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p-Adic Weil Group Representations

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Abstract

We study Weil group representations over the coefficient field $\mathbb{Q}_p$ and establish certain equivalences of categories in the flavor of FONTAINE's classification of p-adic representations of the absolute Galois group. If we restrict to crystalline (or de-Rham) Weil group representations, we can describe the category of these Weil group representations in terms of generators. More precisely it is generated as an abelian tensor category by the full subcategory of Galois group representations and finite unramified inductions of the character $\mathbb{Q}_p(|\cdot|)$ given by ARTIN's reciprocity law.

Contents

Introduction

Let p be a prime, $K/\mathbb{Q}_p$ a finite field extension with ring of integers $\mathcal{O}_K$ and residue field k, denote by $\overline{K}$ an algebraic closure of K. It is a fundamental problem in Number Theory to understand the structure of the (local) absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$. An usual strategy in many fields of mathematics to understand the structure of a group is the study of its representation theory. The (local) Langlands program suggests that there is a deep connection between the representations of G_K and representations of reductive groups. Over time several approaches were made to give this idea a concrete incarnation. The classical local Langlands correspondence provided by Harris-Taylor [HT01] and Henniart [Hen00] for GL_n relates certain (more precisely: irreducible admissible) representations of $\mathrm{GL}_n(K)$ over $\mathbb{C}$ and certain n-dimensional (more precisely: semisimple Weil-Deligne) representations of W_K over $\mathbb{C}$, where the Weil group W_K is (as an abstract group) the subgroup of G_K consisting of all automorphisms whose restriction to the residual Galois group G_k is an integral power of the Frobenius automorphism. Due to Grothendieck's (l-adic) Monodromy Theorem [Tat79, §4] the latter category of Weil-Deligne representations of W_K over $\mathbb{C}$ is equivalent to usual l-adic representations of W_K where $l \neq p$. In contrast to the l-adic case p-adic Hodge theory only deals with Galois representations instead of Weil group representations, which raises the natural question how both concepts can be linked. In this thesis we will study the difference between categories of Galois group representations and Weil group representations over the coefficient field $\mathbb{Q}_p$.

More precisely we modify Fontaine's classification of p-adic Galois representations (given in [Fon90] and [Fon94a]) with the intention to fit Weil group representations into the picture. If we restrict the problem to de-Rham representations, we receive enough structure on the corresponding modules to completely describe the Weil group representations as subquotients of Galois group representations twisted by induced representations of the character given by Artin's reciprocity law.

In **chapter 1** we collect general statements about Weil group representations. It is pointed out that Weil group representations are the same as Galois group representations over the coefficient rings $\mathbb{F}_p, \overline{\mathbb{F}}_p$ and $\mathbb{Z}_p$. If one considers representations with coefficients in $\mathbb{Q}_p$, this is false. The character given by ARTIN's reciprocity law $\mathbb{Q}_p(|\cdot|)$ is a Weil group representation but can't be extended to a Galois group representation. We introduce an axiomatic setting in which we adjust the theory of B-admissible representations (e.g. given in [BC09]) to our purposes. In particular we define a B-admissible Weil group representation by requesting that the restriction to a representation of the inertia group is B-admissible. Afterwards we prove that the category of B-admissible Weil group representations is equivalent to the category of pairs (D, F) where D is the object consisting of "linear algebra data" FONTAINE associates to representations of the inertia group and F is a semilinear operator satisfying certain extra conditions (see Axioms 1.1 to 1.5), essentially the linearization of F has to define an isomorphism.

In **chapter 2** we introduce the period rings, which are required in order to define crystalline, log-crystalline (i.e. semistable) and de-Rham representations. We use the language introduced by SCHOLZE [Sch11] of perfectoid fields and tilts in order to reduce the wild amount of notation to a minimum. During this excursion we recapitulate the basic facts about these rings. We explicitly calculate the $\mathrm{Gal}(\overline{K}/F)$-invariants of B_{dR} (see Theorem 2.16) and B_{st} (see Lemma 2.35) for an algebraic extension F/K such that $\hat{F} \subseteq \mathbb{C}_p$ is a perfectoid field.

In **chapter 3** we apply the theory of B-admissible representations developed in chapter 1 to the period rings mentioned in chapter 2. By checking that the axioms formulated before hold in this situation we receive several equivalences of categories, which describe certain categories of B-admissible (e.g. crystalline, log-crystalline, de-Rham) Weil group representations in terms of linear algebra data. These equivalences (see Theorem 3.12 and Theorem 3.20) are based on the well-known equivalences of categories for (crystalline, log-crystalline, de-Rham) p-adic representations of the inertia group I_K. We endow the objects of linear algebra data with an additional operator F that is highly compatible with the given structures and mimics a lift of the Frobenius in $W_K \subseteq G_K$.

In **chapter 4** we give a complete treatment of the case of (potentially) log-crystalline representations, which is by the p-adic Monodromy Theorem (see [Ber02]) the same as dealing with de-Rham representations. It turns out that a Weil group representation can be lifted to a Galois group representation if and only if the corresponding (admissible filtered φ-)module (D, F)

has Newton slope 0 with respect to F (see Theorem 4.7). Hence we decompose the module (D, F) along the semilinear map F via the Classification Theorem of Dieudonne-Manin. This is possible since such a decomposition is compatible with the additional structures (see Theorem 4.19) on the module D. In the last step we take powers of every summand and then "tilt" it to Newton slope 0 by forming the tensor product with a representation induced from $\mathbb{Q}_p(|\cdot|)$. This leads to the **main result** (see Theorem 4.25): The category of (potentially) log-crystalline Weil group representations is generated (as a tensor category) by the full subcategory of Galois group representations and induced representations of the character $\mathbb{Q}_p(|\cdot|)$.

In **chapter 5** we treat the case of general p-adic representations. We use the main result from chapter 1 once again to construct categories of linear algebra data which classify (general) mod-p representations of W_K (see Theorem 5.7). This construction works out in a similar way in the case of p-adic Weil group representations (see Theorem 5.11).

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Chapter 1

Weil Group Representations

Once and for all we fix the following *Notations:* We denote by

- k the finite field with $q = p^f$ elements.
- $G_k \cong \hat{\mathbb{Z}}$ the absolute Galois group of k.
- K_0 the fraction field of the ring of Witt vectors $W(k)$.
- K/K_0 a purely ramified finite Galois extension contained in a fixed algebraic closure $\overline{K}$ of K.
- $\mathcal{O}_K \subseteq K$ the ring of integral elements with maximal ideal (π).
- G_K the absolute Galois group of K.
- $\deg_K \colon G_K \to G_k \cong \hat{\mathbb{Z}}$ the canonical projection.
- $I_K := \ker(\deg_K)$ the absolute inertia group of K.
- $K^{nr} := \bigcup_{r \in \mathbb{N}} K(\mu_{p^r-1})$ the maximal unramified extension of K.
- P_0 the completion of the maximal unramified extension of $\mathbb{Q}_p$.
- σ_K an element of G_K such that $\deg_K(\sigma_K) = 1$.
- σ the continuous automorphism of P_0 such that $\sigma(x) \equiv x^p \mod p$.
- K_∞ the algebraic extension of K given by adjoining all p-power roots of unity to K.
- $W(\cdot)$ the functor that attaches to a ring R the ring of (unramified) Witt vectors $W(R)$. We denote (multiplicative) Teichmüller map by $\tau_R \colon R \to W(R)$ and neglect the index if no confusion is possible.

1.1 Trivia about the Weil Group

We call $W_K := \deg_K^{-1}(\mathbb{Z})$ the Weil group of K and consider it as a topological group endowed with the coarsest topology such that:

- the subspace topology on I_K is the usual (profinite) topology of I_K.
- I_K is open in W_K.

Then

$$1 \to I_K \overset{\subseteq}{\to} W_K \xrightarrow{\deg_K} \mathbb{Z} \to 1$$

is an exact sequence of topological groups, where $\mathbb{Z}$ is endowed with the discrete topology. W_K is a dense subset of G_K since $\mathbb{Z}$ is dense in $\hat{\mathbb{Z}} \cong G_k$. The Weil group naturally embeds into the context of local class field theory in the following way. For a finite abelian extension L/K the local *norm residue symbol*

$$(\cdot, L/K) \colon K^\times \to \mathrm{Gal}(L/K)$$

is an epimorphism of topological groups with kernel $N_{L/K}(L^\times)$ [Neu86, Chapter III, Theorem (2.1)], which maps $\mathcal{O}_K^\times$ onto $I(L/K)$ and the group $1 + \mathfrak{m}_K^n$ onto the n-th ramification group $G^n(L/K)$ with respect to the upper numbering [Neu86, Chapter III, Theorem (8.10)]. Let K^{ab} denote the maximal abelian extension of K. By passing to the projective limit we obtain that $(\mathcal{O}_K^\times, K^{ab}/K) \subseteq I(K^{ab}/K)$ is dense but since $\mathcal{O}_K^\times$ is compact this actually an equality. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}_K^\times & \xrightarrow{\subseteq} & K^\times & \xrightarrow{\nu_K} & \mathbb{Z} & \longrightarrow & 0\,. \\
 & & \downarrow{\scriptstyle (\cdot, K^{ab}/K)} & & \downarrow{\scriptstyle (\cdot, K^{ab}/K)} & & \downarrow{\scriptstyle =} & & \\
1 & \longrightarrow & I(K^{ab}/K) & \xrightarrow{\subseteq} & W_K^{ab} & \xrightarrow{\deg_K} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Since the outer vertical maps are surjective we obtain that the inner vertical arrow maps onto W_K^{ab}. By the *existence theorem* [Neu86, Chapter III, Theorem (3.1)]

$$\bigcap_{L/K \text{ finite abelian}} N_{L/K}(L^\times) \subseteq \bigcap_{f,n} (\pi_K^f) \times 1 + \mathfrak{m}_K^n = \{1\}.$$

Hence the map $(\cdot, K^{ab}/K) \colon K^\times \overset{\cong}{\to} W_K^{ab}$ is an isomorphism of topological groups. We call its inverse

$$r_K \colon W_K^{ab} \to K^\times$$

the *reciprocity law* of local class field theory.

1.2 p-adic Representations

Definition 1.1 *Let G be a locally compact topological group and E be a normed field. An E-representation of G is a finite dimensional vector space V over E together with a continuous group homomorphism $\rho\colon G \to \mathrm{Aut}_E(V)$ (where $\mathrm{Aut}_E(V) \cong \mathrm{GL}_n(E)$ is endowed with the topology induced by the norm on E, which is independent of the choice of the base of V). We define a morphism of E-representations of G to be an E-linear map that is G-equivariant and denote the corresponding category by $\mathrm{Rep}_E(G)$. The category $\mathrm{Rep}_{\mathbb{Q}_p}(G)$ of p-adic representations of G will be denoted by $\mathrm{Rep}(G)$.*

An important lemma in the case where G is a profinite group is the following.

Lemma 1.2 *Let R be a valuation ring with field of fractions E. For any profinite group G and each object V of $\mathrm{Rep}_E(G)$ there exists a G-stable R-lattice $M \subseteq V$.*

We (literally) imitate the proof of [BC09, Lemma 1.2.6.].

