$, we could also define

$$\mathbb{G}^{(f),\mathbb{F}} := (\mathbb{G}_{\mathbb{F}})^{(\overline{\Phi}_c^{(p)}(j_{\mathbb{F}}(f))},$$

but going through the definitions and with Lemma 2.2.4 one sees that

$$\mathbb{G}^{(f),\mathbb{F}} \cong (\mathbb{G}^{(f)})_{\mathbb{F}},$$

so working with $\mathbb{G}^{(f),\mathbb{F}}$ gives the same results as working with $\mathbb{G}^{(f)}$ in this part for the perfect setting, but in the next part, where we want to compare the perfect with the nonperfect case, we will need to work with $\mathbb{G}^{(f)}$ for both settings.

Since $\mathbb{G}^{(f)}$ is *not* necessarily defined over k, it is

$$\mathbb{G}^{(f)}(\bar{\tau}(\gamma)) : \mathbb{G}^{(f)}(\mathbb{K}) \to \mathbb{G}^{(f)}(\mathbb{K})$$

not a well defined morphism of groups for $\gamma \in \mathbb{O}_K$. But because of (F1),(F2), (F3), f is defined on G_K and takes values in $\mathbb{G}(k)$, we can well define for $\gamma = \mathrm{pr}_{H_K}(g_\gamma)\pi^{n(\gamma)}$, where $g_\gamma \in G_K$ the morphism of groups

$$\gamma * A := \gamma \underset{f}{*} A := f(g_\gamma)\mathbb{G}(\bar{\rho}(g_\gamma)\varphi_L^{n(\gamma)})(A)f(g_\gamma)^{-1} \; \forall A \in \mathbb{G}^{(f)}(\mathbb{K}).$$

Since $H_K \subset G_K$ is normal and (F2),(F3), it is $\gamma * A \in \mathbb{G}^{(f)}(\mathbb{K})$ and since $f(g)$ takes values in $\mathbb{G}(k)$, it is $(\gamma\delta) * A = \gamma * (\delta * A)$.

It follows, that $\mathbb{G}^{(f)}(\mathbb{K})$ is an $\mathbb{O}_K$-group.

Lemma 2.2.5. *Let X be an affine scheme of finite type over a topological ring R. Let $X \subset \mathbb{A}^n$ be a closed immersion into an affine space. Then the induced topology*

$$X(R) \subset R^n$$

is independent of the choice of embedding. Furthermore, if $X \subset \mathrm{GL}_n$ is a closed subgroup and $R^\times$ is a topological group, then $X(R) \subset \mathrm{GL}_n(R)$ is a topological group with the induced topology, which is the same topology as for a closed embedding $X \subset \mathbb{A}^n$.

Proof. Let $\iota_1 : X \to \mathbb{A}^n$ and $\iota_2 : X \to \mathbb{A}^m$ be two closed immersions. Let $\iota_1(X) = \mathrm{Spec}(R[X_1, \ldots, X_n]/I_1)$ and $\iota_2(X) = \mathrm{Spec}(R[X_1, \ldots, X_m]/I_2)$. Then

$$\iota_2 \circ \iota_1^{-1} : \iota_1(X) \to \iota_2(X)$$

is given by a map of R-algebras

$$R[X_1, \ldots, X_m]/I_2 \to R[X_1, \ldots, X_n]/I_1$$

and so induces a polynomial map

$$(\iota_1(X))(R) \to (\iota_2(X))(R)$$

Since R is a topological ring, every polynomial map is continuous and by symmetry the inverse map

$$(\iota_2(X))(R) \to (\iota_1(X))(R)$$

is also continuous.

If $X \subset \mathrm{GL}_n$ is a closed subgroup, we have the closed embedding $\mathrm{GL}_n \subset \mathbb{A}^{n^2+1}$. By the first part of this Lemma, the topology on $X(R)$ via $X \subset \mathbb{A}^{n^2+1}$ is independent of the choice of embedding $X \subset \mathrm{GL}_n \subset \mathbb{A}^{n^2+1}$, but the open embedding $\mathrm{GL}_n \subset \mathbb{A}^{n^2}$ gives the same topology on $\mathrm{GL}_n(R)$ as the embedding $\mathrm{GL}_n \subset \mathbb{A}^{n^2+1}$. It follows that the topology on $X(R)$ is also independent of the choice of embedding $X \subset \mathrm{GL}_n$.

To show that $X(R)$ is a topological group, it suffices to show that $\mathrm{GL}_n(R)$ is a topological group, since then $X(R) \subset \mathrm{GL}_n(R)$ is a subgroup. But this is true by the hypothesis, since multiplication on $\mathrm{GL}_n(R)$ is polynomial and inverting elements is given by polynomial maps and inverting elements in $R^\times$ by Cramer's rule. $\square$

Remark. If R is one of the topological rings with a weak topology we constructed in this work, then we also call the topology of this Lemma the *weak topology on $X(R)$*. Those rings satisfy all conditions made in this Lemma by Remark 1.1.42, Lemma 1.2.9 and Proposition 1.2.12.

We view $\mathbb{G}^{(f)}(\mathbb{K})$ as a topological group with the weak topology. Then $\mathbb{G}^{(f)}(\mathbb{K})$ is a topological Γ_K-group via the action $\gamma * \cdot$ defined above. To see this, one first notices that the weak topology on $\mathbb{K}^{sep}$ induces the discrete topology on k, so f is continuous for the weak topology on $\mathbb{G}(\mathbb{K}^{sep})$. Secondly the action $G_K \times \mathbb{G}(\mathbb{K}^{sep}) \to \mathbb{G}(\mathbb{K}^{sep})$ given by $\mathbb{G}(\bar{\rho}(\cdot))$ is continuous for the weak topology on $\mathbb{G}(\mathbb{K}^{sep})$ by Lemma 1.1.30 and Lemma 2.2.5. Lastly, one has to check that

$$\iota_1 : \mathbb{G}^{(f)}(\mathbb{K}) \subset \mathbb{G}(\mathbb{K}^{sep}) \subset \mathrm{GL}_n(\mathbb{K}^{sep})$$

is continuous for the weak topologies. For this last statement, view $j_{\mathbb{K}}(f) : H_K \to \mathrm{GL}_n(\mathbb{K}^{sep})$. By Hilbert 90, there exists $B \in \mathrm{GL}_n(\mathbb{K}^{sep})$, such that

$$f(h) = B^{-1} \cdot \mathbb{G}(\bar{\rho}(h))(B) \ \forall h \in H_K.$$

From this and (F2),(F3), it follows that

$$\iota_2 : \mathbb{G}^{(f)}(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K}), A \mapsto B \cdot \iota_1(A) \cdot B^{-1}$$

is a well defined embedding, which is conjugate and hence topologically isomorphic to ι_1. Furthermore, ι_2 is the embedding given by

$$\mathbb{G}^{(f)} \subset \mathrm{GL}_n^{(f)} \cong \mathrm{GL}_n,$$

where "$\cong$" is the following isomorphism. Let

$$\mathrm{GL}_{n,\mathbb{K}} = \mathrm{Spec}(R),\ \mathbb{G}_{\mathbb{K}} = \mathrm{Spec}(S) \text{ and } p : R \to S$$

be the projection that induces our fixed embedding $\mathbb{G} \subset \mathrm{GL}_n$. By Remark 2.1.21 the linear algebraic group $\mathrm{GL}_n^{(f)}$ is defined by

$$\mathrm{Spec}((R \underset{\mathbb{K}}{\otimes} \mathbb{K}^{sep})^{H_K})$$

with the H_K-action being induced by the action on $\mathrm{GL}_{n,\mathbb{K}^{sep}}$ via

$$h \mapsto (\mathrm{id}_{\mathrm{GL}_n}, \mathrm{Spec}(\bar{\rho}(h))) \circ (f(h)_*)^{-1} \ \forall h \in H_K$$

with $f(h)_*$ being the map induced by conjugation with $f(h)$ on $\mathrm{GL}_n(T)$ for all $\mathbb{K}^{sep}$-algebras T. By (Gö10, Proposition 12.27 (1)) $\mathrm{GL}_n^{(f)}$ is the quotient of $\mathrm{GL}_{n,\mathbb{K}^{sep}}$ under this H_K-action. Consider the map

$$\phi : \mathrm{GL}_{n,\mathbb{K}^{sep}} \xrightarrow{B_*} \mathrm{GL}_{n,\mathbb{K}^{sep}} \xrightarrow{\mathrm{pr}_{\mathrm{GL}_n}} \mathrm{GL}_n,$$

where B_* is analoguesly defined as $f(h)_*$ above by the induced map of conjugation with B. Then by the following calculation $\phi \circ a(h) = \phi$ for all $h \in H_K$

with $a(h)$ being the map given by the action above. By Lemma 2.1.23, we have

$$B_* \circ (\mathrm{id}_{\mathrm{GL}_n}, \mathrm{Spec}(\bar{\rho}(h))) = (\mathrm{id}_{\mathrm{GL}_n}, \mathrm{Spec}(\bar{\rho}(h))^{-1}) \circ \mathbb{G}(\bar{\rho}(h))(B)_* \ \forall h \in H_K.$$

Furthermore, we have

$$\mathbb{G}(\bar{\rho}(h))(B) \cdot f(h)^{-1} = B \ \forall h \in H_K$$

and

$$\mathrm{pr}_{\mathrm{GL}_n} \circ (\mathrm{id}_{\mathrm{GL}_n}, \mathrm{Spec}(\bar{\rho}(h))^{-1}) = \mathrm{pr}_{\mathrm{GL}_n},$$

so together we get $\phi \circ a(h) = \phi$. By the universal property of the the quotient $\mathrm{GL}_n^{(f)}$, this induces a morphism of groups

$$\bar{\phi} : \mathrm{GL}_n^{(f)} \to \mathrm{GL}_n$$

with $\phi = \bar{\phi} \circ \mathrm{pr}$, where $\mathrm{pr} : \mathrm{GL}_{n,\mathbb{K}^{sep}} \to \mathrm{GL}_n^{(f)}$ is the projection given by the inclusion

$$(R \underset{\mathbb{K}}{\otimes} \mathbb{K}^{sep})^{H_K} \subset R \underset{\mathbb{K}}{\otimes} \mathbb{K}^{sep}.$$

Then $\bar{\phi}$ induces the map

$$\mathrm{GL}_n(\mathbb{K}^{sep}) = \mathrm{GL}_n^{(f)}(\mathbb{K}^{sep}) \to \mathrm{GL}_n(\mathbb{K}^{sep}), A \mapsto BAB^{-1},$$

since $\phi = \bar{\phi} \circ \mathrm{pr}$ and the identification $\mathrm{GL}_n(\mathbb{K}^{sep}) = \mathrm{GL}_n^{(f)}(\mathbb{K}^{sep})$ is induced by the isomorphism $\mathrm{pr} \times \mathrm{pr}_{\mathbb{K}^{sep}}$ given by classical Galois descent, where

$$\mathrm{pr}_{\mathbb{K}^{sep}} : \mathrm{GL}_{n,\mathbb{K}^{sep}} \to \mathrm{Spec}(\mathbb{K}^{sep})$$

is the projection. Furthermore $\bar{\phi}$ is an isomorphism, since by a similar argumentation as above, we obtain that

$$\psi : \mathrm{GL}_{n,\mathbb{K}^{sep}} \overset{B_*^{-1}}{\to} \mathrm{GL}_{n,\mathbb{K}^{sep}} \overset{\mathrm{pr}}{\to} \mathrm{GL}_n^{(f)}$$

induces an inverse $\mathrm{GL}_n \to \mathrm{GL}_n^{(f)}$ to $\bar{\phi}$ by the universal property of

$$\mathrm{GL}_n = \mathrm{GL}_n^{(1)},$$

where $1 \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(k))$ is the trivial morphism.

The closed immersion $\mathbb{G}^{(f)} \subset \mathrm{GL}_n^{(f)}$ is the map induced on the respective H_K-invariants by the projection

$$p \otimes \mathrm{id}_{\mathbb{K}^{sep}} : R \underset{\mathbb{K}}{\otimes} \mathbb{K}^{sep} \to S \underset{\mathbb{K}}{\otimes} \mathbb{K}^{sep}.$$

This is an embedding, since taking H_K-invariants in this setting is exact by the classical Galois descent. Furthermore, this embedding induces the map

$$j_1 : \mathbb{G}^{(f)}(\mathbb{K}) \subset \mathbb{G}^{(f)}(\mathbb{K}^{sep}) \to \mathrm{GL}_n^{(f)}(\mathbb{K}^{sep}) = \mathrm{GL}_n(\mathbb{K}^{sep}), A \mapsto \iota_1(A),$$

since

$$\iota_1 : \mathbb{G}^{(f)}(\mathbb{K}) \subset \mathbb{G}^{(f)}(\mathbb{K}^{sep}) = \mathbb{G}(\mathbb{K}^{sep}) \to \mathrm{GL}_n(\mathbb{K}^{sep})$$

and by definition we have the equality of embeddings

$$(\mathbb{G}^{(f)}_{\mathbb{K}^{sep}} \subset \mathrm{GL}^{(f)}_{n,\mathbb{K}^{sep}} \cong \mathrm{GL}_{n,\mathbb{K}^{sep}}) = (\mathbb{G}^{(f)}_{\mathbb{K}^{sep}} \cong \mathbb{G}_{\mathbb{K}^{sep}} \subset \mathrm{GL}_{n,\mathbb{K}^{sep}}),$$

where the isomorphisms are the ones given by classical Galois descent, which also induce the equalities in both maps j_1 and ι_1 above.

Together, we obtain that

$$\mathbb{G}^{(f)} \subset \mathrm{GL}_n^{(f)} \cong \mathrm{GL}_n$$

induces ι_2.

Definition 2.2.6. We define

$$C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{K})) := \{\alpha : \mathbb{O}_K \to \mathbb{G}^{(f)}(\mathbb{K}) \mid \alpha(\gamma\delta) = \alpha(\gamma)\cdot(\gamma * \alpha(\delta))\ \forall \gamma, \delta, \alpha_{|\Gamma_K} \text{ is continuous}\}.$$

An *etale* (φ_L, Γ_K)*-module over* $\mathbb{K}$ *with values in* $\mathbb{G}^{(f)}$ is an element $\alpha \in C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{K}))$.

We say, that $\alpha, \beta \in C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{K}))$ are *cohomologous* or $\alpha \sim \beta$, if there exists an $A \in \mathbb{G}^{(f)}(\mathbb{K})$, such that $\alpha(\gamma) = A^{-1} \cdot \beta(\gamma) \cdot \gamma \underset{f}{*} A$ for all $\gamma \in \mathbb{O}_K$. We define

$$H^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{K})) := C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{K}))/ \sim .$$

Consider the map

$$\bar{j} : \mathrm{mor}^{cont}(G_K, \mathbb{G}(k)) \to H^1(H_K, \mathbb{G}(\mathbb{E}^{sep}))$$

and fix a family of elements $\{f_i\}_i \subset \mathrm{mor}^{cont}(G_K, \mathbb{G}(k))$, such that

$$\bar{j} : \{f_i\}_i \to \mathrm{im}(\bar{j})$$

is bijective. Now let $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(k))$ be any element. Then there exists an unique i and some $B \in \mathbb{G}(\mathbb{K}^{sep})$, such that

$$f(h) = B^{-1} \cdot f_i(h) \cdot \mathbb{G}(\bar{\rho}(h))(B)\ \forall h \in H_K \tag{$*$}$$

We define a G_K-action of sets on $\mathbb{G}^{(f_i)}(\mathbb{K}^{sep})$ via

$$\begin{aligned} g.A := g \underset{f,f_i}{.} A &:= f(g) \cdot \mathbb{G}(\bar{\rho}(g))(A) \cdot f_i(g)^{-1} \\ \text{and } \pi^n.A := \pi^n \underset{f,f_i}{.} A &:= \mathbb{G}(\varphi_L^n)(A) \ \forall A \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep}), g \in G_K, n \in \mathbb{N}. \end{aligned}$$

Since f and f_i take values in $\mathbb{G}(k)$ and φ_L commutes with $\bar{\rho}(g)$ for all $g \in G_K$, we calculate

$$\begin{aligned} (g_1 g_2).A &= g_1.(g_2.A) \ \forall A \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep}), g_1, g_2 \in G_K \\ g.(\pi^n.A) &= \pi^n.(g.A) \ \forall A \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep}), g \in G_K, n \in \mathbb{N} \\ \pi^{n+m}.A &= \pi^n.(\pi^m.A) \ \forall A \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep}), \ n, m \in \mathbb{N}. \end{aligned} \tag{act}$$

By $(*)$, it is

$$B^{-1} \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep})^{H_K,f} := \{A \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep}) \mid h.A = A \ \forall h \in H_K\},$$

since for any $h \in H_K$, it is

$$\begin{aligned} h.(B^{-1}) &= f(h) \cdot \mathbb{G}(\bar{\rho}(h))(B^{-1}) \cdot f_i(h)^{-1} \\ &= B^{-1} \cdot f_i(h) \cdot \mathbb{G}(\bar{\rho}(h))(B) \cdot \mathbb{G}(\bar{\rho}(h))(B^{-1}) \cdot f_i(h)^{-1} \\ &= B^{-1}. \end{aligned}$$

Remark. If $A_0 \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep})^{H_K,f}$, then

$$\mathbb{G}^{(f_i)}(\mathbb{K}^{sep})^{H_K,f} = A_0 \mathbb{G}^{(f_i)}(\mathbb{K}).$$

Proof. If $A \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep})^{H_K,f}$, then $A = A_0 B_0$ for some $B_0 \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep})$ and for $h \in H_K$ we have

$$A_0 \cdot \mathbb{G}^{(f_i)}(\bar{\rho}(h))(B_0) = (h.A_0) \cdot \mathbb{G}^{(f_i)}(\bar{\rho}(h))(B_0) \overset{(F2)}{=} h.(A_0 B_0) = A_0 B_0,$$

so $B_0 \in \mathbb{G}^{(f_i)}(\mathbb{K})$ and the same calculation shows that if $B_0 \in \mathbb{G}^{(f_i)}(\mathbb{K})$, then $A := A_0 B_0 \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep})^{H_K,f}$. $\square$

If $A_0 \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep})^{H_K,f}$, then because of $(F2), (F3)$ and (act) it makes sense to define $\gamma.A_0$ for $\gamma \in \mathbb{O}_K$ and since $H_K \subset G_K$ is normal and $(F2), (F3)$ it is $\gamma.A_0 \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep})^{H_K,f}$. So by the last remark, we define

$$\alpha_{f,A_0}(\gamma) := A_0^{-1} \cdot \gamma.A_0 \in \mathbb{G}^{(f_i)}(\mathbb{K}).$$

Lemma 2.2.7. *It is*

$$\alpha := \alpha_{f,A_0} \in C^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K}))$$

and if $B \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep})^{H_K,f}$ *is another element, then* $[\alpha_{f,A_0}]_\sim = [\alpha_{f,B}]_\sim \in H^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K}))$.

Proof. If $\gamma = \mathrm{pr}_H(g_\gamma)\pi^{n(\gamma)}$, we calculate

$$\begin{aligned}
\alpha(\gamma\delta) &= A_0^{-1}(\gamma\delta).A_0 \\
&= A_0^{-1} \cdot f(g_\gamma) \cdot f(g_\delta) \cdot \mathbb{G}(\bar{\rho}(g_\gamma)\varphi_L^{n(\gamma)}\bar{\rho}(g_\delta)\varphi_L^{n(\delta)})(A_0) \cdot f_i(g_\delta)^{-1} \cdot f_i(g_\gamma)^{-1} \\
&= A_0^{-1} \cdot f(g_\gamma) \cdot \mathbb{G}(\bar{\rho}(g_\gamma)\varphi_L^{n(\gamma)})(A_0) \cdot f_i(g_\gamma)^{-1} \\
&\cdot f_i(g_\gamma) \cdot \mathbb{G}(\bar{\rho}(g_\gamma)\varphi_L^{n(\gamma)})(A_0^{-1} \cdot f(g_\delta) \cdot \mathbb{G}(\bar{\rho}(g_\delta)\varphi_L^{n(\delta)})(A_0) \cdot f_i(g)^{-1}) f_i(g_\gamma)^{-1} \\
&= \alpha(\gamma) \cdot \gamma \underset{f_i}{*} (\alpha(\delta).
\end{aligned}$$

By the construction of the action (act) $\alpha_{|\Gamma_K}$ is continuous, since f, f_i and $\bar{\rho}$ are continuous for the weak topology on $\mathbb{G}(\mathbb{K}^{sep})$ and by the discussion before Definition 2.2.6. If $B \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep})^{H_K,f}$, then $B = A_0B_0$ for $B_0 \in \mathbb{G}^{(f_i)}(\mathbb{K})$, so we calculate for $\gamma \in \mathbb{O}_K$ that

$$B_0^{-1}\alpha_{f,A}(\gamma)\gamma \underset{f_i}{*} B_0 = \alpha_{f,B}(\gamma).$$

□

Definition 2.2.8. We define a map

$$\mathbb{D} : \mathrm{mor}^{cont}(G_K, \mathbb{G}(k)) \to \coprod_i H^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K})), f \mapsto [\alpha_{f,A_0}]_\sim =: \alpha_f,$$

which is independent of the choice of $A_0 \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep})^{H_K,f}$ by the last Lemma.

Definition 2.2.9. Let G, H be topological groups. We denote for two morphisms of topological groups

$$f, f' \in \mathrm{mor}^{cont}(G, H) := \{a \in \mathrm{mor}_{Grp}(G, H) \mid a \text{ is continuous}\}$$

the relation $f \sim f'$, if they are conjugate, i.e. there exists $B \in H$, such that

$$f(g) = B \cdot f'(g) \cdot B^{-1}$$

for every $g \in G$.

Lemma 2.2.10. *If $f, f' \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(k))$ are conjugate, then*

$$\mathbb{D}(f) = \mathbb{D}(f').$$

Proof. It is

$$A^{-1} \cdot f(g) \cdot A = f'(g)$$

for some $A \in \mathbb{G}(k)$ and all $g \in G_K$. Furthermore, if

$$f(h) = B^{-1} \cdot f_i(h) \cdot \mathbb{G}(\bar{\rho}(h))(B) \ \forall h \in H_K,$$

then

$$f'(h) = (BA)^{-1} \cdot f_i(h) \cdot \mathbb{G}(\bar{\rho}(h))(BA)$$

and

$$A_0 \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep})^{H_K, f}, \text{ if and only if } A^{-1}A_0 \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep})^{H_K, f'}.$$

In that case, we calculate for $\gamma = \mathrm{pr}_H(g_\gamma)\pi^{n(\gamma)}$ that

$$\begin{aligned} \alpha_{f', A^{-1}A_0}(\gamma) &= A_0^{-1}AA^{-1} \cdot f(g_\gamma) \cdot \mathbb{G}(\bar{\rho}(g_\gamma)\varphi_L^{n(\gamma)})(AA^{-1}A_0) \cdot f_i(g_\gamma) \\ &= \alpha_{f, A_0}(\gamma). \end{aligned}$$

□

Let

$$\Psi := \Psi_{\mathbb{K}^{sep}} : \mathbb{G}(\mathbb{K}^{sep}) \to \mathbb{G}(\mathbb{K}^{sep}), A \mapsto A^{-1} \cdot \mathbb{G}(\varphi_L)(A)$$

be the Langmap.

Remark. If $\alpha \in C^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K}))$, such that $\alpha(\pi) = \Psi(A)$ for some $A \in \mathbb{G}(\mathbb{K}^{sep})$, then for every $g \in G_K$, it is

$$f_{\alpha, A^{-1}}(g) := A\alpha(\mathrm{pr}_{H_K}(g)) \cdot f_i(g) \cdot \mathbb{G}(\bar{\rho}(g))(A^{-1}) \in \mathbb{G}(k).$$

Proof. Set $B := A^{-1}$, so it is

$$\alpha(\pi) \cdot \mathbb{G}(\varphi_L)(B) = B. \tag{$*$}$$

Using that α is a cocycle, we calculate

$$\begin{aligned} B \cdot \mathbb{G}(\varphi_L)(f_{\alpha,B}(g)) &\overset{(*)}{=} \alpha(\pi) \cdot \mathbb{G}(\varphi_L)(\alpha(\mathrm{pr}_{H_K}(g)) \cdot f_i(g) \cdot \mathbb{G}(\bar{\rho}(g))(B)) \\ &= \alpha(\mathrm{pr}_{H_K}(g)\pi) \cdot f_i(g) \cdot \mathbb{G}(\bar{\rho}(g)\varphi_L)(B) \\ &= \alpha(\mathrm{pr}_{H_K}(g)) \cdot f_i(g) \cdot \mathbb{G}(\bar{\rho}(g))(\alpha(\pi) \cdot \mathbb{G}(\varphi_L)(B)) \\ &\overset{(*)}{=} B \cdot f_{\alpha,B}(g), \end{aligned}$$

so

$$f_{\alpha,B}(g) \in \mathbb{G}(\mathbb{K}^{sep})^{\varphi_L = 1} = \mathbb{G}(k)$$

by Lemma 2.1.25. □

Lemma 2.2.11. *Let $\alpha \in C^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K}))$, such that $\alpha(\pi) = \Psi(A)$ for some $A \in \mathbb{G}(\mathbb{K}^{sep})$ and $B := A^{-1}$. It is*

$$f_{\alpha,B} \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(k)),$$

which satisfies

$$f_{\alpha,B}(h) = B^{-1} \cdot f_i(h) \cdot \mathbb{G}(\bar{\rho}(h))(B) \ \forall h \in H_K.$$

Proof. It is continuous by construction, since α, $\bar{\rho}$ and f_i are continuous for the weak topology on $\mathbb{G}(\mathbb{K}^{sep})$ by the discussion before Definition 2.2.6 and this weak topology induces the discrete topology on $\mathbb{G}(k)$. Using that α is a cocycle and f_i takes values in $\mathbb{G}(k)$, we calculate

$$\begin{aligned} f_{\alpha,B}(g_1g_2) &= B^{-1}\alpha(\mathrm{pr}_{H_K}(g_1))f_i(g_1)\mathbb{G}(\bar{\rho}(g_1))(\alpha(\mathrm{pr}_{H_K}(g_2)))f_i(g_1)^{-1}f_i(g_1)f_i(g_2)\mathbb{G}(\bar{\rho}(g_1g_2))(B) \\ &= B^{-1}\alpha(\mathrm{pr}_{H_K}(g_1))f_i(g_1)\mathbb{G}(\bar{\rho}(g_1))(BB^{-1}\alpha(\mathrm{pr}_{H_K}(g_2))f_i(g_2) \cdot \mathbb{G}(\bar{\rho}(g_2))(B)) \\ &= f_{\alpha,B}(g_1)\mathbb{G}(\bar{\rho}(g_1))(f_{\alpha,B}(g_2)) \\ &= f_{\alpha,B}(g_1)f_{\alpha,B}(g_2), \end{aligned}$$

where the last equality follows from $f_{\alpha,B}(g_2) \in \mathbb{G}(k)$.

The second part of the statement follows from the definition, since it is $\alpha(1) = 1$ by the cocycle condition of α. □

Lemma 2.2.12. *Let $\alpha \in C^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K}))$. If*

$$\alpha(\pi) \cdot \mathbb{G}(\varphi_L)(B) = B$$

and

$$\alpha(\pi) \cdot \mathbb{G}(\varphi_L)(BB_0) = BB_0$$

for two different elements $B \in \mathbb{G}(\mathbb{K}^{sep})$ and $BB_0 \in \mathbb{G}(\mathbb{K}^{sep})$, then

$$B_0 \in \mathbb{G}(k) \text{ and } B_0^{-1}f_{\alpha,B}(g)B_0 = f_{\alpha,BB_0} \ \forall g \in G_K,$$

so $f_{\alpha,B} \sim f_{\alpha,BB_0}$.

Proof. We calculate

$$BB_0 \cdot \mathbb{G}(\varphi_L)(B_0^{-1}B^{-1}) = \alpha(\pi) = B \cdot \mathbb{G}(\varphi_L)(B^{-1}),$$

so $B_0 \in \mathbb{G}(\mathbb{K}^{sep})^{\varphi_L=1} = \mathbb{G}(k)$ by Lemma 2.1.25.

The second part of the statement follows from this and the definitions of those maps. □

Definition 2.2.13. We obtain maps

$$\mathbb{V}_i : \{[\alpha]_\sim \in H^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K})) \mid \alpha(\pi) \in \mathrm{im}(\Psi)\} \to (\mathrm{mor}^{cont}(G_K, \mathbb{G}(k))/\sim),\ f_\alpha := [f_{\alpha,B}]_\sim$$

for every i, which is independent of the choice of B with $\Psi(B^{-1}) = \alpha(\pi)$.

Lemma 2.2.14. *If $\alpha, \beta \in C^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K}))$, such that*

$$\alpha(\pi) \cdot \mathbb{G}(\varphi_L)(B) = B \text{ and } \beta(\gamma) = A \cdot \alpha(\gamma) \cdot \gamma \underset{f_i}{*} A^{-1}\beta(\gamma),$$

for $B \in \mathbb{G}(\mathbb{K}^{sep})$ and $A \in \mathbb{G}^{(f_i)}(\mathbb{K})$, then

$$\beta(\pi) \cdot \mathbb{G}(\varphi_L)(AB) = AB \text{ and } f_{\beta,AB} = f_{\alpha,B}.$$

Proof. The first statement follows directly from the hypothesis. For the second one, we calculate

$$f_{\beta,AB}(g) = B^{-1}A^{-1}A\alpha(\mathrm{pr}_{H_K})f_i(g)\mathbb{G}(\bar{\rho}(g))(A^{-1})f_i(g)^{-1}f_i(g)\mathbb{G}(\bar{\rho}(g))(AB) = f_{\alpha,B}(g).$$

□

We can now prove the desired correspondence between galois representations with values in $\mathbb{G}$ and certain (φ_L, Γ_K)-modules in the "π-torsion case".

Proposition 2.2.15. *The map*

$$\mathbb{D} : (\mathrm{mor}^{cont}(G_K, \mathbb{G}(k))/\sim) \to \coprod_i H^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K})), [f]_\sim \mapsto \alpha_f$$

is injective and has image

$$\coprod_i \{[\alpha]_\sim \in H^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K})) \mid \alpha(\pi) \in \mathrm{im}(\Psi)\}.$$

The inverse map is given by

$$\mathbb{V} := \coprod_i \mathbb{V}_i : \coprod_i \{[\alpha]_\sim \in H^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K})) \mid \alpha(\pi) \in \mathrm{im}(\Psi)\} \to (\mathrm{mor}^{cont}(G_K, \mathbb{G}(k))/\sim).$$

This bijection identifies

$$\{[a]_\sim \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(k))/\sim \ \mid \bar{\jmath}_{\mathbb{K}}(a) = \bar{\jmath}_{\mathbb{K}}(f_i)\} \cong \{[\alpha] \in H^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K})) \mid \alpha(\pi) \in \mathrm{im}(\Psi)\}$$

for every i.

Proof. For $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(k))/\sim$ with $j(f) \sim j(f_i)$ and $A_0 \in \mathbb{G}^{(f_i)}(\mathbb{K}^{sep})^{H_K,f}$, it is

$$\alpha_{f,A_0}(\pi) = \Psi(A_0).$$

If $\alpha \in C^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K}))$, such that

$$\alpha(\pi)\mathbb{G}(\varphi_L)(B) = B, \tag{$*$}$$

then $f_{\alpha,B} \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(k))$ and $f_{\alpha,B}(h) = B^{-1}f_i(h)\mathbb{G}(\bar{\rho}(h))(B)$, so we calculate for $\gamma = \mathrm{pr}_{H_K}(g_\gamma)$

$$\alpha_{f_{\alpha,B},B^{-1}}(\gamma) = BB^{-1}\alpha(\gamma)f_i(g)\mathbb{G}(\bar{\rho}(g))(B)\mathbb{G}(\bar{\rho}(g))(B^{-1})f_i(g)^{-1} = \alpha(\gamma)$$

and

$$\alpha_{f_{\alpha,B},B^{-1}}(\pi) = B\mathbb{G}(\varphi_L)(B^{-1}) \overset{(*)}{=} \alpha(\pi),$$

and since α and $\alpha_{f_{\alpha,B},B^{-1}}$ are both 1-cocycles in $C^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K}))$, they are equal and hence it is $\mathbb{D}(\mathbb{V}([\alpha]_\sim))$.

For f as in the beginning of the proof, we calculate

$$f_{\alpha_{f,A_0},A_0^{-1}}(g) = A_0A_0^{-1}f(g)\mathbb{G}(\bar{\rho}(g))(A_0)f_i(g)^{-1}f_i(g)\mathbb{G}(\bar{\rho}(g))(A_0^{-1}) = f(g),$$

so we have $\mathbb{V}(\mathbb{D}([f]_\sim)) = [f_\sim]$ and so $\mathbb{D}$ is injective with inverse $\mathbb{V}$ on the image

$$\coprod_i \{[\alpha]_\sim \in H^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K})) \mid \alpha(\pi) \in \mathrm{im}(\Psi)\}.$$

$\square$

Since this bijection is dependent on the choice of $\{f_i\}_i$, the maps $\mathbb{D}$ and $\mathbb{V}$ are in general not "functorial". Under certain conditions, there is still a way to get something like functoriality. For this we first note that, if $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ is a morphism of groups and $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}_1(k))$, then

$$\phi_{\mathbb{K}^{sep}} : \mathbb{G}_1^{(f)}(\mathbb{K}) \to \mathbb{G}_2^{(\phi_k \circ f)}(\mathbb{K})$$

is a well defined morphism of $\mathbb{O}_K$-groups by (F2),(F3), which is continuous, because it is a polynomial map.

Lemma 2.2.16. *Let $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ be a morphism of groupschemes over k, such that the induced map*

$$(\phi_{\mathbb{K}^{sep}})_* : H^1(H_K, \mathbb{G}_1(\mathbb{E}^{sep})) \to H^1(H_K, \mathbb{G}_2(\mathbb{E}^{sep}))$$

is injective on $\mathrm{im}(\bar{j}^{\mathbb{G}_1})$. Then for any choice $\{f_i^{(1)}\}_i \subset \mathrm{mor}^{cont}(G_K, \mathbb{G}_1(k))$, such that

$$\bar{j}^{\mathbb{G}_1} : \{f_i^{(1)}\}_i \to \mathrm{im}(\bar{j}^{\mathbb{G}_1})$$

is bijective, we can complement $\{\phi_k \circ f_i\}_i \subset \mathrm{mor}^{cont}(G_K, \mathbb{G}_2(k))$ *to a subset* $\{f_l^{(2)}\}_l$*, such that*

$$\bar{j}^{\mathbb{G}_2} : \{f_l^{(2)}\}_l \to \mathrm{im}(\bar{j}^{\mathbb{G}_2})$$

is bijective. Furthermore, the following diagram is commutative.

$$\begin{tikzcd}
(\mathrm{mor}^{cont}(G_K, \mathbb{G}_1(k))/\sim) \arrow[r, "\mathbb{D}"] \arrow[d, "(\phi_k)_*"'] & \coprod_i H^1(\mathbb{O}_K, \mathbb{G}_1^{(f_i^{(1)})}(\mathbb{K})) \arrow[d, "(\phi_{\mathbb{K}^{sep}})_*"] \\
(\mathrm{mor}^{cont}(G_K, \mathbb{G}_2(k))/\sim) \arrow[r, "\mathbb{D}"'] & \coprod_l H^1(\mathbb{O}_K, \mathbb{G}_2^{(f_l^{(2)})}(\mathbb{K})).
\end{tikzcd}$$

Proof. The first part of the statement that for any such subset $\{f_i^{(1)}\}_i$, the subsets $\{\phi_k \circ f_i\}_i$ can be complemented, follows directly from the hypothesis that $(\phi_{\mathbb{K}^{sep}})_*$ is injective on $\mathrm{im}(\bar{j}^{\mathbb{G}_1})$ and because the following diagram is commutative, which is commutative since ϕ is a natural transformation.

$$\begin{tikzcd}
\mathrm{mor}^{cont}(G_K, \mathbb{G}_1(k)) \arrow[r, "\bar{j}^{\mathbb{G}_1}"] \arrow[d, "(\phi_k)_*"'] & H^1(H_K, \mathbb{G}_1(\mathbb{K}^{sep})) \arrow[d, "(\phi_{\mathbb{K}^{sep}})_*"] \\
\mathrm{mor}^{cont}(G_K, \mathbb{G}_2(k)) \arrow[r, "\bar{j}^{\mathbb{G}_2}"] & H^1(H_K, \mathbb{G}_2(\mathbb{K}^{sep})).
\end{tikzcd}$$

For the second part of the statement about the commutative diagram, let $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}_1(k))$, such that there exists $B \in \mathbb{G}_1(\mathbb{K}^{sep})$ with

$$f(h) = B \cdot f_i^{(1)}(h) \cdot \mathbb{G}_1(\bar{\rho}(h))(B^{-1}) \ \forall h \in H_K.$$

Since ϕ is a natural transformation of groups, it follows that

$$\phi_k(f(h)) = \phi_{\mathbb{K}^{sep}}(B) \cdot \phi_k(f_i^{(1)}(h) \cdot \mathbb{G}_2(\bar{\rho}(h))(\phi_{\mathbb{K}^{sep}}(B)^{-1}) \ \forall h \in H_K.$$

Now $\mathbb{D}([f]_\sim)$ is given by

$$\alpha_{f,B}(\gamma) = B \cdot f(g_\gamma)\mathbb{G}(\bar{\rho}(g)\varphi_L^{n_\gamma})(B^{-1}) \cdot f_i(g_\gamma)^{-1} \ \forall\, \mathrm{pr}_{H_K}(g_\gamma)\pi^{n_\gamma} = \gamma \in \mathbb{O}_K.$$

On the other hand $\mathbb{D}([\phi_k \circ f]_\sim)$ is given by

$$\alpha_{\phi_k \circ f, \phi_{\mathbb{K}^{sep}}}(\gamma) = \phi_{\mathbb{K}^{sep}}(B) \cdot \phi_k(f(g_\gamma))\mathbb{G}(\bar{\rho}(g)\varphi_L^{n_\gamma})(\phi_{\mathbb{K}^{sep}}(B^{-1})) \cdot \phi_k(f_i(g_\gamma))^{-1}$$

for all γ as above. Again, since ϕ is a natural transformation of groups, it follows that

$$\phi_{\mathbb{K}^{sep}} \circ \alpha_{f,B} = \alpha_{\phi_k \circ f, \phi_{\mathbb{K}^{sep}}}.$$

$\square$

Remark. This condition on ϕ is rather restricitve. For example, if $\phi : \mathbb{G} \to \mathrm{GL}_n$ is a representation in $Rep_k(\mathbb{G})$, then ϕ satisfies the condition in this Lemma, if and only if $\bar{j}^{\mathbb{G}} \equiv 1$ is the trivial map. We will give examples of this in the next section. The problem for the general case lies within the fact that the map $\mathbb{D}$ depends on the choice of representatives $\{f_i\}_i$ as chosen after Definition 2.2.6. In the general case, we can still write down a diagram that is natural up to some twisted conjugation, which depends on the choice of those representatives, as we will see in the following Lemma.

But before that, we get the following technicallity out of the way.

Remark 2.2.17. As always in this part, let $\mathbb{G}$ be a linear algebraic group over k. Let $f_1, f_2, f_3 : G_K \to \mathbb{G}(k)$ be morphisms of groups. As before, we define a G_K-action of sets on $\mathbb{G}(\mathbb{K}^{sep})$ by setting

$$g \underset{f_i,f_j}{\cdot} A := f_i(g)^{-1} \cdot \mathbb{G}(\bar{\rho}(g))(A) f_j(g) \; \forall A \in \mathbb{G}(\mathbb{K}^{sep}), g \in G_K.$$

Then for $A, B \in \mathbb{G}(\mathbb{K}^{sep})$ and $g \in G_K$, we have

$$g \underset{f_1,f_2}{\cdot} A \cdot g \underset{f_2,f_3}{\cdot} B = g \underset{f_1,f_3}{\cdot} (AB).$$

Proof. We calculate

$$\begin{aligned} & g \underset{f_1,f_2}{\cdot} A \cdot g \underset{f_2,f_3}{\cdot} B \\ =& f_1(g)^{-1} \cdot \mathbb{G}(\bar{\rho}(g))(A) \cdot f_2(g) f_2(g)^{-1} \cdot \mathbb{G}(\bar{\rho}(g))(B) f_3(g)^{-1} \\ =& f_1(g)^{-1} \cdot \mathbb{G}(\bar{\rho}(g))(AB) f_3(g)^{-1} \\ =& g \underset{f_1,f_3}{\cdot} (AB), \end{aligned}$$

where the second equality comes from the fact aht $\mathbb{G}(\bar{\rho}(g))$ is an endomorphism of groups. ☐

Lemma 2.2.18. *Let $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ be a morphism of groupschemes over k. Choose some representatives $\{f_i^{(1)}\}_i$ for $\mathbb{G}_1$ and $\{f_l^{(2)}\}_l$ for $\mathbb{G}_2$ as in the last Lemma. By definition of these representatives for any $f_i^{(1)}$ there exists a unique $f_l^{(2)}$ and some (non-unique) $B_i \in \mathbb{G}_2(\mathbb{K}^{sep})$, such that*

$$\phi_k \circ f_i^{(1)}(h) = B_i \cdot f_l^{(2)}(h) \cdot \mathbb{G}_2(\bar{\rho}(h))(B_i^{-1}). \tag{$*$}$$

Then the following diagram is commutative and the right vertical map is independent of the choice of B_i.

$$\begin{tikzcd}
(\mathrm{mor}^{cont}(G_K,\mathbb{G}_1(k))/\sim) \arrow[r, "\mathbb{D}"] \arrow[d, "(\phi_k)_*"'] & \coprod_i H^1(\mathbb{O}_K,\mathbb{G}_1^{(f_i^{(1)})}(\mathbb{K})) \arrow[d, "{\coprod_i[\alpha\mapsto[\gamma\mapsto B_i^{-1}\cdot\phi_{\mathbb{K}^{sep}}\circ\alpha(\gamma)\cdot\gamma.B_i]]}"] \\
(\mathrm{mor}^{cont}(G_K,\mathbb{G}_2(k))/\sim) \arrow[r, "\mathbb{D}"'] & \coprod_l H^1(\mathbb{O}_K,\mathbb{G}_2^{(f_l^{(2)})}(\mathbb{K})).
\end{tikzcd}$$

Here for $\gamma = \mathrm{pr}_{H_L}(g_\gamma)\pi^{n_\gamma} \in \mathbb{O}_K$, we have

$$\gamma.B_i := \gamma \underset{\phi_k\circ f_i^{(1)},f_l^{(2)}}{\cdot} B_i := \phi_k \circ f_i^{(1)}(g_\gamma)\cdot\mathbb{G}_2(\bar\rho(g_\gamma)\circ\varphi_L^{n_\gamma})(B_i)\cdot f_l^{(2)}(g_\gamma)^{-1}.$$

This is well defined by the discussion before Lemma 2.2.7.

Proof. We first show that the right vertical map is well defined. So let $\alpha \in C^1(\mathbb{O}_K)$. First we calculate for $\gamma = \mathrm{pr}_{H_K}(g_\gamma)\pi^{n_\gamma} \in \mathbb{O}_K$ and $h \in H_K$ that

$$\begin{aligned}
& h \underset{f_l^{(2)}}{*} (B_i^{-1}\cdot\phi_{\mathbb{K}^{sep}}\circ\alpha(\gamma)\cdot\gamma.B_i)\\
=& f_l^{(2)}(h)\mathbb{G}_2(\bar\rho(h))(B_i^{-1})\cdot\phi_{\mathbb{K}^{sep}}(\mathbb{G}_1(\bar\rho(h))(\alpha(\gamma)))\cdot\mathbb{G}_2(\bar\rho(h))(\gamma.B_i)\cdot f_l^{(2)}(h)^{-1}\\
\overset{(*)}{=}& B_i^{-1}\phi_{\mathbb{K}^{sep}}\circ(f_i^{(1)}(h)\mathbb{G}_1(\bar\rho(h))(\alpha(\gamma)))\cdot\mathbb{G}_2(\bar\rho(h))(\gamma.B_i)\cdot f_l^{(2)}(h)^{-1}\\
=& B_i^{-1}\cdot\phi_{\mathbb{K}^{sep}}\circ\alpha(\gamma)\cdot\phi_k\circ f_i^{(1)}(h)\cdot\mathbb{G}_2(\bar\rho(h))(\gamma.B_i)\cdot f_l^{(2)}(h)^{-1}\\
=& B_i^{-1}\cdot\phi_{\mathbb{K}^{sep}}\circ\alpha(\gamma)\cdot\gamma.B_i.
\end{aligned}$$

The first equality comes from the fact that ϕ is a natural transformation, the third one follows from the fact that $\alpha(\gamma)\in\mathbb{G}^{(f_i^{(1)})}(\mathbb{K})$, so

$$f_i^{(1)}(h)\cdot\mathbb{G}_1(\bar\rho(h))(\alpha(\gamma)) = \alpha(\gamma)\cdot f_i^{(1)}(h)$$

by (F2) and (F3). The last equality follows from the fact that

$$\gamma.B_i \in \{A\in\mathbb{G}_2(\mathbb{K}^{sep}) \mid h.A = A\ \forall h\in H_K\},$$

which is shown as in the discussion before Lemma 2.2.7. Again by (F2) and (F3), it follows that the right vertical map sends α to a map, which takes values in $\mathbb{G}_2^{(f_l^{(2)})}(\mathbb{K})$.

Next we calculate for $\gamma,\delta\in\mathbb{O}_K$ that

$$\begin{aligned}
& B_i^{-1}\cdot\phi_{\mathbb{K}^{sep}}(\alpha(\gamma\delta))\cdot\gamma\delta.B_i\\
=& B_i^{-1}\cdot\phi_{\mathbb{K}^{sep}}(\alpha(\gamma))\phi_{\mathbb{K}^{sep}}(\gamma\underset{f_i^{(1)}}{*}\alpha(\delta))\cdot\gamma.(\delta.B_i)\\
=& B_i^{-1}\cdot\phi_{\mathbb{K}^{sep}}(\alpha(\gamma))\cdot\gamma.B_i\cdot\gamma\underset{f_l^{(2)},\phi_k\circ f_i^{(1)}}{\cdot}B_i^{-1}\cdot\gamma\underset{\phi_k\circ f_i^{(1)}}{*}\phi_{\mathbb{K}^{sep}}(\alpha(\delta))\cdot\gamma.(\delta.B_i)\\
=& B_i^{-1}\cdot\phi_{\mathbb{K}^{sep}}(\alpha(\gamma))\cdot\gamma.B_i\cdot\gamma\underset{f_l^{(2)}}{*}(B_i^{-1}\phi_{\mathbb{K}^{sep}}(\alpha(\delta))\delta.B_i).
\end{aligned}$$

The first equality comes from the fact that α is a cocycle, the second equality is due to multiplying with $1 = \gamma.B_i \cdot \gamma \underset{f_l^{(2)},\phi_k \circ f_i^{(1)}}{.} B_i^{-1}$ (see Remark 2.2.17) and the fact that

$$\phi_{\mathbb{K}^{sep}}(\gamma \underset{f_i^{(1)}}{*} A) = \gamma \underset{\phi_k \circ f_i^{(1)}}{*} \phi_{\mathbb{K}^{sep}}(A)\ \forall A \in \mathbb{G}_1(\mathbb{K}^{sep}).$$

The third equality follows from Remark 2.2.17. So the right vertical map sends a continuous cocycle to a cocycle, which is continuous by an analogues argument as in Lemma 2.2.7. So the right vertical map sends elements of $C^1(\mathbb{O}_K, \mathbb{G}_1^{(f_i^{(1)})}(\mathbb{K}))$ to $H^1(\mathbb{O}_K, \mathbb{G}_2^{(f_l^{(2)})}(\mathbb{K}))$. So next, we have to show that this map is independent of the choice of cohomology class, but first we show that the map is independent of the choice of B_i. So let $A_i \in \mathbb{G}(\mathbb{K}^{sep})$ be another element satisfying $(*)$. As in the remark before Lemma 2.2.7, we calculate that

$$A_i = B_i B_0,$$

where $B_0 \in \mathbb{G}^{(f_l^{(2)})}(\mathbb{K})$, so we have that

$$A_i^{-1} \cdot \phi_{\mathbb{K}^{sep}} \circ \alpha(\gamma) \cdot \gamma.A_i = B_0^{-1} \cdot B_i^{-1} \cdot \phi_{\mathbb{K}^{sep}} \circ \alpha(\gamma) \cdot \gamma.B_i \cdot \gamma \underset{f_l^{(2)}}{*} B_0\ \forall \gamma \in \mathbb{O}_K$$

by Remark 2.2.17, so those maps are the same on the cohomology. Now let $\alpha, \beta \in C^1(\mathbb{O}_K, \mathbb{G}_1^{(f_i^{(1)})}(\mathbb{K}))$ be in the same cohomology class, i.e. there exists $B \in \mathbb{G}_1^{(f_i^{(1)})}(\mathbb{K})$, such that

$$\alpha(\gamma) = B^{-1} \cdot \beta(\gamma) \cdot \gamma \underset{f_i^{(1)}}{*} B.$$

Then we have

$$\begin{aligned}
&\phi_{\mathbb{K}^{sep}}(B)B_i \cdot f_l^{(2)}(h) \cdot \mathbb{G}_2(\bar{\rho}(h))((\phi_{\mathbb{K}^{sep}}(B)B_i)^{-1}) \\
=&\phi_{\mathbb{K}^{sep}}(B)B_i \cdot f_l^{(2)}(h) \cdot \mathbb{G}_2(\bar{\rho}(h))(B_i^{-1})\phi_{\mathbb{K}^{sep}}(\mathbb{G}_1(\bar{\rho}(h))(B^{-1})) \\
\overset{(*)}{=}&\phi_{\mathbb{K}^{sep}}(B \cdot f_i^{(1)}(h) \cdot \mathbb{G}_1(\bar{\rho}(h))(B^{-1})) \\
=&\phi_k \circ f_i^{(1)}(h).
\end{aligned}$$

Here, the first equality is due to the fact that ϕ is a natural transformation and the last equality comes from the fact that $B \in \mathbb{G}_1^{(f_i^{(1)})}(\mathbb{K})$ and, again, (F2) and (F3). It follows that $\phi_{\mathbb{K}^{sep}}(B)B_i$ is an element satisfying $(*)$ and we have for every $\gamma \in \mathbb{O}_K$

$$\begin{aligned}
&B_i^{-1} \cdot \phi_{\mathbb{K}^{sep}} \circ \beta(\gamma) \cdot \gamma.B_i \\
=&B_i^{-1}\phi_{\mathbb{K}^{sep}}(B^{-1}) \cdot \phi_{\mathbb{K}^{sep}} \circ \beta(\gamma) \cdot \gamma.(\phi_{\mathbb{K}^{sep}}(B)B_i) \\
=&B_i^{-1} \cdot \phi_{\mathbb{K}^{sep}} \circ \alpha(\gamma) \cdot \gamma.B_i.
\end{aligned}$$

Here the first equality follows from the independence of the choice of the element satisfying $(*)$ and the second equality is due to α and β being cohomological and Remark 2.2.17. Thus, the right vertical map does not depend on the cohomology class of α and so the map is well defined. Finally, we calculate that the diagram is commutative. So let $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}_1(k))$ and $f_i^{(1)}$ be the representative, such that there exists an $A_0 \in \mathbb{G}(\mathbb{K}^{sep})$ with

$$f(h) = A_0 \cdot f_i^{(1)}(h) \cdot \mathbb{G}_1(\bar{\rho}(h))(A_0^{-1}) \ \forall h \in H_K.$$

It follows by $(*)$ and the fact that ϕ is a natural transformation that

$$\phi_k \circ f(h) = \phi_{\mathbb{K}^{sep}}(A_0)B_i \cdot f_l^{(2)}(h) \cdot \mathbb{G}_2(\bar{\rho}(h))((\phi_{\mathbb{K}^{sep}}(A_0)B_i)^{-1}).$$

So we calculate

$$\begin{aligned} &\mathbb{D}([\phi_k \circ f]_\sim) \\ =&[\gamma \mapsto B_i^{-1}\phi_{\mathbb{K}^{sep}}(A_0^{-1}) \cdot \gamma \underset{\phi_k\circ f, f_l^{(2)}}{.} (\phi_{\mathbb{K}^{sep}}(A_0)B_i)]_\sim \\ =&[\gamma \mapsto B_i^{-1}\phi_{\mathbb{K}^{sep}}(A_0^{-1}) \cdot \gamma \underset{\phi_k\circ f, \phi_k\circ f_i^{(1)}}{.} \phi_{\mathbb{K}^{sep}}(A_0) \cdot \gamma.B_i]_\sim, \end{aligned} \tag{$**$}$$

where the first equality is by definition and the second one is by Remark 2.2.17. On the other hand, we have that

$$\begin{aligned} &\phi_{\mathbb{K}^{sep}} \circ \mathbb{D}([f]_\sim) \\ =&[\gamma \mapsto \phi_{\mathbb{K}^{sep}}(A_0^{-1}) \cdot \phi_{\mathbb{K}^{sep}}(\gamma \underset{f, f_i^{(1)}}{.} A_0)]_\sim \\ =&[\gamma \mapsto \phi_{\mathbb{K}^{sep}}(A_0^{-1}) \cdot \gamma \underset{\phi_k\circ f, \phi_k\circ f_i^{(1)}}{.} \phi_{\mathbb{K}^{sep}}(A_0)]_\sim, \end{aligned}$$

so the equality $(**)$ before this one gives the desired commutativity. $\square$

The welldefinedness of the map on the right in the last Lemma has the following application.

Proposition 2.2.19. *Let $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(k))$. Then*

$$C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{K})) = \{\alpha : \mathbb{O}_K \to \mathbb{G}^{(f)}(\mathbb{K}) \mid \alpha(\gamma\delta) = \alpha(\gamma) \cdot \gamma \underset{f}{*} \alpha(\delta) \ \forall \gamma, \delta \in \mathbb{O}_K\} =: \mathcal{C}^1,$$

i.e. such an 1*-cocycle is automatically continuous for the weak topology on* $\mathbb{G}^{(f)}(\mathbb{K})$.

Proof. Choose an embedding $\iota : \mathbb{G} \to \mathrm{GL}_n$ and a matrix $B \in \mathrm{GL}_n(\mathbb{K}^{sep})$, such that

$$\iota_k \circ f(h) = B^{-1} \cdot \mathrm{GL}_n(\bar{\rho}(h))(B) \ \forall h \in H_K.$$

This is possible by Hilbert 90. Let $\alpha \in \mathcal{C}^1$, then as in the last Lemma, we see that

$$[\gamma \mapsto B^{-1} \cdot \iota_{\mathbb{K}^{sep}}(\alpha(\gamma)) \cdot \gamma \underset{\iota_k \circ f, 1}{\cdot} B]$$

is a 1-cocycle $\mathbb{O}_K \to \mathrm{GL}_n(\mathbb{K})$, where $\mathrm{GL}_n(\mathbb{K})$ is an $\mathbb{O}_K$-group via $\mathrm{GL}_n(\bar{\tau})$. By the discussion before Definition 2.2.6, the weak topology on $\mathbb{G}^{(f)}(\mathbb{K})$ is the same as the topology induced by the embedding

$$\mathbb{G}^{(f)}(\mathbb{K}) \stackrel{\subseteq}{\to} \mathbb{G}(\mathbb{K}^{sep}) \stackrel{\iota_{\mathbb{K}^{sep}}}{\to} \mathrm{GL}_n(\mathbb{K}^{sep}),$$

where $\mathrm{GL}_n(\mathbb{K}^{sep})$ carries the weak topology induced by the valuation on $\mathbb{K}^{sep}$. So by arguments as in the proof of Lemma 2.2.7, this map is continuous on Γ_K for the weak topology on $\mathrm{GL}_n(\mathbb{K})$, if and only if $\alpha_{|\Gamma_K}$ is continuous for the weak topology on $\mathbb{G}^{(f)}(\mathbb{K})$. So we can reduce ourselves to the case that $f \equiv 1$ is the trivial morphism and $\mathbb{G} = \mathrm{GL}_n$, but in this case the claim follows from a variant of Lemma 1.3.16 for the π-torsion case and Theorem 1.3.12. □

If we assume $\mathbb{G}$ to be connected, then the correspondence "reaches" all (φ_L, Γ_K)-modules.

Theorem 2.2.20. *If $\mathbb{G}$ is connected, we have inverse bijections*

$$\mathbb{D} : (\mathrm{mor}^{cont}(G_K, \mathbb{G}(k))/ \sim) \to \coprod_i H^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K})) : \mathbb{V}.$$

This bijection identifies

$$\{[a]_\sim \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(k))/ \sim \ \mid \bar{j}_{\mathbb{K}}(a) = \bar{j}_{\mathbb{K}}(f_i)\} \cong H^1(\mathbb{O}_K, \mathbb{G}^{(f_i)}(\mathbb{K}))$$

for every i.

Proof. This follows from Proposition 2.2.15 and Corollary 2.1.18. □

2.2.3 The Perfect versus the Nonperfect Case

Let $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(k))$ and $\mathbb{G}^{(f)}$ again be the corresponding form of $\mathbb{G}$ for $j(f)$ over $\mathbb{E}$. Since the correspondence in the last part was possible for $\mathbb{K} \in \{\mathbb{E}, \mathbb{F}\}$, we can calculate for connected $\mathbb{G}$ that

$$\mathbb{D}_{\mathbb{F}} \circ \mathbb{V}_{\mathbb{E}} : H^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{E})) \to H^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{F}))$$

is the map given by the inclusion $\mathbb{E} \subset \mathbb{F}$, which is therefore bijective. But in the "π-torsion" case, we can generalize this for general $\mathbb{G}$ by giving a direct proof. This needs some preparation.

Remark. Let $\mathbb{G}^{(f)} \subset \mathrm{GL}_n$ be an embedding. Since $\mathbb{E}$ is complete as a local field, the induced topology on

$$\mathbb{G}^{(f)}(\mathbb{E}) \subset \mathrm{GL}_n(\mathbb{E})$$

is complete as well, since $\mathbb{G}(\mathbb{E})$ is closed a closed subgroup of the complete subgroup $\mathrm{GL}_n(\mathbb{E})$. The later is complete, since

$$\mathrm{GL}_n(\mathbb{E}) \subset \mathbb{E}^{n^2+1}$$

is closed.

Proposition 2.2.21. *(See Coh00, Propositions 4.2.10 & 4.4.45)*
Let e be the ramification index of $L|\mathbb{Q}_p$. The p-adic logarithm

$$\log : 1 + \mathfrak{m}_L \to \pi^{1-e}\mathcal{O}_L \cong \mathcal{O}_L, 1 + x \mapsto \sum_{n \geq 1} (-1)^{n+1} \frac{x^n}{n}$$

is a well defined continuous morphism of compact topological groups, which is injective on $1 + \mathfrak{m}_L^N$ for some $N >> 0$.

Proof. Using the Propositions of (Coh00), it only remains to show that

$$\sum_{n \geq 1} (-1)^{n+1} \frac{x^n}{n} \in \pi^{1-e}\mathcal{O}_L$$

for every $x \in \mathfrak{m}_L$ and the injectivity statement. For the first statement, we calculate for every $n \in \mathbb{N}$, such that $n = p^r m$ with $(p, m) = 1$ that

$$\frac{|x^n|}{|n|} \leq \frac{|\pi^n|}{|n|} = \frac{|\pi^n|}{|p^r|} = |\pi|^{n-er} \leq |\pi^{r(1-e)}| \leq |\pi|^{1-e}.$$

For the second statement, we have that the only elements in the kernel of log are roots of unity by (Coh00, Propositions 4.4.45). Since $[L : \mathbb{Q}_p]$ is finite, there are only finitely many roots of unity in $1 + \mathfrak{m}_L$. Let $\mu \subset 1 + \mathfrak{m}_L$ be this set of roots of unity. Let $\nu : L \to \mathbb{Z}$ be the valuation with respect to $\mathfrak{m}_L$ and set

$$N := \max\{\nu(x - 1) \mid x \in \mu\} + 1.$$

Then $\mu \cap (1 + \mathfrak{m}_L^N) = \{1\}$, so log is injective on $1 + \mathfrak{m}_L^N$. □

Lemma 2.2.22. *Any closed subgroup $H \subset \mathcal{O}_L$ is a $\mathbb{Z}_p$-module of finite rank with a topological isomorphism $H \cong \mathbb{Z}_p^m$. In particular, it is topologically finitely generated.*

Proof. For every $a \in \mathbb{Z}_p$, it is $a = \lim_n a_n$ with $a_n \in \mathbb{Z}$ and so

$$a \cdot h = \lim_n (a_n h) \in H \; \forall h \in H,$$

since H is closed and the $\mathbb{Z}$-module structure of H is continuous. It follows that $H \subset \mathcal{O}_L$ is a $\mathbb{Z}_p$-submodule. But $\mathcal{O}_L$ is finitely generated and free over $\mathbb{Z}_p$ with a topological isomorphism $\mathcal{O}_L \cong \mathbb{Z}_p^{[L:\mathbb{Q}_p]}$. It follows that there exists an isomorphism of $\mathbb{Z}_p$-modules $H \cong \mathbb{Z}_p^m$ with $m \leq [L : \mathbb{Q}_p]$ by the elementary divisor theorem (See Bos05, 2.9 Theorem 2). Again by the elementary divisor theorem (See Bos05, 2.9 Theorem 2), the isomorphism $\mathbb{Z}_p^m \to H \subset \mathbb{Z}_p^{[L:\mathbb{Q}_p]}$ is given via multiplication with a matrix $A \in Mat_{[L:\mathbb{Q}_p] \times m}(\mathbb{Z}_p)$. This is continuous and hence a topological isomorphism, since $\mathbb{Z}_p^m$ is compact.

So $H \cong \mathbb{Z}_p^m$ is topologically finitely generated, since $\mathbb{Z}_p$ is topologically generated by $1 \in \mathbb{Z}_p$. □

Lemma 2.2.23. *Let G be topological group and $H \subset G$ be a subgroup of finite index. If H is topologically finitely generated as the topological subgroup of G, then so is G.*

Proof. Let $U_H := \langle h_1, \ldots, h_n \rangle \subset H$ be a finitely generated subgroup, such that

$$H \subset \overline{U_H},$$

where $\overline{U_H} \subset G$ denotes the closure of U_H in G. Let $g_1, \ldots, g_m \in G$ be representatives of G/H. Then

$$U := \langle g_1, \ldots, g_m, h_1, \ldots, h_n \rangle$$

is dense in G, since for any $1 \leq i \leq m$ we have that

$$U_i := \langle g_i, h_1, \ldots, h_n \rangle$$

satisfies

$$g_i H \subset g_i \overline{U_H} = \overline{g_i U_H} \subset \overline{U_i} \subset \overline{U}$$

by the fact that multiplication with g_i is a homeomorphism from G to itself. It follows that

$$G = \coprod_i g_i H \subset \overline{U}.$$

□

Corollary 2.2.24. *The group Γ_K is topologically finitely generated.*

Proof. It is
$$\mathcal{O}_L^\times \cong \mathbb{Z}/(q-1)\mathbb{Z} \times (1+\mathfrak{m}_L).$$
First we show
$$\Gamma_K = (\Gamma_K \cap \mathbb{Z}/(q-1)\mathbb{Z}) \times (\Gamma_K \cap (1+\mathfrak{m}_L)). \tag{$*$}$$
We have to show that for $x, y \in \mathcal{O}_L^\times$, such that $x \in \mathbb{Z}/(q-1)\mathbb{Z}$ and $y \in 1+\mathfrak{m}_L$ it is
$$xy \in \Gamma_K \Leftrightarrow x \in \Gamma_K, y \in \Gamma_K.$$
The direction from right to left is obvious. If $xy \in \Gamma_K$, it follows from $x \in \mathbb{Z}/(q-1)\mathbb{Z}$ that
$$(xy)^{q-1} = x^{q-1}y^{q-1} = y^{q-1},$$
so $y^{q-1} \in \Gamma_K$. The group $1+\mathfrak{m}_L$ is a pro-p-group by Proposition 1.1.23.a), so the $\mathbb{Z}$-module structure extends to a $\mathbb{Z}_p$-module structure via taking limits similar as in the proof of Lemma 2.2.22. It follows that
$$U_K := \Gamma_K \cap (1+\mathfrak{m}_L)$$
is a $\mathbb{Z}_p$-module by an argument as in the proof of Lemma 2.2.22, since $U_K \subset 1+\mathfrak{m}_L$ is open and hence closed. But $q-1$ is a unit in $\mathbb{Z}_p$, so
$$y^{q-1} \in \Gamma_K \Rightarrow y = (y^{q-1})^{\frac{1}{(q-1)}} \in \Gamma_K.$$
It follows that
$$xy \in \Gamma_K \Rightarrow y \in \Gamma_K \text{ and } x = (xy)y^{-1} \in \Gamma_K.$$
By $(*)$ it suffices to show that U_K is topologically finitely generated, but $U_K \subset (1+\mathfrak{m}_L)$ is open and hence closed in the compact topological group $1+\mathfrak{m}_L$. By Proposition 2.2.21 and Lemma 2.2.22, it is
$$1+\mathfrak{m}_L^N \cong \mathbb{Z}_p^m$$
for some $N, m \geq 1$. We deduce via Proposition 2.2.21 that the group
$$U_{K,N} := U_K \cap (1+\mathfrak{m}_L^N) = \Gamma_K \cap (1+\mathfrak{m}_L^N)$$
is topologically isomorphic to a closed subgroup of $\mathcal{O}_L$ and so $U_{K,N}$ is topologically finitely generated by Lemma 2.2.22. So U_K is topologically finitely generated by Lemma 2.2.23, since $U_{K,N} \subset U_K$ is of finite index. This last statement follows from the fact that
$$U_K/U_{K,N} \cong \mathrm{Gal}(K_N|K_1)$$
by Proposition 1.1.23, where $K_r = KL_r$ for $r \in \mathbb{N}$. $\square$

Proposition 2.2.25. *The inclusion $\mathbb{E} \subset \mathbb{F}$ induces a bijection*

$$\iota : H^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{E})) \tilde{\to} H^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{F})).$$

Proof. By Lemma 2.1.16 and (F3) we have that for any $A \in \mathbb{G}^{(f)}(\mathbb{F})$, there is an $n \in \mathbb{N}$, such that

$$\mathbb{G}(\varphi_L^n)(A) \in \mathbb{G}^{(f)}(\mathbb{E}).$$

It is $\mathbb{O}_K \cong \pi^{\mathbb{N}} \times \Gamma_K$. Since Γ_K is topologically finitely generated, every $\alpha \in C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{F}))$ is continuous on Γ_K and $\mathbb{G}^{(f)}(\mathbb{E})$ is complete, we have that there exists an $n \in \mathbb{N}$, such that

$$\mathbb{G}(\varphi_L^n) \circ \alpha \in C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{E})).$$

Here $\mathbb{G}(\varphi_L^n) \circ \alpha$ is still a cocycle since $\mathbb{G}(\varphi_L^n)$ commutes with the $\mathbb{O}_K$-action on $\mathbb{G}^{(f)}(\mathbb{F})$ as it is the action given by π^n.