Proof: Let $\rho\colon G \to \mathrm{Aut}_E(V)$ be the continuous group homomorphism that defines V. Take an arbitrary R-lattice $M_0 \subseteq V$ and obtain the commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}_R(M_0) & \hookrightarrow & \mathrm{Aut}_E(V) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{GL}_d(R) & \hookrightarrow & \mathrm{GL}_d(E), \end{array}$$

where $d = \dim_E(V)$. Since $\mathrm{GL}_d(R)$ is an open subgroup of $\mathrm{GL}_d(E)$ the preimage $G_0 := \rho^{-1}(\mathrm{Aut}_R(M_0))$ is open in G, in particular G/G_0 is finite. Therefore

$$M := \sum_{gG_0 \in G/G_0} \rho(g)(M_0)$$

is a well-defined R-lattice in V that is G-stable. □

Example 1.3 *Consider the continuous homomorphism of groups given by*

$$\rho\colon W_K \twoheadrightarrow W_K^{ab} \xrightarrow{r_K} K^\times \xrightarrow{|\cdot|_K} p^{\mathbb{Z}} \subseteq \mathbb{Q}_p^\times,$$

where the first arrow is the canonical projection. This defines a one-dimensional p-adic representation of W_K, which we will denote by $\mathbb{Q}_p(|\cdot|_K)$ in the following. The map ρ does not extend (continuously) to a map $\hat{\rho}\colon G_K \to \mathbb{Q}_p^\times$.

Assume this would be the case. As a profinite group G_K is compact and thus its image $\hat{\rho}(G_K) \subseteq \mathbb{Q}_p^\times$ would be compact, in particular bounded. But $\hat{\rho}(G_K)$ would contain $p^{\mathbb{Z}}$ which is unbounded. For another way to see this, we apply Lemma 1.2. Since there is an element $\sigma_K \in W_K$ such that $\rho(\sigma_K) = p^{-1}$ there can be no $\mathbb{Z}_p$-lattice which is invariant under a (hypothetical) action of G_K.

Remark 1.4 *Let E be a normed field. Then any E-representation of G_K restricts to an E-representation of W_K since the topology on W_K is finer than the subspace topology on W_K inherited from G_K. On the other hand any E-representation of W_K that extends to an E-representation of G_K does this in an unique way since W_K is dense in G_K. Therefore we consider $\mathrm{Rep}_E(G_K)$ as a full subcategory of $\mathrm{Rep}_E(W_K)$. By the preceding example these categories are not equivalent via restriction in the case of $E = \mathbb{Q}_p$.*

1.3 Mod-p- and $\mathbb{Z}_p$-Representations

Let E now be a local field with finite residue field. We remark that both exact sequences in the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_E & \xrightarrow{\subseteq} & W_E & \xrightarrow{\deg_E} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{\subseteq} & & \downarrow{\subseteq} & & \\ 1 & \longrightarrow & I_E & \xrightarrow{\subseteq} & G_E & \xrightarrow{\deg_E} & \hat{\mathbb{Z}} & \longrightarrow & 0 \end{array}$$

split via choosing an element $\sigma_E \in \deg_E^{-1}(1)$. We see that

$$G_E = I_E \rtimes \hat{\mathbb{Z}} \text{ and } W_E = I_E \rtimes \mathbb{Z}$$

as topological groups. This emphasizes the significance of the following statement [Bou71, III, Prop. 28], which will be used to prove that certain actions of W_E extend continuously to G_E-actions if the corresponding $\mathbb{Z}$-action extends continuously to a $\hat{\mathbb{Z}}$-action.

Proposition 1.5 *Let L, N be topological groups and $\tau\colon L \to \mathrm{Aut}(N)$ a group homomorphism such that*

$$N \times L \to N, (x, y) \mapsto \tau(y)(x)$$

is continuous (with respect to the product topology on the source). For continuous group homomorphisms $f\colon N \to G$ and $g\colon L \to G$ into a topological group G, such that $f(\tau(y)(x)) = g(y)f(x)g(y^{-1})$ holds for all $x \in N$ and $y \in L$, the group homomorphism $N \rtimes L \to G$ given by $(x, y) \mapsto f(x)g(y)$ is continuous. In particular $N \rtimes L$ endowed with the product topology is a topological group.

According to [RZ00, Chapter 4] (or more generally Lemma 4.1) every group homomorphism $\mathbb{Z} \to G$ into a profinite group G extends continuously to a group homomorphism $\hat{\mathbb{Z}} \to G$. This has the following consequences.

Corollary 1.6 *Let q be a power of p. The forgetful functor*

$$\mathcal{F}\colon \mathrm{Rep}_{\mathbb{F}_q}(G_E) \to \mathrm{Rep}_{\mathbb{F}_q}(W_E)$$

is an equivalence of categories.

Proof: It is enough to show that any representation $\rho\colon W_E \to \mathrm{Aut}_{\mathbb{F}_q}(V) \cong \mathrm{GL}_d(\mathbb{F}_q)$ can be lifted to a representation of G_E. Choose $\sigma_E \in \deg_E^{-1}(1)$ and obtain a group homomorphism $f\colon \mathbb{Z} \to \mathrm{GL}_d(\mathbb{F}_q)$ given by $1 \mapsto \rho(\sigma_E)$. Since $\mathrm{GL}_d(\mathbb{F}_q)$ is finite f extends to a continuous homomorphism $f\colon \hat{\mathbb{Z}} \to \mathrm{GL}_d(\mathbb{F}_q)$ and we use Proposition 1.5 to extend ρ via f to a continuous group homomorphism $\hat{\rho}\colon G_E \to \mathrm{GL}_d(\mathbb{F}_q)$. □

Let R be a complete discrete valuation ring with finite residue field and maximal ideal (t). We remark that the functor GL_d from the category of rings to the category of sets is representable and therefore preserves projective limits by [ML78, V.4. Theorem 1]. Then the same argument as above still works if we consider free R-representations (i.e. finitely generated free R-modules equipped with a continuous linear action of G_E) since

$$\mathrm{GL}_d(R) \cong \varprojlim_n \mathrm{GL}_d(R/(t^n)) \text{ (as topological groups)}$$

is profinite. (Hence any group homomorphism $\mathbb{Z} \to \mathrm{GL}_d(R)$ lifts to a continuous group homomorphism $\hat{\mathbb{Z}} \to \mathrm{GL}_d(R)$, see for example [RZ00, §4.1.]).

Corollary 1.7 *The forgetful functor*

$$\mathcal{F}\colon \mathrm{Rep}_R(G_E) \to \mathrm{Rep}_R(W_E)$$

is an equivalence of categories.

This holds for $R = \mathbb{Z}_p$ in particular.

1.4 Formalism of Admissibility

In this section we extend the formalism of admissibility (see e.g. [BC09, I.5.]) in order to extend it to representations of Weil groups.

Let G denote a profinite group, $I \subseteq G$ a closed normal subgroup such that $G/I \cong \hat{\mathbb{Z}}$ and denote by $\deg\colon G \twoheadrightarrow \hat{\mathbb{Z}}$ the composition of this isomorphism with the canonical projection $G \twoheadrightarrow G/I$. Choose an element

$$\varsigma \in \deg^{-1}(1) \subseteq G.$$

The group homomorphism

$$\mathbb{Z} \to \mathrm{Aut}(I) \text{ given by } n \mapsto (u \mapsto \varsigma^n u \varsigma^{-n})$$

is continuous. Set $W := I \rtimes \mathbb{Z}$ (with respect to the map above), which we understand as a subgroup of G via $(u, n) \mapsto u\varsigma^n$, and endow it with the product topology of I (which carries the topology inherited by G) and $\mathbb{Z}$ (which carries the discrete topology).

Let (F, σ) denote either the pair $(\mathbb{F}_{p^s}, \bar{\sigma})$, where $\bar{\sigma}\colon x \mapsto x^{p^r}$ is the r-th power of the usual Frobenius map, or the pair $(W(\mathbb{F}_{p^s})[\frac{1}{p}], \sigma)$, where $\sigma = W(\bar{\sigma})[\frac{1}{p}]$ for some $\mathbb{N} \ni r \leq s \in \mathbb{N} \cup \{\infty\}$. Assume that $B \supseteq F$ is a topological ring that carries an action of G such that $B^G \subset B^I$ are fields endowed with a Frobenius endomorphism σ which extends the Frobenius on F and commutes with the action of G. In the following the term φ-module refers to modules endowed with a σ-semilinear map φ.

Now we want to introduce the concept of admissibility (with respect to an (E, G)-regular ring B). Hence let E be the fixed field of F with respect to σ and B be an (E, G)-*regular ring*, i.e. $B \supseteq F$ is an E-domain that carries an action of G such that $\mathrm{Frac}(B)^G = B^G$ is a field and for all $b \in B$ such that $E \cdot b$ is G stable we have $b \in B^\times$. We also assume B to be (E, I)-regular. Recall [BC09, §5.2.] that an E-representation of G (resp. I) is called *admissible* if

$$\dim_{B^G}(B \otimes_E V)^G = \dim_E(V) \text{ (resp. } \dim_{B^I}(B \otimes_E V)^I = \dim_E(V)).$$

For the group W we vary this kind of definition for our purposes as follows.

Definition 1.8 *An E-representation V of W is called B-admissible if the restriction $V|_I$ is a B-admissible representation of I. We denote the full subcategory of* $\mathrm{Rep}_E(W)$ *containing only the B-admissible E-representations of W by* $\mathrm{Rep}_E^B(W)$.

One may consider the assignments $V \mapsto (B \otimes_E V)^G$ (resp. $V \mapsto (B \otimes_E V)^I$) as functors from the category $\mathrm{Rep}_E^B(G)$ (resp. $\mathrm{Rep}_E^B(I)$) to the category of φ-modules over B^G (resp. B^I). As usual we denote the "comparison morphisms" as follows. Let

$$\tilde{\alpha}'_\bullet \colon B \otimes_{B^I} (B \otimes_E (\bullet))^I \to B \otimes_E (\bullet)$$

denote the natural transformation given by

$$\tilde{\alpha}'_V \colon \sum_{i,j} b_i \otimes b_{ij} \otimes v_j \mapsto \sum_j (\sum_i b_i b_{ij}) \otimes v_j$$

for all objects V in $\mathrm{Rep}_E^B(I)$ and let

$$\tilde{\beta}'_\bullet \colon B \otimes_{B^I} (B \otimes_{B^I} (\bullet))^{\varphi=\mathrm{id}} \to B \otimes_{B^I} (\bullet)$$

denote the natural transformation given by

$$\tilde{\beta}'_M \colon \sum_{i,j} b_i \otimes b_{ij} \otimes m_j \mapsto \sum_j (\sum_i b_i b_{ij}) \otimes m_j$$

for all φ-modules M over B^I. These are natural transformations of E-linear additive tensor functors which means the following:

Definition 1.9 *Let $\mathcal{C}$ and $\mathcal{D}$ be E-linear abelian tensor categories and let $F, G \colon \mathcal{C} \to \mathcal{D}$ denote E-linear additive tensor functors. We call a natural transformation $t_\bullet \colon F \dashrightarrow G$ a natural transformation of E-linear additive tensor functors if the diagram*

$$\begin{array}{ccc} F(X_1 \otimes_{\mathcal{C}} X_2) & \xrightarrow{t_{X_1 \otimes_{\mathcal{C}} X_2}} & G(X_1 \otimes_{\mathcal{C}} X_2) \\ \downarrow \cong & & \downarrow \cong \\ F(X_1) \otimes_{\mathcal{D}} F(X_2) & \xrightarrow{t_{X_1} \otimes_{\mathcal{D}} t_{X_2}} & G(X_1) \otimes_{\mathcal{D}} G(X_2) \end{array}$$

of vector spaces over E commutes for all objects X_1 and X_2 in $\mathcal{C}$. (The vertical arrows are given by the natural isomorphisms making the functors F and G tensor functors.)

Denote by $j_\bullet$ the (canonical) natural injective transformation

$$\mathrm{id}_{\mathrm{Rep}_E^B(I)} \dashrightarrow B^{\varphi=\mathrm{id}} \otimes (\bullet)$$

which is given by $j_V(v) = 1 \otimes v$.

We adopted the definitions and notations from [DM82] and assume that the following axiom holds:

Axiom 1.1 *There exists an E-linear abelian tensor category $\mathcal{C}_I$ together with an E-linear faithful additive tensor functor*

$$T\colon \mathcal{C}_I \to (\varphi\text{-modules over } B^I)$$

such that there exist two mutually inverse E-linear additive tensor functors

$$\tilde{\mathbb{D}}_B\colon \mathrm{Rep}_E^B(I) \to \mathcal{C}_I \text{ and } \tilde{\mathbb{V}}_B\colon \mathcal{C}_I \to \mathrm{Rep}_E^B(I),$$

in particular there exist natural isomorphisms of E-linear additive tensor functors

$$\tilde{\alpha}_\bullet\colon \tilde{\mathbb{V}}_B \circ \tilde{\mathbb{D}}_B \overset{\cong}{\dashrightarrow} \mathrm{id}_{\mathrm{Rep}_E^B(I)}$$

and

$$\tilde{\beta}_\bullet\colon \tilde{\mathbb{D}}_B \circ \tilde{\mathbb{V}}_B \overset{\cong}{\dashrightarrow} \mathrm{id}_{\mathcal{C}_I}.$$

We furthermore require these data to satisfy the following properties:

- *There exists an injective natural transformation of E-linear additive tensor functors*
$$\eta_\bullet\colon \tilde{\mathbb{V}}_B \dashrightarrow (B \otimes_{B^I} T(\bullet))^{\varphi=\mathrm{id}}$$
- *and there exists a natural isomorphism of E-linear additive tensor functors*
$$\xi_\bullet\colon T \circ \tilde{\mathbb{D}}_B \overset{\cong}{\dashrightarrow} (B \otimes_E (\bullet))^I$$

such that

$$\tilde{\alpha}''_\bullet := (\tilde{\alpha}'_\bullet)^{\varphi=\mathrm{id}} \circ (B \otimes_{B^I} \xi_\bullet)^{\varphi=\mathrm{id}} \circ \eta_{\tilde{\mathbb{D}}_B(\bullet)}\colon \tilde{\mathbb{V}}_B(\tilde{\mathbb{D}}_B(\bullet)) \dashrightarrow (B \otimes_E (\bullet))^{\varphi=\mathrm{id}}$$

is an injective natural transformation of E-linear additive tensor functors satisfying $j_\bullet \circ \tilde{\alpha}_\bullet = \tilde{\alpha}''_\bullet$ and

$$\tilde{\beta}''_\bullet := (\tilde{\beta}'_\bullet)^I \circ (B \otimes_E \eta_\bullet)^I \circ \xi_{\tilde{\mathbb{V}}_B(\bullet)}\colon T(\tilde{\mathbb{D}}_B(\tilde{\mathbb{V}}_B(\bullet))) \overset{\cong}{\dashrightarrow} (B \otimes_{B^I} T(\bullet))^I \ (\overset{\cong}{\dashrightarrow} T(\bullet))$$

is a natural isomorphism of E-linear additive tensor functors satisfying

$$T(\tilde{\beta}_\bullet) = \tilde{\beta}''_\bullet.$$

Remark 1.10 $T(\mathrm{Hom}_{\mathcal{C}_I}(D_1, D_2)) \subseteq \mathrm{Hom}_{\varphi\text{-mod.}/B^I}(T(D_1), T(D_2))$ *is an E-subspace since T is E-linear and faithful.*

Remark 1.11 *Let (D, φ) be a φ-module over B^I. We define (B^I, ς) to be the B^I-module which is B^I as an abelian group and scalar multiplication is given by $\mu.b := \varsigma(\mu) \cdot b$ for all $\mu, b \in B^I$. Set*

$$\varsigma^*(D) := B^I \otimes_{B^I,\varsigma} D := (B^I, \varsigma) \otimes_{B^I} D.$$

We receive a map

$$\varsigma^*(\varphi)\colon \varsigma^*(D) \to \varsigma^*(D), \mu \otimes d \mapsto \sigma(\mu) \otimes \varphi(d).$$

Since σ commutes with the G-action on B the map is well-defined and it is σ-semilinear:

$$\begin{aligned}\varsigma^*(\varphi)(\mu \cdot \sum_i \mu_i \otimes m_i) &= \varsigma^*(\varphi)(\sum_i \mu\mu_i \otimes m_i)\\ &= \sum_i \sigma(\mu)\sigma(\mu_i) \otimes \varphi(m_i)\\ &= \sigma(\mu) \cdot \varsigma^*(\varphi)(\sum_i \mu_i \otimes m_i)\end{aligned}$$

holds for all $\sum_i \mu_i \otimes m_i \in \varsigma^*(D)$ and $\mu \in B^I$. This construction is functorial. Take a morphism

$$f\colon (D_1, \varphi_1) \to (D_2, \varphi_2)$$

of φ-modules over B^I and define

$$\varsigma^*(f)\colon \varsigma^*(D_1) \to \varsigma^*(D_2), \mu \otimes d \mapsto \mu \otimes f(d).$$

This map is B^I-linear and satisfies $\varsigma^*(f) \circ \varsigma^*(\varphi_1) = \varsigma^*(\varphi_2) \circ \varsigma^*(f)$, hence ς^* is a self-equivalence of categories. Furthermore

$$\iota(D)\colon \varsigma^*(D) \to D, 1 \otimes d \mapsto d$$

induces a ς^{-1}-semilinear bijection and for all $\mu \in B^I$ and $d \in D$

$$\begin{aligned}\iota(D)(\varsigma^*(\varphi)(\mu \otimes d)) &= \iota(D)(\sigma(\mu) \otimes \varphi(d))\\ &= (\varsigma^{-1} \circ \sigma)(\mu)\varphi(d)\\ &= (\sigma \circ \varsigma^{-1})(\mu)\varphi(d)\\ &= \varphi(\varsigma^{-1}(\mu)d)\\ &= \varphi(\iota(D)(1 \otimes \varsigma^{-1}(\mu)(d)))\\ &= \varphi(\iota(D)(\mu \otimes d))\end{aligned}$$

holds, i.e. $\iota(D) \circ \varsigma^*(\varphi) = \varphi \circ \iota(D)$ has been verified. In the same manner we check

$$\iota(D_2) \circ \varsigma^*(f) = f \circ \iota(D_1). \tag{1.1}$$

Lemma 1.12 *Let (D, φ) be a φ-module over B^I. Then there exists an isomorphism $\varsigma^*(\sigma^*(D)) \cong \sigma^*(\varsigma^*(D))$ of vector spaces.*

Proof: Consider the linear map given by

$$\varsigma^*(\sigma^*(D)) \to ((\varsigma \circ \sigma)^{-1})^*(D), \lambda \otimes \mu \otimes d \mapsto \lambda\varsigma(\mu) \otimes d.$$

This is well-defined and the inverse is given by $x \otimes d \mapsto x \otimes 1 \otimes d$. One receives an isomorphism $\sigma^*(\varsigma^*(D)) \cong ((\sigma \circ \varsigma)^{-1})^*(D)$ by interchanging the roles of σ and ς. But σ and ς commute and we obtain the claim. $\square$

In order to modify the category $\mathcal{C}_I$ such that it captures the structure of $\mathrm{Rep}_E^B(W)$ instead of $\mathrm{Rep}_E^B(I)$ we need the following axiom.

Axiom 1.2 *Assume that ς^* lifts to an equivalence of categories on $\mathcal{C}_I$, i.e. there exists an equivalence of categories from $\mathcal{C}_I$ to itself which we also denote by ς^* making the diagram*

$$\begin{array}{ccc} \mathcal{C}_I & \xrightarrow{\varsigma^*} & \mathcal{C}_I \\ \downarrow T & & \downarrow T \\ (\varphi\text{-mod. over } B^I) & \xrightarrow{\varsigma^*} & (\varphi\text{-mod. over } B^I) \end{array}$$

commutative.

This allows us to state the following definition.

Definition 1.13 *Denote by $\mathcal{C}_W$ the following category:*

- *The objects are pairs (D, F), where D is an object of $\mathcal{C}_I$ and F is a self-map of $T(D)$ such that $F^{\mathrm{lin},\varphi} := F \circ \iota(D)$ lifts (uniquely) to an isomorphism F^{lin} in $\mathcal{C}_I$, i.e. $T(F^{\mathrm{lin}}) = F^{\mathrm{lin},\varphi}$ holds.*
- *A morphism $(D_1, F_1) \to (D_2, F_2)$ in $\mathcal{C}_W$ consists of a morphism $f\colon D_1 \to D_2$ in $\mathcal{C}_I$ such that $f \circ F_1^{\mathrm{lin}} = F_2^{\mathrm{lin}} \circ \varsigma^*(f)$ holds.*
- *The composition of morphisms is the usual composition of maps.*

In order to define a functor from the category $\mathrm{Rep}_E^B(W)$ to $\mathcal{C}_W$ we need to define a self-map F_V of $T(\tilde{\mathbb{D}}_B(V))$ for any object V of $\mathrm{Rep}_E^B(W)$. Let F_V be the map determined by the commutative diagram

$$\begin{array}{ccc} T(\tilde{\mathbb{D}}_B(V)) & \xrightarrow{F_V} & T(\tilde{\mathbb{D}}_B(V)) \\ \cong \downarrow \xi_V & & \cong \downarrow \xi_V \\ (B \otimes_E V)^I & \xrightarrow{F_V^{\varphi}} & (B \otimes_E V)^I \end{array}$$

where the bottom map is given by

$$F_V^{\varphi}\colon \sum_i b_i \otimes v_i \mapsto \sum_i \varsigma.b_i \otimes \varsigma.v_i.$$

Remark that this map is well-defined since I is a normal subgroup of W and hence

$$u.F_V^{\varphi}(x) = \sum_i (u \circ \varsigma).b_i \otimes (u \circ \varsigma).v_i = F_V^{\varphi}((\varsigma^{-1} \circ u \circ \varsigma).x) = F_V^{\varphi}(x)$$

for all $u \in I_K$ and $x = \sum_i b_i \otimes v_i \in (B \otimes_E V)^I$. Furthermore

$$\begin{aligned}(\varphi \circ F_V^{\varphi})(x) &= \sum_i \varphi(\varsigma.b_i) \otimes \varsigma.v_i \\ &= \sum_i \varsigma.\varphi(b_i) \otimes \varsigma.v_i \\ &= (F_V^{\varphi} \circ \varphi)(x)\end{aligned}$$

holds for all $x = \sum_i b_i \otimes v_i \in (B \otimes_E V)^I$. Let $F_V^{\mathrm{lin},\varphi} := F_V \circ \iota(\tilde{\mathbb{D}}_B(V))$ denote the linearization of F_V. In order to show that this construction is indeed functorial we need to enforce the existence of a (unique) lift of $F_V^{\mathrm{lin},\varphi}$:

Axiom 1.3 *Assume that there exists a (unique) lift* $F_V^{\mathrm{lin}} \in \mathrm{Isom}_{\mathcal{C}_I}(\tilde{\mathbb{D}}_B(V))$ *such that* $T(F_V^{\mathrm{lin}}) = F_V^{\mathrm{lin},\varphi}$ *for any object* V *in* $\mathrm{Rep}_E^B(I)$.