Now let $\alpha_0 \in C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{E}))$ and $m \in \mathbb{N}$ be arbitrary. Then we calculate using that $\mathbb{G}(\varphi_L) = \mathbb{G}^{(f)}(\pi)$ and α_0 is a 1-cocycle

$$\begin{aligned}
\mathbb{G}(\varphi_L^m)(\alpha_0(\gamma)) &= \alpha_0(\pi^m)^{-1}\alpha_0(\pi^m)\mathbb{G}(\varphi_L^m)(\alpha_0(\gamma)) \\
&= \alpha_0(\pi^m)^{-1}\alpha_0(\gamma\pi^m) \\
&= \alpha_0(\pi^m)^{-1}\alpha_0(\gamma) \cdot \gamma \underset{f}{*} \alpha_0(\pi)^m.
\end{aligned}$$

So

$$\alpha_0 \sim \mathbb{G}(\varphi_L^m) \circ \alpha_0$$

are cohomological and thus

$$C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{F})) \to H^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{E})), \alpha \mapsto [\mathbb{G}(\varphi_L^n) \circ \alpha]_\sim, \ n >> 0$$

is independent on the choice of n, if chosen big enough. Let

$$\alpha, \beta \in C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{F}))$$

cohomological, so there is a $B \in \mathbb{G}^{(f)}(\mathbb{F})$, such that

$$B^{-1} \cdot \alpha(\gamma) \cdot \gamma \underset{f}{*} B = \beta(\gamma) \ \forall \gamma \in \mathbb{O}_K.$$

Now choose n big enough, so that

$$\mathbb{G}(\varphi_L^n) \circ \alpha \in C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{E})) \text{ and } \mathbb{G}(\varphi_L^n)(B) \in \mathbb{G}^{(f)}(\mathbb{E}).$$

Then

$$\mathbb{G}(\varphi_L^n)(\beta(\gamma)) = \mathbb{G}(\varphi_L^n)(B^{-1}) \cdot \mathbb{G}(\varphi_L^n)(\alpha(\gamma)) \cdot \gamma \underset{f}{*} ((\mathbb{G}(\varphi_L^n)(B)) \ \forall \gamma \in \mathbb{O}_K,$$

so $\mathbb{G}(\varphi_L^n) \circ \beta \in C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{E}))$ and

$$\mathbb{G}(\varphi_L^n) \circ \alpha \sim \mathbb{G}(\varphi_L^n) \circ \beta$$

are cohomological, so we get an induced map

$$\phi : H^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{F})) \to H^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{E})), [\alpha]_\sim \mapsto [\mathbb{G}(\varphi_L^n) \circ \alpha]_\sim, \ n >> 0.$$

Since we can choose $n = 0$ for $\alpha_0 \in C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{E}))$, we have

$$\phi(\iota([\alpha_0]_\sim)) = [\alpha_0]_\sim.$$

On the other hand for any $\alpha \in C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{F}))$ and any m, it is $\mathbb{G}(\varphi_L^m) \circ \alpha \in C^1(\mathbb{O}_K, \mathbb{G}^{(f)}(\mathbb{F}))$ cohomological to α. This is shown just as for α_0 above. Thus, we obtain

$$\iota(\phi([\alpha]_\sim)) = [\alpha]_\sim$$

□

2.3 Examples

In this section, we will give a few examples for the image of the map

$$\bar{j} : \mathrm{mor}^{cont}(G_K, \mathbb{G}(k)) \to H^1(H_K, \mathbb{G}(\mathbb{E}^{sep})).$$

Example 2.3.1. i) For $\mathbb{G} = \mathrm{GL}_n$, we have $\bar{j} \equiv 1$ by Hilbert 90 (See Ser79, chapter X §1 Proposition 3), and so we "recover" the classical correspondence of representations and (φ, Γ)-modules. (In the characteristic 0 case, we will see a proof that the maps defined in the last section coincide with the maps induced by the Functor of Theorem 1.3.4 for GL_n, if $L = K$.) The same is true for the affine space $\mathbb{G} = \mathbb{A}^n$ by the additive version of Hilbert 90.

ii) Since $\mathbb{E}$ is a local field, if $\mathbb{G}_{\mathbb{E}}$ is semisimple and simply connected, then we also have $\bar{j} \equiv 1$, since then

$$H^1(H_K, \mathbb{G}(\mathbb{E}^{sep})) = 1$$

by (Ser97, chapter III §3.1 b) on p. 139).

iii) Let $\mathbb{G} = \mathrm{PGL}_n$ the quotient $\mathrm{GL}_n / \mathrm{GL}_1$. It exists and is a linear algebraic group by (Mil17, 5.c) Proposition 5.18) and is smooth by (Mil17, 1.e) Proposition 1.62.b)). Then for every field extension $E|k$, we have that

$$\mathrm{PGL}_n(E) = \mathrm{GL}_n(E)/E^\times$$

as we will prove later in Corollary 2.3.24. We will use this fact implicitedly many times in the following.

In this case, we also have $\bar{j} \equiv 1$, if $K_\infty|K$ contains a (galois) extension $E|K$, such that $[E : K] = q - 1$, for example if $L_\infty \cap K = L$, since then $K_1|K$ has degree $q - 1$, see Proposition 1.1.23.i). This is non trivial, since

$$H^1(H_K, \mathrm{PGL}_n(\mathbb{E}^{sep})) \cong \mathbb{Z}/n\mathbb{Z}$$

by local classfield theory, see (Ser79, chapter X §5 Proposition 9 and Lemma 1) and (Ser67, 1.6 Proposition 4). For this, we calculate

$$H^2(G_K, k^\times) \cong \mathbb{Z}/(q-1)\mathbb{Z}$$

via the short exact sequence

$$1 \to k^\times \hookrightarrow \overline{L}^\times \overset{(\cdot)^{q-1}}{\to} \overline{L}^\times \to 1.$$

and the fact that we have an isomorphism

$$inv_K : H^2(G_K, \bar{L}^\times) \overset{(inf)^{-1}}{\to} H^2(\hat{\mathbb{Z}}, (K^{nr})^\times) \overset{(v)_*}{\to} H^2(\hat{\mathbb{Z}}, \mathbb{Z}) \overset{\delta^{-1}}{\to} H^1(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \overset{\phi \mapsto \phi(1)}{\to} \mathbb{Q}/\mathbb{Z}$$

by the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0,$$

where $v : K^{nr} \to \mathbb{Z}$ is the valuation with respect to a uniformizer $\pi_K \in \mathfrak{m}_K$, inf is the inflation map, $\hat{\mathbb{Z}} \cong \mathrm{Gal}(K^{nr}|K)$, see Proposition 1.2.29.ii) and (Ser67, 1.1 Theorem 1 & Corollary before Theorem 3). Now consider the commutative diagram, where res denotes the restriction map

$$\begin{tikzcd}
H^2(\mathrm{Gal}(E|K), E^\times) \arrow[r, "inf"] \arrow[dr, "inf"] & H^2(\Gamma_K, K_\infty^\times) \arrow[d, "inf"] \\
H^2(G_K, k^\times) \arrow[r, "(\subset)_*"'] \arrow[d, "res"] & H^2(G_K, \overline{L}^\times) \arrow[d, "res"] \\
H^2(H_K, k^\times) \arrow[r, "(\subset)_*"'] & H^2(H_K, \overline{L}^\times).
\end{tikzcd}$$

Here the maps $(\subset)_*$ are injective by Hilbert 90. Furthermore

$$inf : H^2(\mathrm{Gal}(E|K), E^\times) \to H^2(G_K, \overline{L}^\times)$$

is injective and

$$inv_K \circ inf : H^2(\mathrm{Gal}(E|K), E^\times) \to \mathbb{Q}/\mathbb{Z}$$

has image $\mathbb{Z}/(q-1)\mathbb{Z}$ by (Neu15, chapter I § 6 (1.6.7) Proposition) and (Ser67, 1.6 Proposition 4). By (Neu15, chapter I § 6 (1.6.7) Proposition) and Hilbert 90 it furthermore follows that

$$H^2(\Gamma_K, K_\infty^\times) \overset{inf}{\to} H^2(G_K, \overline{L}^\times) \overset{res}{\to} H^2(H_K, \overline{L}^\times)$$

is exact. So since

$$(\subset)_* : H^2(G_K, k^\times) \to H^2(G_K, \overline{L}^\times) \text{ and } inf : H^2(\mathrm{Gal}(E|K), E^\times) \to H^2(G_K, \overline{L}^\times)$$

have the same image and

$$(\subset)_* : H^2(H_K, k^\times) \to H^2(H_K, \overline{L}^\times)$$

is injective it follows that

$$res : H^2(G_K, k^\times) \to H^2(H_K, k^\times)$$

is the trivial map 0. Now consider the following commutative diagram, which has an exact horizontal sequence.

$$\begin{tikzcd}
 & H^1(G_K, \mathrm{PGL}_n(k)) \arrow[r, "\delta"] \arrow[d, "res"'] \arrow[dr, "0"] & H^2(G_K, k^\times) \arrow[d, "res=0"] \\
H^1(H_K, \mathrm{GL}_n(k)) \arrow[r, "(\mathrm{pr})_*"] \arrow[d, "(\subset)_*"'] & H^1(H_K, \mathrm{PGL}_n(k)) \arrow[r, "\delta"'] \arrow[d, "(\subset)_*"] & H^2(H_K, k^\times) \\
H^1(H_K, \mathrm{GL}_n(\mathbb{E}^{sep})) \arrow[r, "(\mathrm{pr})_*"] & H^1(H_K, \mathrm{PGL}_n(\mathbb{E}^{sep})) &
\end{tikzcd}$$

It follows by the diagonal 0 that

$$res : H^1(G_K, \mathrm{PGL}_n(k)) \to H^1(H_K, \mathrm{PGL}_n(k))$$

has image in the image of $(\mathrm{pr})_*$ and so

$$\bar{j} : \mathrm{mor}^{cont}(G_K, \mathrm{PGL}_n(k)) \overset{\mathrm{pr}}{\to} H^1(G_K, \mathrm{PGL}_n(k)) \overset{(\subset)_* \circ res}{\longrightarrow} H^1(H_K, \mathrm{PGL}_n(\mathbb{E}^{sep}))$$

is the trivial map since

$$H^1(H_K, \mathrm{GL}_n(\mathbb{E}^{sep})) = 0$$

by Hilbert 90.

iv) Let again $\mathbb{G} = \mathrm{PGL}_n$. In the last example, we have seen that $\bar{j}$ is trivial, if K_∞ contains a field extension $E|K$, such that $[E : K] = q - 1$. We now show that this condition is necessary for the triviality of $\bar{j}$, if $n >> 0$ is chosen suitably. In particular, if $q \neq 2$ and $K = L_1$, then $K_\infty|K$ doesn't contain such a field by Proposition 1.1.23.i), so we see that then $\bar{j}$ is *not* trivial for some suitable $n >> 0$ by the following argumentation. In particular, since the only cohomology class in $H^1(H_K, \mathrm{PGL}_n(\mathbb{E}^{sep}))$ that induces the isomorphism class of the trivial $\mathbb{E}$-form $\mathrm{PGL}_{n,\mathbb{E}}$ is the trivial cohomology class by (Har68, §3.3), in this case there is a nontrivial (pure) inner form for the correspondence of Galois representations and (φ_L, Γ_K)-module.

We consider the commutative diagram

$$
\begin{tikzcd}
H^1(G_K, \mathrm{PGL}_n(k)) \arrow[r, "\delta_1^{(n)}"] \arrow[d, "res"'] & H^2(G_K, k^\times) \arrow[d, "res"] \\
H^1(H_K, \mathrm{PGL}_n(k)) \arrow[d, "(\subset)_*"'] & H^2(H_K, k^\times) \arrow[d, "(\subset)_*"] \\
H^1(H_K, \mathrm{PGL}_n(\mathbb{E}^{sep})) \arrow[r, "\delta_2"'] & H^2(H_K, (\mathbb{E}^{sep})^\times)
\end{tikzcd}
$$

By (Ser79, chapter X §5 Proposition 9) the map δ_2 is injective. Consider the short exact sequence

$$0 \to k^\times \overset{\subsetneq}{\to} (\mathbb{E}^{sep})^\times \overset{(\cdot)^{q-1}}{\to} (\mathbb{E}^{sep})^\times \to 0,$$

where the right arrow is indeed surjective, since the polynomials $X^{q-1} - a \in \mathbb{E}^{sep}[X]$ are separable for $a \neq 0$. As in the last example, we see via Hilbert 90 that

$$H^2(H_K, k^\times) \overset{(\subset)_*}{\to} H^2(H_K, (\mathbb{E}^{sep})^\times)$$

is injective. If $m|n$, i.e. $n = rm$ for $n, r, m \geq 1$ we have an embedding of G_K-groups

$$\phi_{m,n} : \mathrm{PGL}_m(k) \to \mathrm{PGL}_n(k), A \mapsto \begin{pmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{pmatrix},$$

which comes from an embedding $\tilde{\phi}_{m,n}\, \mathrm{GL}_m(k) \to \mathrm{GL}_n(k)$ given by the definition. We have the following commutative diagram of short exact sequences.

$$
\begin{tikzcd}
0 \arrow[r] & k^\times \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathrm{GL}_m(k) \arrow[r] \arrow[d, "\tilde{\phi}_{m,n}"] & \mathrm{PGL}_m(k) \arrow[r] \arrow[d, "\phi_{m,n}"] & 0 \\
0 \arrow[r] & k^\times \arrow[r] & \mathrm{GL}_n(k) \arrow[r] & \mathrm{PGL}_n(k) \arrow[r] & 0
\end{tikzcd}
$$

By functoriality of the linking morphisms $\delta_1^{(m)}, \delta_2^{(n)}$, we obtain the commutativity

$$
\begin{tikzcd}
H^1(G_K, \mathrm{PGL}_m(k)) \arrow[d, "(\phi_{m,n})^*"'] \arrow[dr, "\delta_1^{(m)}"] & \\
 & H^2(G_K, k^\times) \\
H^1(G_K, \mathrm{PGL}_n(k)) \arrow[ur, "\delta_1^{(n)}"'] &
\end{tikzcd}
$$

It follows that for $m|n$, we have that

$$\mathrm{im}(\delta_1^{(m)}) \subset \mathrm{im}(\delta_1^{(n)}). \tag{div}$$

In the last example, we calculated that $H^2(G_K, k^\times) \cong \mathbb{Z}/(q-1)\mathbb{Z}$ is finite. We write

$$\{1, \ldots, q-1\} = H^2(G_K, k^\times).$$

Furthermore the union of the image of the maps $\delta_1^{(n)}$ going through all $n \geq 1$ is all of $H^2(G_K, k^\times)$ by (Hup67, V 24.2 Hilfssatz c)). So, for every $i \in H^2(G_K, k^\times)$, there exists an $n_i \geq 1$, such that $i \in \mathrm{im}(\delta_1^{(n_i)})$. Then by (div), we have that $\delta_1^{(n_1 n_2 \cdots n_{q-1})}$ is surjective.

The map $\bar{j}$ is the one on the left (post composed with the projection $\mathrm{mor}^{cont}(G_K, \mathrm{PGL}_n(k)) \to H^1(G_K, \mathrm{PGL}_n(k)))$ in the first diagram of this example iv), so $\bar{j}$ is trivial for some $n >> 0$, such that $\delta_1^{(n)}$ is surjective, if and only if

$$res : H^2(G_K, k^\times) \to H^2(H_K, k^\times)$$

is the trivial map. But in the last example, we calculated that the kernel of res is the image of the inflation

$$inf : H^2(\Gamma_K, K_\infty^\times) \to H^2(G_K, \overline{L}^\times)$$

intersected with $H^2(G_K, k^\times) \cong \mathbb{Z}/(q-1)\mathbb{Z}$. So res is zero, if and only if K_∞ contains a field extension $E|K$, such that $[E : K] = q-1$ by (Ser67, 1.6 Proposition 4).

v) Let S_2 be the group with 2 elements. For $\mathbb{G} = \underline{S_2}_k = \mathrm{Spec}(\prod_{x \in S_2} k)$, we have that

$$\bar{j} : \mathrm{mor}^{cont}(G_K, S_2) \to \mathrm{mor}^{cont}(H_K, S_2), f \mapsto f_{|H_K}.$$

By Galoistheory this corresponds to

$$\{\overline{L}|E|K \mid [E : K] \leq 2\} \to \{\overline{L}|E_\infty|E_\infty \mid [E_\infty : K_\infty] \leq 2\}, E \mapsto EK_\infty.$$

But this map is nontrivial, e.g. choose K to be the unramified extension of L of degree 2. On the other hand S_2 is commutative, so the only inner form of $\mathbb{G}$ is the trival one. That means that we go through "multiple copies" of $H^1(\mathbb{O}_K, \mathbb{G}(\mathbb{K}))$ on the right hand side of the correspondence, although the different copies of $\mathbb{G}(\mathbb{K})$ might have different $\mathbb{O}_K$-actions. Also consider that $\Psi \equiv 1$ in this case.

vi) Let S_3 be the non commutative group with 6 elements. For $\mathbb{G} = \underline{S_3}_k$, we have $\underline{S_2}_k \subset \underline{S_3}_k$ and so $\bar{j}$ is non trivial, but S_3 also has trivial center and only inner automorphisms, so we have to go through at least one non trivial (pure) inner form for the correspondence of Galois representations and (φ_L, Γ_K)-module. Here we also have $\Psi \equiv 1$.

In our examples, $\bar{j}$ was always trivial, when $\mathbb{G}$ was connected (or reductive, or semisimple) and $L = K$. In the proof for PGL_n, we used "both parts" of the map $\bar{j}$, i.e. that it is the composition of the map induced by the inclusion $k \subset \mathbb{E}^{sep}$ and the restriction $H_K \subset G_K$. We show in the following that this was necessary: For the rest of this section, we assume

$$\mathbb{G} = \mathrm{PGL}_2 \text{ and } K = L = \mathbb{Q}_3$$

and we show that the map induced by the inclusion $k \subset \mathbb{E}^{sep}$

$$H^1(H_K, \mathrm{PGL}_2(k)) \to H^1(H_K, \mathrm{PGL}_2(\mathbb{E}^{sep}))$$

is non trivial. So, if one wants to go prove that $\bar{j}$ is trivial for a bigger class of linear algebraic groups, one has to use both the inclusion and restriction part.

Lemma 2.3.2. *For a totally ramified field $E|\mathbb{E}$, there exist exactly two cyclic galois extensions*

$$E_0|E \textit{ with } [E_0 : E] = 4,$$

which are in $\mathbb{E}^{sep}$.

Proof. Denote by $E^{ab} \subset \mathbb{E}^{sep}$ the maximal abelian extension of F. By local classfield theory, it is

$$E^{ab} = E^{nr} E_\infty.$$

So it is

$$\mathrm{Gal}(E^{ab}|E) \cong \hat{\mathbb{Z}} \times \mathbb{Z}/2\mathbb{Z} \times (1 + \mathfrak{m}_E).$$

By Proposition 1.1.23.i), the group $(1 + \mathfrak{m}_E)$ is a pro-3-group. It follows by Galois theory that the cyclic extensions $E_0|E$ of degree 4 correspond to the continuous projections

$$\hat{\mathbb{Z}} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$$

up to having the same kernel. There are exactly two of those, namely the map $(\mathrm{pr}, triv)$ and (pr, ι), where $\mathrm{pr} : \hat{\mathbb{Z}} \to \mathbb{Z}/4\mathbb{Z}$ is the projection, $triv : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ is the trivial map and $\iota : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ is the inclusion. □

One of these extensions is obviously $E^{(4)}|E$, where $E^{(4)}$ denotes the unramified extension of degree 4 over E. The other extension is the following.

Proposition 2.3.3. *Let $E|\mathbb{E}$ be a totally ramified field extension. Then $E \cong \mathbb{F}_3((Y))$. It is $\mathbb{F}_9 = \mathbb{F}_3[i]$ for $i^2 = -1$. We set*

$$u := \sqrt{(1+i)Y} \text{ and } v := \sqrt{(1-i)Y}.$$

Then

$$E_0 := E(u) = E(u, v)$$

is a cyclic galois extension of degree 4 over E with ramification index $e = 2$ and inertia index $f = 2$.

Proof. We calculate

$$uv = \pm iY,$$

so

$$E(u) = E(u, v), \text{ since } i = \frac{u^2}{Y} - 1 \in E(u).$$

Furthermore, it is

$$uv \notin K, u^2 \notin K, v^2 \notin K.$$

On the other hand we have

$$P(T) := (T^2 - u^2)(T^2 - v^2) = T^4 + YT^2 - Y^2 \in E[T],$$

since

$$u^2 + v^2 = -Y.$$

This implies that $P(T)$ is the minimal polynomial of u and that $E(u) = E(u, v)$ is its splitting field. It is $b := -Y^2$ not a square in E, since $i \notin E$, but for $a := Y$, it is

$$b(a^2 - 4b) = -Y^2(Y^2 + Y^2) = -2Y^4 = Y^4$$

a square. By (Hun00, chapter V.4 exercise 9.(b)), it follows that $E(u)|E$ is cyclic of degree 4. Since $i \in E(u)$, it is $\mathbb{F}_9 \subset E(u)$, so $2 \mid f$, but on the other hand it is

$$u^2 = (1+i)Y,$$

so $2 \mid e$. Since $[E(u) : E] = 4$, it is $e = 2 = f$. □

Recall that we set $K = L = \mathbb{Q}_3$. Consider the exact, commutative diagram

$$\begin{tikzcd}
H^1(H_K, \mathrm{GL}_2(k)) \arrow[r, "(\mathrm{pr})_*"] & H^1(H_K, \mathrm{PGL}_2(k)) \arrow[r, "\delta_1"] \arrow[d, "(\subset)_*=\phi_1"'] & H^2(H_K, k^\times) \arrow[d, "(\subset)_*=\phi_2"] \\
& H^1(H_K, \mathrm{PGL}_2(\mathbb{E}^{sep})) \arrow[r, "\delta_2"'] & H^2(H_K, (\mathbb{E}^{sep})^\times).
\end{tikzcd}$$

By Hilbert 90, ϕ_2 and δ_2 are injective. It follows for $[f] \in H^1(H_K, \mathrm{PGL}_2(k))$ that

$$\phi_1([f]) = 1, \text{ if and only if } [f] \in \mathrm{im}((\mathrm{pr})_*).$$

Since H_L acts trivially on $\mathrm{GL}_2(k)$, it is

$$H^1(H_L, \mathrm{GL}_2(k)) = \mathrm{mor}^{cont}(H_L, \mathrm{GL}_n(k))/\sim .$$

The same holds for PGL_2.

Consider $K^{(2)}|\mathbb{E}$ to be the unramified extension of degree 2 and $K_2|\mathbb{E}$ to be a totally ramified extension of degree 2 given by taking an uniformizer $X \in \mathbb{E}$ and setting

$$Y := \sqrt{X},\ K_2 := \mathbb{E}(Y).$$

Then $K^{(2)}K_2$ is galois with galois group $S_2 \oplus S_2$, where S_2 denotes the group with two elements.

Proposition 2.3.4. *Let $Z \subset \mathrm{GL}_n(k)$ be the center. We define a morphism $f \in \mathrm{mor}^{cont}(H_K, \mathrm{PGL}_2(k))$ by setting*

$$f : H_K \to H_K/G_{K^{(2)}K_2} \to \mathrm{PGL}_2(k), \begin{cases} (1,0) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot Z \\ (0,1) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot Z \end{cases},$$

where $(1,0)$ corresponds to the Frobenius φ on $K^{(2)}|\mathbb{E}$ and $(0,1)$ corresponds to the non trivial automorphism on $K_2|\mathbb{E}$. Then $[f] \notin \mathrm{im}((\mathrm{pr})_*)$.

Proof. Assume there exists $B \in \mathrm{PGL}_2(k)$ and a continuous morphism

$$\tilde{f} : H_L \to \mathrm{GL}_2(k),$$

such that $\mathrm{pr} \circ \tilde{f}(h) = B^{-1} f(h) B$ for all $h \in H_L$. Since there exists $\tilde{B} \in \mathrm{GL}_2(k)$ with $\mathrm{pr}(\tilde{B}) = B$, we can assume that $B = 1$.

It is $\ker(\tilde{f}) \subset \tilde{f}^{-1}(Z) = \ker(f)$ open and normal and so $\ker(\tilde{f}) = G_E$ for a finite galois extension $E|\mathbb{E}$, such that $K^{(2)}K_2 \subset E$. Since $Z = \{\mathrm{id}_2, -id_2\}$, we see, by making a case study, that any lift of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot Z$ and any lift of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot Z$ generate

$$U := \mathrm{pr}^{-1}(\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot Z, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot Z \rangle).$$

It follows that $\mathrm{im}(\tilde{f}) = U$ and so $E|K^{(2)}K_2$ has degree 2, since

$$U \to \langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot Z, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot Z \rangle$$

has kernel $Z = S_2$. It is easy to see that any lift of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot Z$ has order 4 and any lift of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot Z$ has order 2. If $E|K^{(2)}K_2$ would be totally ramified, then $E|K_2$ is a galois extension of degree 4 with inertia degree $f = 2$ and ramification index $e = 2$, since $K^{(2)} \subset E$ and $E|K^{(2)}K_2$ is totally ramified. If $E|K_2$ has galois group $S_2 \oplus S_2$, then

$$E = K^{(2)}K_{tot}$$

for a totally ramified extension $K_{tot}|\mathbb{E}$ of degree 4 with $K_2 \subset K_{tot}$. Then

$$E \cong K^{(2)} \bigotimes_{\mathbb{E}} K_{tot},$$

since $K^{(2)} \cap K_{tot} = \mathbb{E}$. It follows that $\varphi \otimes \mathrm{id}_{K_{tot}}$ is an $\mathbb{E}$-algebra automorphism on E, which lifts the automorphism of $K^{(2)}K_2$ corresponding to $(1, 0)$, since $K_2 \subset K_{tot}$. This element has order 2, but $\tilde{f}(\varphi \otimes \mathrm{id}_{K_{tot}})$ has order 4. This is a contradiction.

So $E|K_2$ is cyclic. By Lemma 2.3.2 and Proposition 2.3.3, we have that $E = K_2(u)$, where $u := \sqrt{(1+i)Y}$ and $i \in \mathbb{F}_9$ with $i^2 = -1$. Then

$$P(T) := T^4 - (1+i)^2 X \in K^{(2)}[T]$$

is the minimal polynomial of u over $K^{(2)}$. It is $b := -(1+i)^2 X$ not a square in $K^{(2)}$, since X is a uniformizer in $K^{(2)}$, but for $a = 0$, we have that

$$b(a - 4b) = -4b^2$$

is a square in $K^{(2)}$, since $i \in K^{(2)}$. So $E|K^{(2)}$ is cyclic of degree 4 by (Hun00, chapter V.4 exercise 9.(b)). Let

$$\tilde{\phi} \in \mathrm{Gal}(E|K^{(2)})$$

be a lift of the nontrivial element $\phi \in \mathrm{Gal}(K_2|\mathbb{E})$ via

$$\mathrm{Gal}(E|K^{(2)}) \to \mathrm{Gal}(K_2|\mathbb{E}), a \mapsto a_{|K_2}.$$

Since $\mathrm{Gal}(E|K^{(2)})$ is cyclic of degree 4, it is $\tilde{\phi}$ of order 4, but $\tilde{f}(\tilde{\phi})$ is of order 2. This is a contradiction, since $\tilde{f} : \mathrm{Gal}(E|\mathbb{K}^{(2)}) \to U$ is injective.

It follows that $E|K^{(2)}K_2$ is unramified, so $E = K^{(4)}K_2$, where $K^{(4)}|\mathbb{E}$ is the unramified extension of degree 4, which is an abelian galois extension over $\mathbb{E}_L$ with Galois group

$$\mathbb{Z}/4\mathbb{Z} \oplus S_2,$$

but U is not abelian. Since $U \cong H_K/G_E$, this is yet another contradiction, so there can't be such a $\tilde{f}$. □

Corollary 2.3.5. *The map*

$$H^1(H_K, \mathrm{PGL}_2(k)) \to H^1(H_K, \mathrm{PGL}_2(\mathbb{E}^{sep}))$$

induced by the inclusion $k \subset \mathbb{E}^{sep}$ is surjective and in particular not constant.

Proof. Since

$$H^1(H_K, \mathrm{PGL}_2(\mathbb{E}^{sep}) \cong \mathbb{Z}/2\mathbb{Z}$$

by (Ser79, chapter X §5 Proposition 9 and Lemma 1) and (Ser67, 1.6 Proposition 4), it suffices to show that the map is not constant. But this follows from Proposition 2.3.4 and the discussion before it. □

2.3.1 Some Calculations for Semisimple Groups

Definition 2.3.6. We define

$$[K_\infty : K]$$

to be the set of all degrees $[E : K]$, where $E|K$ is a finite extension with $E \subset K_\infty$. We say that an $n \geq 1$ satisfies that n *divides* $[K_\infty : K]$ *or* $n|[K_\infty : K]$, if n divides one of the degrees $[E : K]$ in $[K_\infty : K]$.

Let

$$\mu_n := \mathrm{Spec}(k[X]/\langle X^n - 1\rangle)$$

be the groupscheme of n-the roots of unity.

The argumentation in Example 2.3.1.iii) has the following generalization.

Proposition 2.3.7. *Let $\bar{k} \subset \mathbb{C}_p^\flat$ be the algebraic closure of k. Let $\mathbb{H}, \mathbb{G}$ be two linear algebraic groups over k, such that there exists a commutative diagram of short exact sequences*

$$\begin{tikzcd} 1 \arrow[r] & \bigoplus_{i=1}^{m}\mu_{n_i}(\overline{k}) \arrow[r] \arrow[d, "(\subset)_*"'] & \mathbb{H}(\overline{k}) \arrow[r] \arrow[d, "(\subset)_*"'] & \mathbb{G}(\overline{k}) \arrow[r] \arrow[d, "(\subset)_*"'] & 1 \\ 1 \arrow[r] & \bigoplus_{i=1}^{m}\mu_{n_i}(\overline{\mathbb{F}}) \arrow[r] & \mathbb{H}(\overline{\mathbb{F}}) \arrow[r] & \mathbb{G}(\overline{\mathbb{F}}) \arrow[r] & 1 \end{tikzcd}$$

of G_K-groups, where the G_K-action is the one induced by the natural action on $\bar{k}$ resp. $\overline{\mathbb{F}}$. If the $\bigoplus_{i=1}^{m}\mu_{n_i}(E)$ lie in the center of $\mathbb{H}(E)$ for $E \in \{\overline{k}, \overline{\mathbb{F}}\}$, $n_i|[K_\infty : K]$ for all $1 \le i \le m$ and $H^1(H_K, \mathbb{H}(\overline{\mathbb{F}})) = 1$, then

$$\bar{j} : \mathrm{mor}^{cont}(G_K, \mathbb{G}(k)) \to H^1(H_K, \mathbb{G}(\overline{\mathbb{F}}))$$

is the trivial map.

Proof. Since $\mathrm{char}(k) = p$, we can without loss of generality assume that $p \nmid n_i$ for alle $1 \le i \le m$. Since $\bar{k}$ is algebraically closed, we have a short exact sequence of G_K-groups

$$0 \to \mu_n(\overline{k}) \to \overline{L} \overset{(\cdot)^n}{\to} \overline{L} \to 0,$$

if $p \nmid n$. Beware that the G_K-action of $\overline{k}$ as the residue field of $\overline{L}$ and as the subfield $\overline{k} \subset \mathbb{C}_p^\flat$ coinside, since

$$\overline{k} = \lim_{\underset{(\cdot)^q}{\leftarrow}}(\mathcal{O}_{L^{nr}}/\pi\mathcal{O}_{L^{nr}}) \subset \mathcal{O}_{\mathbb{C}_p^\flat}.$$

As in Example 2.3.1.iii), we calculate that

$$H^2(G_K, \mu_n(\overline{k})) \cong \mathbb{Z}/n\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$

and that

$$inf : H^2(\Gamma_K, K_\infty^\times) \to H^2(G_K, \overline{L}^\times) \cong \mathbb{Q}/\mathbb{Z}$$

has image containing $\mathbb{Z}/n\mathbb{Z}$, if $n|[K_\infty : K]$. It follows that

$$res : H^2(G_K, \mu_n(\overline{k})) \to H^2(H_K, \mu_n(\overline{k}))$$

is the zero map, if $n|[K_\infty : K]$, since $\mathrm{im}(inf) \subset res^{-1}(\{1\})$ already on cocycle level by definition of these maps and since

$$(\subset)_* : H^2(H_K, \mu_n(\overline{k})) \to H^2(H_K, \overline{L}^\times)$$

is injective by Hilbert 90. We have the following commutative diagram.

$$\begin{tikzcd}
H^1(G_K, \mathbb{G}(k)) \arrow[d, "(\subset)_*"'] & \\
H^1(G_K, \mathbb{G}(\overline{k})) \arrow[r, "\delta_1"] \arrow[d, "res"'] & \bigoplus_{i=1}^m H^2(G_K, \mu_{n_i}(\overline{k})) \arrow[d, "\oplus res"] \\
H^1(H_K, \mathbb{G}(\overline{k})) \arrow[d, "(\subset)_*"'] & \bigoplus_{i=1}^m H^2(H_K, \mu_{n_i}(\overline{k})) \arrow[d, "\oplus(\subset)_*"] \\
H^1(H_K, \mathbb{G}(\overline{\mathbb{F}})) \arrow[r, "\delta_2"'] & \bigoplus_{i=1}^m H^2(H_K, \mu_{n_i}(\overline{\mathbb{F}}))
\end{tikzcd}$$

Beware here that $\overline{k}$ is a topological G_K-group with the discrete topology, so the second row makes sense. Furthermore, we could pull out the $\bigoplus$ in the second column by (Neu13, (3.7) Proposition). By assumption and the argument above the map $\oplus res$ is the zero map and δ_2 has trivial fiber $\delta_2^{-1}(\{1\}) = \{1\}$, so the map on the left column is the trivial map. But this map is the map $\bar{j}$, since $(\subset)_*$ commutes with the restriction res. $\square$

Remark. Recall that by Remark 2.2.3 the map

$$\bar{j}_{\mathbb{E}} : \mathrm{mor}^{cont}(G_K, \mathbb{G}(k)) \to H^1(H_K, \mathbb{G}(\mathbb{E}^{sep}))$$

is trivial, if and only if the map

$$\bar{j}_{\mathbb{F}} : \mathrm{mor}^{cont}(G_K, \mathbb{G}(k)) \to H^1(H_K, \mathbb{G}(\overline{\mathbb{F}}))$$

is trivial.

We will apply this Proposition to the universal covering of a simple reductive group over k. We recall some facts about the classification of reductive groups via root data that we need for this.