Now consider a morphism $f\colon V_1 \to V_2$ in $\mathrm{Rep}_E^B(W)$. We see that

$$\begin{aligned}(F_{V_2}^{\varphi} \circ (B \otimes_E f)^I)(x) &= F_{V_2}^{\varphi}(\sum_i b_i \otimes f(v_i)) \\ &= \sum_i \varsigma.b_i \otimes \varsigma.f(v_i) \\ &= \sum_i \varsigma.b_i \otimes f(\varsigma.v_i) \\ &= ((B \otimes_E f)^I \circ F_{V_1}^{\varphi})(x)\end{aligned}$$

holds for all $x = \sum_i b_i \otimes v_i \in \tilde{\mathbb{D}}_B(V_1)$. Hence one obtains

$$\begin{aligned}
T(\tilde{\mathbb{D}}_B(f) \circ F^{\text{lin}}_{V_1} &= T(\tilde{\mathbb{D}}_B(f)) \circ F^{\text{lin},\varphi}_{V_1} \\
&= \xi^{-1}_{V_2} \circ (B \otimes_E f)^I \circ \xi_{V_1} \circ F_{V_1} \circ \iota(\tilde{\mathbb{D}}_B(V_1)) \\
&= \xi^{-1}_{V_2} \circ (B \otimes_E f)^I \circ F^{\varphi}_{V_1} \circ \xi_{V_1} \circ \iota(\tilde{\mathbb{D}}_B(V_1)) \\
&= \xi^{-1}_{V_2} \circ F^{\varphi}_{V_2} \circ (B \otimes_E f)^I \circ \xi_{V_1} \circ \iota(\tilde{\mathbb{D}}_B(V_1)) \\
&= F_{V_2} \circ T(\tilde{\mathbb{D}}_B(f)) \circ \iota(\tilde{\mathbb{D}}_B(V_1)) \\
&\overset{(1.1)}{=} F_{V_2} \circ \iota(\tilde{\mathbb{D}}_B(V_2)) \circ \varsigma^*(T(\tilde{\mathbb{D}}_B(f))) \\
&= T(F^{\text{lin}}_{V_2} \circ \varsigma^*(\tilde{\mathbb{D}}_B(f)))
\end{aligned}$$

and therefore $\tilde{\mathbb{D}}_B(f) \circ F^{\text{lin}}_{V_1} = F^{\text{lin}}_{V_2} \circ \varsigma^*(\tilde{\mathbb{D}}_B(f))$ holds since T is faithful.

For the case that $\eta_\bullet$ is not surjective (i.e. no natural isomorphism), we need to assume two more axioms. In the cases where this natural transformation is a natural isomorphism these axioms are satisfied automatically.

Axiom 1.4 *Assume that for all object D in $\mathcal{C}_I$ there exists a (unique) bijective map $\hat{\varsigma}_D$ making the following diagram commutative:*

$$\begin{array}{ccc}
\tilde{\mathbb{V}}_B(D) & \xrightarrow{\hat{\varsigma}_D} & \tilde{\mathbb{V}}_B(D) \\
\downarrow{\scriptstyle \eta_D} & & \downarrow{\scriptstyle \eta_D} \\
(B \otimes_{B^I} T(D))^{\varphi=\text{id}} & \xrightarrow{\hat{\varsigma}^{\varphi}_D} & (B \otimes_{B^I} T(D))^{\varphi=\text{id}}
\end{array}$$

where the bottom map is given by

$$\hat{\varsigma}^{\varphi}_D \colon \sum_i b_i \otimes d_i \mapsto \sum_i \varsigma.b_i \otimes d_i.$$

Remark 1.14 *The latter axiom makes sense since the bottom map is well-defined:*

$$\begin{aligned}
\varphi(\hat{\varsigma}^{\varphi}_D(x)) &= \varphi(\sum_i \varsigma.b_i \otimes d_i) \\
&= \sum_i \varphi(\varsigma.b_i) \otimes \varphi(d_i) \\
&= \sum_i \varsigma.\varphi(b_i) \otimes \varphi(d_i) \\
&= \hat{\varsigma}^{\varphi}_D(\varphi(x)) = \hat{\varsigma}^{\varphi}_D(x),
\end{aligned}$$

holds for all $x = \sum_i b_i \otimes d_i \in (B \otimes_{B^I} T(D))^{\varphi=\text{id}}$. If $\eta_\bullet$ is a natural isomorphism the axiom is satisfied automatically.

Axiom 1.5 *Assume that for all objects (D, F) in $\mathcal{C}_W$ there exists a (unique) bijective map $\hat{F}_D$ making the following diagram commutative:*

$$\begin{array}{ccc} \tilde{\mathbb{V}}_B(D) & \xrightarrow{\hat{F}_D} & \tilde{\mathbb{V}}_B(D) \\ \downarrow \eta_D & & \downarrow \eta_D \\ (B \otimes_{B^I} T(D))^{\varphi=\mathrm{id}} & \xrightarrow{\hat{F}_D^{\varphi}} & (B \otimes_{B^I} T(D))^{\varphi=\mathrm{id}} \end{array}$$

where the bottom map is given by

$$\hat{F}_D^{\varphi} \colon \sum_i b_i \otimes d_i \mapsto \sum_i b_i \otimes F(d_i).$$

Remark 1.15 *The latter axiom makes sense since the bottom map is well-defined:*

$$\begin{aligned} \varphi(\hat{F}_D^{\varphi}(x)) &= \varphi(\sum_i b_i \otimes F(d_i)) \\ &= \sum_i \varphi(b_i) \otimes (\varphi \circ F)(d_i) \\ &= \sum_i \varphi(b_i) \otimes (F \circ \varphi)(d_i) \\ &= \hat{F}_D^{\varphi}(\varphi(x)) = \hat{F}_D^{\varphi}(x), \end{aligned}$$

holds for all $x = \sum_i b_i \otimes d_i \in (B \otimes_{B^I} T(D))^{\varphi=\mathrm{id}}$. If $\eta_\bullet$ is a natural isomorphism the axiom is satisfied automatically.

We need the following relations in order to prove the theorem below.

Remark 1.16 *Let (D, F) denote an object of $\mathcal{C}_W$. The following relations are immediate from the definitions:*

- $\hat{\varsigma}_D^{\varphi} \circ \hat{F}_D^{\varphi} = \hat{F}_D^{\varphi} \circ \hat{\varsigma}_D^{\varphi}$ *holds and hence we also have* $\hat{\varsigma}_D \circ \hat{F}_D = \hat{F}_D \circ \hat{\varsigma}_D$.
- $\hat{F}_D(ux) = u.\hat{F}_D(x)$ *for all* $u \in I$ *and* $x \in \tilde{\mathbb{V}}_B(D)$.
- $\hat{\varsigma}(ux) = (\varsigma u \varsigma^{-1}).\hat{\varsigma}_D(x)$ *holds for all* $x \in \tilde{\mathbb{V}}_B(D)$.

Now we are set to prove that the category $\mathcal{C}_W$ constructed above is indeed equivalent to $\mathrm{Rep}_E^B(W)$.

Theorem 1.17 *The additive E-linear tensor functor*

$$\mathrm{Rep}_E^B(W) \to \mathcal{C}_W \text{ given by } V \mapsto (\tilde{\mathbb{D}}_B(V), F_V),$$

provides an equivalence of categories.

Proof: Let (D, F) be an object of $\mathcal{C}_W$. We set $V := \tilde{\mathbb{V}}_B(D)$, which is then an E-representation of I. In order to define a W-action on V let

$$W \times V \to V \text{ be given by } (g, v) \mapsto g.v := (\hat{F}_D \circ \hat{\varsigma}_D)^{\deg(g)}(u.v).$$

For all $v \in V$ and $g_1 = \varsigma^{\deg(g_1)} \cdot u_1, g_2 = \varsigma^{\deg(g_2)} u_2 \in W$ such that $u_1, u_2 \in I$ we have

$$\begin{aligned} g_1.(g_2.v) &= g_1.((\hat{F}_D \circ \hat{\varsigma}_D)^{\deg(g_2)}(u_2.v)) \\ &= (\hat{F}_D \circ \hat{\varsigma}_D)^{\deg(g_1)}(u_1.(\hat{F}_D \circ \hat{\varsigma}_D)^{\deg(g_2)}(u_2.v)) \\ &= (\hat{F}_D \circ \hat{\varsigma}_D)^{\deg(g_1 g_2)}((\varsigma^{-\deg(g_2)} u_1 \varsigma^{\deg(g_2)} u_2).v) \\ &= (g_1 g_2).v \end{aligned} \tag{1.2}$$

since $g_1 g_2 = \varsigma^{\deg(g_1 g_2)}(\varsigma^{-\deg(g_2)} u_1 \varsigma^{\deg(g_2)} u_2)$ holds. Thus we indeed defined an W-action and it remains to check that the map above is continuous. By assumption its restriction to $I \times V \to V$ is continuous and I is open in W. Hence $I \times V$ is open in $W \times V$ and $W \times V \to V$ is therefore continuous. We claim that this procedure defines a functor which we also denote by $\tilde{\mathbb{V}}_B$ by slight abuse of notation. Consider a morphism $f \colon (D_1, F_1) \to (D_2, F_2)$ in $\mathcal{C}_W$ and for sake of brevity write

$$f^* := (B \otimes_{B^I} T(f))^{\varphi = \mathrm{id}}.$$

For all $v = \sum_i b_i \otimes d_i \in (B \otimes_{B^I} T(D_1))^{\varphi = \mathrm{id}}$ and $g \in W$ we have:

$$\begin{aligned} f^*(g.v) &= f^*(\sum_i g.b_i \otimes F_1^{\deg(g)}(d_i)) \\ &= \sum_i g.b_i \otimes (f \circ F_1^{\deg(g)})(d_i) \\ &= \sum_i g.b_i \otimes (F_2^{\deg(g)} \circ f)(d_i) \\ &= g.f^*(v). \end{aligned}$$

This enables the following calculation:

$$\begin{aligned}(\eta_{D_2} \circ \tilde{\mathbb{V}}_B(f))(g.v) &= (f^* \circ \eta_{D_1})(g.v) \\ &= f^* \circ \eta_{D_1} \circ (\hat{F}_{D_1} \circ \hat{\varsigma}_{D_1})^{\deg(g)}(u.v) \\ &= f^* \circ (\hat{F}^{\varphi}_{D_1} \circ \hat{\varsigma}^{\varphi}_{D_1})^{\deg(g)}(u.\eta_{D_1}(v)) \\ &= f^*(g.\eta_{D_1}(v)) \\ &= g.((f^* \circ \eta_{D_1})(v)) \\ &= g.(\eta_{D_2} \circ \tilde{\mathbb{V}}_B(f))(v) \\ &= (\hat{F}^{\varphi}_{D_2} \circ \hat{\varsigma}^{\varphi}_{D_2})^{\deg(g)}(u.(\eta_{D_2} \circ \tilde{\mathbb{V}}_B(f))(v)) \\ &= \eta_{D_2} \circ (\hat{F}_{D_2} \circ \hat{\varsigma}_{D_2})^{\deg(g)}(u.\tilde{\mathbb{V}}_B(f)(v)) \\ &= \eta_{D_2}(g.\tilde{\mathbb{V}}_B(f)(v))\end{aligned}$$

holds for all $g = \varsigma^{\deg(g)}u \in W$ and $v \in \tilde{\mathbb{V}}_B(D_1)$ and since η_{D_2} is injective we have

$$\tilde{\mathbb{V}}_B(f)(g.v) = g.\tilde{\mathbb{V}}_B(f)(v)$$

in particular. Thus $\tilde{\mathbb{V}}_B(f)$ is $E[W]$-linear indeed. In the last step of the proof we show that $\tilde{\mathbb{D}}_B$ and $\tilde{\mathbb{V}}_B$ are quasi-inverse functors (between $\mathrm{Rep}_E^B(W)$ and $\mathcal{C}_W$). It suffices to check that the comparison isomorphisms from Axiom 1.1 lift to isomorphisms in the current situation, i.e. we need to prove $E[W]$-linearity of the $E[I]$-linear natural isomorphism

$$\tilde{\alpha}_\bullet \colon \tilde{\mathbb{V}}_B \circ \tilde{\mathbb{D}}_B \overset{\cong}{\dashrightarrow} \mathrm{id}_{\mathrm{Rep}_E^B(I)}$$

and compatibility with F of the natural isomorphism

$$\tilde{\beta}_\bullet \colon \tilde{\mathbb{D}}_B \circ \tilde{\mathbb{V}}_B \overset{\cong}{\dashrightarrow} \mathrm{id}_{\mathcal{C}_I}.$$