Lemma 2.3.8. *Let E be a field of characteristic* $\mathrm{char}(E) = p > 0$. *Let $\mathbb{H}$ be a linear algebraic group over E. Then*

$$\{A \in \mathbb{H}(E) \mid \exists n \geq 1 : A^{p^n} = 1\} = \{1\}.$$

Proof. Fix an embedding $\mathbb{H} \subset \mathrm{GL}_n$. It suffices to show the statement for $\mathrm{GL}_n(E)$, since $\mathbb{H}(E) \subset \mathrm{GL}_n(E)$ is an embedding of groups. Let $A = (a_{ij})_{ij} \in \mathrm{GL}_n(E)$. Since $(\cdot)^p : E \to E$ is a morphism of rings, we have

$$A^{p^n} = (a_{ij}^{p^n}),$$

so $A^{p^n} = 1$ holds, if and only if if and only if $A = 1$, because $(\cdot)^p : E \to E$ is injective. $\square$

We will follow the convention that every reductive (and in particular semisimple) group over k are connected and smooth.

Definition 2.3.9. Let E be a field. A smooth and connected linear algebraic group $\mathbb{H}$ over E is called *reductive* (resp. *semisimple*), if there exists no non-zero Zariski-closed, Zariski-connected, normal and unipotent (resp. solvable) subgroup in $\mathbb{H}(\overline{E})$ for some algebraic closure $\overline{E}|E$.

Definition 2.3.10. Let X be a free abelian group of finite rank with a finite subset $R \subset X$ and $Q \subset X$ be the subgroup generated by R. If $(Q \otimes \mathbb{R}, R)$ is a root system (See Spr98, part 7.4.1), we will write (X, R) to be an *integral root system*. Let $\mathbb{H}$ be a reductive group over some field E with algebraic closure $\overline{E}$. Let $T \subset \mathbb{H}(\overline{E})$ be a maximal torus, i.e. $T \cong \prod_{i=1}^{n} \overline{E}^{\times}$ is isomorphic as varieties over $\overline{E}$. We define the *characters of* T to be

$$X^*(T) := \{\phi : T \to \overline{E}^{\times} \mid \phi \text{ is a morphism of group varieties}\}.$$

Let $(X, R) := (X^*(T), R)$ be the integral root system corresponding to $\mathbb{H}(\overline{E})$ (See Spr98, part 7.4.3). This is independent of the choice of T. We define the *cocenter of* $\mathbb{H}$ *or of* (X, R) to be

$$C^* := C^*(\mathbb{H}) := C^*(X, R) := X/Q,$$

where Q is the subgroup generated by $R \subset X$.

From now on $\mathbb{G}$ is always a semisimple group over k.

Lemma 2.3.11. *Let $C \subset \mathbb{G}(\overline{k})$ be the center. Then there exists a (non canonical) embedding of groups*

$$C \subset C^*$$

from the center to the cocenter.

Proof. Let $G := \mathbb{G}(\overline{k})$. The group C is a group variety over $\overline{k}$, since it is the intersection of all Zariski-closed subsets

$$C_x(G) := \{g \in G \mid gxg^{-1} = x\}$$

for all $x \in G$. It is finite by (Spr98, 1.2.4 Proposition, 7.3.1 Proposition & 8.1.5.ii) Theorem). Let $\mathrm{mor}_{\overline{k}-var}(C, \overline{k}^{\times})$ be the maps of group varieties. Then

$$\mathrm{mor}_{\overline{k}-var}(C, \overline{k}^{\times}) = \mathrm{mor}_{Grp}(C, \overline{k}^{\times})$$

are the morphisms of groups, since any map from a finite subset of a variety into a variety is polynomial. Furthermore $\overline{k}^{\times} \subset \mathbb{Q}/\mathbb{Z}$ is the subgroup $\varinjlim_{p \nmid n} \mathbb{Z}/n\mathbb{Z}$. It follows by Lemma 2.3.8 and the structure theorem of finite abelian groups that there exists a non canonical isomorphism

$$C \cong \mathrm{mor}_{\overline{k}-var}(C, \overline{k}^{\times}).$$

Now by (Spr98, 8.1.12.(8) Exercises), there exists an embedding

$$\mathrm{mor}_{\overline{k}-var}(C, \overline{k}^{\times}) \subset C^*.$$

□

Definition 2.3.12. Let (V, R) be a rootsystem. Denote $R^\vee \subset V^\vee := \mathrm{mor}_{\mathbb{R}-mod}(V, \mathbb{R})$ the dual to the root system R. We define the *weight lattice of* (V, R) to be the free abelian group of finite rank

$$P := \{v \in V \mid \langle v, R^\vee \rangle \subset \mathbb{Z}\}.$$

Let $Q \subset V$ be the subgroup generated by R. It is $Q \subset P$ and we define the *fundamental group of* (V, R) to be

$$\pi_1(V, R) := P/Q.$$

Remark 2.3.13. If (X, R) is the integral root system of a semisimple group $\mathbb{H}$ over some field E, we set $V := X \otimes \mathbb{R}$. Then (V, R) is a root system and for this root system we have

$$Q \subset X \subset P$$

by (Spr98, 8.1.8.ii) Proposition & part 7.4.3) and so

$$C^*(\mathbb{H}) = X/Q \subset P/Q = \pi_1(V, R).$$

If $C \subset \mathbb{H}(\overline{E})$ is the center, then we furthermore have

$$C \subset C^*(\mathbb{H}) \subset \pi_1(V, R)$$

by Lemma 2.3.11.

Beware though, that in general it is

$$\pi_1(V, R) \neq \pi_1(\mathbb{H}(\overline{E}), 1),$$

where $\pi_1(\mathbb{G}(\overline{E}), 1)$ is the kernel of the universal covering of $\mathbb{H}(\overline{E})$, since the latter is dependent on the lattice $X \subset V$ of characters of a maximal torus in $\mathbb{H}(\overline{E})$, but the former is not.

We follow the notation of (Spr98, part 9.5.1) for the connected Dynkin diagrams. We have the following table of fundamental groups for the the root system corresponding to the Dynkin diagram by (Spr98, 17.1-17.8).

A_{n-1}	B_n	C_n	D_{2n}	D_{2n+1}	E_6	E_7	E_8	F_4	G_2
$\mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	1	1

Definition 2.3.14. We say that a linear algebraic group $\mathbb{H}$ over some field E is *quasi-simple*, if it is semisimple and $\mathbb{H}(\overline{E})$ doesn't contain a non-zero Zariski-connected, Zariski-closed and normal subgroup, where $\overline{E}|E$ is an algebraic closure.

We set

$$S_{type} \in \{A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2\}$$

to be a type of a connected Dynkin diagram and its corresponding root system.

Remark 2.3.15. (See Spr98, 8.1.12.(4) Exercises & part 9.5.1)

Let $\mathbb{H}$ be a semisimple group over some field E with corresponding integral root system (X, R). By Remark 2.3.15, the pair $(X \otimes \mathbb{R}, R)$ is a root system. Then $\mathbb{H}$ is quasi-simple, if and only if $(X \otimes \mathbb{R}, R)$ is one of the types S_{type}.

Let $\phi : \mathbb{H}_1 \to \mathbb{H}_2$ be a morphism between reductive groups over some field E, such that the induced map

$$\phi_{\overline{E}} : \mathbb{H}_1(\overline{E}) \to \mathbb{H}_2(\overline{E})$$

is surjective and has finite kernel for an algebraic closure $\overline{E}|E$. This is equivalent to ϕ being surjective and having finite kernel by the discussion before (Con15, Example A.1.12). Let (X_i, R_i) be the integral root system corresponding to $\mathbb{H}_i$, $i = 1, 2$. We set $l = \mathrm{char}(E)$, if $\mathrm{char}(E) > 0$ or $l = 1$, if $\mathrm{char}(E) = 0$. By (Spr98, part 9.6.3) the isogeny ϕ induces a tripple (f, b, q), where

$$f := f(\phi) : X_2 \to X_1$$

is an embedding of groups with finite cokernel,

$$b := b(\phi) : R_1 \to R_2$$

is a bijection and

$$q := q(\phi) : R_1 \to \{l^n \mid n \geq 0\}$$

is a map satisfying

$$f(b(\alpha)) = q(\alpha) \cdot \alpha, f^\vee(\alpha^\vee) = q(\alpha) \cdot b(\alpha)^\vee, \forall \alpha \in R_1,$$

where $\alpha^\vee \in R_1^\vee$ is the coroot corresponding to α and

$$f^\vee : \cdot \circ f : X_1^\vee := \mathrm{mor}_{Grp}(X_1, \mathbb{Z}) \to \mathrm{mor}_{Grp}(X_2, \mathbb{Z}) =: X_2^\vee.$$

A triple (f, b, q) satisfying these conditions is called an *l-morphism*.

Definition 2.3.16. Let E be a field

i) Let $\mathbb{H}_1, \mathbb{H}_2$ be two semisimple algebraic groups over E. A morphism of groups

$$\phi : \mathbb{H}_1 \to \mathbb{H}_2$$

is called an *isogeny*, if it is surjective and has finite kernel. An isogeny is called *central*, if $q(\phi) \equiv 1$ is the trivial map. By (Con15, Theorem A.4.10), this is equivalent to the kernel of ϕ being contained in some scheme theoretic center, which we define later.

ii) A semisimple group $\mathbb{H}$ over E is called *simply connected*, if for every connected $\tilde{\mathbb{H}}$, we have that every central isogeny

$$\phi : \tilde{\mathbb{H}} \to \mathbb{H}$$

is an isomorphism.

We elaborate on the scheme theoretic center.

Proposition 2.3.17. *(See Mil17, 1.k) Proposition 1.92)*

Let $\mathbb{H}$ be a linear algebraic group over a field E and let $H \subset \mathbb{H}$ be a closed subgroup. Then the functor

$$C_{\mathbb{H}}(H)(R) := \{h \in \mathbb{H}(R) \mid hgh^{-1} = g \ \forall g \in H(S), \ \forall R\text{-algebras } S\}$$

for every E-algebra R is represented by a closed subgroup $C_{\mathbb{H}}(H) \subset \mathbb{H}$ called the centralizer of H in $\mathbb{H}$. *If $H = \mathbb{H}$, we define the* center of $\mathbb{H}$ *to be*

$$Z(\mathbb{H}) := C_{\mathbb{H}}(\mathbb{H}).$$

Semisimple groups have universal coverings.

Proposition 2.3.18. *(See Con15, Corollary A.4.11)*
Let $\mathbb{H}$ be a semisimple group over some field E. Then up to unique isomorphism there exists a unique simply connected semisimple group $\tilde{\mathbb{H}}$ and a central isogeny

$$\tilde{\mathbb{H}} \to \mathbb{H}.$$

We call $\tilde{\mathbb{H}}$ the universal cover *of $\mathbb{H}$. This Formation is stable under basechange to another basefield. Furthermore, if (X, R) is the integral root system corresponding to $\mathbb{H}$, then the integral root system to $\tilde{\mathbb{H}}$ is (P, R), where $P \subset X \otimes \mathbb{R}$ is the weight lattice as in Remark 2.3.13. In particular, if $\mathbb{H}$ is quasi-simple, then so is $\tilde{\mathbb{H}}$.*

Proof. The statement about the weight lattice is in the proof of (Con15, Corollary A.4.11). The statement about being quasi-simple is this statement about the weight lattice together with Remark 2.3.15. □

Proposition 2.3.19. *(See Mil17, 2.e) part 2.31 & 21.e) Theorem 21.51)*
If $\mathbb{H}$ is semisimple over some field E, then we have a central isogeny

$$\prod_{i=1}^{n} \mathbb{H}_i \to \mathbb{H},$$

where the $\mathbb{H}_i$ are quasi-simple, linear algebraic groups over E.

Corollary 2.3.20. *If $\mathbb{H}$ is semisimple over some field E, then we have a central isogeny*

$$\prod_{i=1}^{n} \mathbb{H}_i \to \mathbb{H},$$

where the $\mathbb{H}_i$ are quasi-simple, simply connected linear algebraic groups over E.

Proof. Proposition 2.3.18 and Proposition 2.3.19, where we use the fact that the product of central isogenies is still a central isogeny. □

Definition 2.3.21. Let $\mathbb{H}$ be a linear algebraic group over some field E. Furthermore let E^{sep} be the separable closure in some algebraically closed field containing E.

i) We call $\mathbb{H} = T$ a *torus*, if $\mathbb{H}_{E^{sep}} \cong \prod_{i=1}^{n} \mathrm{GL}_{1,E^{sep}}$.

ii) We call a torus $\mathbb{H} = T$ *split over E*, if $\mathbb{H} \cong \prod_{i=1}^{n} \mathrm{GL}_{1,E}$.

iii) We call a reductive group $\mathbb{H}$ *split over* E, if there exists a maximal torus $T \subset \mathbb{H}$, such that T is split over E.

Remark. By the classification of reductive groups $\mathbb{H}$ split over E (See Con15, Theorem A.4.6), we can deduce by the construction of the universal covering given in the proof of (Con15, Corollary A.4.11) that if $\mathbb{H}$ is a semisimple group split over E, then so is its universal covering $\tilde{\mathbb{H}}$. In the situation of Proposition 2.3.19, we have by (Bor91, 22.9 Proposition) that the dimension of a torus $T \subset \mathbb{H}$, which is maximal for the property of being split over E is the sum of the dimensions of such tori $T_i \subset \mathbb{H}_i$ for all $1 \leq i \leq n$ and furthermore that $\prod_{i=1}^{n} T_i$ is such a torus in $\prod_{i=1}^{n} \mathbb{H}_i$. Thus, if $\mathbb{H}$ is split over E, then so is the product $\prod_{i=1}^{n} \mathbb{H}_i$ by the following argumentation. A surjective map onto a reduced linear algebraic group over E is faithfully flat by (Mil17, 1.g) Summary 1.71). So the central isogeny is still a central isogeny after basechange to an algebraic closure $\overline{E}$ of E. Since $\mathbb{H}$ is E-split, the dimension of a maximal E-split torus is equal to a maximal torus in $\mathbb{H}_{\overline{E}}$. So the same is true for $\prod_{i=1}^{n} \mathbb{H}_i$ by the statements above from (Bor91, 22.9 Proposition). But by (Gö10, Lemma 5.7 (1)) the dimension of a closed subset of a torus is equal to the dimension of the torus, if and only if the closed subset is equal to the torus. It follows that $\prod_{i=1}^{n} T_i$ is a maximal torus in $\prod_{i=1}^{n} \mathbb{H}_i$.

Together, we obtain that in the situation of Corollary 2.3.20, we have that $\prod_{i=1}^{n} \mathbb{H}_i$ is split over E, if $\mathbb{H}$ is split over E.

Lemma 2.3.22. *Let $\mathbb{H}$ be semisimple and split over E. We consider*

$$\phi : \prod_{i=1}^{n} \mathbb{H}_i \to \mathbb{H}$$

as in Corollary 2.3.20. Then

$$\ker(\phi) = \prod_{j=1}^{m} \mu_{k_j}$$

is a product of roots of unity μ_{k_j}.

Proof. Set

$$\mathbb{H}^{(qs)} := \prod_{i=1}^{n} \mathbb{H}_i$$

Since ϕ is a central isogeny, $\ker(\phi_{\bar{k}})$ is finite and in the scheme theoretic center $Z(\mathbb{H}^{(qs)})$. Since $\mathbb{H}^{(qs)}$ is reductive as the product of semisimple schemes, the center $Z(\mathbb{H}^{(qs)})$ is contained in every maximal torus $T \subset \mathbb{H}^{(qs)}$ by (Mil17, 21.b) Proposition 21.7). Since one of those T is split over E by assumption and the Remark above, the finite kernel $\ker(\phi)$ is a product of roots of unity by the classification of diagonalizable algebraic groups via finitely generated abelian groups, see (Mil17, 12.c) Theorem 12.8 & 12.d) Theorem 12.12). □

Recall that a scheme S over some field E is called *geometrically reduced*, if the basechange S_F to a perfect field F containing E is reduced or equivalently if the basechange S_{E_0} to every field extension $E_0|E$ is reduced (See Gö10, Proposition 5.49).

Proposition 2.3.23. *(Inspired by Mil17, 3.k) Proposition 3.45)*
Let $\pi : \mathbb{H}_1 \to \mathbb{H}_2$ be a faithfully flat map of linear algebraic groups over some field E. Let $E_0|E$ be some field extension, $\overline{E_0}$ be an algebraic closure and $E_0^{sep} \subset \overline{E_0}$ be the separable closure.

i) The sequence

$$1 \to \ker(\pi)(\overline{E_0}) \to \mathbb{H}_1(\overline{E_0}) \to \mathbb{H}_2(\overline{E_0}) \to 1$$

is exact.

ii) If $\ker(\pi)$ is geometrically reduced, then

$$1 \to \ker(\pi)(E_0^{sep}) \to \mathbb{H}_1(E_0^{sep}) \to \mathbb{H}_2(E_0^{sep}) \to 1$$

is exact.

Proof. Set $N := \ker(\pi)$. By (Mil17, 5.e) Corollary 5.48), the sequence

$$1 \to N(\overline{E_0}) \to \mathbb{H}_1(\overline{E_0}) \xrightarrow{\pi_a} \mathbb{H}_2(\overline{E_0}) \to 1$$

is exact, where $\pi_a := \pi_{\overline{E_0}}$ is the map induced by π. This shows i) Since

$$N(R) = \ker(\pi_R : \mathbb{H}_1(R) \to \mathbb{H}_2(R))$$

for every E-algebra R, we only need to show that

$$\pi_a : \mathbb{H}_1(E_0^{sep}) \to \mathbb{H}_2(E_0^{sep})$$

is surjective, if N is geometrically reduced. Let $f \in \mathbb{H}_2(E_0^{sep}) \subset \mathbb{H}_2(\overline{E_0})$. By the exactness above the variety over $\overline{E_0}$

$$P := \pi_a^{-1}(\{f\}) \subset \mathbb{H}_1(\overline{E_0}) \neq \emptyset$$

is non empty, so let $x \in P$. Define $\tilde{P} := \pi^{-1}(f)$ with $\pi^{-1}(f)$ being the fiber of f under $\pi : \mathbb{H}_1 \to \mathbb{H}_2$, where we view f as an element of the topological space underlying $\mathbb{H}_2$. Multiplying with x yields a functorial bijection

$$N(R) \to \tilde{P}(R)$$

for any $\overline{E_0}$-algebra R, so by the Yoneda Lemma have an isomorphism of $\overline{E_0}$-schemes $N_{\overline{E_0}} \cong \tilde{P}_{\overline{E_0}}$. It follows by assumption on N that $\tilde{P}$ is geometrically reduced, and since it is non empty, it is $\tilde{P}(E_0^{sep}) \neq \emptyset$ by (Gö10, Proposition 6.21). But by Definition of the fiber or Lemma 2.1.15, it is

$$\tilde{P}(E_0^{sep}) = P \cap \mathbb{H}_1(E_0^{sep}),$$

so there exists an inverse image of f under π_a in $\mathbb{H}_1(E_0^{sep})$. $\square$

Remark. (See Mil17, 1.g) Summary 1.71)

If $\mathbb{H}_2$ is reduced, then a morphism of groups

$$\phi : \mathbb{H}_1 \to \mathbb{H}_2$$

is faithfully flat, if and only if ϕ is surjective, i.e. a surjective ϕ onto a reduced group is automatically flat.

Corollary 2.3.24. *We are in the situation as in the previous Proposition. If* $\ker(\pi)$ *is geometrically reduced, then for any field extension* $E_0|E$*, we have the following exact sequence.*

$$1 \to \ker(\pi)(E_0) \to \mathbb{H}_1(E_0) \to \mathbb{H}_2(E_0) \to H^1(G_{E_0}, \ker(\pi)(E_0^{sep})).$$

If E_0 *is perfect, we can drop the assumption on* $\ker(\pi)$.

Proof. For any linear algebraic group $\mathbb{H}$ over E_0, we have

$$H^0(G_{E_0}, \mathbb{H}(E_0^{sep})) = \mathbb{H}(E_0^{sep})^{G_{E_0}} = \mathbb{H}(E_0)$$

by Lemma 2.1.25 and so the statements follow from the usual cohomology sequence (of non abelian groups) (See Ser97, I §5.4 Proposition 36) applied to the short exact sequences in Proposition 2.3.23. $\square$

Proposition 2.3.25. *Let* $\mathbb{G}$ *be a simply connected, semisimple group over* k*. Then*

$$H^1(H_K, \mathbb{G}(\overline{\mathbb{F}})) = 1.$$

Proof. Let $c \in C^1(H_K, \mathbb{G}(\overline{\mathbb{F}}))$. Since c is continuous, we can assume that $c \in C^1(G, \mathbb{G}(\overline{\mathbb{F}}))$ for some finite galois extension $F|\mathbb{F}$ with Galois group $G := \mathrm{Gal}(F|\mathbb{F})$. We have that $\overline{\mathbb{F}}$ is the perfect hull of $\mathbb{E}^{sep}$ and G is finite. So by Lemma 2.1.16, there exists an $N \geq 1$, such that

$$\mathbb{G}(\varphi_L^N) \circ c \in C^1(H_K, \mathbb{G}(\mathbb{E}^{sep})),$$

which is also a cocycle since φ_L^N commutes with $\bar{\rho}(h)$ for all $h \in H_K$. By the statement about basechange in Proposition 2.3.18, $\mathbb{G}_{\mathbb{E}}$ is still simply connected and semisimple over $\mathbb{E}$. Then by Example 2.3.1.ii), it is

$$H^1(H_K, \mathbb{G}(\mathbb{E}^{sep})) = 1.$$

So there exists an $A \in \mathbb{G}(\mathbb{E}^{sep})$, such that

$$\mathbb{G}(\varphi_L^N)(c(h)) = A^{-1} \cdot \mathbb{G}(\overline{\rho}(h))(A).$$

Since $\overline{\mathbb{F}}$ is perfect, it is

$$B := \mathbb{G}(\varphi_L^{-N})(A) \in \mathbb{G}(\overline{\mathbb{F}}).$$

Since φ_L^{-N} commutes with $\overline{\rho}(h)$ for every h, we obtain

$$c(h) = B^{-1} \cdot \mathbb{G}(\overline{\rho}(h))(B).$$

So c is a coboundary. □

Theorem 2.3.26. *Let $[K_\infty : K] = [L_\infty : L]$, e.g. $K = L$.*

The map

$$\bar{j} : \mathrm{mor}^{cont}(G_K, \mathbb{G}(k)) \to H^1(\mathbb{O}_K, \mathbb{G}(\overline{\mathbb{F}}))$$

is trivial $\bar{j} \equiv 1$ for semisimple $\mathbb{G}$ split over k in the following cases.

i) If $\mathbb{G}$ is quasi-simple of type $S_{type} \notin \{D_{2n+1}, E_6 \mid n \geq 2\}$. If $\mathrm{char}(k) = 2$ or $4 \mid q-1$, we can also allow quasi-simple groups of type $S_{type} = D_{2n+1}$ and if $\mathrm{char}(k) = 3$ or $3 \mid q-1$, we can also allow quasi-simple groups of type $S_{type} = E_6$.

ii) If $\mathbb{G}$ is isomorphic to the product of groups as in i).

iii) Let

$$\phi : \prod_{i=1}^{n} \mathbb{G}_i \to \mathbb{G}$$

be as in Corollary 2.3.20. The map $\bar{j}$ for $\mathbb{G}$ is trivial, if the $\mathbb{G}_i$ are of type $S_{type} \notin \{A_{n-1}, D_{2m+1}, E_6 \mid m \geq 2, n \nmid (q-1)p^r \ \forall r \geq 1\}$. If $\mathrm{char}(k) = 2$ or $4 \mid q-1$, we can also allow $\mathbb{G}_i$ to be of type $S_{type} = D_{2m+1}$ and if $\mathrm{char}(k) = 3$ or $3 \mid q-1$, we can also allow $\mathbb{G}_i$ to be of type $S_{type} = E_6$.

In particular, the condition on q in i) or iii) is always satisfied, if $k|\mathbb{F}_{p^2}$, i.e. in these cases, every quasi-simple group split over k has trivial $\bar{j}$ and in iii) only $\mathbb{G}_i$ for certain types $S_{type} = A_{n-1}$ might be problematic for the property of $\bar{j}$ being trivial for semisimple $\mathbb{G}$ split over k.

Proof. The statement ii) follows from i), since for two linear algebraic groups $\mathbb{G}_1, \mathbb{G}_2$ over k, the map

$$\bar{j} : \mathrm{mor}^{cont}(G_K, \mathbb{G}_1(k) \times \mathbb{G}_2(k)) \to H^1(H_K, \mathbb{G}_1(\overline{\mathbb{F}}) \times \mathbb{G}_2(\overline{\mathbb{F}}))$$

corresponds to

$$\bar{j}\times\bar{j} : \mathrm{mor}^{cont}(G_K, \mathbb{G}_1(k))\times\mathrm{mor}^{cont}(G_K, \mathbb{G}_2(k)) \to H^1(H_K, \mathbb{G}_1(\overline{\mathbb{F}}))\times H^1(H_K, \mathbb{G}_2(\overline{\mathbb{F}}))$$

via the universal property of the product and the canonical isomorphism

$$H^1(H_K, \mathbb{G}_1(\overline{\mathbb{F}})) \times H^1(H_K, \mathbb{G}_2(\overline{\mathbb{F}})) \cong H^1(H_K, \mathbb{G}_1(\overline{\mathbb{F}}) \times \mathbb{G}_2(\overline{\mathbb{F}}))$$

of (Neu13, (3.7) Proposition).

Let $\mathbb{G}$ be semisimple and split over k with universal covering $\tilde{\mathbb{G}}$ and we consider

$$\phi : \prod_{i=1}^{n} \mathbb{G}_i \to \mathbb{G}$$

as in Corollary 2.3.20. We set

$$\mathbb{H} := \prod_{i=1}^{n} \mathbb{G}_i.$$

By Proposition 2.3.25, we have $H^1(H_K, \prod_i^n \mathbb{G}_i(\overline{\mathbb{F}})) = 1$, since the product can be pulled out of the H^1 by (Neu13, (3.7) Proposition). Furthermore, we have the following commutative diagram of short exact sequences of G_K-groups

$$\begin{tikzcd}
1 \arrow[r] & \bigoplus_{i=1}^{m} \mu_{n_i}(\overline{k}) \arrow[r] \arrow[d, "(\subset)_*"'] & \mathbb{H}(\overline{k}) \arrow[r] \arrow[d, "(\subset)_*"'] & \mathbb{G}(\overline{k}) \arrow[r] \arrow[d, "(\subset)_*"'] & 1 \\
1 \arrow[r] & \bigoplus_{i=1}^{m} \mu_{n_i}(\overline{\mathbb{F}}) \arrow[r] & \mathbb{H}(\overline{\mathbb{F}}) \arrow[r] & \mathbb{G}(\overline{\mathbb{F}}) \arrow[r] & 1
\end{tikzcd}$$

by Proposition 2.3.23.i) and Lemma 2.3.22. By Lemma 2.3.8, we can assume that $p \nmid n_i$ for all i. The exact sequences above are central, i.e. the image of the left arrow is in the center of the middle group. Since the center of $\mathbb{H}(\overline{k})$ is contained in the fundamental group of its corresponding Dynkin diagram by Remark 2.3.13, we consider the following table of fundamental groups

A_{n-1}	B_n	C_n	D_{2n}	D_{2n+1}	E_6	E_7	E_8	F_4	G_2
$\mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	1	1

Because of $p \nmid n_i$, the n_i are uniquely determined by the values $\mu_{n_i}(\overline{k})$, since $\overline{k}$ is algebraically closed of characteristic p. So the n_i are divisors of the exponents of the groups in the table. Making a case study gives us almost all the statements of i) and iii) by Proposition 2.3.7, since every divisor of $q-1$ is in $[L_\infty : L]$ by Proposition 1.1.23.i).

The only case, we still need to settle is, if $\mathbb{G}$ is quasi-simple split over k of type A_{n-1}. Beware that $A_3 = D_3$, which is why we excluded D_3 in i) and iii). By (Spr98, 17.1) and the uniqueness in Proposition 2.3.18, the unique simply connected group split over E of type A_{n-1} is SL_n. It has center μ_n. Furthermore the quotient SL_n/μ_n is isomorphic to PGL_n by (Mil17, 5.e) Example 5.49). In Example 2.3.1.iii), we have already seen this case. Now let $\mathbb{G}$ be split over k and arbitrary of this type. Since μ_m for $m \mid n$ are the only closed subgroups of μ_n by the classification of diagonizable algebraic groups via finitely generated abelian groups (See Mil17, 12.c) Theorem 12.8 & 12.d) Theorem 12.12), we have a commutative diagram of short exact central sequences

$$\begin{tikzcd}
1 \arrow[r] & \mu_m(\overline{\mathbb{F}}) \arrow[r] \arrow[d, "\subset"] & \mathrm{SL}_n(\overline{\mathbb{F}}) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathbb{G}(\overline{\mathbb{F}}) \arrow[r] \arrow[d, "\mathrm{pr}"] & 1 \\
1 \arrow[r] & \mu_n(\overline{\mathbb{F}}) \arrow[r] & \mathrm{SL}_n(\overline{\mathbb{F}}) \arrow[r] & \mathrm{PGL}_n(\overline{\mathbb{F}}) \arrow[r] & 1
\end{tikzcd}$$

by the universal covering for $\mathbb{G}$ in Proposition 2.3.18 and Proposition 2.3.23. Thus we obtain a commutative diagram

$$\begin{tikzcd}
H^1(H_K, \mathbb{G}(\overline{\mathbb{F}})) \arrow[r, "pr_*"] \arrow[d, "\delta_1"] & H^1(H_K, \mathrm{PGL}_n(\overline{\mathbb{F}})) \arrow[d, "\delta_2"] \\
H^2(H_K, \mu_m(\overline{\mathbb{F}})) \arrow[r, "(\subset)_*"'] & H^2(H_K, \mu_n(\overline{\mathbb{F}})).
\end{tikzcd}$$

Here δ_1, δ_2 have trivial fiber $\delta_i^{-1}(\{1\}) = \{1\}$, since $H^1(H_K, \mathrm{SL}_n(\overline{\mathbb{F}})) = 1$. Let A be an abelian group. We define for $r \geq 1$ the torsion subgroup

$$A[r] := \{a \in A \mid ra = 0\}.$$

The map

$$H^2(H_K, \mu_m(\overline{\mathbb{F}})) \overset{(\subset)_*}{\to} H^2(H_K, \mu_n(\overline{\mathbb{F}}))$$

is by calculations as in Example 2.3.1.iii) the inclusion

$$H^2(H_K, \overline{\mathbb{F}}^\times)[m^{(p)}] \subset H^2(H_K, \overline{\mathbb{F}}^\times)[n^{(p)}],$$

where $n^{(p)} \geq 1$ is the unique natural number with $p \nmid n^{(p)}$ and $n = n^{(p)}p^r$ and $m^{(p)}$ analoguesly defined. It follows that

$$H^1(H_K, \mathbb{G}(\overline{\mathbb{F}})) \overset{pr_*}{\to} H^1(H_K, \mathrm{PGL}_n(\overline{\mathbb{F}}))$$

has trivial fiber

$$\mathrm{pr}_*^{-1}(\{1\}) = \{1\}.$$

By the homomorphism theorem (See Mil17, 5.e) Remark 5.39), it is $\mathbb{G} \cong \mathrm{SL}_n / \mu_m$ for the quotient SL_n / μ_m, so by the universal property of the quotient, we have a map $\pi : \mathbb{G} \to \mathrm{PGL}_n$ as algebraic groups over k, which induces the projection $\mathrm{pr} : \mathbb{G}(\overline{\mathbb{F}}) \to \mathrm{PGL}_n(\overline{\mathbb{F}})$. It follows that $\bar{j}$ is trivial for $\mathbb{G}$, since $\bar{j}$ is trivial for PGL_n and pr_* has trivial fiber of 1 considering the following commutative diagram.

$$\begin{tikzcd}
\mathrm{mor}^{cont}(G_K, \mathbb{G}(k)) \arrow[r, "\pi_{k,*}"] \arrow[d, "\bar{j}"'] & \mathrm{mor}^{cont}(G_K, \mathrm{PGL}_n(k)) \arrow[d, "\bar{j}"] \\
H^1(H_K, \mathbb{G}(\overline{\mathbb{F}})) \arrow[r, "\mathrm{pr}_*"'] & H^1(H_K, \mathrm{PGL}_n(\overline{\mathbb{F}}))
\end{tikzcd}$$

In the table of fundamental groups, we see that the only groups with exponents unequal to two other than of type A_{n-1} are $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$. Now if $p = 2$, we have $3 \mid p^2 - 1$ and if $p = 3$, we have $4 \mid p^2 - 1$. Furthermore, if $p \geq 5$, we have $12 \mid p^2 - 1$, since $p^2 - 1 = (p+1)(p-1)$. This proves the addendum to this Theorem for $k|\mathbb{F}_{p^2}$ by using Proposition 2.3.7 with $(p^2 - 1) \mid (q - 1) \in [L_\infty : L]$, see Proposition 1.1.23.i). □

3 The Case of Characteristic 0 Coefficients

In this chapter, we will give the correspondences analogues to the ones in the last chapter, but for a linear algebraic group over $\mathcal{O}_L$ with (φ_L, Γ_K)-modules over the rings

$$(\mathcal{R}, \mathcal{R}^{nr}) \in \{(\mathbb{A}_K, \mathbb{A}), (W(\mathbb{F})_L, W(\overline{\mathbb{F}})_L)\}$$

instead of their residue fields

$$(\mathbb{K}, \mathbb{K}^{sep}) \in \{(\mathbb{E}, \mathbb{E}^{sep}), (\mathbb{F}, \overline{\mathbb{F}})\}.$$

Let

$$\mathbb{G}(\mathcal{O}_L) \cong \lim_{\leftarrow} \mathbb{G}(\mathcal{O}_L/\pi^n\mathcal{O}_L)$$

carry the profinite topology and let

$$\mathbb{G}(\mathcal{R}^{nr}) \cong \lim_{\leftarrow} \mathbb{G}(\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr})$$

carry the prodiscrete topology. By using the map

$$\bar{\jmath}_{\mathcal{R}} : \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L)) \to H^1(H_K, \mathbb{G}(\mathcal{R}^{nr})),$$

we can get a correspondence as in chapter 2.2.2.