We begin with the latter. For sake of brevity we denote the map $(B \otimes_E \eta_D)^I$ by η_D^*. Then take an element $x = \sum_i b_i \otimes v_i \in (B \otimes_E \tilde{\mathbb{V}}_B(D))^I$, expand the image $\eta_D(v_i) = \sum_j b_{ij} \otimes d_j \in (B \otimes_E (B \otimes_{B^I} T(D))^{\varphi=\mathrm{id}})^I$ and obtain:

$$\begin{aligned}
(\tilde{\beta}'_D)^I \circ \eta_D^* \circ \hat{F}^{\varphi}_{\tilde{\mathbb{V}}_B(D)}(x) &= (\tilde{\beta}'_D)^I \circ \eta_D^*(\sum_i \varsigma.b_i \otimes \varsigma.v_i) \\
&= (\tilde{\beta}'_D)^I(\sum_i \varsigma.b_i \otimes \eta_D(\varsigma.v_i)) \\
&= (\tilde{\beta}'_D)^I(\sum_i \varsigma.b_i \otimes \eta_D(\hat{F}_D \circ \hat{\varsigma}_D)(v_i))) \\
&= (\tilde{\beta}'_D)^I(\sum_i \varsigma.b_i \otimes \underbrace{(\hat{F}_D \circ \hat{\varsigma}_D)(\eta_D(v_i)))}_{=\sum_j \varsigma.b_{ij} \otimes F(d_j)}) \\
&= \sum_{i,j} \varsigma.b_i \varsigma.b_{ij} F(d_j) \\
&= (F \circ (\tilde{\beta}'_D)^I \circ \eta_D^*)(x).
\end{aligned}$$

Now we can conclude that $\tilde{\beta}_D$ is a morphism in $\mathcal{C}_W$:

$$\begin{aligned}
T(\tilde{\beta}_D) \circ F_{\tilde{\mathbb{V}}_B(D)} &= (\tilde{\beta}'_D)^I \circ \eta_D^* \circ \xi_{\tilde{\mathbb{V}}_B(D)} \circ \hat{F}_{\tilde{\mathbb{V}}_B(D)} \\
&= (\tilde{\beta}'_D)^I \circ \eta_D^* \circ \hat{F}^{\varphi}_{\tilde{\mathbb{V}}_B(D)} \circ \xi_{\tilde{\mathbb{V}}_B(D)} \\
&= F \circ (\tilde{\beta}'_D)^I \circ \eta_D^* \circ \xi_{\tilde{\mathbb{V}}_B(D)} \\
&= F \circ T(\tilde{\beta}_D)
\end{aligned}$$

implies

$$\begin{aligned}
T(\tilde{\beta}_D \circ F^{\mathrm{lin}}_{\tilde{\mathbb{V}}_B(D)}) &= T(\tilde{\beta}_D) \circ F_{\tilde{\mathbb{V}}_B(D)} \circ \iota(\tilde{\mathbb{D}}_B(V)) \\
&= F \circ T(\tilde{\beta}_D) \circ \iota(\tilde{\mathbb{D}}_B(V)) \\
&\overset{(1.1)}{=} F \circ \iota(\tilde{\mathbb{D}}_B(V)) \circ \varsigma^* T(\tilde{\beta}_D) \\
&= T(F^{\mathrm{lin}}_{\tilde{\mathbb{V}}_B(D)} \circ \varsigma^* \tilde{\beta}_D).
\end{aligned}$$

Therefore $\tilde{\beta}_D \circ F^{\mathrm{lin}}_{\tilde{\mathbb{V}}_B(D)} = F^{\mathrm{lin}}_{\tilde{\mathbb{V}}_B(D)} \circ \varsigma^* \tilde{\beta}_D$ holds since T is faithful. It is left to check that $\tilde{\alpha}_V$ is $E[W]$-linear for any V in $\mathrm{Rep}_E^B(I)$. Since j_V is $E[W]$-linear and injective it suffices to show that $\tilde{\alpha}''_V$ is $E[W]$-linear. Take an element $x \in (\tilde{\mathbb{V}}_B \circ \tilde{\mathbb{D}}_B)(V)$, write $\eta_{\tilde{\mathbb{D}}_B(V)}(x) = \sum_i b_i \otimes d_i \in (B \otimes_{B^I} T(\tilde{\mathbb{D}}_B(V)))^{\varphi=\mathrm{id}}$ as well as $\xi_V(d_i) = \sum_j b_{ij} \otimes v_j \in (B \otimes_E V)^I$ and abbreviate $\xi_V^* := (B \otimes_{B^I} \xi_V)^{\varphi=\mathrm{id}}$.

Then we have

$$\begin{aligned}
\tilde{\alpha}''_V(\varsigma.x) &= ((\tilde{\alpha}')^{\varphi=\mathrm{id}} \circ \xi^*_V \circ \eta_{\tilde{\mathbb{D}}_B(V)})(\varsigma.x) \\
&= ((\tilde{\alpha}')^{\varphi=\mathrm{id}} \circ \xi^*_V \circ \hat{F}^{\varphi}_{\tilde{\mathbb{D}}_B(V)} \circ \hat{\varsigma}^{\varphi}_{\tilde{\mathbb{D}}_B(V)}(\sum_i b_i \otimes d_i) \\
&= ((\tilde{\alpha}')^{\varphi=\mathrm{id}} \circ \xi^*_V(\sum_i \varsigma.b_i \otimes F_V(d_i)) \\
&= ((\tilde{\alpha}')^{\varphi=\mathrm{id}}(\sum_i \varsigma.b_i \otimes (\xi_V \circ F_V)(d_i)) \\
&= ((\tilde{\alpha}')^{\varphi=\mathrm{id}}(\sum_i \varsigma.b_i \otimes (F_V \circ \xi_V)(d_i)) \\
&= ((\tilde{\alpha}')^{\varphi=\mathrm{id}}(\sum_{i,j} \varsigma.b_i \otimes \varsigma.b_{ij} \otimes \varsigma.v_j) \\
&= \varsigma.(\sum_{i,j} b_i b_{ij} v_j) \\
&= \varsigma.(\tilde{\alpha}')^{\varphi=\mathrm{id}}(\sum_{i,j} b_i \otimes b_{ij} \otimes v_j) \\
&= \varsigma.\tilde{\alpha}''_V(x).
\end{aligned}$$

We conclude that $\tilde{\alpha}$ is an $E[W]$-linear isomorphism which finishes the proof. □

Example 1.18 *Take $E = \mathbb{F}_p$ (i.e. $r = 1$) and $B = \overline{\mathbb{F}}_p$. Then Theorem 1.17 recovers the fact that Galois group representations over $\mathbb{F}_p$ are just the same as Weil group representations over $\mathbb{F}_p$ as follows:*

$$\mathrm{Rep}_{\mathbb{F}_p}(W_{\mathbb{F}_p}) \sim \mathcal{C}_{W_{\mathbb{F}_p}} \sim (\varphi\text{-modules over } \mathbb{F}_p) \sim \mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{F}_p}).$$

The equivalence in the middle is given by $D \mapsto D^{F=\mathrm{id}}$ in one direction and by given by $M \mapsto (\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} M, F)$ in the opposite direction where F is given by

$$\sum_i \mu_i \otimes m_i \mapsto \sum_i \sigma(\mu_i) \otimes m_i$$

for all $x = \sum_i \mu_i \otimes m_i \in \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} M$.

Chapter 2

Period Rings

In this chapter we will introduce the so called period rings constructed by FONTAINE (see for example [Fon94a]) that serve well in order to give a hierarchy of p-adic Galois representations. We will give a slight generalization by constructing these period rings from a perfectoid field F that is contained in $\mathbb{C}_p$ rather than just starting with $\mathbb{C}_p$ itself. It will be proven that this variation behaves well with taking invariants under $\mathrm{Aut}(\mathbb{C}_p/F)$ (see Proposition 2.34 and 2.16).

2.1 Perfectoid Fields

Definition 2.1 *Let L be a valued field with respect to a nonarchimedian absolute value $|\cdot|\colon L \to \mathbb{R}_{\geq 0}$. We call L perfectoid if the following conditions are satisfied:*

1. *L is complete and the value group $|L^\times|$ is dense in $\mathbb{R}_{>0}$.*
2. *The ring homomorphism $\mathcal{O}_L/p \to \mathcal{O}_L/p$, $\bar{x} \to \bar{x}^p$ is surjective.*

We call this ring homomorphism the "mod p"-Frobenius of L.

Example 2.2 *Let $\mu_l \in \overline{\mathbb{Q}}_p$ denote the subgroup of l-th roots of unity for any $l \in \mathbb{N}$.*

- *The completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$ is perfectoid.*
- *The completion of $\mathbb{Q}_{p,\infty} := \bigcup_{n\geq 1} \mathbb{Q}_p(\mu_{p^n})$ is perfectoid.*
- *The completion of $\mathbb{Q}_p(p^{p^{-\infty}}) := \bigcup_{n\geq 1} \mathbb{Q}_p(p^{\frac{1}{p^n}})$ is perfectoid.*

- *The completion of* $\mathbb{Q}_p^{nr} := \mathrm{Frac}(W(\overline{\mathbb{F}}_p))$ *is not perfectoid, since its value group is discrete.*
- *The completion of* $\mathbb{Q}_p^{tr} := \bigcup_{p\nmid e} \mathbb{Q}_p^{nr}(p^{\frac{1}{e}})$ *is not perfectoid, since the "mod p"-Frobenius is not surjective.*
- *The completion of of the separable closure of* $\mathbb{F}_p((t))$ *is perfectoid.*
- *The completion of* $\mathbb{F}_p((t))(t^{p^{-\infty}}) := \bigcup_{n\geq 1} \mathbb{F}_p((t))(t^{\frac{1}{p^n}})$ *is perfectoid.*

Remark 2.3 *Any perfectoid field is perfect.*

2.2 Tilting

The concept of tilting was basically already introduced by FONTAINE in [Fon94a]. It turned out that this construction gives a deep connection between finite Galois extensions of perfectoid field in mixed characteristic $(0, p)$ and their 'tilts' in equal characteristic p (compare Theorem 2.7). We will stick to the notations introduced by SCHOLZE in [Sch11] who denoted the tilting functor by $F \mapsto F^\flat$. A detailed exposition can be found in [Sch17].

Let L be a perfectoid field such that $K \subseteq L$ and $\varpi \in L$ a pseudo uniformizer, i.e. ϖ satisfies $|\pi| \leq |\varpi| < 1$. Furthermore we set

$$\mathcal{O}_L := \{x \in L \mid |x| \leq 1\}.$$

Definition 2.4 *The map*

$$\phi\colon \mathcal{O}_L/\varpi\mathcal{O}_L \to \mathcal{O}_L/\varpi\mathcal{O}_L \text{ given by } \bar{x} \mapsto \bar{x}^q$$

is a ring homomorphism and we obtain a projective system $(\mathcal{O}_L/\varpi\mathcal{O}_L, \phi)_n$. *We define*

$$\mathcal{O}_{L^\flat} := \varprojlim_n (\mathcal{O}_L/\varpi\mathcal{O}_L, \phi)_n \text{ and } L^\flat := \mathrm{Frac}(\mathcal{O}_{L^\flat})$$

and call $L^\flat$ *the tilt of* L. *For* $\alpha = (\alpha_i)_i \in \mathcal{O}_{L^\flat}$ *choose representatives* $a_i \in \mathcal{O}_L$ *and set*

$$\alpha^\sharp := \lim_{i\to\infty} a_i^{q^i} \in \mathcal{O}_L.$$

Denote the composition of $\sharp$ *and* $|\cdot|$ *by*

$$|\cdot|_\flat\colon \mathcal{O}_{L^\flat} \to \mathbb{R}_{\geq 0}, \alpha \mapsto |\alpha^\sharp|.$$

It is not yet clear if all definitions make sense but this is covered by:

Theorem 2.5 *1. $\alpha^\sharp$ is independent of the choices of liftings of the α_i's.*

2. $\mathcal{O}_{L^\flat}$ is a valuation ring with respect to $|\cdot|_\flat$.