In the first section, we will establish several techniques to lift some statements from the characteristic p case in chapter 2 to the characteristic 0.

In the second section, we will then prove the correspondences in the characteristic 0 case, which are analogues to the ones made in chapter 2.

3.1 General Theories

In this section, we will establish techniques for smooth group schemes of finite type over a discrete valuation ring and some technique regarding the injectivity of certain maps in the theory of general group cohomology.

3.1.1 Injectivity on H^1

In this part, we will show that for some monoid M and an M-group A with a filtration

$$A := A_0 \supset A_1 \supset A_2 \supset \dots$$

the projection $A \to A/A_1$ induces an injective map on the first cohomology, if the filtration satisfies certain conditions.

Definition 3.1.1. Let M be a monoid. Then an *M-group* is a group A together with a morphism of monoids

$$f : M \to \mathrm{End}_{Grp}(A),$$

where $\mathrm{End}_{Grp}(A)$ denotes the group endomorphisms of A. If M is a topological monoid and A is a topological group, then A is called a *topological M-group*, if the induced map

$$M \times A \to A, (m, a) \mapsto m * a := f(m)(a)$$

is continuous.

Definition 3.1.2. Let A be a M-group (resp. be a topological M-group). Then

$$C^1(M, A) := \{c : M \to A \mid c(m_1 m_2) = c(m_1) \cdot (m_1 * c(m_2)) \ \forall m_1, m_2\}$$

(resp.

$$C^1(M, A) := \{c : M \to A \,|\, c(m_1 m_2) = c(m_1) \cdot (m_1 * c(m_2)) \ \forall m_1, m_2, c \text{ is continuous.}\})$$

is called the set of 1-*cocycles of M in A* (resp. *continuous* 1-*cocycles of M in A*). We say $c_1, c_2 \in C^1(M, A)$ are *cohomological* $c_1 \sim c_2$, if there exists $a \in A$, such that

$$c_1(m) = a^{-1} \cdot c_2(m) \cdot (m * a) \ \forall m \in M.$$

This relation is an equivalence relation and we set

$$H^1(M, A) := C^1(M, A)/\sim .$$

We fix a monoid (respectively topological monoid) M and a M-group (respectively topological M-group) A. Beware that A is not necessarily commutative.

Let $c \in C^1(M, A)$. By the cocycle condition (and continuity). Then we can define the (topological) M-group

$$A_c := A, \ m \underset{c}{*} a := c(m)(m * a)c(m)^{-1} \ \forall m \in M, a \in A.$$

We calculate

$$\begin{aligned}(mm') \underset{c}{*} a &= c(m)(m * c(m'))(m * (m' * a))(m * c(m')^{-1})c(m)^{-1} \\ &= c(m)(m * (c(m')(m' * a)c(m')^{-1}))c(m)^{-1} \\ &= c(m)(m * (m' \underset{c}{*} a))c(m)^{-1} \\ &= m \underset{c}{*} (m' \underset{c}{*} a).\end{aligned}$$

So this is really an action. The action is continuous, if A is a topological M-group, since the action is the composition of the continuous maps

$$M \times A \xrightarrow{\phi} A \times A \times A \xrightarrow{\prod} A,$$

where

$$\phi((m,a)) = (c(m), m * a, c(m)^{-1}).$$

Remark. Let $b, c \in C^1(M, A)$. Then

$$bc^{-1} \in C^1(M, A_c).$$

Furthermore, it is

$$b \sim c \Leftrightarrow [bc^{-1}] = 1 \in H^1(M, A_c).$$

Proof. We calculate for $m, m' \in M$ that

$$\begin{aligned}
(bc^{-1})(mm') &= b(mm')c(mm')^{-1} \\
&= b(m)(m * b(m'))(m * c(m')^{-1})c(m)^{-1} \\
&= b(m)c(m)^{-1}c(m)(m * (b(m')c(m')^{-1}))c(m)^{-1} \\
&= bc^{-1}(m)(m \underset{c}{*} bc^{-1}(m')).
\end{aligned}$$

Furthermore, if

$$a^{-1}b(m)(m * a) = c(m) \ \forall m \in M,$$

we equivalently have

$$a^{-1}(bc^{-1})(m)(m \underset{c}{*} a) = 1 \ \forall m \in M.$$

□

Let $f : A \to B$ be a morphism of (topological) M-groups. Then for every $c \in C^1(M, A)$, it is

$$f_*(c) := f \circ c \in C^1(M, B)$$

and since

$$f(m \underset{c}{*} a) = f(c(m))(m * f(a))f(c(m))^{-1} = m \underset{f_*(c)}{*} f(a),$$

the same map

$$f : A_c \to B_{f_*(c)}$$

is a map of (topological) M-groups.

Lemma 3.1.3. *Let $f : A \to B$ be a morphism of M-groups. Then the induced map*

$$H^1(M, A) \to H^1(M, B)$$

is injective, if and only if the sequence

$$1 \to H^1(M, A_c) \to H^1(M, B_{f*(c)})$$

is exact for every $c \in C^1(M, A)$.

Proof. Let $b, c \in C^1(M, A)$. Then we have by the last Remark and since f is a morphism of groups that

$$f_*(b) \sim f_*(c) \Leftrightarrow [f_*(b) f_*(c)^{-1}] = 1 \Leftrightarrow [f_*(bc^{-1})] = 1$$

and again by the last Remark that

$$b \sim c \Leftrightarrow [bc^{-1}] = 1.$$

If follows that

$$(f_*(b) \sim f_*(c) \Rightarrow b \sim c) \Leftrightarrow ([f_*(bc^{-1})] = 1 \Rightarrow [bc^{-1}] = 1).$$

□

Now let A have a filtration of normal M-invariant subgroups

$$A := A_0 \supset A_1 \supset A_2 \supset \dots,$$

such that the canonical map $A \to \varprojlim A/A_n$ is an isomorphism. Since the A_n are normal, they are also invariant under the action twisted by a cocycle $c \in C^1(M, A)$, so it makes sense to define $A_{n,c}$. By (Bou66, III §2.4 Lemma 2), the projection $A \to A/A_n$ is an open map. It follows by an easy calculation (see for example (Kle16, Lemma 2.1.21)) that, if A is a topological M-group, then A/A_n is a topological M-group via the induced topology and action and the projection

$$p_n : A \to A/A_n$$

is a morphism of topological M-groups. If $c \in C^1(M, A)$, we define

$$\bar{c} := p_{n,*}(c) \in C^1(M, A/A_n).$$

Then we have

$$A_c/A_{n,c} = (A/A_n)_{\bar{c}}.$$

Proposition 3.1.4. *(Inspired by Ser67, 1.2 Lemma 3)*
Suppose that

$$H^1(M,(A_n/A_{n+1})_{\bar{c}}) = 1$$

for every $c \in C^1(M,A)$ and every $n \geq 1$. Then the map

$$H^1(M,A) \to H^1(M,A/A_1)$$

induced by the projection $\mathrm{pr} : A \to A/A_1$ *is injective.*

Proof. By Lemma 3.1.3, we have to show that the sequence

$$1 \to H^1(M,A_c) \to H^1(M,(A/A_1)_{\bar{c}}),$$

which is induced by the projection p is exact for every $c \in C^1(M,A)$. So let $d \in H^1(M,A_c)$, such that there exists $\bar{a}_1 \in A/A_1$ satisfying

$$p(d(m)) = \bar{a}_1(m \underset{\bar{c}}{*} \bar{a}_1) \ \forall m \in M.$$

Let $a_1 \in A$ be a lift of $\bar{a}$ and define the 1-coboundary

$$b_1(m) := a_1^{-1}(m \underset{c}{*} a_1) \ \forall m \in M.$$

Then we define

$$d_1 := db_1^{-1}$$

and we calculate

$$\begin{aligned} d(mm')b_1(mm')^{-1} &= d(m)c(m)(m * d(m'))c(m)^{-1}c(m)(m * b_1(m')^{-1})c(m)^{-1}b_1(m)^{-1} \\ &= db_1^{-1}(m)b_1(m)c(m)(m * (d(m')b_1(m')^{-1}))(b_1(m)c(m))^{-1} \\ &= db_1^{-1}(m)(m \underset{b_1c}{*} db_1^{-1}(m')), \end{aligned}$$

so

$$d_1 \in C^1(M, A_{1,b_1c}), \text{ since } p(d(m)) = p(b_1(m)) \ \forall m \in M.$$

This makes sense, since

$$\begin{aligned} b_1c(mm') &= b_1(mm')c(mm') \\ &= b_1(m)c(m)(m * b(m'))c(m)^{-1}c(m)(m * c(m')) \\ &= b_1c(m)(m * (b_1c)(m)), \end{aligned}$$

so $b_1c \in C^1(M,A)$. We have

$$d = d_1b_1$$

By the hypothesis, there exists $\bar{a}_2 \in A_1/A_2$

$$p_2(d_1(m)) = \bar{a}_2^{-1}(m \underset{b_1 c}{*} \bar{a}_2).$$

Choose a lift $a_2 \in A_1$ of $\bar{a}_2$ and define the 1-coboundary

$$b_2(m) := a_2^{-1} \underset{b_1 c}{*} a_2.$$

Then

$$\begin{aligned} b_2 b_1(m) &= a_2^{-1}(m \underset{b_1 c}{*} a_2) a_1^{-1}(m \underset{c}{*} a_1) \\ &= a_2^{-1} a_1^{-1}(m \underset{c}{*} a_1)(m \underset{c}{*} a_2)(m \underset{c}{*} a_1^{-1}) a_1 a_1^{-1}(m \underset{c}{*} a_1) \\ &= a_1 a_2^{-1}(m \underset{c}{*} (a_1 a_2)). \end{aligned}$$

Furthermore, as before there is

$$d_2 \in C^1(M, A_{2, b_2 b_1 c}), \text{ such that } d_1 = d_2 b_2.$$

Successively, we find for all $n \geq 1$ an $a_n \in A_n$, such that

$$b_n = a_n^{-1}(m \underset{b_{n-1} \cdots b_1 c}{*} a_n)$$

and

$$d_n \in C^1(M, A_{n, b_n \cdots b_1 c})$$

which satisfy

$$d = d_n b_n \cdots b_1$$

and

$$b_n \cdots b_1(m) = (a_1 \cdots a_n)^{-1}(m \underset{c}{*} (a_1 \cdots a_n))).$$

Since

$$A \cong \lim_{\leftarrow} A/A_n,$$

there exists $a \in A$, such that

$$p_n(a) = a_1 \cdots a_{n-1}.$$

Furthermore, since

$$d_n(m) \in A_n, \text{ it is } d = \lim_n (p_n \circ b_n \cdots b_1),$$

so

$$d(m) = a^{-1}(m \underset{c}{*} a) \; \forall m \in M.$$

Thus, d is a 1-coboundary. $\square$

In this thesis, we will work with $M = H_K$ as a topological monoid and $M = \mathbb{N}$ as an additive monoid, in particular with $0 \in \mathbb{N}$. We now explain, what the latter case is about.

Let G be any group together with an endomorphism of groups

$$\psi : G \to G.$$

Then we can make G into an $\mathbb{N}$-group by setting

$$f_\psi : \mathbb{N} \to \mathrm{End}_{Grp}(G), n \mapsto \psi^n.$$

We define the *Langmap associated to* ψ to be the map

$$\Psi : G \to G, g \mapsto g^{-1}\psi(g).$$

Lemma 3.1.5. *The following maps are inverse bijections.*

$$\begin{aligned} C^1(\mathbb{N}, G) &\to G \\ c &\mapsto c(1) \\ [n \mapsto \begin{cases} 1, \ n = 0 \\ g \cdot \psi(g) \cdots \psi^{n-1}(g), \ n \geq 1 \end{cases}] =: c_g &\leftarrow\!\shortmid g \end{aligned}$$

Furthermore, these maps induce bijections

$$B^1(\mathbb{N}, G) := \{c \in C^1(\mathbb{N}, G) \mid c \sim 1\} \leftrightarrow \mathrm{im}(\Psi).$$

Proof. We calculate that c_g is a cocycle for every $g \in G$. We have

$$c_g(n + 0) = c_g(n) = c_g(n) \cdot 1 = c_g(n) \cdot \psi^n(c_g(0))$$

for every $n \in \mathbb{N}$. Furthermore, we have

$$c_g(0 + m) = c_g(m) = 1 \cdot c_g(m) = c_g(0)\psi^0(c_g(m))$$

for every $m \in \mathbb{N}$.

Now let $n, m \in \mathbb{N}$ with $n \neq 0 \neq m$, then

$$c_g(n + m) = g \cdot \psi(g) \cdots \psi^{n-1}(g)\psi^n(g \cdot \psi(g) \cdots \psi^{m-1}(g)) = c_g(n)(n * c_g(m)).$$

By definition, we have

$$c_g(1) = g$$

and if $c \in C^1(\mathbb{N}, G)$, then we successively calculate for every $n \geq 1$ that

$$c(n) = c(n - 1 + 1) = c(n - 1)\psi^{n-1}(c(1)) = \cdots = c(1)\psi(c(1)) \cdots \psi^{n-1}(c(1)).$$

If $b \in B^1(\mathbb{N}, G)$, then it is

$$b(1) = g^{-1} \cdot \psi(g) = \Psi(g).$$

If on the other hand

$$g_0 := g^{-1}\psi(g) \in \mathrm{im}(\Psi),$$

then for all $n \geq 1$, we calculate

$$c_{g_0}(n) = (g^{-1}\psi(g))(\psi(g)^{-1}\psi^2(g))\psi^2(g)^{-1} \cdots \psi^{n-1}(g)(\psi^{n-1}(g)^{-1}\psi^n(g)) = g^{-1}\psi^n(g),$$

so

$$c_{g_0} \in B^1(\mathbb{N}, G).$$

□

Corollary 3.1.6. *It is Ψ surjective, if and only if $H^1(\mathbb{N}, G) = 1$.*

Proof. It is $H^1(\mathbb{N}, G) = 1$, if and only if

$$B^1(\mathbb{N}, G) = C^1(\mathbb{N}, G),$$

so by the above Lemma, if and only if

$$\mathrm{im}(\Psi) = G.$$

□

3.1.2 (Formal) Groups over Discrete Valuation Rings

In this part, we will accumulate some facts about linear algebraic groups over a complete discrete valuation ring, in particular over smooth ones. Later on, we will recall the theory of formal schemes over such a ring.

Definition 3.1.7. Let R be an arbitrary ring. Then a *linear algebraic group over R* is a group scheme $\mathbb{G}$ over R, such that there exists a closed immersion

$$\mathbb{G} \subset \mathrm{GL}_n$$

as groups over R.

Proposition 3.1.8. *(See Con17, Remark 1.1.6)*
If R is a Dedekind Domain, then a group scheme over R is a linear algebraic group, if and only if it is an affine group scheme of finite type over R.

For the rest of this part, R is a complete discrete valuation ring with uniformizer ϖ and residue field E. We fix a linear algebraic group $\mathbb{G}$ over R and an embedding $\mathbb{G} \subset \mathrm{GL}_n$ over R.

Lemma 3.1.9. *It is*

$$\mathbb{G}(R) \cong \lim_{\leftarrow} \mathbb{G}(R/\varpi^n R).$$

The same is true for any affine scheme X over R instead of $\mathbb{G}$.

Proof. Since $\mathbb{G} = \mathrm{Spec}(A)$ is affine as a closed subgroup of the affine group GL_n, it is

$$\mathbb{G}(R) = \mathrm{mor}_R(A, R) \cong \lim_{\leftarrow} \mathrm{mor}_R(A, R/\varpi^n R) = \lim_{\leftarrow} G(R/\varpi^n R)$$

by the universal property of the projective limit and because of

$$R \cong \lim_{\leftarrow} R/\varpi^n R.$$

□

We take the following statement as a motivation to make the constructions that follow.

Proposition 3.1.10. *(See Ser92, Part II Chapter IV.9 Corollary 1)*
Let $\mathbb{G}$ be smooth over R of relative dimension d. Then for every $n \geq 1$

$$\ker(\mathbb{G}(\mathrm{pr}) : \mathbb{G}(R/\varpi^{n+1}) \to \mathbb{G}(R/\varpi^n)) \cong E^d$$

carries the structure of a d-dimensional E-vector space.

Now let for any $m \geq 1$

$$\mathrm{pr}_{m,m+1} : R/\varpi^{m+1} \to R/\varpi^m.$$

Consider $\mathbb{G} = \mathrm{GL}_n$. Then for any $m \geq 1$ the morphism

$$\mathrm{GL}_n(\mathrm{pr}_{m,m+1}) : \mathrm{GL}_n(R/(\varpi^{m+1})) \to \mathrm{GL}_n(R/(\varpi^m))$$

has kernel

$$\ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1})) \subset 1 + Mat_{n\times n}(\varpi^m R/(\varpi^{m+1})).$$

Lemma 3.1.11. *We have a well defined isomorphism of groups*

$$\Phi_m : Mat_{n\times n}(E) \cong Mat_{n\times n}((R/(\varpi^{m+1}))/(\varpi)) \to \ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1})), A \mapsto 1+\varpi^m A.$$

Proof. Let $A, B \in Mat_{n\times n}(R/(\varpi^{m+1}))$. Then we calculate that

$$\Phi_m(A+B) = 1+\varpi^m(A+B) = 1+\varpi^m A+\varpi^m B+\varpi^{2m}AB = (1+\varpi^m A)(1+\varpi^m B).$$

This also shows that $1 + Mat_{n\times n}(\varpi^m R/(\varpi^{m+1})) \subset \mathrm{GL}_n(R/(\varpi^{m+1}))$, since for any $1 + \varpi^m A \in 1 + Mat_{n\times n}(\varpi^m R/(\varpi^{m+1}))$, the element $1 - \varpi^m A$ is a multiplicative inverse. It follows that

$$\ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1})) = 1 + Mat_{n\times n}(\varpi^m R/(\varpi^{m+1})).$$

This morphism of groups induces a well defined morphism of groups

$$Mat_{n\times n}(E) \cong Mat_{n\times n}((R/(\varpi^{m+1}))/(\varpi)) \to \ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1})),$$

since for $\varpi A \in \varpi R/(\varpi^{m+1})$, we have

$$\Phi_m(\varpi A) = 1 + \varpi^{m+1}A = 1 \in \mathrm{GL}_n(R/(\varpi^{m+1})).$$

Lastly, it is easy to see that $A \mapsto 1$, if and only if $A \in Mat_{n\times n}(\varpi R/(\varpi^{m+1}))$, so Φ_m is injective. It is surjective by construction and since

$$\ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1})) = 1 + Mat_{n\times n}(\varpi^m R/(\varpi^{m+1})).$$

$\square$

Now consider $\mathbb{G} \subset \mathrm{GL}_n$, so that $\mathbb{G} = \mathrm{Spec}(R[\{X_{ij}\}_{ij}][\frac{1}{det(X_{ij})}]/(P_1, \ldots, P_n))$ for some $P_k \in R[\{X_{ij}\}_{ij}]$. Since $\mathbb{G}$ is a group, we have

$$P_k(1) = 0$$

for $1 \in \mathrm{GL}_n(S)$, where S is any R-algebra.

We have

$$\ker(\mathbb{G}(\mathrm{pr}_{m,m+1})) \subset \ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1})),$$

since $\mathbb{G} \subset \mathrm{GL}_n$ is a natural transformation. We set

$$X^{(m)} := \Phi_m^{-1}(\ker(\mathbb{G}(\mathrm{pr}_{m,m+1}))).$$

Lemma 3.1.12. *It is*

$$X^{(m)} = \mathcal{T}_{(P_k)_k}(1) := \{\overline{A} \in Mat_{n\times n}(E) \mid \sum_{ij} \frac{\partial P_k}{X_{ij}}(1) \cdot \overline{A}_{ij} = 0 \ \forall k\}$$

for all $m \geq 1$. In particular

$$\ker(\mathbb{G}(\mathrm{pr}_{m,m+1}))$$

carries the structure of a finite dimensional E-vector space via Φ_m.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{G}(R/(\varpi^{m+1})) \arrow[r, "\mathbb{G}(\mathrm{pr}_{m,m+1})"] \arrow[d, "\subset"'] & \mathbb{G}(R/(\varpi^m)) \arrow[d, "\subset"] \\
\mathrm{GL}_n(R/(\varpi^{m+1})) \arrow[r, "\mathrm{GL}_n(\mathrm{pr}_{m,m+1})"'] & \mathrm{GL}_n(R/(\varpi^m)).
\end{tikzcd}$$

It follows that

$$\ker(\mathbb{G}(\mathrm{pr}_{m,m+1})) = \ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1})) \cap \mathbb{G}(R/(\varpi^{m+1}))$$

and so it is $\overline{A} \in X^{(m)}$ if and only if $P_k(1+\varpi^m A) = 0 \in R/(\varpi^{m+1})$ for a lift $A \in Mat_{n\times n}(R/(\varpi^{m+1}))$ of $\overline{A}$ and every k. We consider the Taylorexpansion of P_k at $1 \in Mat_{n\times n}(R/(\varpi^{m+1}))$.

$$P_k(1+\varpi^m A) = P_k(1)+\varpi^m \cdot \sum_{i,j} \frac{\partial P_k}{X_{ij}}(1)\cdot A_{ij} + \varpi^{2m} z \text{ for some } z \in R/(\varpi^{m+1}).$$

Since $\varpi^{2m} = 0$ and $P_k(1) = 0$, it is $P_k(1+\varpi^m A) = 0$ if and only if

$$\sum_{i,j} \frac{\partial P_k}{X_{ij}}(1)\cdot A_{ij} \in \varpi R/(\varpi^{m+1}).$$

□

We close this part with a geometric version of Hensel's Lemma.

Proposition 3.1.13. *(Hensel's Lemma)(See Gro67, Theorem 18.5.17 & Proposition 18.5.4) and (See Gro60, Corollary 5.1.8)*

If $(A, \mathfrak{m})$ is a local Henselian ring and X is a smooth scheme over A, then for every $n \geq 1$ the map

$$X(\mathrm{pr}_n) : X(A) \to X(A/\mathfrak{m}^n)$$

induced by the projection $A \to A/\mathfrak{m}^n$ is surjective.

Formal ϖ-adic Schemes

We continue with our notation of this part, so R is a complete discrete valuation ring with uniformizer ϖ.

Definition 3.1.14. Let B be an R-algebra. We view B with the ϖ-adic topology and assume that B is ϖ-adically complete. Then we define the *formal (ϖ-adic) spectrum* of B to be

$$\mathrm{Spf}(B) := \{\mathfrak{p} \in \mathrm{Spec}(B) \mid \mathfrak{p} \subset B \text{ is open.}\}.$$

We view $\mathrm{Spf}(B) \subset \mathrm{Spec}(B)$ with the subset topology of the Zariski topology, i.e. the subsets

$$D_f(B) := \{\mathfrak{p} \in \mathrm{Spf}(B) \mid f \notin \mathfrak{p}\}$$

for every $f \in B$ form a basis of the topology. We define a structure of a locally ringed space (of R-algebras) on $\mathrm{Spf}(B)$ via

$$\mathcal{O}_{\mathrm{Spf}(B)}(D_f) := \hat{B}_f,$$

where $\hat{B}_f$ denotes the ϖ-adic completion of the localisation of B at f denoted by B_f. This really gives the structure of a locally ringed space by (Gro60, Propositions (10.1.3), (10.1.4) & (10.1.6))

We say a locally ringed space of R-algebras $(X, \mathcal{O}_X)$ is a *(complete) formal (ϖ-adic) scheme over R*, if it has an open covering of subspaces, which are isomorphic formal (ϖ-adic) spectra $(\mathrm{Spf}(B), \mathcal{O}_{\mathrm{Spf}(B)})$ as above.

A morphism between formal (ϖ-adic) schemes is a morphism of locally ringed spaces of R-algebras.

We call a formal (ϖ-adic) scheme $(X, \mathcal{O}_X)$ *affine*, if $(X, \mathcal{O}_X) \cong (\mathrm{Spf}(B), \mathcal{O}_{\mathrm{Spf}(B)})$.

Remark. Normally, you would define a morphism of formal schemes to be a morphism of locally ringed spaces of R-algebras $f : X \to Y$, such that the induced map $f_U^\sharp : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is continuous for the ϖ-adic topology for every open $U \subset Y$, such that U and $f^{-1}(U)$ are affine. But for the ϖ-adic topology every morphism of R-algebras is automatically continuous.

Since we are only interested in the ϖ-adic case, we will by abuse of notation call formal ϖ-adic schemes over R just *formal schemes.*

Definition 3.1.15. We define

$$FSch^{comp} := FSch^{comp}_{\varpi,R} \subset (\text{locally ringed spaces of } R-\text{algebras})$$

to be the full subcategory of formal schemes.

We have the following adjointness property.

Proposition 3.1.16. *(See Gro60, Proposition 10.4.6)*

Let X be a complete formal scheme and B be a ϖ-adically complete R-algebra. Then we have a bijection

$$\mathrm{mor}_{FSch^{comp}}(X, \mathrm{Spf}(B)) \tilde{\to} \mathrm{mor}_{R-Alg}(B, \mathcal{O}_X(X)), f \mapsto f_X^\sharp,$$

which is natural in X and B.

Lemma 3.1.17. *If H is a Hopfalgebra over R, then the ϖ-adic completion $\hat{H}$ is a* complete formal Hopfalgebra *over R, i.e.* $\mathrm{Spf}(\hat{H})$ *is a group object in the category $FSch^{comp}$ and for any ϖ-adically complete R-algebra S, we have an isomorphism of groups*

$$\mathrm{mor}_{R-Alg}(\hat{H}, S) \cong \mathrm{mor}_{R-Alg}(H, S),$$

which is natural in S.

Proof. By Proposition 3.1.16 and the gluing property of morphisms of locally ringed spaces (See Gö10, Proposition 3.5), it suffices to show that

$$\mathrm{mor}_{R-Alg}(\hat{H}, S)$$

carries a group structure, which is natural in S, where S is a ϖ-adically complete R-algebra. Since any morphism of R-algebras is ϖ-adically continuous, we have a natural identification

$$\mathrm{mor}_{R-Alg}(\hat{H}, S) \cong \mathrm{mor}_{R-Alg}(H, S)$$

via the natural map $can : H \to \varprojlim H/\varpi^n H = \hat{H}$. To prove this, let $f : H \to S$ be a morphism of R-Algebras. Then for every $n \in \mathbb{N}$ there exists a unique morphism of R-algebras $f_n : H/\varpi^n H \to S/\varpi^n S$ with

$$f_n \circ \mathrm{pr}_{\varpi^n H} = \mathrm{pr}_{\varpi^n S} \circ f.$$

Since S is ϖ-adically complete, there exists a unique

$$\hat{f} := \varprojlim f_n : \hat{H} \to \hat{S} \cong S,$$

such that $\hat{f} \circ can = f$.

It follows that $\mathrm{mor}_{R-Alg}(\hat{H}, S)$ carries the group structure of $\mathrm{mor}_{R-Alg}(H, S)$, which is natural in S. □

If $\mathbb{G} = \mathrm{Spec}(H)$ is a groupscheme over R, we denote by

$$\hat{\mathbb{G}} := \mathrm{Spf}(\hat{H})$$

the object in $FSch^{comp}$, which is a group object by the last Lemma. We call such an object $\mathrm{Spf}(A)$, where A is a complete formal Hopfalgebra an *affine complete formal group over R.*

Lemma 3.1.18. *Let $(A_n, \psi_n : A_{n+1} \to A_n)_{n\geq 1}$ be a projective system, where A_n is a Hopfalgebra over $R/\varpi^n R$ and the ψ_n are morphisms of R-algebras, such that the induced maps*

$$\mathrm{mor}_{R/\varpi^n R-Alg}(A_n, S) \to \mathrm{mor}_{R/\varpi^{n+1} R-Alg}(A_{n+1}, S)$$

are morphisms of groups for every $R/\varpi^n R$-algebra S. If every ψ_n is surjective and satisfies

$$\psi_n^{-1}(\varpi^m A_n) = \varpi^m A_{n+1}$$

for every $1 \leq m \leq n$, then

$$A := \lim_{\leftarrow} A_n$$

is a complete formal Hopfalgebra over R and the projection $\mathrm{pr}_n : A \to A_n$ induces an isomorphism

$$\mathrm{pr}_n : A/\varpi^n A \tilde{\to} A_n$$

for every $n \geq 1$.

Furthermore, if S is a ϖ-adically complete R-algebra, then there exists an isomorphism of groups

$$\mathrm{mor}_{R-Alg}(A, S) \cong \lim_{\leftarrow} \mathrm{mor}_{R/\varpi^n R-Alg}(A_n, S/\varpi^n S),$$

which is natural in S. This projective limit is given by the projections $S/\varpi^{n+1} S \to S/\varpi^n S$. The details that this makes sense are given in the proof.

Proof. Since ψ_n is surjective for every $n \geq 1$, the projection $\mathrm{pr}_m : A \to A_m$ is surjective for any $m \geq 1$. An element $(a_n)_n \in A$ is in the kernel of pr_m, if $a_n = 0$ for all $n \leq m$. Since $\psi_n^{-1}(\varpi^m A_n) = \varpi^m A_{n+1}$ for every $n \geq m$, we have that $a_n \in \varpi^m A_n$ for every $n \geq m$, so $\ker(\mathrm{pr}_m) = \varpi^m A$. It follows that we have an isomorphism

$$\mathrm{pr}_n : A/\varpi^n A \to A_n \tag{$*$}$$

for every $n \geq 1$, so A is ϖ-adically complete. Let S be a ϖ-adically complete R-algebra. Then we have natural bijections

$$\begin{aligned}
\mathrm{mor}_{R-Alg}(A, S) &\cong \lim_{\leftarrow} \mathrm{mor}_{R-Alg}(A, S/\varpi^n S) \\
&\cong \lim_{\leftarrow} \mathrm{mor}_{R/\varpi^n-Alg}(A/\varpi^n A, S/\varpi^n S) \\
&\overset{(*)}{\cong} \lim_{\leftarrow} \mathrm{mor}_{R/\varpi^n-Alg}(A_n, S/\varpi^n S),
\end{aligned}$$

where the first bijection follows from the universal property of the projective limit and the second bijection follows from the fact that every morphism of

R-algebras $f : B \to C$ with $\varpi^n C = 0$ induces a unique $R/\varpi^n R$-algebra morphism $f_n : B/\varpi^n B \to C$ with $f_n \circ \mathrm{pr}_{\varpi^n B} = f$.

The map

$$\mathrm{mor}_{R/\varpi^{n+1}-Alg}(A_{n+1}, S/\varpi^{n+1}S) \to \mathrm{mor}_{R/\varpi^{n+1}-Alg}(A_{n+1}, S/\varpi^n S)$$

induced by the projection is a morphism of groups, since A_{n+1} is a Hopfalgebra over $R/\varpi^{n+1}R$. Notice that $\psi_n : A_{n+1}/\varpi^n A_{n+1} \to A_n$ is an isomorphism by assumption. It follows that $\mathrm{mor}_{R-Alg}(A, S)$ carries the structure of a group, which is natural in S, since

$$\begin{aligned} &\mathrm{mor}_{R/\varpi^{n+1}R-Alg}(A_{n+1}, S/\varpi^n S) \\ \cong\, &\mathrm{mor}_{R/\varpi^n R-Alg}(A_{n+1}/\varpi^n A_{n+1}, S/\varpi^n S) \xrightarrow{\cdot\circ\psi_n^{-1}} \mathrm{mor}_{R/\varpi^n R-Alg}(A_n, S/\varpi^n S) \end{aligned}$$

is an isomorphism of groups by assumption, from which follows that

$$\lim_{\leftarrow} \mathrm{mor}_{R/\varpi^n-Alg}(A_n, S/\varpi^n S)$$

carries a structure of a group, which is natural in S. We deduce that $\mathrm{Spf}(A)$ is a group object in $FSch^{comp}$ by Proposition 3.1.16 and the gluing property of morphisms of locally ringed spaces (See Gö10, Proposition 3.5). $\square$

Although this next statements don't involve formal schemes, we will use this later on in the context of working with affine complete formal groups.

Proposition 3.1.19. *(Inspired by Bri09, Lemma 3.2.6)*

Let $R' := \widehat{R^{nr}}$ be the ϖ-adic completion of the maximal unramified extension of R. Then a separable closure E^{sep} of E in an algebraically closed field containing E is the residue field of R' and $Aut_{R-Alg}(R') = G_E$ by Proposition 1.2.29. Let M be an R'-module, such that there exists an $n \in \mathbb{N}$, such that $\varpi^n M = 0$. If there is a ϖ-adically continuous and semilinear G_E-action of R-algebras on M, then the natural map induced by scalar multiplication

$$R' \underset{R}{\otimes} M^{G_E} \to M$$

is bijective.

Proof. We assumed that there exists an $n \in \mathbb{N}$, such that $\varpi^n M = 0$. We prove this statement by induction for such n. For $n = 1$, this is classical Galois descent (Compare to Sil09, II Lemma 5.8.1). Now let $n \geq 1$ be such that the statement correct for all $1 \leq m \leq n$ and let M be a R'-module, such that $\pi^{n+1}M = 0$. We consider the short exact sequence

$$0 \to \pi^n M \to M \to M/\pi^n M \to 0.$$

Since $M_n := \pi^n M$ satisfies $\pi M_n = 0$, we know from the case $n = 1$ that there is a G_E-equivariant isomorphism of E^{sep}-vector spaces $M_n \cong \bigoplus_{i\in I} E^{sep}$, where $\bigoplus_{i\in I} E^{sep}$ carries the natural G_E-action. It follows that

$$H^1(G_E, M_n) \cong \varinjlim_{N\subset G \text{ open}} H^1(G_E/N, M_n^N) \cong \varinjlim_{N\subset G \text{ open}} \bigoplus_{i\in I} H^1(G_E/N, (E^{sep})^N) = 0,$$

where the first isomorphism is (Neu15, (1.2.5) Proposition) and the second one is (Neu13, (3.7) Proposition). The equality $H^1(G_E/N, (E^{sep})^N) = 0$ is the additive Hilbert 90. It follows that we have the following commutative diagram of exact sequences of the canonical maps since R' is flat over R by (Bou72, I §2.4 Proposition 3.ii)).

$$\begin{tikzcd}
0 \arrow[r] & R' \otimes_R M_n \arrow[r] \arrow[d] & R' \otimes_R M \arrow[r] \arrow[d] & R' \otimes_R M/M_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_n \arrow[r] & M \arrow[r] & M/M_n \arrow[r] & 0
\end{tikzcd}$$

By the inductive hypothesis, we have that the arrows left and right are isomorphisms, so it follows that the one in the center is an isomorphism by the five-Lemma. $\square$

Corollary 3.1.20. *Let $R' = \widehat{R^{nr}}$ be as in the last Proposition. Let $R' - Mod_{G_E}^{ann}$ be the category of those R'-modules M with a semilinear G_E-action on M and such that there exists $n \geq 0$, such that $\pi^n M = 0$. Morphisms in $R' - Mod_{G_E}^{ann}$ are those R'-linear morphisms, which are compatible with the G_E-action. Then the functor*

$$(\cdot)^{G_E} : R' - Mod_{G_E}^{ann} \to R - Mod$$

is exact.