3. $L^\flat$ is a perfect and complete nonarchimedean field with respect to $|\cdot|_\flat$ of characteristic p.

Proof: [Sch17, Proposition 1.4.7.] □

Example 2.6 *• The tilt of the completion of $\mathbb{Q}_p(p^{p^{-\infty}})$ is isomorphic to the completion of $\mathbb{F}_p((t))(t^{p^{-\infty}})$.*

• The tilt of a perfectoid field of characteristic p is the field itself.

Theorem 2.7 *There is a bijection*

$$\{\hat{K}_\infty \subseteq L \subseteq \mathbb{C}_p \mid L \text{ perfectoid }\} \leftrightarrow \{\hat{K}_\infty^\flat \subseteq F \subseteq \mathbb{C}_p^\flat \mid F \text{ perfectoid }\}$$

given by

$$L \mapsto L^\flat.$$

Furthermore any finite extension L_1/L is mapped to a finite extension $L_1^\flat/L^\flat$ of the same degree, i.e. $[L_1 : L] = [L_1^\flat : L^\flat]$.

Proof: [Sch17, Theorem 1.4.24.] and [Sch17, Proposition 1.6.8.]. □

2.3 The map θ

Theorem 2.8 *The map*

$$\theta_L \colon W(\mathcal{O}_{L^\flat}) \to \mathcal{O}_L \text{ (resp. } \Theta_L \colon W(\mathcal{O}_{L^\flat}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to L)$$

given by

$$\sum_{n\geq 0} \tau(\alpha_n)p^n \mapsto \sum_{n\geq 0} \alpha_n^\sharp p^n$$

is a surjective homomorphisms of $\mathbb{Z}_p$-algebras (resp. $\mathbb{Q}_p$-algebras) and its kernel is a principal ideal.

Proof: [Sch17, Lemma 1.4.18],[Sch17, Lemma 1.4.19] and [Sch17, Proposition 2.1.19.]. □

2.4 The Crystalline Period Ring ($B_{\mathbf{crys}}$)

For the remainder of chapter 2 let L be an intermediate field $\hat{K}_\infty \subseteq L \subseteq \mathbb{C}_p$ such that L is a perfectoid field. We denote by $\check{L}$ the intermediate field $K_\infty \subseteq \check{L} \subseteq \overline{K}$ such that the completion of $\check{L}$ is L.

Definition 2.9 *Let $\mathbb{A}_{\text{crys}}(\mathbb{C}_p)$ denote the p-adic completion of the divided power envelope of $W(\mathcal{O}_{L^\flat})$ with respect to the ideal $\ker(\theta_{\mathbb{C}_p})$ over $\mathbb{Z}_p$. In formulas:*

$$\mathbb{A}^0_{\text{crys}}(\mathbb{C}_p) := \mathcal{D}_{(\mathbb{Z}_p,(p),\gamma)}(W(\mathcal{O}_{\mathbb{C}_p^\flat}), \ker(\theta_{\mathbb{C}_p}))$$

and

$$\mathbb{A}_{\text{crys}}(\mathbb{C}_p) := \varprojlim_n \mathbb{A}^0_{\text{crys}}(\mathbb{C}_p)/p^n.$$

We denote

$$B^+_{\text{crys}}(L) := (\mathbb{A}_{\text{crys}}(\mathbb{C}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{\text{Gal}(\overline{\mathbb{Q}}_p/\check{L})}$$

and abbreviate $A_{\text{crys}} := \mathbb{A}_{\text{crys}}(\mathbb{C}_p)$ and $B^+_{\text{crys}} := \mathbb{B}^+_{\text{crys}}(\mathbb{C}_p)$.

For the definition of the divided power envelope see section A.2. Since this definition is rather abstract we will give an explicit description.

Lemma 2.10 *Let $\mathfrak{c}$ be a generator of $\ker(\theta_{\mathbb{C}_p})$. Then we have*

$$\mathbb{A}^0_{\text{crys}}(\mathbb{C}_p) = \{\sum_{m=0}^{l} b_m \frac{\mathfrak{c}^m}{m!} \mid l \in \mathbb{N}, b_m \in W(\mathcal{O}_{\mathbb{C}_p^\flat}) \text{ for all } 0 \leq m \leq l\}.$$

In particular $\mathbb{A}^0_{\text{crys}}(\mathbb{C}_p)$ is an integral domain containing $W(k)$.

Proof: [BC09, §9.1.]. □

Proposition 2.11 *Let $m \geq 1$ be an integer. Then there exists an exact sequence*

$$0 \to K_m \hookrightarrow \bigoplus_{n=0}^{m} W(\mathcal{O}_{\mathbb{C}_p^\flat}) \cdot X^n \xrightarrow{s} M_m \to 0,$$

where

$$M_m := \sum_{n=0}^{m} W(\mathcal{O}_{\mathbb{C}_p^\flat}) \frac{\mathfrak{c}^n}{n!} \subseteq \mathbb{A}^0_{\text{crys}}(\mathbb{C}_p)$$

and $K_m \subseteq W(\mathcal{O}_{\mathbb{C}_p^\flat})[X]$ denotes the $W(\mathcal{O}_{\mathbb{C}_p^\flat})$-submodule generated by

$$\mathfrak{c}X^{n-1} - nX^n \text{ for } 1 \leq n \leq m.$$

The map s is given by $X^n \mapsto \frac{\mathfrak{c}^n}{n!}$. In particular M_m is of finite presentation.

Proof: We have $s(K_m) = 0$ since

$$s(\mathfrak{c}X^{n-1} - nX^n) = \mathfrak{c}\frac{\mathfrak{c}^{n-1}}{(n-1)!} - n\frac{\mathfrak{c}^n}{n!} = 0$$

holds for any $1 \leq n \leq m$. Take an element

$$a = \sum_{n=0}^{m} r_n X^n \in \bigoplus_{n=0}^{m} W(\mathcal{O}_{\mathbb{C}_p^\flat}) \cdot X^n \text{ such that } s(a) = 0.$$

One obtains

$$r_0 = -\sum_{n=1}^{m} r_n \frac{\mathfrak{c}^n}{n!}$$

and hence

$$m!a = \sum_{n=1}^{m} r_n \frac{m!}{n!}(n!X^n - \mathfrak{c}^n) \in K_m$$

by the relation

$$n!X^n - \mathfrak{c}^n \equiv (n-1)!\mathfrak{c}X^{n-1} - \mathfrak{c}^n \equiv \cdots \equiv \mathfrak{c}^n - \mathfrak{c}^n \equiv 0 \mod K_m.$$

We claim that K_m is $\mathbb{Z}$-saturated and conclude $a \in K_m$. Without loss of generality we take $p \cdot f \in K_m$ and remark that

$$\sum_{n=1}^{m} a_n(\mathfrak{c}X^{n-1} - nX^n) = pf \equiv 0 \mod p$$

for some $a_1, \ldots, a_m \in W(\mathcal{O}_{\mathbb{C}_p^\flat})$. Comparing coefficients in the residue ring $\mathcal{O}_{\mathbb{C}_p^\flat}[X]$ delivers

$$a_1\mathfrak{c} \equiv 0 \mod p$$

and

$$a_{n+1}\mathfrak{c} \equiv a_n n \mod p$$

for all $1 \leq n \leq m-1$. Since $\mathfrak{c} \notin pW(\mathcal{O}_{\mathbb{C}_p^\flat})$ and $\mathcal{O}_{\mathbb{C}_p^\flat}$ is a domain, we see that $p \mid a_n$ for all $1 \leq n \leq m$ by induction, hence $f \in K_m$. □

Corollary 2.12 *Every finitely generated $W(\mathcal{O}_{\mathbb{C}_p^\flat})$-submodule of $\mathbb{A}^0_{\mathrm{crys}}(\mathbb{C}_p)$ is contained in a finitely presented submodule.*

Remark 2.13 *$\mathbb{A}^0_{\mathrm{crys}}(\mathbb{C}_p)$ is p-adically separated by [BC09, Explanation after (9.1.2.)]. Therefore we have inclusions*

$$W(k) \subseteq W(\mathcal{O}_{\mathbb{C}_p^\flat}) \subseteq \mathbb{A}^0_{\mathrm{crys}}(\mathbb{C}_p) \subseteq \mathbb{A}_{\mathrm{crys}}(\mathbb{C}_p).$$

Hence $\mathbb{B}^+_{\mathrm{crys}}(L)$ is a K_0-algebra. If the residue field of L is algebraically closed the same argument shows that $\mathbb{B}^+_{\mathrm{crys}}(L)$ is a P_0-algebra.

Another feature of the ring $\mathbb{B}^+_{\mathrm{crys}}(L)$ is that there exists a G_K-equivariant Frobenius endomorphism Φ on $\mathbb{B}^+_{\mathrm{crys}}(L)$ which extends the natural Frobenius endomorphism

$$\Phi\colon W(\mathcal{O}_{L^\flat}) \to W(\mathcal{O}_{L^\flat})$$

coming from the theory of Witt vectors [Sch17, Section 1.1.].

Proposition 2.14 *There exists a G_K-equivariant Frobenius endomorphism on $\mathbb{A}_{\mathrm{crys}}(\mathbb{C}_p)$ extending Φ.*

Proof: [BC09, Lemma 9.1.7.] □

2.5 The Ring of p-adic Periods ($B_{\mathbf{dR}}$)

Definition 2.15 *Let $\mathbb{B}^+_{\mathrm{dR}}(L)$ denote the $\ker(\Theta_L)$-adic completion of $W(\mathcal{O}_{L^\flat})$ localized with respect to the element p. Or short:*

$$\mathbb{B}^+_{\mathrm{dR}}(L) := \varprojlim_n (W(\mathcal{O}_{L^\flat})[\frac{1}{p}])/\ker(\Theta_L)^n.$$

We abbreviate $B^+_{\mathrm{dR}} := \mathbb{B}^+_{\mathrm{dR}}(\mathbb{C}_p)$. Furthermore we call $\mathbb{B}_{\mathrm{dR}}(L) := \mathrm{Frac}(\mathbb{B}^+_{\mathrm{dR}}(L))$ the (field of) p-adic periods with respect to L and $B_{\mathrm{dR}} := \mathbb{B}_{\mathrm{dR}}(\mathbb{C}_p)$ the p-adic periods.

Proposition 2.16

$$\mathbb{B}^+_{\mathrm{dR}}(\mathbb{C}_p)^{G_{\check{L}}} \cong \mathbb{B}^+_{\mathrm{dR}}(L)$$

Proof: $W(\mathcal{O}_{L^\flat})[\frac{1}{p}] \cap \ker(\Theta_{\mathbb{C}_p})^n = \ker(\Theta_L)^n$ holds for all $n \geq 1$ by the commutative diagram after [Sch17, Lemma 1.4.19.]. Thus we obtain a canonical inclusion $\mathbb{B}^+_{\mathrm{dR}}(L) \to B^+_{\mathrm{dR}}$ by the universal property of the projective limit. We have $g.(\ker(\Theta_{\mathbb{C}_p})) \subseteq \ker(\Theta_{\mathbb{C}_p})$ for all $g \in G_K$ since $\Theta_{\mathbb{C}_p}$ is G_K-equivariant. Thus we obtain an injective map

$$\iota_n\colon W(\mathcal{O}_{L^\flat})[\frac{1}{p}]/\ker(\Theta_L)^n \to (W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}]/\ker(\Theta_{\mathbb{C}_p})^n)^{G_{\check{L}}}$$

for each $n \geq 1$. It is enough to prove surjectivity of this map and we do this by induction on n. For $n = 1$ we proceed as follows. Take $x \in W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}]$ such that $g.x - x \in \ker(\Theta_{\mathbb{C}_p})$ for all $g \in G_{\check{L}}$. Then $g.\Theta_{\mathbb{C}_p}(x) - \Theta_{\mathbb{C}_p}(x) = 0$ for all $g \in G_{\check{L}}$, i.e. $\Theta_{\mathbb{C}_p}(x) \in \mathbb{C}_p^{G_{\check{L}}} = L$. Since Θ_L is surjective we receive an $y \in W(\mathcal{O}_{L^\flat})[\frac{1}{p}]$ such that $\Theta_{\mathbb{C}_p}(x) = \Theta_L(y)$. Hence

$$\iota_1(y + \ker(\Theta_L)) = y + \ker(\Theta_{\mathbb{C}_p}) = x + \ker(\Theta_{\mathbb{C}_p})$$

and ι_1 is surjective. Now consider the case $n \geq 1$ and the commutative diagram with exact columns

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\ker(\Theta_L)^{n-1}W(\mathcal{O}_{L^\flat})[\tfrac{1}{p}]/\ker(\Theta_L)^n & \xrightarrow{\cong} & (\ker(\Theta_L)^{n-1}W(\mathcal{O}_{\mathbb{C}_p^\flat})[\tfrac{1}{p}]/\ker(\Theta_{\mathbb{C}_p})^n)^{G_{\breve{L}}} \\
\downarrow & & \downarrow \\
W(\mathcal{O}_{L^\flat})[\tfrac{1}{p}]/\ker(\Theta_L)^n & \xrightarrow{\iota_n} & (W(\mathcal{O}_{\mathbb{C}_p^\flat})[\tfrac{1}{p}]/\ker(\Theta_{\mathbb{C}_p})^n)^{G_{\breve{L}}} \\
\downarrow & & \downarrow \\
W(\mathcal{O}_{L^\flat})[\tfrac{1}{p}]/\ker(\Theta_L)^{n-1} & \xrightarrow{\iota_{n-1}} & (W(\mathcal{O}_{\mathbb{C}_p^\flat})[\tfrac{1}{p}]/\ker(\Theta_{\mathbb{C}_p})^{n-1})^{G_{\breve{L}}} \\
\downarrow & & \\
0 & &
\end{array},$$

where the vertical arrows are the canonical maps. The top arrow is an isomorphism by the case $n = 1$ since $\ker(\Theta_L)$ is a principal ideal and hence

$$\ker(\Theta_L)^{n-1}W(\mathcal{O}_{L^\flat})[\frac{1}{p}]/\ker(\Theta_L)^n \cong W(\mathcal{O}_{L^\flat})[\frac{1}{p}]/\ker(\Theta_L)$$

holds. The bottom arrow is an isomorphism by induction and therefore ι_n is an isomorphism by the five lemma. □

Proposition 2.17 $\mathbb{B}_{\mathrm{dR}}^+(L)$ *is a complete discrete valuation ring with residue field L and* $(\mathbb{B}_{\mathrm{dR}}^+(L))^\times$ *contains* $W(\mathcal{O}_{L^\flat})[\frac{1}{p}] \setminus \{0\}$.

Proof: This is literally [BC09, Proposition 4.4.6.] if one replaces $\mathbb{C}_p$ with L since the proof does not use that $\mathbb{C}_p$ is algebraically closed. □

An immediate consequence of this is that there exists a multiplicative G_K-equivariant Teichmüller map

$$\tau_{\mathrm{dR}} \colon (L^\flat)^\times \to (\mathbb{B}_{\mathrm{dR}}^+(L))^\times$$

given by

$$\frac{a}{b} \mapsto \tau(a)\tau(b)^{-1}$$

for all $a, b \in \mathcal{O}_{L^\flat}$ and $b \neq 0$.

Proposition 2.18 *There exists a $G_{\check{L}}$-equivariant section*

$$s_{\mathrm{dR},\check{L}} \colon \check{L} \to \mathbb{B}_{\mathrm{dR}}^{+}(L)$$

of the $G_{\check{L}}$-equivariant projection map

$$\Theta_{\mathrm{dR},L} \colon \mathbb{B}_{\mathrm{dR}}^{+}(L) \twoheadrightarrow L.$$

Via this section $\mathbb{B}_{\mathrm{dR}}^{+}(L)$ contains a unique copy of $\check{L}$ as a subfield over K_0 and the action of G_{K_0} is compatible with this inclusion.

Proof: For the case $L = \mathbb{C}_p$ see [BC09, Lemma 4.4.10.], then take $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\check{L})$-invariants. □

Warning: The section $s_{\mathrm{dR},\overline{\mathbb{Q}}_p}$ is not continuous and therefore it does not extend to a G_K-equivariant section $\mathbb{C}_p \to \mathbb{B}_{\mathrm{dR}}^{+}(\mathbb{C}_p)$. By [Ser79, Chapter II, §4, Theorem 2] there exists an isomorphism $\mathbb{B}_{\mathrm{dR}}^{+}(\mathbb{C}_p) \cong \mathbb{C}_p[[T]]$ of rings but this map is neither G_K-equivariant nor continuous.

Proposition 2.19 *There exists a continuous G_K-equivariant isomorphism of rings from $\mathbb{A}_{\mathrm{crys}}(\mathbb{C}_p)$ to the subring*

$$\{\sum_{n\geq 0} b_n \cdot \frac{\mathfrak{c}^n}{n!} \in \mathbb{B}_{\mathrm{dR}}^{+}(\mathbb{C}_p) \mid (b_n)_n \text{ sequence in } W(\mathcal{O}_{\mathbb{C}_p^\flat}) \text{ converging to } 0\}$$

of $\mathbb{B}_{\mathrm{dR}}^{+}(\mathbb{C}_p)$ which is compatible with the inclusion $W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}] \subseteq \mathbb{B}_{\mathrm{dR}}^{+}(\mathbb{C}_p)$.

Proof: [BC09, §9.1.] □

2.6 The Tilted p-adic Logarithm

First of all we give a short reminder about the usual versions of p-adic logarithms.

Definition 2.20 *Let B be a complete valuation ring of characteristic 0 and denote the valuation on B by ν_B. We call B logarithmic if*

$$\lim_{n\to\infty} n\gamma - \nu_B(n) = \infty$$

holds for all $\gamma \in \nu(B) \setminus \{\infty\}$ such that $\gamma > 0$.

Lemma 2.21 *Let B be a logarithmic valuation ring and let $\mathfrak{m}_B$ denote its maximal ideal. Then*

$$\log_B : 1 + \mathfrak{m}_B \to B, x \mapsto \sum_{n \geq 1} (-1)^{n+1} \frac{(x-1)^n}{n}$$

is a continuous group homomorphism.

Proof: Since B is complete we only have to check that $\frac{(x-1)^n}{n}$ converges to zero but this is covered by definition. □

Example 2.22 1. *$\mathcal{O}_K$ is logarithmic (see [Neu99, Chapter II, Proposition (5.4)]) and we can extend $\log_{\mathcal{O}_K}$ uniquely to a continuous group homomorphism*

$$\log_K : K^\times \to K$$

such that $\log_K(p) = 0$.

2. *$\mathcal{O}_{\check{L}}$ does not need to be logarithmic since it is not necessarily complete but every element of $\check{L}$ is contained in a finite extension of K. Thus we obtain a $\mathrm{Gal}(\check{L}/\mathbb{Q}_p)$-equivariant group homomorphism*

$$\log_{\mathcal{O}_{\check{L}}} : 1 + \mathfrak{m}_{\check{L}} \to \mathcal{O}_{\check{L}}$$

which can be uniquely extended to a $\mathrm{Gal}(\check{L}/\mathbb{Q}_p)$-equivariant group homomorphism

$$\log_{\check{L}} : \check{L}^\times \to \check{L}$$

such that $\log_{\check{L}}(p) = 0$ (see [BC09, Lemma 9.2.6.]).

3. *$\mathcal{O}_L$ is logarithmic (see [Was82, Proposition 5.4]) and we can extend $\log_{\mathcal{O}_L}$ uniquely to a continuous group homomorphism*

$$\log_L : L^\times \to L$$

such that $\log_L(p) = 0$. Furthermore this map is G_K-equivariant since each $g \in G_K$ extends to a continuous automorphism of $\mathbb{C}_p$. We call

$$\log_p := \log_{\mathbb{C}_p} : \mathbb{C}_p^\times \to \mathbb{C}_p$$

the p-adic logarithm. Furthermore $\ker(\log_p) = p^{\mathbb{Q}} \cdot \mu$ holds by [Was82, Proposition 5.6], where $\mu \subseteq \mathbb{C}_p^\times$ is the multiplicative subgroup consisting of roots of arbitrary roots of unity.

4. $\mathbb{B}_{\mathrm{dR}}^+(L)$ *is logarithmic since* $\nu_{\mathrm{dR}}(n) = 0$ *for all* $n \in \mathbb{N}$ *and we denote*

$$\log_{\mathrm{dR}} := \log_{\mathbb{B}_{\mathrm{dR}}^+(L)} \colon 1 + \mathfrak{m}_{\mathbb{B}_{\mathrm{dR}}^+(L)} \to \mathbb{B}_{\mathrm{dR}}^+(L).$$

We now give a construction of a tilted version of the usual p-adic logarithm $\log_p$ following [BC09, §9.2.]. Since a map defined by the logarithm formula can't have values in a field (or ring) of characteristic p we substitute $\mathbb{C}_p^\flat$ with the ring of p-adic periods B_{dR}^+. Therefore we construct more generally a G_K-invariant group homomorphism

$$\log_L^\flat \colon L^\flat \to \mathbb{B}_{\mathrm{dR}}^+(L).$$

Lemma 2.23 *Each element in* $(L^\flat)^\times/(\mathcal{O}_{L^\flat})^\times$ *can be represented by an element* z *such that* $z^\sharp \in \check{L}^\times$.

Proof: By [Sch11, Lemma 3.4.] we know that $\nu_L(L^\times) = \nu_L^\flat((L^\flat)^\times)$ and since completing a non-archimedian field does not change the value group we have $\nu_{\check{L}}(\check{L}^\times) = \nu_L(L^\times)$. Therefore we obtain

$$(L^\flat)^\times/(\mathcal{O}_{L^\flat}^\times) \cong L^\times/\mathcal{O}_L^\times \cong \check{L}^\times/\mathcal{O}_{\check{L}}^\times.$$

□

Proposition 2.24 *Let* $\kappa_{L^\flat}$ *denote the residue field of* $L^\flat$*. The map*

$$\log_{\mathrm{crys},L}^\flat \colon \mathcal{O}_{L^\flat}^\times \cong \kappa_{L^\flat}^\times \times 1 + \mathfrak{m}_{L^\flat} \to \mathbb{B}_{\mathrm{crys}}^+(L)$$

given by

$$(\lambda, x) \mapsto \sum_{n \geq 1} (-1)^{n+1} \frac{(\tau(x) - 1)^n}{n}$$

is a G_K*-equivariant group homomorphism.*

Proof: For $L = \mathbb{C}_p$ this is [BC09, Lemma 9.2.2.], then take $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\check{L})$-invariants. □

Fix an element $\varepsilon \in \mathcal{O}_{L^\flat}^\times$ such that $\varepsilon^\sharp = 1$ and $(\varepsilon^{1/p})^\sharp \neq 1$. Then the action of G_K on ε is given by $g.\varepsilon = \chi(g) \cdot \varepsilon$ for all $g \in G_K$, where

$$\chi \colon G_K \to \mathbb{Z}_p^\times$$

is the cyclotomic character. Now apply the tilted p-adic logarithm to obtain an element

$$t := \log_L^\flat(\varepsilon) \in \mathbb{B}_{\mathrm{crys}}^+(L)$$

and

$$g.t = \log_L^\flat(g.\varepsilon) = \log_L^\flat(\chi(g) \cdot \varepsilon) = \log_{\mathbb{Z}_p}(\chi(g)) + t$$

for all $g \in G_K$ by the G_K-equivariance of $\log_L^\flat$.

Definition 2.25 *We define* $\mathbb{B}_{\mathrm{crys}}(L) := \mathbb{B}^{+}_{\mathrm{crys}}(L)[\frac{1}{t}]$ *and* $B_{\mathrm{crys}} := \mathbb{B}_{\mathrm{crys}}(\mathbb{C}_p)$.