Proof. This follows since the isomorphism in Proposition 3.1.19 is natural and since R' is faithfully flat over R by (Bou72, I §2.4 Proposition 3.ii)) and (Mat86, Theorem 7.2). $\square$

3.1.3 Lifting Lang-Steinberg

In this part, we will have a first use of the results established in the last two parts. We will show that the surjectivity of the Lang map established in part 2.1.2 can be lifted to a surjectivity of the Lang map on the points of a linear algebraic group over a complete discrete valuation ring with separable algebraically closed residue field. Let R be a complete discrete valuation ring, which is a $\mathcal{O}_L$-algebra and has uniformizer π. Let E be the residue field of R. Assume that there exists a lift

$$(\varphi_L : R \to R) \in \mathrm{End}_{\mathcal{O}_L - Alg}(R)$$

of the q-Frobenius

$$\varphi_L : E \to E, x \mapsto x^q$$

and that E is separably algebraically closed. Let $\mathbb{G}$ be a linear algebraic group over $\mathcal{O}_L$. This next statement serves as motivation for the rest of the part.

Proposition 3.1.21. *(See Gre63, 3. Proposition 3)*
If $\mathbb{G}$ is smooth and has connected special fiber $\mathbb{G}_k$, then the Langmap

$$\Psi_{L^{nr}} : \mathbb{G}(\mathcal{O}_{\widehat{L^{nr}}}) \to \mathbb{G}(\mathcal{O}_{\widehat{L^{nr}}}), A \mapsto A^{-1}\mathbb{G}(\varphi_L)(A)$$

is surjective.

In (Gre61) and (Gre63), Greenberg works with Wittvectors over perfect fields, so following his method, we would only be able to generalize this statement in the perfect setup for a complete discrete valuationg ring with algebraically closed residue field. But since we also want to have an analogue result in the nonperfect case, we will need to develop a new technique. We will need a further application of Theorem 2.1.13.

Lemma 3.1.22. *Recall that we assumed that E is separably algebraically closed. Let V be finite dimensional E-vector space, together with a φ_L-semilinear and etale endomorphism*

$$\varphi_V : V \to V.$$

Then the map

$$\varphi - \mathrm{id} : V \to V, v \mapsto \varphi_L(v) - v$$

is surjective.

Proof. Let $(v_i)_i \subset V$ be a φ_V-invariant E-basis of V, see Theorem 2.1.13 for the existence. Then for any $v \in V$ with

$$v = \sum_i a_i v_i \quad \text{with } a_i \in E.$$

We calculate

$$\varphi_L(v) - v = \sum_i (a_i^q - a_i) v_i.$$

Let $w \in V$ be another element and

$$w = \sum_i b_i v_i \quad \text{with } b_i \in E.$$

So to say that $w \in \operatorname{im}(\varphi_L - \mathrm{id})$, we have to find solutions for $a_i^q - a_i = b_i$ in E for every i. But $X^q - X - b_i \in E[X]$ is a separable polynomial, so there exists such a_i. $\square$

We view $G := \mathbb{G}(R)$ as a $\mathbb{N}$-group via $\mathbb{G}(\varphi_L)$. Furthermore, we denote the induced maps

$$\varphi_L : R/\pi^n R \to R/\pi^n R$$

and

$$\mathbb{G}(\varphi_L) : \mathbb{G}(R/\pi^n R) \to \mathbb{G}(R/\pi^n R).$$

Let

$$\mathrm{pr}_n : R \to R/\pi^n R$$

and

$$\mathrm{pr}_{n,n+1} : R/\pi^{n+1} R \to R/\pi^n R$$

be the projections. Since $\varphi_L \circ \mathrm{pr}_n = \mathrm{pr}_n \circ \varphi_L$, it is

$$G_n := \ker(\mathbb{G}(\mathrm{pr}_n)) \subset G$$

a $\mathbb{N}$-invariant subgroup. Assume $\mathbb{G}$ is smooth over $\mathcal{O}_L$. Then by Hensel's Lemma Proposition 3.1.13, we have that it is

$$G/G_n \cong \mathbb{G}(R/\pi^n R).$$

Furthermore, it is

$$G_n/G_{n+1} \cong \ker(\mathbb{G}(\mathrm{pr}_{n,n+1}))$$

and by Lemma 3.1.9, it is

$$G \cong \varprojlim G/G_n.$$

First, we consider $\mathbb{G} = \mathrm{GL}_n$. We view $Mat_{n\times n}(E)$ as $\mathbb{N}$-Group via $Mat_{n\times n}(\varphi_L)$.

Lemma 3.1.23. *For any $m \geq 1$, the isomorphism*

$$\Phi_m : Mat_{n\times n}(E) \cong Mat_{n\times n}((R/(\pi^{m+1}))/(\pi)) \to \ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1})), A \mapsto 1 + \pi^m A$$

of Lemma 3.1.11 is $\mathbb{N}$-equivariant.

Proof. This follows from the assumption that φ_L is an $\mathcal{O}_L$-algebra homomorphism. $\square$

Remark 3.1.24. Let $c \in C^1(\mathbb{N}, \mathrm{GL}_n(R))$. Then

$$c_1 := \mathrm{GL}_n(\mathrm{pr}_1) \circ c \in C^1(\mathbb{N}, \mathrm{GL}_n(E)),$$

and for $m \geq 1$

$$c_{m+1} := \mathrm{GL}_n(\mathrm{pr}_{m+1}) \circ c \in C^1(\mathbb{N}, \mathrm{GL}_n(R/\pi^{m+1}R))$$

since $\varphi_L \circ \mathrm{pr}_r = \mathrm{pr}_r \circ \varphi_L$ for all $r \in \mathbb{N}$. Let $Mat_{n\times n}(E)_{c_1}$ denote the $\mathbb{N}$-group

$$Mat_{n\times n}(E)_{c_1} := Mat_{n\times n}(E),\ n \underset{c_1}{*} A := c_1(n) \cdot Mat_{n\times n}(\varphi_L^n)(A) \cdot c_1(n)^{-1}.$$

Let $\ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1}))_{c_{m+1}}$ be as in part 3.1.1, which is the analogue definition to the one made for $Mat_{n\times n}(E)_{c_1}$. Then

$$\Phi_m : Mat_{n\times n}(E)_{c_1} \to \ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1}))_{c_{m+1}}$$

is $\mathbb{N}$-equivariant, by Lemma 3.1.23 and since conjugation in $Mat_{n\times n}(E)$ with an element of $\mathrm{GL}_n(E)$ is additive and preserves elements in the center.

Remark 3.1.25. By Corollary 2.1.18 and Corollary 3.1.6, it is that

$$H^1(\mathbb{N}, \mathrm{GL}_n(E)) = 1.$$

So for any $\bar{c} \in C^1(\mathbb{N}, \mathrm{GL}_n(E))$, there exists a Matrix $A \in \mathrm{GL}_n(E)$ satisfying

$$\bar{c}(n) = A^{-1}\, \mathrm{GL}_n(\varphi_L^n)(A)\ \forall n \in \mathbb{N}.$$

It follows that

$$Mat_{n\times n}(E)_{\bar{c}} \to Mat_{n\times n}(E), B \mapsto ABA^{-1}$$

is an isomorphism of $\mathbb{N}$-groups since

$$A\bar{c}(n) Mat_{n\times n}(\varphi_L^n)(B)\bar{c}(n)^{-1}A^{-1} = Mat_{n\times n}(\varphi_L^n)(ABA^{-1}),$$

which holds because of

$$\mathrm{GL}_n(\varphi_L) = Mat_{n\times n}(\varphi_L)_{|\,\mathrm{GL}_n(E)}.$$

Proposition 3.1.26. *If $\mathbb{G}$ is smooth over $\mathcal{O}_L$ and the special fiber $\mathbb{G}_k$ is connected, then the Lang map*

$$\Psi_R : \mathbb{G}(R) \to \mathbb{G}(R), A \mapsto A^{-1}\mathbb{G}(\varphi_L)(A)$$

is surjective.

Proof. We continue with the notation in the discussion before Lemma 3.1.23. In particular, it is

$$G \cong \lim_{\leftarrow} G/G_n,$$

since $\mathbb{G}$ is smooth. By Corollary 3.1.6, we need to show that

$$H^1(\mathbb{N}, G) = 1.$$

But since the special fiber is connected, we have to show by Corollary 2.1.18 and Proposition 3.1.4 that for

$$(G_m/G_{m+1})_{c_{m+1}} \cong \ker(\mathbb{G}(\mathrm{pr}_{m,m+1}))_{c_{m+1}}$$

it is

$$H^1(\mathbb{N}, (G_m/G_{m+1})_{c_{m+1}}) = 1$$

for all $c \in C^1(\mathbb{N}, G),\ m \geq 1$ and

$$c_{m+1} := \mathbb{G}(\mathrm{pr}_{m+1}) \circ c \in C^1(\mathbb{N}, G/G_{m+1}).$$

Note that

$$Mat_{n\times n}(\varphi_L) : Mat_{n\times n}(E) \to Mat_{n\times n}(E)$$

is etale, since it fixes the standard E-basis. By Remark 3.1.24 the map

$$(G_m/G_{m+1})_{c_{m+1}} \subset \ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1}))_{\subset\circ c_{m+1}} \cong Mat_{n\times n}(E)_{\subset\circ c_1}$$

is an $\mathbb{N}$-equivariant embedding. Here $\subset\colon \mathbb{G} \to \mathrm{GL}_n$ means the chosen immersion. By Remark 3.1.25, the right hand side of this embedding is isomorphic to $Mat_{n\times n}(E)$ as an $\mathbb{N}$-group, so by Lemma 3.1.12 this $\mathbb{N}$-equivariant embedding is onto an E-subvector space of $Mat_{n\times n}(E)$ with the $\mathbb{N}$-action given by $Mat_{n\times n}(\varphi_L)$. So it suffices to show that every $\mathbb{N}$-invariant E-subvector space $V \subset Mat_{n\times n}(E)$ is etale, since then $H^1(\mathbb{N}, V) = 1$ by Lemma 3.1.22 and Corollary 3.1.6. Since $\varphi_L : E \to E$ is flat as an extension of fields, it is

$$E \underset{\varphi_L, E}{\otimes} V \overset{\mathrm{id}\otimes\subset}{\to} E \underset{\varphi_L, E}{\otimes} Mat_{n\times n}(E)$$

an injective map. Consider the commutative diagram

$$\begin{tikzcd}
E \underset{\varphi_L,E}{\otimes} Mat_{n\times n}(E) \arrow[r, "Mat_{n\times n}(\varphi_L)^{lin}"] & Mat_{n\times n}(E) \\
E \underset{\varphi_L,E}{\otimes} V \arrow[u, "(\mathrm{id}\otimes\subset)"] \arrow[r, "Mat_{n\times n}(\varphi_L)^{lin}"] & V. \arrow[u, "\subset"']
\end{tikzcd}$$

Since the upper arrow is an isomorphism and the vertical arrows are injective, the botton arrow is injective. But the E-vector spaces on the bottom have the same dimension, so the arrow is an isomorphism. $\square$

3.2 The Correspondences

In this section, we will construct the bijections to generalize Theorem 1.3.18 in the characteristic 0 case. In the first part, we will show that H_K-invariants as established in Part 2.2.2 can be lifted to the characteristic 0 case.

In the second part, we will give a correspondence for those representations to (φ_L, Γ_K)-modules with values in Forms of the linear algebraic group we start with and a correspondence of the nonperfect and the perfect setting. For $\mathbb{G} = \mathrm{GL}_n$, we then show that our correspondence is the same as the one that is induced by Fontaine's functor.

In the last part, we will give a discussion of the theory one can build in the case of the quotientfields of our discrete valuation rings.

3.2.1 Lifting H_K-Invariants

Let in this part $\mathbb{G}$ be a linear algebraic group over $\mathcal{O}_L$ and

$$(\mathcal{R}, \mathcal{R}^{nr}, \mathbb{K}) \in \{(\mathbb{A}_L, \mathbb{A}, \mathbb{E}), (W(\mathbb{F})_L, W(\overline{\mathbb{F}})_L, \mathbb{F})\}.$$

It is

$$\mathbb{G}(\mathcal{R}^{nr}) \cong \lim_{\leftarrow} \mathbb{G}(\mathcal{R}^{nr}/\pi^n \mathcal{R}^{nr})$$

by Lemma 3.1.9. In this part, we view $\mathbb{G}(\mathcal{R})$ with the prodiscrete topology. Let

$$c, d \in C^1(H_K, \mathbb{G}(\mathcal{R}^{nr})).$$

Define an action on $\mathbb{G}(\mathcal{R}^{nr})$ by setting

$$h.A := h \underset{c,d}{.} A := c(h) \cdot \mathbb{G}(\rho(h))(A) d(h)^{-1} \;\forall A \in \mathbb{G}(\mathcal{R}^{nr}), h \in H_K.$$

We set

$$\mathbb{G}(\mathcal{R}^{nr})^{c,d,H_K} := \{A \in \mathbb{G}(\mathcal{R}^{nr}) \mid h.A = A \;\forall h \in H_K\}.$$

Let

$$\mathrm{pr}_n : \mathcal{R}^{nr} \to \mathcal{R}^{nr}/\pi^n \mathcal{R}^{nr}, \mathrm{pr}_{n,n+1} : \mathcal{R}^{nr}/\pi^{n+1}\mathcal{R}^{nr} \to \mathcal{R}^{nr}/\pi^n \mathcal{R}^{nr}$$

be the projections and

$$c_n := \mathbb{G}(\mathrm{pr}_n) \circ c \in C^1(H_K, \mathbb{G}(\mathcal{R}^{nr}/\pi^n \mathcal{R}^{nr})).$$

Let

$$\mathbb{G}(\mathcal{R}^{nr}/(\pi^n))^{c_n, d_n, H_L}$$

be defined by an analoguesly defined action on $\mathbb{G}(\mathcal{R}^{nr}/(\pi^n))$. Then the projections induce maps

$$\mathrm{pr}_{n,c,d} : \mathbb{G}(\mathcal{R}^{nr}/(\pi^{n+1}))^{c_{n+1},d_{n+1},H_L} \to \mathbb{G}(\mathcal{R}^{nr}/(\pi^n))^{c_n,d_n,H_L},$$

since we have

$$h \underset{c_n,d_n}{\cdot} (\mathbb{G}(\mathrm{pr}_n)(A)) = \mathbb{G}(\mathrm{pr}_n)(h \underset{c_{n+1},d_{n+1}}{\cdot} A).$$

Furthermore, it is

$$\mathbb{G}(\mathcal{R}^{nr})^{c,d,H_L} \cong \varprojlim \mathbb{G}(\mathcal{R}^{nr}/(\pi^n))^{c_n,d_n,H_L}, \qquad (*)$$

since $h \underset{c,d}{\cdot} A$ corresponds to

$$\lim_n (h \underset{c_n,d_n}{\cdot} (\mathbb{G}(\mathrm{pr}_n)(A)))$$

via

$$\mathbb{G}(\mathcal{R}^{nr}) \cong \varprojlim \mathbb{G}(\mathcal{R}^{nr}/(\pi^n)).$$

We want to have

$$\mathbb{G}(\mathcal{R}^{nr})^{c,d,H_L} \neq \emptyset$$

if and only for their images under the projection

$$\mathbb{G}(\mathbb{K}^{sep})^{c_1,d_1,H_L} \neq \emptyset.$$

We show that we can also use Proposition 3.1.4, but we will make a more direct approach, so that the reader hopefully can get a better feel for these invariants. We view $\mathbb{G}(\mathcal{R}^{nr}/(\pi^m))$ as a topological H_L-group via the action ρ. This makes sense, since $\rho_{|H_K}$ is continuous for the π-adic topology on $\mathcal{R}$, see Lemma 1.2.7.ii). We furthermore view $Mat_{n\times n}(\mathbb{K}^{sep})$ as a topological H_L-group. We consider the case of GL_n.

Lemma 3.2.1. *For any $m \geq 1$, the isomorphism*

$$\Phi_m : Mat_{n\times n}(\mathbb{K}^{sep}) \cong Mat_{n\times n}((\mathcal{R}^{nr}/(\pi^{m+1}))/(\pi)) \to \ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1})), A \mapsto 1+\pi^m A$$

of Lemma 3.1.11 is H_K-equivariant.

Proof. This follows from the fact that $\rho(h)$ is an $\mathcal{O}_L$-algebra homomorphism for every $h \in H_K$. □

Remark 3.2.2. Let $c \in C^1(H_K, \mathrm{GL}_n(\mathcal{R}^{nr}))$ and let $Mat_{n\times n}(\mathbb{K}^{sep})_{c_1}$ denote the H_K-group

$$Mat_{n\times n}(\mathbb{K}^{sep})_{c_1} := Mat_{n\times n}(\mathbb{K}^{sep}),\ h \underset{c_1}{*} A := c_1(h) \cdot Mat_{n\times n}(\bar{\rho}(h))(A) \cdot c_1(h)^{-1}.$$

Let $ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1}))_{c_{m+1}}$ denote the H_K-group as in part 3.1.1. Then

$$\Phi_m : Mat_{n\times n}(\mathbb{K}^{sep})_{c_1} \to \ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1}))_{c_{m+1}}$$

is H_K-equivariant, by Lemma 3.2.1 and since conjugation in $Mat_{n\times n}(\mathbb{K}^{sep})$ with an element of $\mathrm{GL}_n(\mathbb{K}^{sep})$ is additive and preserves elements in the center.

Remark 3.2.3. By Hilbert 90, it is that

$$H^1(H_K, \mathrm{GL}_n(\mathbb{K}^{sep})) = 1.$$

So for any $\bar{c} \in C^1(H_K, \mathrm{GL}_n(\mathbb{K}^{sep}))$, there exists a Matrix $A \in \mathrm{GL}_n(\mathbb{K}^{sep})$ satisfying

$$\bar{c}(h) = A^{-1}\, \mathrm{GL}_n(\rho(h))(A)\ \forall h \in \mathbb{N}.$$

It follows that

$$Mat_{n\times n}(\mathbb{K}^{sep})_{\bar{c}} \to Mat_{n\times n}(\mathbb{K}^{sep}), B \mapsto ABA^{-1}$$

is an isomorphism of H_K-groups since

$$A\bar{c}(h) Mat_{n\times n}(\rho(h))(B)\bar{c}(n)^{-1}A^{-1} = Mat_{n\times n}(\rho(h))(ABA^{-1}),$$

which holds because of

$$\mathrm{GL}_n(\rho(h)) = Mat_{n\times n}(\rho(h))_{|\,\mathrm{GL}_n(\mathbb{K}^{sep})}\ \forall h \in H_K.$$

Now we consider a closed subgroup $\mathbb{G} \subset \mathrm{GL}_n$ again.

Lemma 3.2.4. *It is*

$$H^1(H_K, \ker(\mathbb{G}(\mathrm{pr}_{m,m+1}))_{c_{m+1}}) = 1$$

for all $m \geq 1$ *and* $c \in C^1(H_K, \mathbb{G}(\mathcal{R}^{nr}))$.

Proof. By Remark 3.2.2 the map

$$\ker(\mathbb{G}(\mathrm{pr}_{n,n+1}))_{c_{m+1}} \subset \ker(\mathrm{GL}_n(\mathrm{pr}_{m,m+1}))_{\subset\circ c_{m+1}} \cong Mat_{n\times n}(\mathbb{K}^{sep})_{\subset\circ c_1}$$

is an H_K-equivariant embedding. By Remark 3.2.3 the right hand side of this embedding is isomorphic to $Mat_{n\times n}(\mathbb{K}^{sep})$ as an H_K-group, so by Lemma 3.1.12, this embedding is onto a $\mathbb{K}^{sep}$-subvector space of $Mat_{n\times n}(\mathbb{K}^{sep})$ with the H_K-action given by $Mat_{n\times n}(\rho(h))$ for $h \in H_K$. It follows that the additive Hilbert 90 gives the desired triviality of the first cohomology. $\square$

Remark. Let $c, d \in C^1(H_K, \mathbb{G}(\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr}))$ for $1 \leq n \leq \infty$, where $\pi^\infty := 0$. Then for every $h \in H_K$ and $A, B \in \mathbb{G}(\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr})$, we have

$$(h \underset{c,d}{.} A) h \underset{d,d}{.} B = h \underset{c,d}{.} (AB).$$

Proof. We calculate

$$\begin{aligned}(h \underset{c,d}{.} A) h \underset{d,d}{.} B &= c(h)\mathbb{G}(\rho(h))(A)d(h)^{-1}d(h)\mathbb{G}(\rho(h))(B)d(h^{-1}) \\ &= h \underset{c,d}{.} (AB).\end{aligned}$$

□

Proposition 3.2.5. *If $\mathbb{G}$ is smooth, then $\mathrm{pr}_{m,c,d}$ is surjective for all $m \geq 1$ and all $c, d \in C^1(H_K, \mathbb{G}(\mathcal{R}^{nr}))$.*

Proof. Let $B \in \mathbb{G}(\mathcal{R}^{nr}/(\pi^m))^{c_m, d_m, H_L}$. By Hensel's Lemma Proposition 3.1.13, choose

$$A \in \mathbb{G}(\mathcal{R}^{nr}/(\pi^{m+1})),$$

such that

$$\mathbb{G}(\mathrm{pr}_{m,m+1})(A) = B.$$

Then define a cocycle

$$a(h) := A^{-1}h.A \in \ker(\mathbb{G}(\mathrm{pr}_{m,m+1}))_{d_{m+1}}.$$

We calculate that this is actually a cocycle.

$$\begin{aligned}&a(h_1)d_{m+1}(h_1)\mathbb{G}(\rho(h_1))(a(h_2))d_{m+1}(h_1)^{-1} \\ =&A^{-1}c_{m+1}(h_1)\mathbb{G}(\rho(h_1))(A)d_{m+1}^{-1}(h_1)d_{m+1}(h_1)\mathbb{G}(\rho(h_1))(a(h_2))d_{m+1}(h_1)^{-1} \\ =&A^{-1}c_{m+1}(h_1)\mathbb{G}(\rho(h_1))(Aa(h_2))d_{m+1}(h_1)^{-1} \\ =&A^{-1}c_{m+1}(h_1)\mathbb{G}(\rho(h_1))(AA^{-1}h_2.A)d_{m+1}(h_1)^{-1} \\ =&A^{-1}c_{m+1}(h_1)\mathbb{G}(\rho(h_1))(c_{m+1}(h_2))\mathbb{G}(\rho(h_1h_2))(A)\mathbb{G}(\rho(h_1))(d_{m+1}(h_2)^{-1})d_{m+1}(h_1)^{-1} \\ =&A^{-1}c_{m+1}(h_1h_2)\mathbb{G}(\rho(h_1h_2))(A)d_{m+1}(h_1h_2)^{-1} \\ =&a(h_1h_2)\end{aligned}$$

By Lemma 3.2.4, we get $A_0 \in \ker(\mathbb{G}(\mathrm{pr}_{m,m+1}))$, such that

$$A^{-1}h.A = A_0 \cdot \mathbb{G}^{(d_{m+1})}(h)(A_0^{-1}),$$

where

$$\mathbb{G}^{(d_{m+1})}(h)(A_0^{-1}) := d_{m+1}(h) \cdot \mathbb{G}(\rho(h))(A_0^{-1}) \cdot d_{m+1}(h)^{-1} \; \forall h \in H_K.$$

It follows by the previous Remark that

$$h.(AA_0) = AA_0 \text{ and } \mathbb{G}(\mathrm{pr}_{m,m+1})(AA_0) = B \ \forall h \in H_K,$$

so AA_0 is an element in the inverse image of B under $\mathrm{pr}_{m,c,d}$. □

Corollary 3.2.6. *If $\mathbb{G}$ is smooth, then it is*

$$\mathbb{G}(\mathcal{R}^{nr})^{c,d,H_L} \neq \emptyset,$$

if and only if

$$\mathbb{G}(\mathbb{K}^{sep})^{c_1,d_1,H_L} \neq \emptyset.$$

In particular

$$H^1(H_K, \mathbb{G}(\mathcal{R}^{nr})) \to H^1(H_K, \mathbb{G}(\mathbb{K}^{sep}))$$

is injective.

Proof. This follows from successively lifting an Element in $\mathbb{G}(\mathbb{K}^{sep})^{c_1,d_1,H_L}$ via the last Proposition and the correspondence in $(*)$ before Lemma 3.2.1. The second claim follows from the fact that, it is

$$B \in \mathbb{G}(\mathcal{R}^{nr})^{c,d,H_L},$$

if and only if

$$B \cdot d(h) \cdot \mathbb{G}(\rho(h))(B^{-1}) = c(h) \ \forall h \in H_L.$$

□

3.2.2 Galois Representations and Etale (φ_L, Γ_K)-Modules

In this part, $\mathbb{G}$ is a smooth linear algebraic group over $\mathcal{O}_L$ together with a closed immersion of groups $\mathbb{G} \subset \mathrm{GL}_n$. and

$$(\mathcal{R}, \mathcal{R}^{nr}, \mathbb{K}) \in \{(\mathbb{A}_L, \mathbb{A}, \mathbb{E}), (W(\mathbb{F})_L, W(\overline{\mathbb{F}})_L, \mathbb{F})\}.$$

It is

$$\mathbb{G}(\mathcal{O}_L) \cong \varprojlim \mathbb{G}(\mathcal{O}_L/\pi^n\mathcal{O}_L)$$

by Lemma 3.1.9. We view it with the profinite topology. We call it the *π-adic topology on* $\mathbb{G}(\mathcal{O}_L)$.

Remark. The π-adic topology on $\mathbb{G}(\mathcal{O}_L)$ is the same as the topology induced by

$$\mathbb{G}(\mathcal{O}_L) \subset \mathrm{GL}_n(\mathcal{O}_L)$$

via the π-adic topology on $\mathcal{O}_L$.

Let
$$\bar{j}_{\mathcal{R}} : \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L)) \to H^1(H_K, \mathbb{G}(\mathcal{R}^{nr}))$$
be the map given by the restriction $H_K \subset G_K$ and $\mathcal{O}_L \subset \mathcal{R}^{nr}$.

Remark 3.2.7. The map
$$\mathrm{im}(j_{\mathbb{A}_K}) \to \mathrm{im}(j_{W(\mathbb{F})_L})$$
induced by inclusion
$$\mathbb{A} \subset W(\overline{\mathbb{F}})_L$$
is bijective.

Proof. This follows from the injectivity statements in Corollary 3.2.6 and Remark 2.2.3. □

From here on out we view $\mathbb{G}(\mathcal{S})$, where
$$\mathcal{S} \in \{\mathcal{R}, \mathcal{R}^{nr}, \mathbb{K}, \mathbb{K}^{sep}\}$$
with the weak topology, see Lemma 2.2.5 and the Remark following it. Let $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))$ and define
$$\mathbb{G}_{\mathcal{R}}^{(f)} := \{A \in \mathbb{G}(\mathcal{R}^{nr}) \mid A = f(h)\mathbb{G}(\rho(h))(A)f(h)^{-1}\ \forall h \in H_K\}.$$
We view $\mathbb{G}_{\mathcal{R}}^{(f)} \subset \mathbb{G}(\mathcal{R}^{nr})$ with the subset topology. Then $\mathbb{G}_{\mathcal{R}}^{(f)}$ can be made into an $\mathbb{O}_K$-group, which is a topological Γ_K-group as in part 2.2.2 by setting for $\gamma = \mathrm{pr}_{H_K}(g_\gamma)\pi^{n_\gamma} \in \mathbb{O}_K$, where $g_\gamma \in G_K$
$$\gamma * A := \gamma \underset{f}{*} A := f(g_\gamma)\mathbb{G}(\rho(g)\varphi_L^{n_\gamma})(A)f(g_\gamma)^{-1}.$$
For this beware that we need that weak topology on $\mathcal{R}^{nr}$ induces the π-adic topology on $\mathcal{O}_L$ and Lemma 1.2.7.ii).

Since $\mathbb{G} \subset \mathrm{GL}_n$, there exists by Hilbert 90 and Corollary 3.2.6 a $B \in \mathrm{GL}_n(\mathcal{R}^{nr})$, such that
$$f(h) = B^{-1}\,\mathrm{GL}_n(\rho(h))(B)\ \forall h \in H_K.$$
Then we have two embeddings
$$\iota_1 : \mathbb{G}_{\mathcal{R}}^{(f)} \subset \mathbb{G}(\mathcal{R}^{nr}) \subset \mathrm{GL}_n(\mathcal{R}^{nr}), A \mapsto A$$
and
$$\iota_2 : \mathbb{G}_{\mathcal{R}}^{(f)} \to \mathrm{GL}_n(\mathcal{R}), A \mapsto B\iota_1(A)B^{-1}.$$
Viewing $\mathrm{GL}_n(\mathcal{R})$ and $\mathrm{GL}_n(\mathcal{R}^{nr})$ with the with the weak topology, ι_1 and ι_2 induce the same topology on $\mathbb{G}_{\mathcal{R}}^{(f)}$, since the weak topology on $\mathcal{R}$ is the topology induced via $\mathcal{R} \subset \mathcal{R}^{nr}$, where $\mathcal{R}^{nr}$ carries the weak topology.

Definition 3.2.8. We define

$$C^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{R}}^{(f)}) := \{\alpha : \mathbb{O}_K \to \mathbb{G}_{\mathcal{R}}^{(f)} \mid \alpha(\gamma\delta) = \alpha(\gamma)\gamma \underset{f}{*} \alpha(\delta)\ \forall \gamma, \delta,\ \alpha_{|\Gamma_K} \text{ is continuous}\}$$

and call $\alpha \in C^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{R}}^{(f)})$ an *etale* (φ_L, Γ_K)*-module over* $\mathcal{R}$ *with values in* $\mathbb{G}^{(f)}$.

Fix a subset

$$\{f_i\}_i \subset \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L)),$$

such that the map $\bar{j}_{\mathbb{A}_K}$ induces a bijection

$$\bar{j}_{\mathbb{A}_K} : \{f_i\}_i \to \mathrm{im}(\bar{j}_{\mathbb{A}_K}).$$

Recall the Langmap

$$\Psi := \Psi_{\mathcal{R}^{nr}} : \mathbb{G}(\mathcal{R}^{nr}) \to \mathbb{G}(\mathcal{R}^{nr}), A \mapsto A^{-1}\mathbb{G}(\varphi_L)(A).$$

Analogues to the case in part 2.2.2, we can make constructions, which give us maps

$$\mathbb{D} : (\mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))/\sim) \to \coprod_i H^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{R}}^{(f_i)})$$

and

$$\mathbb{V} : \coprod_i \{[\alpha] \in H^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{R}}^{(f_i)}) \mid \alpha(\pi) \in \mathrm{im}(\Psi_{\mathcal{R}^{nr}})\} \to (\mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))/\sim).$$

We briefly recall the construction of $\mathbb{D}$, since we will need it later on. Let $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))$. Then there exists a unique i, such that $\bar{j}_{\mathcal{R}}(f) = \bar{j}_{\mathcal{R}}(f_i)$. This means that

$$\mathbb{G}(\mathcal{R}^{nr})^{f, f_i, H_L} \neq \emptyset.$$

So let $A_0 \in \mathbb{G}(\mathcal{R}^{nr})^{f, f_i, H_L}$. Then we have the well defined cocycle

$$\alpha_{f, A_0}(\gamma) := A_0^{-1} f(g_\gamma)\mathbb{G}(\rho(g_\gamma)\varphi_L^{n_\gamma})(A_0) f_i(g_\gamma)^{-1} \in \mathbb{G}_{\mathcal{R}}^{(f)},$$

if $\gamma = \mathrm{pr}_{H_K}(g_\gamma)\pi^{n_\gamma} \in \mathbb{O}_K$. Then

$$\mathbb{D}([f]_\sim) := \alpha_f := [\alpha_{f, A_0}]_\sim.$$

They satisfy the following correspondence.

Theorem 3.2.9. *We have inverse bijections*

$$\mathbb{D} : (\mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))/\sim) \leftrightarrows \coprod_i \{[\alpha] \in H^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{R}}^{(f_i)}) \mid \alpha(\pi) \in \mathrm{im}(\Psi)\} : \mathbb{V}$$

and if the special fiber $\mathbb{G}_k$ *is connected, the right hand side is* $\coprod_i H^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{R}}^{(f_i)})$.
This bijection identifies

$$\{[a]_\sim \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))/\sim \mid \bar{j}_{\mathcal{R}}(a) = \bar{j}_{\mathcal{R}}(f_i)\} \cong \{[\alpha] \in H^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{R}}^{(f_i)}) \mid \alpha(\pi) \in \mathrm{im}(\Psi)\}.$$

for every i.

Proof. The first and third part of the statement works completely analogues as in part 2.2.2. The second part is Proposition 3.1.26. □

Since this bijection is dependent on the choice of $\{f_i\}_i$, the maps $\mathbb{D}$ and $\mathbb{V}$ are in general not "functorial". Under certain conditions, there is still a way to get something like functoriality. For this we first note that, if $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ is a morphism of groups and $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}_1(\mathcal{O}_L))$, then

$$\phi_{\mathcal{R}^{nr}} : \mathbb{G}_{1\mathcal{R}}^{(f)} \to \mathbb{G}_{2\mathcal{R}}^{(\phi_k \circ f)}$$

is a well defined morphism of $\mathbb{O}_K$-groups by definition, which is continuous, because it is a polynomial map.

Lemma 3.2.10. *Let* $\phi : \mathbb{G}_1 \to \mathbb{G}_2$*, such that the induced map*

$$(\phi_{\mathbb{A}})_* : H^1(H_K, \mathbb{G}_1(\mathbb{A})) \to H^1(H_K, \mathbb{G}_2(\mathbb{A}))$$

is injective on $\mathrm{im}(\bar{j}_{\mathbb{A}_L}^{\mathbb{G}_1})$*. Then for any choice* $\{f_i^{(1)}\}_i \subset \mathrm{mor}^{cont}(G_K, \mathbb{G}_1(\mathcal{O}_L))$*, such that*

$$\bar{j}_{\mathbb{A}_L}^{\mathbb{G}_1} : \{f_i^{(1)}\}_i \to \mathrm{im}(\bar{j}^{\mathbb{G}_1})$$

is bijective, we can complement $\{\phi_{\mathcal{O}_L} \circ f_i\}_i \subset \mathrm{mor}^{cont}(G_K, \mathbb{G}_2(\mathcal{O}_L))$ *to a subset* $\{f_l^{(2)}\}_l$*, such that*

$$\bar{j}_{\mathbb{A}_L}^{\mathbb{G}_2} : \{f_l^{(2)}\}_l \to \mathrm{im}(\bar{j}^{\mathbb{G}_2})$$

is bijective. Furthermore, the following diagram is commutative.

$$\begin{tikzcd}
(\mathrm{mor}^{cont}(G_K, \mathbb{G}_1(\mathcal{O}_L))/\sim) \arrow[r, "\mathbb{D}"] \arrow[d, "(\phi_{\mathcal{O}_L})_*"'] & \coprod_i H^1(\mathbb{O}_K, \mathbb{G}_{1\mathcal{R}}^{(f_i^{(1)})}) \arrow[d, "(\phi_{\mathcal{R}^{nr}})_*"] \\
(\mathrm{mor}^{cont}(G_K, \mathbb{G}_2(\mathcal{O}_L))/\sim) \arrow[r, "\mathbb{D}"'] & \coprod_l H^1(\mathbb{O}_K, \mathbb{G}_{2\mathcal{R}}^{(f_l^{(2)})}).
\end{tikzcd}$$

Proof. As for Lemma 2.2.16. $\square$

In this case, we also have a generalization of the previous Lemma.

Lemma 3.2.11. *Let $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ be a morphism of groupschemes over $\mathcal{O}_L$. Choose some representatives $\{f_i^{(1)}\}_i$ for $\mathbb{G}_1$ and $\{f_l^{(2)}\}_l$ for $\mathbb{G}_2$ as in the last Lemma. By definition of these representatives for any $f_i^{(1)}$ there exists a unique $f_l^{(2)}$ and some (non-unique) $B_i \in \mathbb{G}_2(\mathcal{R}^{nr})$, such that*

$$\phi_k \circ f_i^{(1)}(h) = B_i \cdot f_l^{(2)}(h) \cdot \mathbb{G}_2(\rho(h))(B_i^{-1}). \tag{$*$}$$

Then the following diagram is commutative and the right vertical map is independent on the choice of B_i.

$$\begin{tikzcd}
(\mathrm{mor}^{cont}(G_K, \mathbb{G}_1(\mathcal{O}_L))/\sim) \arrow[r, "\mathbb{D}"] \arrow[d, "(\phi_{\mathcal{O}_L})_*"'] & \coprod_i H^1(\mathbb{O}_K, \mathbb{G}_{1,\mathcal{R}}^{(f_i^{(1)})}) \arrow[d, "\coprod_i [\alpha \mapsto [\gamma \mapsto B_i^{-1} \cdot \phi_{\mathcal{R}^{nr}} \circ \alpha(\gamma) \cdot \gamma.B_i]]"] \\
(\mathrm{mor}^{cont}(G_K, \mathbb{G}_2(\mathcal{O}_L))/\sim) \arrow[r, "\mathbb{D}"'] & \coprod_l H^1(\mathbb{O}_K, \mathbb{G}_{2,\mathcal{R}}^{(f_l^{(2)})}).
\end{tikzcd}$$

Here for $\gamma = \mathrm{pr}_{H_L}(g_\gamma)\pi^{n_\gamma} \in \mathbb{O}_K$, we have

$$\gamma.B_i := \gamma \underset{\phi_{\mathcal{O}_L} \circ f_i^{(1)}, f_l^{(2)}}{.} B := \phi_{\mathcal{O}_L} \circ f_i^{(1)}(g_\gamma) \cdot \mathbb{G}_2(\rho(g_\gamma) \circ \varphi_L^{n_\gamma})(B_i) \cdot f_l^{(2)}(g_\gamma)^{-1}.$$

This is well defined by arguments as in the discussion before Lemma 2.2.7.

Proof. As for Lemma 2.2.18. $\square$

Proposition 3.2.12. *Let $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))$. Then*

$$C^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{R}}^{(f)}) = \{\alpha : \mathbb{O}_K \to \mathbb{G}_{\mathcal{R}}^{(f)} \mid \alpha(\gamma\delta) = \alpha(\gamma) \cdot \gamma \underset{f}{*} \alpha(\delta) \ \forall \gamma, \delta \in \mathbb{O}_K\},$$

i.e. such an 1*-cocycle is automatically continuous for the weak topology on $\mathbb{G}_{\mathcal{R}}^{(f)}$.*

Proof. As for Proposition 2.2.19. $\square$

We futhermore obtain the following correspondence between non-perfect and perfect (φ_L, Γ_K)-modules.

Theorem 3.2.13. *The inclusion $\mathbb{A}_K \subset W(\mathbb{F})_L$ induces a bijection*

$$\{[\alpha] \in H^1(\mathbb{O}_K, \mathbb{G}_{\mathbb{A}_K}^{(f)}) \mid \alpha(\pi) \in \mathrm{im}(\Psi_{\mathbb{A}})\} \tilde{\to} \{[\alpha] \in H^1(\mathbb{O}_K, \mathbb{G}_{W(\mathbb{F})_L}^{(f)}) \mid \alpha(\pi) \in \mathrm{im}(\Psi_{W(\overline{\mathbb{F}})_L})\}$$

and, if $\mathbb{G}_k$ is connected, it induces a bijection

$$H^1(\mathbb{O}_K, \mathbb{G}_{\mathbb{A}_K}^{(f)}) \tilde{\to} H^1(\mathbb{O}_K, \mathbb{G}_{W(\mathbb{F})_L}^{(f)}).$$

Proof. Extend f to a subset $\{f_i\}_i$, which satisfies the condition that

$$\bar{j}_{\mathbb{A}_K} : \{f_i\}_i \to \operatorname{im}(\bar{j}_{\mathbb{A}_K})$$

is bijective. By Remark 3.2.7 and Theorem 3.2.9, we have the bijections

$$\begin{aligned}\mathbb{V}_{\mathbb{A}_K} :& \{[\alpha] \in H^1(\mathbb{O}_K, \mathbb{G}^{(f)}_{\mathbb{A}_K}) \mid \alpha(\pi) \in \operatorname{im}(\Psi_{\mathbb{A}})\} \\ &\to \{[a]_\sim \in \operatorname{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))/\sim \ \mid \bar{j}_{\mathbb{A}_K}(a) = \bar{j}_{\mathbb{A}_K}(f)\}\end{aligned}$$

and

$$\begin{aligned}\mathbb{D}_{W(\mathbb{F})_L} :& \{[a]_\sim \in \operatorname{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))/\sim \ \mid \bar{j}_{\mathbb{A}_K}(a) = \bar{j}_{\mathbb{A}_K}(f)\} \\ &\to \{[\alpha] \in H^1(\mathbb{O}_K, \mathbb{G}^{(f)}_{W(\mathbb{F})_L}) \mid \alpha(\pi) \in \operatorname{im}(\Psi_{W(\bar{\mathbb{F}})_L})\}.\end{aligned}$$

Now one shows that

$$\begin{aligned}\mathbb{D}_{W(\mathbb{F})_L} \circ \mathbb{V}_{\mathbb{A}_K} :& \{[\alpha] \in H^1(\mathbb{O}_K, \mathbb{G}^{(f)}_{\mathbb{A}_K}) \mid \alpha(\pi) \in \operatorname{im}(\Psi_{\mathbb{A}})\} \\ &\tilde{\to} \{[\alpha] \in H^1(\mathbb{O}_K, \mathbb{G}^{(f)}_{W(\mathbb{F})_L}) \mid \alpha(\pi) \in \operatorname{im}(\Psi_{W(\bar{\mathbb{F}})_L})\}\end{aligned}$$

is the map induced by the inclusion $\mathbb{A}_K \subset W(\mathbb{F})_L$, just as one calculates that

$$\mathbb{D}_{\mathbb{K}} \circ \mathbb{V}_{\mathbb{K}} = \mathrm{id}$$

in the proof of Proposition 2.2.15. The second part is Proposition 3.1.26. □

Let $A_{\mathcal{R}^{nr}} := \operatorname{Aut}_{\mathcal{R}^{nr}}(\mathbb{G}_{\mathcal{R}^{nr}})$ be the group of automorphisms of the group scheme $\mathbb{G}_{\mathcal{R}^{nr}}$ over $\mathcal{R}^{nr}$. As in part 2.1.3, we obtain a H_K-action of groups $A_{\mathcal{R}^{nr}}$ by conjugating $f \in A_{\mathcal{R}^{nr}}$ with $(\mathrm{id}_{\mathbb{G}}, \operatorname{Spec}(h^{-1}))$ for $h \in H_K$. As in Lemma 2.1.23, we obtain a H_K-equivariant map

$$\Phi : \mathbb{G}(\mathcal{R}^{nr}) \to A_{\mathcal{R}^{nr}}$$

by sending $g \in \mathbb{G}(\mathcal{R}^{nr})$ to $[\mathbb{G}(S) \ni x \mapsto gxg^{-1} \in \mathbb{G}(S)]$ for all $\mathcal{R}^{nr}$-algebras S. Here, we view $\mathbb{G}(\mathcal{R}^{nr})$ as an H_K-group via $\mathbb{G}(\rho(h))$ for every $h \in H_K$.

Let $\mathbb{G}_{\mathcal{R}^{nr}} = \operatorname{Spec}(H)$ for some Hopfalgebra H (of finite type) over $\mathcal{R}^{nr}$, then by right exactness of the tensor product, it is

$$\mathbb{G}_{\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr}} = \operatorname{Spec}(H/\pi^n H)$$

for all $n \geq 1$. Let

$$A_{\mathcal{R}^{nr},n} := \operatorname{Aut}_{\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr}}(\mathbb{G}_{\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr}}).$$

Analoguesly to above, we obtain a H_K-equivariant map

$$\Phi_n : \mathbb{G}(\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr}) \to A_{\mathcal{R}^{nr},n}.$$

For every cocycle $c \in C^1(H_K, \mathbb{G}(\mathcal{R}^{nr}))$, which is continuous for the prodiscrete topology, we define

$$c_n := \mathbb{G}(\mathrm{pr}_n) \circ c \in C^1(H_K, \mathbb{G}(\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr}))$$

a cocycle, which is continuous for the discrete topology on $\mathbb{G}(\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr})$. We furthermore obtain a cocycle $c_n^{(a)} := \Phi_n \circ c_n \in C^1(H_K, A_{\mathcal{R}^{nr},n})$, which is continuous for the discrete topology on $A_{\mathcal{R}^{nr},n}$. As described in Remark 2.1.21 the cocycle $c_n^{(a)}$ induces a H_K-semilinear action of Hopfalgebras over $\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr}$ on $H/\pi^n H$, which is continuous for the discrete topology on $H/\pi^n H$ via $h \mapsto (\mathrm{id}_{\mathbb{G}}, \mathrm{Spec}(h)) \circ c_n^{(a)}(h)^{-1},\ h \in H_K$. We get a Hopfalgebra

$$H^{(c_n)} := (H/\pi^n H)^{H_K}$$

over $\mathcal{R}/\pi^n\mathcal{R}$ for the invariants under the action defined by $c_n^{(a)}$ above. Then

$$\mathbb{G}^{(c_n)} := \mathrm{Spec}(H^{(c_n)})$$

is an $(\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr})|(\mathcal{R}/\pi^n\mathcal{R})$-Form of $\mathbb{G}_{\mathcal{R}/\pi^n\mathcal{R}}$. This means that the multiplication

$$\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr} \underset{\mathcal{R}/\pi^n\mathcal{R}}{\otimes} H^{(c_n)} \to H/\pi^n H$$

is an isomorphism, so we have an identification $\mathbb{G}^{(c_n)}_{\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr}} \cong \mathbb{G}_{\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr}}$. To see this, we have by Proposition 3.1.19 that the multiplication

$$\mathcal{R}^{nr} \underset{\mathcal{R}}{\otimes} H^{(c_n)} \to H/\pi^n H$$

is an isomorphism, but by right exactness of the tensor product and since $\pi^n H^{(c_n)} = 0$, we also have the isomorphism

$$\mathcal{R}^{nr} \underset{\mathcal{R}}{\otimes} H^{(c_n)} \to \mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr} \underset{\mathcal{R}/\pi^n\mathcal{R}}{\otimes} H^{(c_n)}, x \otimes y \mapsto \mathrm{pr}_{\pi^n\mathcal{R}^{nr}}(x) \otimes y,$$

so we obtain that the map induced by multiplication

$$\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr} \underset{\mathcal{R}/\pi^n\mathcal{R}}{\otimes} H^{(c_n)} \to H/\pi^n H$$

is an isomorphism. As calculated for Remark 2.1.21.(1), this identification gives and identification

$$\mathbb{G}^{(c_n)}(\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr}) \cong \mathbb{G}(\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr}),$$

which satisfies

$$\mathbb{G}^{(c_n)}(\rho(h))(A) = c_n(h) \cdot \mathbb{G}(\rho(h))(A) \cdot c_n(h)^{-1}\ \forall h \in H_K, A \in \mathbb{G}(\mathcal{R}^{nr}/\pi^n\mathcal{R}^{nr}). \tag{T}$$

Proposition 3.2.14. *We use the same notation as in the discussion above. Let*

$$\psi_n : H^{(c_{n+1})} \to H^{(c_n)}$$

be the map induced by the projection $\mathrm{pr}_n : H/\pi^{n+1}H \to H/\pi^n H$*. Then the projective system* $(H^{(c_n)}, \psi_n)_{n\geq 1}$ *satisfies the conditions of Lemma 3.1.18. In particular*

$$H^{(c)} := \lim_{\leftarrow} H^{(c_n)}$$

is a complete formal Hopfalgebra over $\mathcal{R}$*. We set* $\hat{\mathbb{G}}^{(c)} := \mathrm{Spf}(H^{(c)})$*. We then furthermore have that* $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))$ *seen as an element* $C^1(H_K, \mathbb{G}(\mathcal{R}^{nr}))$ *via restriction* $H_K \subset G_K$ *and inclusion* $\mathcal{O}_L \subset \mathcal{R}^{nr}$ *satisfies*

$$\hat{\mathbb{G}}^{(f)}(\mathcal{R}) \cong \mathbb{G}_{\mathcal{R}}^{(f)}.$$

Proof. By Corollary 3.1.20 the functor $(\cdot)^{H_K}$ is exact on those modules $\mathcal{R}^{nr}$-modules M with a semilinear and π-adically continuous H_K-action, such that there exists an $n \geq 1$, such that $\pi^n M = 0$. It follows from the surjectivity of the projection $\mathrm{pr}_n : H/\pi^{n+1}H \to H/\pi^n H$ that $\psi_n : H^{(c_{n+1})} \to H^{(c_{n+1})}$ is surjective.

Consider for every $n \geq 1$ and $1 \leq m \leq n$ the right exact sequence

$$H/\pi^{n+1}H \overset{\pi^m \cdot}{\to} H/\pi^{n+1}H \to H/\pi^m H \to 0.$$

It follows by the exactness of $(\cdot)^{H_K}$ of Corollary 3.1.20 that the map

$$\psi_{n+1,m} : H^{(c_{n+1})} \to H^{(c_m)}$$

induced by the projection

$$H/\pi^{n+1}H \to H/\pi^m H$$

has kernel

$$\ker(\psi_{n+1,m}) = \pi^m H^{(c_{n+1})}. \qquad (K_{n+1,m})$$

By induction for $n \geq 1$, we show that

$$\psi_n^{-1}(\pi^m H^{(c_n)}) = \pi^m H^{(c_{n+1})}$$

for every $1 \leq m \leq n$ For $n = 1$, we have to show that

$$\psi_1^{-1}(\pi H^{(c_1)}) = \pi H^{(c_2)}.$$

But $\psi_1^{-1}(\pi H^{(c_1)}) = \ker(\psi_1)$, so this follows from $(K_{2,1})$. Let the statement be correct for some $n-1 \geq 1$ and every $1 \leq m \leq n-1$. Then for $n = m$ we have to show

$$\psi_n^{-1}(\pi^n H^{(c_n)}) = \pi^n H^{(c_{n+1})}.$$

But as above we have $\psi_n^{-1}(\pi^n H^{(c_n)}) = \ker(\psi_n)$, so this follows from $(K_{n+1,n})$. So let $m < n$. Then by induction hypothesis, we have the chain of equalities

$$\begin{aligned}
\psi_n^{-1}(\pi^m H^{(c_n)}) &= \psi_n^{-1}(\psi_{n-1}^{-1}(\pi^m H^{(c_{n-1})})) \\
&= \ldots \\
&= \psi_n^{-1}(\psi_{n-1}^{-1}(\ldots(\psi_m^{-1}(\pi^m H^{(c_m)}))\ldots) \\
&= \ker(\psi_{n+1,m}),
\end{aligned}$$

so we have

$$\psi_n^{-1}(\pi^m H^{(c_n)}) = \pi^m H^{(c_{n+1})}$$

by $(K_{n+1,m})$.

Lastly for the conditions on the projective systems, we have to show that for every $\mathcal{R}/\pi^n\mathcal{R}$-algebra S, we have that ψ_n induces a morphism of groups

$$\mathrm{mor}_{\mathcal{R}/\pi^n\mathcal{R}-Alg}(H^{(c_n)}, S) \to \mathrm{mor}_{\mathcal{R}/\pi^{n+1}\mathcal{R}}(H^{(c_{n+1})}, S).$$

But the projection

$$H/\pi^{n+1}H \to H/\pi^n H$$

respects the structure of a Hopfalgebra, since this structure is for all $H/\pi^n H$ induced by the one on H. Since $H^{(c_n)}$ carries the Hopfalgebra structure induced by $H/\pi^n H$, it follows that ψ_n also respects the structure of a Hopfalgebra, so the induced map

$$\mathrm{mor}_{\mathcal{R}/\pi^n\mathcal{R}-Alg}(H^{(c_n)}, S) \to \mathrm{mor}_{\mathcal{R}/\pi^{n+1}\mathcal{R}}(H^{(c_{n+1})}, S)$$

is indeed a morphism of groups.

We have

$$\begin{aligned}
\hat{\mathbb{G}}^{(f)}(\mathcal{R}) &\cong \varprojlim \mathbb{G}^{(f_n)}(\mathcal{R}/\pi^n\mathcal{R}) \\
&\cong \varprojlim \{A \in \mathbb{G}(\mathcal{R}^{nr}/\pi^n\mathcal{R}) \mid f_n(h) \cdot \mathbb{G}(\rho(h))(A) \cdot f_n(h)^{-1} \ \forall h \in H_K\} \\
&\cong \mathbb{G}_{\mathcal{R}}^{(f)},
\end{aligned}$$

where the first isomorphism is from the isomorphism in Lemma 3.1.18, the second isomorphism is the identification (T) before this Proposition together with Lemma 2.1.25 and the last isomorphism is the bijection $(*)$ in the discussion before Lemma 3.2.1 for $c = d = f$, which is an isomorphism for $c = d$, since the action defined there is an action of groups, if $c = d$. $\square$

This Proposition says that the (φ_L, Γ_K)-module side of the correspondence doesn't just take values in abstract groups, but $\mathcal{R}$-valued points of complete formal groups, which are related to the π-adic completion $\hat{\mathbb{G}}_{\mathcal{R}^{nr}} := \mathrm{Spf}(\hat{H})$, where $\hat{H}$ is the complete formal Hopfalgebra over $\mathcal{R}^{nr}$ obtained by π-adic completion of the Hopfalgebra H over $\mathcal{R}^{nr}$ with $\mathbb{G}_{\mathcal{R}^{nr}} = \mathrm{Spec}(H)$, see Lemma 3.1.17 that this is indeed a complete formal Hopfalgebra. The relation of $H^{(c)}$ of the previous Proposition to $\hat{H}$ is as follows.

Definition 3.2.15. Let R be a complete discrete valuation ring with uniformizer ϖ and A, B be two R-algebras. We define the *completed tensor product of* A *and* B *over* R to be

$$A \hat{\otimes}_R B := \lim_{\leftarrow} (A \otimes_R B)/\pi^n (A \otimes_R B)$$

Lemma 3.2.16. *We continue the notation from above, the previous Proposition and the discussion before it. Let H_0 be the Hopfalgebra over $\mathcal{R}$ with $\mathbb{G}_{\mathcal{R}} = \mathrm{Spec}(H_0)$, so $H = H_0 \otimes_{\mathcal{R}} \mathcal{R}^{nr}$. Let $\hat{H}_0$ be the π-adic completion of H_0 and $\hat{\mathbb{G}}_{\mathcal{R}} := \mathrm{Spf}(\hat{H}_0)$. Then*

$$\hat{H} = H_0 \hat{\otimes}_{\mathcal{R}} \mathcal{R}^{nr} \cong \hat{H}_0 \hat{\otimes}_{\mathcal{R}} \mathcal{R}^{nr}$$

and

$$\hat{H} \cong H^{(c)} \hat{\otimes}_{\mathcal{R}} \mathcal{R}^{nr}.$$

*We say that $\hat{\mathbb{G}}^{(c)}$ is an $\mathcal{R}^{nr}|\mathcal{R}$-*Form of $\hat{\mathbb{G}}_{\mathcal{R}}$.

Proof. The equality

$$\hat{H} = H_0 \hat{\otimes}_{\mathcal{R}} \mathcal{R}^{nr}$$

follows by definition of the π-adic completion. Furthermore, we have by right exactness of the tensorproduct and $\hat{H}_0/\pi^n \hat{H}_0 \cong H_0/\pi^n H_0$ that

$$\lim_{\leftarrow} (\hat{H}_0 \otimes_{\mathcal{R}} \mathcal{R}^{nr})/(\pi^n) \cong \lim_{\leftarrow} (H_0/\pi^n H_0 \otimes_{\mathcal{R}} \mathcal{R}^{nr}) \cong \lim_{\leftarrow} (H_0 \otimes_{\mathcal{R}} \mathcal{R}^{nr})/(\pi^n)$$

By Lemma 3.1.18 the projection $H^{(c)} \to H^{(c_n)}$ induces an isomorphism $H^{(c)}/\pi^n H^{(c)} \cong H^{(c_n)}$, so we have by the right exactness of the tensor product that

$$(H^{(c)} \otimes_{\mathcal{R}} \mathcal{R}^{nr})/\pi^n (H^{(c)} \otimes_{\mathcal{R}} \mathcal{R}^{nr}) \cong H^{(c_n)} \otimes_{\mathcal{R}} \mathcal{R}^{nr},$$

so it is

$$(H^{(c)} \otimes_{\mathcal{R}} \mathcal{R}^{nr})/\pi^n (H^{(c)} \otimes_{\mathcal{R}} \mathcal{R}^{nr}) \cong H/\pi^n H$$

by Proposition 3.1.19. It follows that

$$\hat{H} \cong H^{(c)} \hat{\underset{\mathcal{R}}{\otimes}} \mathcal{R}^{nr}.$$

□

If $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))$ is the trivial morphism, we don't have to go to the completion of $\mathbb{G}$ by the following Remark.

Remark. By Lemma 2.1.25, it is

$$\mathbb{G}^{(1)}_{\mathcal{R}} = \mathbb{G}(\mathcal{R}),$$

when $1 \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))$ is the trivial map.

It follows by Hilbert 90 and Corollary 3.2.6 that we get a map, which satisfies the properties in Theorem 1.3.18.i).

We now show that this map is the one induced by the Fontaine functor. So let $K = L$. Let $f \in \mathrm{mor}^{cont}(G_L, \mathrm{GL}_n(\mathcal{O}_L))$. Then we define

$$\underline{\rho_f} := (\mathcal{O}_L^n, \rho_f : [g \mapsto [v \mapsto f(g) \cdot v]])$$

as an element in $Rep^{(n)}_{\mathcal{O}_L}(G_L)$. On the other hand, if $M \in \Gamma_L \Phi^{et,(n)}_{\mathbb{A}_L}$ is an etale (φ_L, Γ_L)-module, which is free of rank n together with an $\mathbb{A}_L$-basis $\underline{x} := (x_i)_i$, then by Lemma 1.3.16 we have the cocycle

$$\alpha_{M,\underline{x}}(\gamma) := c_{\underline{x}}(\gamma) := A_{\gamma,\underline{x}}$$

in $C^1(\mathcal{O}_L^\bullet, \mathrm{GL}_n(\mathbb{A}_L))$, where $A := A_{\gamma,\underline{x}} \in \mathrm{GL}_n(\mathbb{A}_L)$ is the Matrix, which satisfies

$$\gamma * x_i = \sum_{j \leq n} A_{ji} x_j.$$

Proposition 3.2.17. *Let $f \in \mathrm{mor}^{cont}(G_L, \mathrm{GL}_n(k))$. For every morphism of groups $\sigma : \mathrm{GL}_n \to \mathrm{GL}_m$ over $\mathcal{O}_L$ and every $\mathbb{A}_L$-basis $(x_i)_i$ of $\mathbb{D}(\underline{\rho_f})$ there exists an $\mathbb{A}_L$-basis $(y_k)_{k \leq m} \subset \mathbb{D}(\underline{\rho_{\sigma_{\mathcal{O}_L} \circ f}})$, such that the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{O}_L^\bullet \arrow[r, "\alpha_{\mathbb{D}(\underline{\rho_f}),(x_i)_i}"] \arrow[dr, "\alpha_{\mathbb{D}(\underline{\rho_{\sigma_{\mathcal{O}_L} \circ f}}),(y_k)_k}"'] & \mathrm{GL}_n(\mathbb{A}_L) \arrow[d, "\sigma_{\mathbb{A}_L}"] \\
 & \mathrm{GL}_m(\mathbb{A}_L)
\end{tikzcd}$$

Furthermore, for every such basis $(x_i)_i$, there exists $A_0 \in \mathrm{GL}_n(\mathbb{A})^{f,1,H_L}$, such that

$$\alpha_{\mathbb{D}(\underline{\rho_f}),(x_i)_i} = \alpha_{f,A_0}.$$

Proof. For every $r \geq 1$ we observe the inverse isomorphisms

$$\begin{aligned}
(\mathbb{A})^r &\to (\mathcal{O}_L^r \underset{\mathcal{O}_L}{\otimes} \mathbb{A}) \\
(a_i)_i &\mapsto \sum_i e_i \otimes a_i \\
(b_i a)_i &\leftarrow\!\shortmid (b_i)_i \otimes a,
\end{aligned}$$

where $e_i \in \mathcal{O}_L^r$ denotes the i-th standard vector. Let $\rho_{\mathcal{O}_L^r}$ be any continuous G_L representation on $\mathcal{O}_L^r$, in the sense, that we write

$$\rho_{\mathcal{O}_L^r}(g) \in \mathrm{GL}_r(k).$$

Then the above isomorphisms give rise to an identification

$$\mathbb{D}(\underline{\rho_{\mathcal{O}_L^r}}) \cong ((\mathbb{A})^r)^{H_L},$$

where the G_L-action on the right-hand side is given via

$$g.((a_i)_i) = (\sum_i \rho_{\mathcal{O}_L^r}(g)_{ji}\rho(g)(a_i))_j \ \forall g \in G_L. \tag{$*$}$$

The π-action is given by

$$\pi.((a_i)_i) = (\varphi_L(a_i))_i.$$

By (Sch17, Proposition 3.3.7) we know, that the $\mathbb{A}_L$-bases for $((\mathbb{A})^r)^{H_L}$ correspond to the $\mathbb{A}$-bases in $(\mathbb{A})^r$, which are in the H_L-invariants. By

$$(\overline{x_i})_i \subset ((\mathbb{A})^r)^{H_L}$$

we denote the $\mathbb{A}_L$-basis, which corresponds to $(x_i)_i$ via the above isomorphism and write $\overline{x}$ as the corresponding element of $\mathrm{GL}_n(\mathbb{A})$, i.e.

$$\overline{x}_{ij} := (\overline{x}_j)_i.$$

Define $(\overline{y_k})_k$ as the corresponding $\mathbb{A}$-basis to $\sigma_{\mathbb{A}}(\overline{x})$. We need to show, that $\overline{y_k}$ is H_L-invariant for it to correspond to a $\mathbb{A}_L$-basis $(y_k)_k$ as desired. For $A \in \mathrm{GL}_r(\mathbb{A})$, we define

$$g.A := (g.((A_{i1})_i), \ldots, g.((A_{ir})_i)) \forall g \in G_L \text{ and } \pi.A := (\pi.((A_{i1})_i), \ldots, \pi.((A_{ir})_i)).$$

Using $(*)$, one calculates, that

$$g.A = \rho_{\mathcal{O}_L^r}(g) \cdot \mathrm{GL}_r(\rho(g))(A) \text{ and } \pi.A = \mathrm{GL}_r(\varphi_L)(A). \qquad (**)$$

Since $\rho_{\mathcal{O}_L^r}(g) \in \mathrm{GL}_n(k)$ for all $g \in G_L$, we have that

$$\mathrm{GL}_r(\psi)(\rho_{\mathcal{O}_L^r}(g)) = \rho_{\mathcal{O}_L^r}(g)$$

for any $\psi \in \mathrm{End}_{\mathcal{O}_L-Alg}(\mathbb{A})$. With this and since φ_L commutes with $\rho(g)$ for every $g \in G_L$ one calculates, that

$$\begin{aligned}(g_1g_2).A &= g_1.(g_2.A),\\ \pi.(g.A) &= g.(\pi.A),\\ \pi^{n+m}.A &= \pi^n.(\pi^m.A)\ \forall g, g_1, g_2 \in G_L,\ A \in \mathrm{GL}_r(\mathbb{A}),\ n, m \in \mathbb{N}.\end{aligned}$$

With this it makes sense to define $\gamma.A$ for every $\gamma \in \mathcal{O}_L^\bullet$ and

$$A \in \mathrm{GL}_r(\mathbb{A})^{H_L, \rho_{\mathcal{O}_L^r}} := \{A \in \mathrm{GL}_r(\mathbb{A}) \mid h.A = A\ \forall h \in H_L\}$$

and it is

$$\gamma.A = (\gamma.((A_{i1})_i), \ldots, \gamma.((A_{ir})_i))$$

for all such γ, where $\gamma.((A_{i1})_i)$ denotes the $\mathcal{O}_L^\bullet$-action on

$$((\mathbb{A})^r)^{H_L} \cong \mathbb{D}(\underline{\rho_{\mathcal{O}_L^r}}).$$

Finally we calculate for $A \in \mathrm{GL}_n(\mathbb{A})$ and $g \in G_L$, that

$$g.(\sigma_{\mathbb{A}}(A)) = \sigma_{\mathbb{A}}(f(g)) \cdot \mathrm{GL}_m(\rho(g))(\sigma_{\mathbb{A}}(A)) = \sigma_{\mathbb{A}}(f(g)) \cdot \sigma_{\mathbb{A}}(\mathrm{GL}_n(\rho(g))(A)) = \sigma_{\mathbb{A}}(g.A).$$

For the first equality, we used, that

$$\sigma_k = \sigma_{\mathbb{A} \mid \mathrm{GL}_n(k)},$$

for the second equality, we used, that σ is a natural transformation between GL_n and GL_m and for the last equality, we used, that $\sigma_{\mathbb{A}}$ is a morphism of groups. With this, we have shown, that $(\overline{y_k})_k$ is H_L-invariant. Analogues, it is

$$\pi.(\sigma_{\mathbb{A}}(A)) = \sigma_{\mathbb{A}}(\pi.A).$$

Let $\gamma \in \mathcal{O}_L^\bullet$ and

$$A_\gamma := \alpha_{\mathbb{D}(\rho_f), (\overline{x_i})_i}(\gamma).$$

For the given diagram to commute we need to show, that

$$\sigma_{\mathbb{A}}(A_\gamma) = \alpha_{\mathbb{D}(\rho_{\sigma_{\mathcal{O}_L}} \circ f), (\overline{y_k})_k}(\gamma),$$

again using, that

$$\sigma_{\mathbb{A}|\,\mathrm{GL}_n(\mathbb{A}_L)} = \sigma_{\mathbb{A}_L}.$$

For that, we calculate

$$\overline{y}\alpha_{\mathbb{D}(\rho_{f,\sigma}),(\overline{y_k})_k}(\gamma) = \gamma.\overline{y} = \gamma.\sigma_{\mathbb{A}}(\overline{x}) = \sigma_{\mathbb{A}}(\gamma.\overline{x}) = \sigma_{\mathbb{A}}(\overline{x})\sigma_{\mathbb{A}}(A_\gamma) = \overline{y}\sigma_{\mathbb{A}}(A_\gamma).$$

Furthermore, because of $(**)$, we have that for $A_0 := \bar{x}$, it is

$$\alpha_{\mathbb{D}(\underline{\rho_f}),(x_i)_i} = \bar{x}^{-1}\gamma.\bar{x} = \alpha_{f,A_0}.$$

$\square$

Remark. This Proposition can also be proven for $\mathcal{R}$ instead of $\mathbb{A}_L$. Furthermore, one can generalize this for $\sigma : \mathbb{G}_1 \to \mathbb{G}_2$ with finite $K|L$ instead of $K = L$, but one has to begin with $f, f' \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))$, such that

$$\bar{j}_{\mathcal{R}}(f) = \bar{j}_{\mathcal{R}}(f')$$

and then, instead of using bases $(x_i)_i, (y_i)_i$, one has to work with

$$X \in \mathbb{G}_1(\mathcal{R}^{nr})^{f,f',H_K} \text{ and } Y \in \mathbb{G}_2(\mathcal{R}^{nr})^{\sigma_{\mathcal{O}_L}\circ f,\sigma_{\mathcal{O}_L}\circ f',H_K}$$

and instead of using

$$\alpha_{\mathbb{D}(\rho_f),(x_i)_i}, \alpha_{(\mathbb{D}\rho_{\sigma_{\mathcal{O}_L}\circ f}),(x_i)_i},$$

one has to work with with

$$\alpha_{f,X} \text{ and } \alpha_{\sigma_{\mathcal{O}_L}\circ f,Y},$$

so that we obtain that for every such X there exists such an Y (e.g. $Y := \sigma_{\mathcal{R}^{nr}}(X)$) giving a commutative diagram

$$\begin{tikzcd}
\mathbb{O}_K \arrow[r, "\alpha_{f,X}"] \arrow[dr, "\alpha_{\sigma_{\mathcal{O}_L}\circ f,Y}"'] & \mathbb{G}_1^{(f')}(\mathcal{R}) \arrow[d, "\sigma_{\mathcal{R}^{nr}}"] \\
& \mathbb{G}_2^{(\sigma_{\mathcal{O}_L}\circ f')}(\mathcal{R})
\end{tikzcd}$$

The second part of the last Proposition together with Lemma 3.2.10 closes the proof of Theorem 1.3.18.

Faithfully Flat Descent

We have seen that the (φ_L, Γ_K)-side of the correspondence takes values in points of formal group schemes. But for the trivial map $1 \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))$, we have seen that the corresponding group were points of the linear algebraic group $\mathbb{G}$ itself. We can generalize this for those $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))$, which are also continuous for the discrete topology on $\mathbb{G}(\mathcal{O}_L)$. This needs some statements from the general theory of faithfully flat descent, which we will recall now. We will follow (Gö10, section (14.20)). Let Γ be a group and S be a scheme. We define

$$\Gamma \times S := \coprod_{\gamma \in \Gamma} S$$

the scheme given by glueing along the disjoint union. The canonical map

$$\Gamma \times S \to S$$

is faithfully flat (See Gö10, Remark 14.8). $\Gamma \times S$ is the constant group scheme of Γ over S. It follows that an action of Γ on an S-scheme S' via S-automorphisms corresponds to a morphism

$$(\Gamma \times S) \underset{S}{\times} S' \to S',$$

via $(\gamma, s') \mapsto \gamma \cdot s'$ on the T-valued points for a S'-scheme T.

Definition 3.2.18. Let Γ be finite. A *Galois covering with Galois group* Γ is a finite faithfully flat morphism $p : S' \to S$ together with an action of Γ on S' via S-automorphisms, such that the morphism

$$\sigma : \Gamma \times S' \to S' \underset{S}{\times} S',$$

given by $(\gamma, s') \mapsto (s', \gamma s')$ on the T-valued points for a S'-scheme T is an isomorphism.

Example 3.2.19. Let R be a complete discrete valuation ring and R_0 be a finite unramified extension of R, such that

$$\mathrm{Quot}(R_0) | \mathrm{Quot}(R)$$

is a Galois extension. Then we have the finite Galois group

$$G_{R_0|R} := \mathrm{Aut}_{R-Alg}(R_0) \cong \mathrm{Gal}(k_{R_0} | k_R),$$

where the isomorphism follows from Proposition 1.2.29.ii). It is R_0 fully faithful over R by (Bou72, I §2.4 Proposition 3.ii)) and (Mat86, Theorem 7.2). Furthermore, we have that

$$R_0 \cong R[X]/P(X)$$

for some separable polynomial $P(X) \in R[X]$ with $\#G_{R_0|R}$ distinct zeroes by Lemma 1.2.26 and the Theorem of Gauß for factorial rings (See Bos05, section 2.7 Satz 7). We deduce that

$$R_0 \underset{R}{\otimes} R_0 \cong \prod_{s \in G_{R_0|R}} R_0$$

by the Chinese Remainder Theorem, so the inclusion $\iota : R \to R_0$ is a Galois covering with Galois group $G_{R_0|R}$.

Definition 3.2.20. Let $S' \to S$ be a Galois covering with Galois group Γ and X' be an S'-scheme. Then a Γ-action on X' via S'-automorphisms is called *compatible*, if the following diagram commutes for every $\gamma \in \Gamma$.

$$\begin{tikzcd} X' \arrow[r, "\gamma"] \arrow[d] & X' \arrow[d] \\ S' \arrow[r, "\gamma"'] & S' \end{tikzcd}$$

Theorem 3.2.21. *(See Gö10, Theorem 14.84)*
Let S be an affine scheme and $S' \to S$ be a Galois covering with Galois group Γ. Then the functor

$$(\textit{quasi-projective } S\textit{-schemes}) \to (\textit{quasi-projective } S'\textit{-schemes with compatible } \Gamma\textit{-action})$$
$$X \mapsto (X \underset{S}{\times} S', (\mathrm{id}_{X'}, \phi_{can}))$$

is an quasi equivalence of categories. Here ϕ_{can} denotes the Γ-action on S' given by the Galois covering. Let $S' = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ both be affine and $X' = \mathrm{Spec}(R)$ be an affine scheme of finite type over S' with an compatible Γ-action, i.e. a Γ-action on R, which is semilinear for the Γ-action on B given by the Galois covering. Then the descent is given by the invariants R^Γ and the natural isomorphism for the quasi equivalence is the multiplication

$$B \underset{A}{\otimes} R^\Gamma \to R.$$

Proof. For the second part about the affine case, look into how a compatible Γ-action gives a descent datum in the discussion before (Gö10, Theorem 14.84) and look into step (i) in the proof of (Gö10, Theorem 14.66) for the explicit form of the descent given by a descent datum in this affine case and that the multiplication induces an isomorphism. $\square$

Let $\mathcal{R}$ and $\mathcal{R}^{nr}$ be again as defined in the beginning of this part. Let furthermore $\mathcal{R}^{un}$ be the maximal unramified extension of $\mathcal{R}$ in $W(\mathbb{C}_p^\flat)_L$.

Remark 3.2.22. The ring $\mathcal{R}^{nr}$ is the π-adic completion of $\mathcal{R}^{un}$.

Proof. This is just by definition for the case $\mathcal{R} = \mathbb{A}_L$.

For $\mathcal{R} = W(\mathbb{F})_L$, let $F|\mathbb{F}$ be a finite extension in $\overline{F}$. By Proposition 1.2.29.ii), there exists a finite unramified extension C of the quotient field $\mathrm{Quot}(W(\mathbb{F})_L)$ with residue field F. By the universal property of the maximal unramified extension (See Kle16, Satz 2.1.10.ii)) or by a variant of (Sch17, Lemma 3.1.2) there exists a lift of the q-Frobenius $(\cdot)^q : F \to F$ on $\mathcal{O}_C$, which we denote by $\varphi_C : \mathcal{O}_C \to \mathcal{O}_C$. It follows that we can deduce from Lemma 1.1.13 that

$$\mathcal{O}_C \cong W(F)_L,$$

in particular $W(F)_L \subset W(\overline{\mathbb{F}})_L$ is the finite unramified extension of $W(\mathbb{F})_L$ with residue field F, which is unique by Proposition 1.2.29.ii). It follows that

$$W(\mathbb{F}_L)^{un} = \bigcup_{\substack{F|\mathbb{F} \\ finite}} W(F)_L.$$

So let $x = (x_n)_n \in W(\overline{\mathbb{F}})_L$. Then for every $n \geq 1$ there exists a finite extension $F_n|F$ with $F_n \subset \overline{\mathbb{F}}$, such that $x_i \in F_n$ for every $i \leq n$. Define a sequence

$$y^{(n)} \in W(F_n)_L \subset W(\mathbb{F}_L)^{un}$$

by setting

$$y_i^{(n)} := \begin{cases} x_i, & \text{if } i \leq n \\ 0 & \text{elsewhere} \end{cases} .$$

Then in the π-adic topology, we have

$$\lim_n y^{(n)} = x.$$

So $W(\mathbb{F}_L)^{un} \subset W(\overline{\mathbb{F}})_L$ is π-adically dense. $\square$

We obtain the following special case for our example above.

Proposition 3.2.23. *Let A be a $\mathcal{R}^{un}$-algebra of finite type with a semilinear and discrete H_K-action of $\mathcal{R}$-algebras. Then the map given by multiplication*

$$\mathcal{R}^{un} \bigotimes_{\mathcal{R}} A^{H_K} \to A$$

is an isomorphism.

Proof. For any open normal subgroup $N \subset H_K$, we define

$$\mathcal{R}_N := (\mathcal{R}^{un})^N,$$

which is a finite unramified extension over $\mathcal{R}$. Choose an isomorphism

$$f : \mathcal{R}^{un}[X_1, \ldots, X_n]/I \to A.$$

Choose generators $I = \langle P_1, \ldots, P_m \rangle$. Since the H_K-action is discrete, finitely many elements are fixed by an open normal subgroup. It follows that the set $\{h \cdot f(X_i) \mid h \in H_K,\ 1 \leq i \leq n\}$ is finite. So we can choose N small enough such that N fixes all the $f(X_i),\ 1 \leq i \leq n$,

$$P_i \in \mathcal{R}_N[X_1, \ldots, X_n]\ \forall 1 \leq i \leq m \tag{P}$$

and

$$h \cdot f(X_j) \in A_0 := R_N[f(X_1), \ldots, f(X_n)]\ \forall h \in H_K, 1 \leq j \leq n. \tag{G}$$

From (P) it follows that $(I \cap \mathcal{R}_N[X_1, \ldots, X_n]) \cdot \mathcal{R}^{un}[X_1, \ldots, X_n] = I$, so by right exactness of the tensor product, we have that the multiplication induces an isomorphism

$$\mathcal{R}^{un} \bigotimes_{\mathcal{R}_N} A_0 \to A.$$

Since $\mathcal{R}^{un}$ is faithfully flat over $\mathcal{R}_N$ by (Bou72, I §2.4 Proposition 3.ii)) and (Mat86, Theorem 7.2), it follows that the $\mathcal{R}_N$-algebra A_0 is of finite type (See Gö10, Proposition 14.46). It furthermore follows that we are reduced to show that the map induced by the multiplication

$$\mathcal{R}_N \bigotimes_{\mathcal{R}} A_0^{H_K/N} \to A_0$$

is an isomorphism. By (G) the $\mathcal{R}_N$-algebra A_0 is H_K-invariant and of finite type over $\mathcal{R}_N$. So by Example 3.2.19 we can use Theorem 3.2.21, by which we obtain that the multiplication induces an isomorphism as desired. $\square$

Let $\mathbb{G}$ again be a linear algebraic group over $\mathcal{O}_L$ with fixed embedding $\mathbb{G} \subset \mathrm{GL}_n$ and $\mathbb{G}_{\mathcal{R}^{un}} = \mathrm{Spec}(H)$ for a Hopfalgebra (of finite type) over $\mathcal{R}^{un}$. Let $A_{\mathcal{R}^{un}} := \mathrm{Aut}_{\mathcal{R}^{un}}(\mathbb{G}_{\mathcal{R}^{un}})$ be the group of automorphisms of the group scheme $\mathbb{G}_{\mathcal{R}^{un}}$ over $\mathcal{R}^{un}$. As in part 2.1.3, we obtain a H_K-action of groups $A_{\mathcal{R}^{un}}$ by conjugating $f \in A_{\mathcal{R}^{un}}$ with $(\mathrm{id}_{\mathbb{G}}, \mathrm{Spec}(h^{-1}))$ for $h \in H_K$. As in Lemma 2.1.23, we obtain an H_K-equivariant map

$$\Phi : \mathbb{G}(\mathcal{R}^{un}) \to A_{\mathcal{R}^{un}}$$

by sending $g \in \mathbb{G}(\mathcal{R}^{un})$ to $[\mathbb{G}(S) \ni x \mapsto gxg^{-1} \in \mathbb{G}(S)]$ for all $\mathcal{R}^{un}$-algebras S. Here, we view $\mathbb{G}(\mathcal{R}^{un})$ as an H_K-group via $\mathbb{G}(\rho(h))$ for every $h \in H_K$.

For every cocycle $c \in C^1(H_K, \mathbb{G}(\mathcal{R}^{un}))$, which is continuous for the discrete topology, we obtain a cocycle $c^{(a)} := \Phi \circ c \in C^1(H_K, A_{\mathcal{R}^{un}})$, which is continuous for the discrete topology on $A_{\mathcal{R}^{un}}$. As described in Remark 2.1.21, the cocycle $c^{(a)}$ induces an H_K-semilinear action of Hopfalgebras over $\mathcal{R}^{un}$ on H, which continuous for the discrete topology on H via $h \mapsto (\mathrm{id}_{\mathbb{G}}, \mathrm{Spec}(h)) \circ c^{(a)}(h)^{-1},\ h \in H_K$. We get a Hopfalgebra

$$H^{(c),alg} := H^{H_K}$$

over $\mathcal{R}$ for the invariants under the action defined by $c^{(a)}$ above. Then

$$\mathbb{G}^{(c)} := \mathrm{Spec}(H^{(c),alg})$$

is an $\mathcal{R}^{un}|\mathcal{R}$-Form of $\mathbb{G}_{\mathcal{R}}$ by Proposition 3.2.23. This means that the multiplication

$$\mathcal{R}^{un} \underset{\mathcal{R}}{\otimes} H^{(c),alg} \to H$$

is an isomorphism, so we have an identification $\mathbb{G}^{(c)}_{\mathcal{R}^{un}} \cong \mathbb{G}_{\mathcal{R}^{un}}$. As calculated for Remark 2.1.21.(1), this identification gives and identification

$$\mathbb{G}^{(c)}(\mathcal{R}^{un}) \cong \mathbb{G}(\mathcal{R}^{un}),$$

which satisfies

$$\mathbb{G}^{(c)}(\rho(h))(A) = c(h) \cdot \mathbb{G}(\rho(h))(A) \cdot c(h)^{-1}\ \forall h \in H_K, A \in \mathbb{G}(\mathcal{R}^{un}). \qquad (T_2)$$

We obtain the following identification of the groups, which we have on the (φ_L, Γ_K)-side of the correspondence.

Lemma 3.2.24. *We continue the notation from the discussion above. The group $\mathbb{G}^{(c)}$ is of finite type over $\mathcal{R}$ and for every $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))$, which is also continuous for the discrete topology on $\mathbb{G}(\mathcal{O}_L)$, we have*

$$\mathbb{G}^{(f)}(\mathcal{R}) = \{A \in \mathbb{G}(\mathcal{R}^{nr}) \mid A = f(h)\mathbb{G}(\rho(h))(A)f(h)^{-1}\ \forall h \in H_K\} = \mathbb{G}^{(f)}_{\mathcal{R}}.$$

Here we view $f \in C^1(H_K, \mathbb{G}(\mathcal{R}^{nr}))$ via the restriction $H_K \subset G_K$ and the inclusion $\mathcal{O}_L \subset \mathcal{R}^{nr}$.

Proof. Since $\mathcal{R}^{un} \underset{\mathcal{R}}{\otimes} H^{(c),alg} \cong H$ and $\mathcal{R}^{un}$ is faithfully flat over $\mathcal{R}$ by (Bou72, I §2.4 Proposition 3.ii)) and (Mat86, Theorem 7.2), we have that $H^{(c),alg}$ is of finite type (See Gö10, Proposition 14.46).

Since $\mathcal{R}^{nr}$ is the π-adic completion of $\mathcal{R}^{un}$ by Remark 3.2.22, we have that

$$\mathbb{G}(\mathcal{R}^{un}) \subset \mathbb{G}(\mathcal{R}^{nr}) \cong \varprojlim \mathbb{G}(\mathcal{R}^{un}/\pi^n\mathcal{R}^{un})$$

is dense for the pro discrete topology on the right hand side. Futhermore, the topological group $\mathbb{G}(\mathcal{R}^{nr})$ is complete for the pro discrete topology. To see this, one has to check the two following facts. Firstly, the pro discrete topology is the subset topology $\mathbb{G}(\mathcal{R}^{nr}) \subset \mathrm{GL}_n(\mathcal{R}^{nr})$, where the topology on $\mathrm{GL}_n(\mathcal{R}^{nr})$ is induced by the π-adic topology on $\mathcal{R}^{nr}$, which is π-adically complete. Secondly, the subset $\mathbb{G}(\mathcal{R}^{nr}) \subset \mathrm{GL}_n(\mathcal{R}^{nr})$ is closed for this topology as a subset of zeroes of polynomials with coefficients in $\mathcal{O}_L$. It follows that two continuous endomorphisms on $\mathbb{G}(\mathcal{R}^{nr})$, which are equal on $\mathbb{G}(\mathcal{R}^{un})$ are already equal everywhere. In particular, we have

$$\mathbb{G}^{(f)}(\rho(h))(A) = f(h) \cdot \mathbb{G}(\rho(h))(A) \cdot f(h)^{-1} \; \forall h \in H_K, A \in \mathbb{G}(\mathcal{R}^{nr})$$

by (T_2) and since $\mathbb{G}^{(f)}(\rho(h))$ and the map on the righthandside of this equality are continuous for the pro discrete topology of $\mathbb{G}(\mathcal{R}^{nr})$, since $\rho(h)$ is a continuous automorphism for the π-adic topology on $\mathcal{R}^{nr}$ and f is continuous for the (pro) discrete topology of $\mathbb{G}(\mathcal{R}^{nr})$. Thus, the equality

$$\mathbb{G}^{(f)}(\mathcal{R}) = \{A \in \mathbb{G}(\mathcal{R}^{nr}) \mid A = f(h)\mathbb{G}(\rho(h))(A)f(h)^{-1} \; \forall h \in H_K\}$$

holds by Lemma 2.1.25. □

3.2.3 Thoughts on the Quotientfield Case

In the last part of this thesis, we will give a short discussion how the results in the last part can be lifted to the case of the quotientfields of our discrete valuation rings. We now introduce some more notation. We define

$$\mathcal{E}_K := \mathrm{Quot}(\mathbb{A}_L),\; \mathcal{E} := \mathrm{Quot}(\mathbb{A}).$$

and

$$\mathcal{F}_K := \mathrm{Quot}(W(\mathbb{F})_L),\; \mathcal{F} := \mathrm{Quot}(W(\overline{F})_L).$$

We set

$$(\mathcal{S}, \mathcal{S}^{nr}) \in \{(\mathcal{E}_K, \mathcal{E}), (\mathcal{F}_K, \mathcal{F})\}$$

and its ring of integers $(\mathcal{R}, \mathcal{R}^{nr})$, so again

$$(\mathcal{R}, \mathcal{R}^{nr}) \in \{(\mathbb{A}_K, \mathbb{A}), (W(\mathbb{F})_L, W(\overline{\mathbb{F}})_L)\}$$

As over their ring of integers, we have

$$G_{\mathcal{S}^{nr}|\mathcal{S}} := \mathrm{Aut}_{\mathcal{S}}(\mathcal{S}^{nr}) \overset{\cong}{\to} H_K.$$

Let

$$\rho : G_K \to \mathrm{Aut}_{L-Alg}(\mathcal{S}^{nr})$$

be the continuation of $\rho : G_K \to \mathrm{Aut}_{\mathcal{O}_L-Alg}(\mathcal{R}^{nr})$ and analoguesly denote

$$\varphi_L \in \mathrm{End}_{L-alg}(\mathcal{S}^{nr}).$$

Then $\rho_{|H_K}$ coincides with the natural $G_{\mathcal{E}_{\overline{\mathbb{F}}}}$- action via the above isomorphism. It follows, that we get a $\mathbb{O}_K$-action on $\mathcal{S}$ denoted by $\tau(\gamma)$ for every $\gamma \in \mathbb{O}_K$. This is the continuation of the $\mathbb{O}_K$-action on $\mathcal{R}$.

Let $\mathbb{G}$ be a linear algebraic group over $\mathcal{O}_L$ with a fixed embedding $\mathbb{G} \subset \mathrm{GL}_n$ and set for $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))$ the groups

$$\mathbb{G}_{\mathcal{S}}^{(f)} := \{A \in \mathbb{G}(\mathcal{S}^{nr}) \mid A = f(h)\mathbb{G}(\rho(h))(A)f(h)^{-1}\}.$$

This can be made into an $\mathbb{O}_K$-group as in the integral case. We denote the action by

$$\gamma \underset{f}{*} A \text{ for } \gamma \in \mathbb{O}_K, A \in \mathbb{G}(\mathcal{S}^{nr}).$$

Definition 3.2.25. We set

$$C^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{S}}^{(f)}) := \{c : \mathbb{O}_K \to \mathbb{G}_{\mathcal{S}}^{(f)} \mid c(\gamma\delta) = c(\gamma)\gamma \underset{f}{*} c(\delta) \ \forall \gamma, \delta\}.$$

We set $c \sim d$ to be the usual cohomology equivalence for $c, d \in C^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{S}}^{(f)})$

$$C^{1,Int}(\mathbb{O}_K, \mathbb{G}_{\mathcal{S}}^{(f)}) := \{c \in C^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{S}}^{(f)}) \mid \exists c_0 \in C^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{R}}^{(f)}) : c \sim c_0\}.$$

Take caution that with $C^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{R}}^{(f)})$, we mean those cocycles, who are continuous on Γ_K.

An element $\alpha \in C^{1,Int}(\mathbb{O}_K, \mathbb{G}_{\mathcal{S}}^{(f)})$ is called an *etale (φ_L, Γ_K)–module over $\mathcal{S}$ with values in $\mathbb{G}^{(f)}$.*

Since $\mathbb{G} \subset \mathrm{GL}_n$, there exists by Hilbert 90 and Corollary 3.2.6 a $B \in \mathrm{GL}_n(\mathcal{R}^{nr})$, such that

$$f(h) = B^{-1}\,\mathrm{GL}_n(\rho(h))(B) \ \forall h \in H_K.$$

Then we have two embeddings

$$\iota_1 : \mathbb{G}_{\mathcal{S}}^{(f_i)} \subset \mathbb{G}(\mathcal{S}^{nr}) \subset \mathrm{GL}_n(\mathcal{S}^{nr}), A \mapsto A$$

and

$$\iota_2 : \mathbb{G}_{\mathcal{S}}^{(f_i)} \to \mathrm{GL}_n(\mathcal{S}), A \mapsto B\iota_1(A)B^{-1}.$$

This at least gives us that the subgroups $\mathbb{G}_{\mathcal{S}}^{(f_i)}$ can be seen as subgroups of matrices with entries in the smaller ring $\mathcal{R}$.

Definition 3.2.26. We define

$\mathrm{mor}^{Int}(G_K, \mathbb{G}(L)) :=$
$\{f \in mor(G_K, \mathbb{G}(L)) \mid \exists B \in \mathbb{G}(L), f' \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L)) : \ f(g) = B^{-1}f'(g)B \ \forall g \in G_K\}.$

Remark. By definition we have canonical bijections

$$(\mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))/ \underset{L}{\sim}) \to \mathrm{mor}^{Int}(G_K, \mathbb{G}(L))/ \sim$$

and

$$(C^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{R}}^{(f)})/ \underset{\mathcal{S}}{\sim}) \to H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{\mathcal{S}}^{(f)}),$$

where the conjugation (resp. cohomology) relations are those over the quotientfield, i.e. given by conjugation (resp. $\mathbb{O}_K$-twisted conjugation) with $B \in \mathbb{G}(L)$ (resp. $B \in \mathbb{G}_{\mathcal{S}}^{(f)}$).

Let

$$\bar{\jmath}_{\mathcal{S}} : \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L)) \to H^1(H_K, \mathbb{G}(\mathcal{S}^{nr}))$$

be the map induced by restriction to $H_K \subset G_K$ and inclusion $\mathcal{O}_L \subset \mathcal{S}^{nr}$. We fix a subset $\{f_i\}_i \subset \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))$, such that

$$\bar{\jmath}_{\mathcal{S}} : \{f_i\}_i \to \mathrm{im}(\bar{\jmath}_{\mathcal{S}})$$

is bijective.

Warning: Since we have no comparison from $H^1(H_K, \mathbb{G}(\mathcal{S}^{nr}))$ to the characteristic p case, it might happen here that for $\mathcal{S} = \mathcal{E}_K$ and for $\mathcal{S} = \mathcal{F}_K$ there are different subsets $\{f_i^{(\mathcal{E})}\}_i$ and $\{f_i^{(\mathcal{F})}\}_i$ satisfying their respective condition.

Let $f \in mor^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))$ and f_i such that

$$\bar{\jmath}_{\mathcal{S}}(f) = \bar{\jmath}_{\mathcal{S}}(f_i).$$

We define for every $A \in \mathbb{G}(\mathcal{S}^{nr}), g \in G_K$ and $n \geq 1$

$$g.A := f(g) \cdot \mathbb{G}(\rho(g))(A) f_i(g)^{-1}, \pi^n.A := \mathbb{G}(\varphi_L^n)(A).$$

We furthermore define

$$\mathbb{G}(\mathcal{S}^{nr})^{H_L,f,f_i} := \{A \in \mathbb{G}(\mathcal{S}^{nr}) \mid h.A = A \ \forall A\}.$$

This is not empty because of

$$\bar{j}_{\mathcal{S}}(f) = \bar{j}_{\mathcal{S}}(f_i).$$

We get a $\mathbb{O}_K$-action on $\mathbb{G}(\mathcal{S}^{nr})^{H_L,f}$, which we denote by $\gamma.A$. For every $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L))$, we choose $A_0 \in \mathbb{G}(\mathcal{S}^{nr})^{H_L,f,f_i}$ and define

$$\alpha_{f,A_0}(\gamma) := A_0^{-1}\gamma.A_0 \in \mathbb{G}(\mathcal{E}_{\mathbb{F}}).$$

As in the integral case we see, that

$$\alpha_{f,A_0}(\gamma) \in C^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{S}}^{(f_i)})$$

and is up to cohomology independent on the choice of A_0. We get a map

$$\mathbb{D} : \mathrm{mor}^{Int}(G_K, \mathbb{G}(L)) \to \coprod_i H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{\mathcal{S}}^{(f_i)}), f \mapsto [\alpha_{f,A_0}]_\sim =: \alpha_f,$$

which induces a map on the set of conjugacyclasses

$$\mathrm{mor}^{Int}(G_K, \mathbb{G}(L))/\sim.$$

As in the characteristic p and the integral case, we get the following results.

Proposition 3.2.27. *The map*

$$\mathbb{D} : (\mathrm{mor}^{Int}(G_K, \mathbb{G}(L))/\sim) \to \coprod_i H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{\mathcal{S}}^{(f_i)})$$

is injective with image

$$\coprod_i \{[\alpha]_\sim \in H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{\mathcal{S}}^{(f_i)}) \mid \alpha \in C^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{R}}^{(f_i)}),\ \alpha(\pi) \in \mathrm{im}(\Psi_{\mathcal{R}^{nr}})\}.$$

This bijection identifies

$$\begin{aligned}&\{[a]_\sim \in \mathrm{mor}^{Int}(G_K, \mathbb{G}(L))/\sim \ \mid a \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L)),\ \bar{j}_{\mathcal{S}}(a) = \bar{j}_{\mathcal{S}}(f_i)\}\\ \cong&\{[\alpha]_\sim \in H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{\mathcal{S}}^{(f_i)}) \mid \alpha \in C^1(\mathbb{O}_K, \mathbb{G}_{\mathcal{R}}^{(f_i)}),\ \alpha(\pi) \in \mathrm{im}(\Psi_{\mathcal{R}^{nr}})\}\end{aligned}$$

for every i.

Lemma 3.2.28. *Let $\phi : \mathbb{G}_1 \to \mathbb{G}_2$, such that the induced map*

$$(\phi_{\mathcal{S}^{nr}})_* : H^1(H_K, \mathbb{G}_1(\mathcal{S}^{nr})) \to H^1(H_K, \mathbb{G}_2(\mathcal{S}^{nr}))$$

is injective on $\mathrm{im}(\bar{j}^{\mathbb{G}_1}_{\mathcal{R}})$. *Then for any choice* $\{f_i^{(1)}\}_i \subset \mathrm{mor}^{cont}(G_K, \mathbb{G}_1(\mathcal{O}_L))$, *such that*

$$\bar{j}^{\mathbb{G}_1}_{\mathcal{S}} : \{f_i^{(1)}\}_i \to \mathrm{im}(\bar{j}^{\mathbb{G}_1})$$

is bijective, we can complement $\{\phi_{\mathcal{O}_L} \circ f_i\}_i \subset \mathrm{mor}^{cont}(G_K, \mathbb{G}_2(\mathcal{O}_L))$ *to a subset* $\{f_l^{(2)}\}_l$, *such that*

$$\bar{j}^{\mathbb{G}_2}_{\mathcal{S}} : \{f_l^{(2)}\}_l \to \mathrm{im}(\bar{j}^{\mathbb{G}_2})$$

is bijective. Furthermore, the following diagram is commutative.

$$\begin{tikzcd}
(\mathrm{mor}^{Int}(G_K, \mathbb{G}_1(L))/\sim) \arrow[r, "\mathbb{D}"] \arrow[d, "(\phi_L)_*"'] & \coprod_i H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{1,\mathcal{S}}{}^{(f_i^{(1)})}) \arrow[d, "(\phi_{\mathcal{S}^{nr}})_*"] \\
(\mathrm{mor}^{Int}(G_K, \mathbb{G}_2(L))/\sim) \arrow[r, "\mathbb{D}"'] & \coprod_l H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{2,\mathcal{S}}{}^{(f_l^{(2)})}).
\end{tikzcd}$$

In this case, we also have a generalization of the previous Lemma.

Lemma 3.2.29. *Let* $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ *be a morphism of groupschemes over* $\mathcal{O}_L$. *Choose some representatives* $\{f_i^{(1)}\}_i$ *for* $\mathbb{G}_1$ *and* $\{f_l^{(2)}\}_l$ *for* $\mathbb{G}_2$ *as in the last Lemma. By definition of these representatives for any* $f_i^{(1)}$ *there exists a unique* $f_l^{(2)}$ *and some (non-unique)* $B_i \in \mathbb{G}_2(\mathcal{S}^{nr})$, *such that*

$$\phi_k \circ f_i^{(1)}(h) = B_i \cdot f_l^{(2)}(h) \cdot \mathbb{G}_2(\rho(h))(B_i^{-1}). \qquad (*)$$

Then the following diagram is commutative and the right vertical map is independent on the choice of B_i.

$$\begin{tikzcd}
(\mathrm{mor}^{Int}(G_K, \mathbb{G}_1(L))/\sim) \arrow[r, "\mathbb{D}"] \arrow[d, "(\phi_L)_*"'] & \coprod_i H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{1,\mathcal{S}}^{(f_i^{(1)})}) \arrow[d, "\coprod_i [\alpha \mapsto [\gamma \mapsto B_i^{-1} \cdot \phi_{\mathcal{S}^{nr}} \circ \alpha(\gamma) \cdot \gamma . B_i]]"] \\
(\mathrm{mor}^{Int}(G_K, \mathbb{G}_2(L))/\sim) \arrow[r, "\mathbb{D}"'] & \coprod_l H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{2,\mathcal{S}}^{(f_l^{(2)})}).
\end{tikzcd}$$

Here for $\gamma = \mathrm{pr}_{H_L}(g_\gamma)\pi^{n_\gamma} \in \mathbb{O}_K$, *we have*

$$\gamma . B_i := \gamma \underset{\phi_{\mathcal{O}_L} \circ f_i^{(1)}, f_l^{(2)}}{.} B := \phi_{\mathcal{O}_L} \circ f_i^{(1)}(g_\gamma) \cdot \mathbb{G}_2(\rho(g_\gamma) \circ \varphi_L^{n_\gamma})(B_i) \cdot f_l^{(2)}(g_\gamma)^{-1}.$$

This is well defined by arguments as in the discussion before Lemma 2.2.7.

Theorem 3.2.30. *If $\mathbb{G}$ is smooth and $\mathbb{G}_k$ is connected, then*

$$\mathbb{D} : (\mathrm{mor}^{Int}(G_K, \mathbb{G}(L))/\sim) \to \coprod_i H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{\mathcal{S}}^{(f_i)})$$

is bijective. This bijection identifies

$$\{[a]_\sim \in \mathrm{mor}^{Int}(G_K, \mathbb{G}(L))/\sim \ \mid a \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_L)),\ \bar{j}_{\mathcal{S}}(a) = \bar{j}_{\mathcal{S}}(f_i)\}$$
$$\cong H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{\mathcal{S}}^{(f_i)})$$

for every i.

Furthermore, we obtain the following statement for the comparison between the nonperfect and the perfect case.

Theorem 3.2.31. *Let the map*

$$\mathrm{im}(\bar{j}_{\mathcal{E}_K}) \to \mathrm{im}(\bar{j}_{\mathcal{F}_K})$$

induced by the inclusion $\mathcal{E} \subset \mathcal{F}$ be bijective. Then for any $f \in \mathrm{mor}^{cont}(G_K, \mathbb{G}(\mathcal{O}_K))$ the inclusion

$$\mathbb{A}_K \subset W(\mathbb{F})_L$$

induces a bijection between the sets

$$\{[\alpha]_\sim \in H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{\mathcal{E}_K}^{(f)}) \mid \alpha \in C^1(\mathbb{O}_K, \mathbb{G}_{\mathbb{A}_K}^{(f)}), \alpha(\pi) \in \mathrm{im}(\Psi_{\mathbb{A}})\}$$
$$\tilde{\to}\{[\alpha]_\sim \in H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{\mathcal{F}_K}^{(f)}) \mid \alpha \in C^1(\mathbb{O}_K, \mathbb{G}_{W(\mathbb{F})_L}^{(f)}), \alpha(\pi) \in \mathrm{im}(\Psi_{W(\mathbb{F})_L})\}$$

If $\mathbb{G}$ is smooth and $\mathbb{G}_k$ is connected then the inclusion induces a bijection

$$H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{\mathcal{E}_K}^{(f)}) \tilde{\to} H^{1,Int}(\mathbb{O}_K, \mathbb{G}_{\mathcal{F}_K}^{(f)}).$$

Proof. As in the integral case, one uses the statements before this one together with the assumption. $\square$

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