Proposition 2.26 *There exists an unique injective continuous* G_K*-equivariant map*

$$j\colon \mathbb{A}_{\mathrm{crys}}(\mathbb{C}_p) \to \mathbb{B}^{+}_{\mathrm{dR}}(\mathbb{C}_p)$$

such that the diagram

$$\begin{array}{ccc}
\mathbb{A}^{0}_{\mathrm{crys}}(\mathbb{C}_p) & \xrightarrow{\subseteq} & W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}] \\
\downarrow{\scriptstyle\subseteq} & & \downarrow{\scriptstyle\subseteq} \\
\mathbb{A}_{\mathrm{crys}}(\mathbb{C}_p) & \xrightarrow{j} & \mathbb{B}^{+}_{\mathrm{dR}}(\mathbb{C}_p)
\end{array}$$

commutes. In particular $\mathbb{B}_{\mathrm{crys}}(L)$ *may be viewed as a subring of* $\mathbb{B}_{\mathrm{dR}}(L)$.

Proof: The map j is unique since $\mathbb{A}^{0}_{\mathrm{crys}}(\mathbb{C}_p)$ is dense in $\mathbb{A}_{\mathrm{crys}}(\mathbb{C}_p)$. The existence is proven in [BC09, §9.1.] which gives an inclusion of $\mathbb{B}_{\mathrm{crys}}(\mathbb{C}_p)$ into $\mathbb{B}_{\mathrm{dR}}(\mathbb{C}_p)$. For the relative case take $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\check{L})$-invariants and apply Proposition 2.16. □

Proposition 2.27 *The map*

$$K \otimes_{K_0} \mathbb{B}_{\mathrm{crys}}(L) \to \mathbb{B}_{\mathrm{dR}}(L), \lambda \otimes b \mapsto \lambda \cdot b$$

is injective. If the residue field κ_L *of* L *is algebraically closed the map*

$$(K \cdot P_0) \otimes_{P_0} \mathbb{B}_{\mathrm{crys}}(L) \to \mathbb{B}_{\mathrm{dR}}(L), \lambda \otimes b \mapsto \lambda \cdot b$$

is also injective.

Proof: $\mathbb{B}_{\mathrm{crys}}(L)$ is a K_0-algebra (resp. P_0-algebra if κ_L is algebraically closed) by Remark 2.13. The case $L = \mathbb{C}_p$ is known due to [Fon94a, Théorème 4.2.4.] resp. [BC09, Theorem 9.1.5.]. Take $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\check{L})$-invariants and apply Proposition 2.16 to obtain the relative statement. □

From all above we obtain the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{O}_{L^\flat} & \xrightarrow{\tau} & W(\mathcal{O}_{L^\flat}) & \xrightarrow{\subseteq} & \mathbb{B}^{+}_{\mathrm{dR}}(L) & \xleftarrow{s_{\mathrm{dR},\check{L}}} & \check{L} \\
& \searrow^{\sharp} & \downarrow{\scriptstyle\theta_L} & & \downarrow{\scriptstyle\Theta_{\mathrm{dR},L}} & \swarrow_{\subseteq} & \\
& & \mathcal{O}_L & \xrightarrow{\subseteq} & L & &
\end{array} \tag{2.1}$$

and with Lemma 2.23 we are ready to define the tilted p-adic logarithm as follows.

Theorem 2.28 *Let z be an element of $(L^\flat)^\times$. By Lemma 2.23 we can write $z = u \cdot y$ with $u \in \mathcal{O}_{L^\flat}^\times$ and $y \in (L^\flat)^\times$ such that $y^\sharp \in \check{L}$. Then the map*

$$\log_L^\flat \colon (L^\flat)^\times \to \mathbb{B}_{\mathrm{dR}}^+(L)$$

given by

$$z = u \cdot y \mapsto \log_{dR}(\frac{\tau_{\mathrm{dR}}(y)}{s_{\mathrm{dR}}(y^\sharp)}) + s_{\mathrm{dR}}(\log_{\check{L}}(y^\sharp)) + \log_{\mathrm{crys},L}^\flat(u)$$

is a G_K-equivariant group homomorphism that extends $\log_{\mathrm{crys}}^\flat$. We call

$$\log_p^\flat := \log_{\mathbb{C}_p}^\flat$$

the tilted p-adic logarithm.

Proof: The first thing we have to check is $\frac{\tau_{\mathrm{dR}}(y)}{s_{\mathrm{dR}}(y^\sharp)} \in 1 + \mathfrak{m}_{\mathrm{dR}}$ but

$$\Theta_{\mathrm{dR}}(\frac{\tau_{\mathrm{dR}}(y)}{s_{\mathrm{dR}}(y^\sharp)} - 1) = \frac{y^\sharp}{y^\sharp} - 1 = 0$$

holds by (2.1). For further details consult [BC09, Lemma 9.2.7.]. □

This gives us the following commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_{L^\flat}^\times & \xrightarrow{\log_{\mathrm{crys},L}^\flat} & \mathbb{B}_{\mathrm{crys}}^+(L) \\ \downarrow \subseteq & & \downarrow \subseteq \\ (L^\flat)^\times & \xrightarrow{\log_L^\flat} & \mathbb{B}_{\mathrm{dR}}^+(L). \end{array} \tag{2.2}$$

2.7 G_K-Invariants of Period Rings

With the diagram (2.2) in mind we are now able to calculate the Galois invariants of $\mathbb{B}_{\mathrm{crys}}(L)$. We assume that the residue field of L is algebraically closed to assure that $B_{\mathrm{dR}}(L)$ contains P_0.

Proposition 2.29 *The element t is an uniformizer in $\mathbb{B}_{\mathrm{dR}}^+(L)$.*

Proof: [BC09, Proposition 4.4.8]. □

Proposition 2.30 $\mathbb{B}_{\mathrm{dR}}(L)^{G_K} = K$ *and* $\mathbb{B}_{\mathrm{dR}}(L)^{I_K} = K \cdot P_0$.

Proof: The G_K-action on $\mathbb{B}_{\mathrm{dR}}(L)$ is compatible with the filtration

$$\{t^i \cdot \mathbb{B}_{\mathrm{dR}}^+(L) \mid i \in \mathbb{Z}\}$$

since Θ_L is G_K-equivariant. We take G_K-invariants of the sequence

$$0 \to t^{i+1}\mathbb{B}_{\mathrm{dR}}^+(L) \to t^i\mathbb{B}_{\mathrm{dR}}^+(L) \to \mathbb{C}_p(i) \to 0$$

and obtain that the induced sequence

$$0 \to (t^{i+1}\mathbb{B}_{\mathrm{dR}}^+(L))^{G_K} \to (t^i\mathbb{B}_{\mathrm{dR}}^+(L))^{G_K} \to (\mathbb{C}_p(i))^{G_K}$$

is exact for all $i \in \mathbb{Z}$. But $(\mathbb{C}_p(i))^{G_K} = 0$ for all $i \in \mathbb{Z} \setminus \{0\}$ by the Theorem of Tate and Sen [Tat67, §(3.3), Theorem 2]. Hence

$$(t^{i+1}\mathbb{B}_{\mathrm{dR}}^+(L))^{G_K} = (t^i\mathbb{B}_{\mathrm{dR}}^+(L))^{G_K}$$

holds for all $\in \mathbb{Z} \setminus \{0\}$. Since $\mathbb{B}_{\mathrm{dR}}^+(L)$ is a complete and separated discrete valuation ring we have

$$\bigcap_{i\geq 1} t^i\mathbb{B}_{\mathrm{dR}}^+(L)^{G_K} \subseteq \bigcap_{i\geq 1} t^i\mathbb{B}_{\mathrm{dR}}^+(L) = 0$$

and therefore $(t^i\mathbb{B}_{\mathrm{dR}}^+(L))^{G_K} = 0$ for all $i \geq 1$. Hence $\mathbb{B}_{\mathrm{dR}}(L)^{G_K} = \mathbb{B}_{\mathrm{dR}}^+(L)^{G_K}$ and for $i = 0$ the second exact sequence implies $\mathbb{B}_{\mathrm{dR}}^+(L)^{G_K} \subseteq \mathbb{C}_p^{G_K} = K$. But since $\mathbb{B}_{\mathrm{dR}}^+(L)$ contains K we have $\mathbb{B}_{\mathrm{dR}}^+(L)^{G_K} = K$. We replace G_K with I_K in the argument and obtain $\mathbb{B}_{\mathrm{dR}}(L)^{I_K} = K \cdot P_0$ as well. □

Proposition 2.31 $\mathbb{B}_{\mathrm{crys}}(L)^{G_K} = K_0$ *and* $\mathbb{B}_{\mathrm{crys}}(L)^{I_K} = P_0$.

Proof: We have inclusions $K_0 \subseteq \mathbb{B}_{\mathrm{crys}}(L) \subseteq \mathbb{B}_{\mathrm{dR}}(L)$ and taking G_K-invariants gives us $\mathbb{B}_{\mathrm{crys}}(L)^{G_K} \subseteq K$. By Proposition 2.27 we obtain

$$\dim_{K_0}(\mathbb{B}_{\mathrm{crys}}(L)^{G_K}) = 1.$$

Again, replace G_K by I_K and receive $\mathbb{B}_{\mathrm{crys}}(L)^{I_K} = P_0$ as well. □

2.8 The Log-crystalline Period Ring (B_{st})

Originally the term "semistable" was used instead of "log-crystalline", since one may define this property as being semistable with respect to the difference slope given by the degree function $t_H - t_N$ (see [CF00, §3.4. & Theoreme A]) in the sense of Appendix B. Since the terms "stable" or "semistable" are used to oblivion in many contexts (see for example Appendix B), we will stick to the notation 'log-crystalline' (as in [FF11]). However, in order to avoid an increase of names for certain rings, we will keep the name B_{st} (instead of switching to $B_{\log}$).

Fix an element $p_L^\flat \in \mathcal{O}_{L^\flat}$ such that $|(p_L^\flat)^\sharp|_L = |p|_L$. Such an element exists by [Sch11, Lemma 3.4.(ii)] and set

$$u_L := \log_L^\flat(p_L^\flat) \in \mathbb{B}_{\mathrm{dR}}^+(L).$$

For the case $L = \mathbb{C}_p$ we fix an element $\tilde{p} \in \mathcal{O}_{\mathbb{C}_p^\flat}$ such that $\tilde{p}^\sharp = p$ and set

$$\begin{aligned} u &:= \log_L^\flat(\tilde{p}) \\ &= \log_{\mathrm{dR}}(\frac{\tau_{\mathrm{dR}}(\tilde{p})}{p}) + s_{\mathrm{dR}}(\underbrace{\log_{\breve{L}}(p)}_{=0}) \\ &= \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \left(\frac{\tau(\tilde{p})}{p} - 1\right)^n \in B_{\mathrm{dR}}^+. \end{aligned}$$

We remark that the element u_L (resp. u) depends on the choice of $p_L^\flat$ (resp. $\tilde{p}$) and the ring $\mathbb{B}_{\mathrm{st}}(L)$ will also depend on this choice. Fortunately the image of B_{st} in B_{dR} is independent of this choice, see [BC09, §9.2.]. Furthermore we will show that $B_{\mathrm{st}}^{G_{\breve{L}}} = \mathbb{B}_{\mathrm{st}}(L)$ holds and therefore the image of $\mathbb{B}_{\mathrm{st}}(L)$ in $\mathbb{B}_{\mathrm{dR}}(L)$ is independent of the choice of u_L.

Proposition 2.32 *The element u is transcendental over* $\mathrm{Frac}(B_{\mathrm{crys}})$.

Proof: [Fon94a, (Proof of) Théorème 4.2.4.]. □

Definition 2.33 *We define the log-crystalline period ring to be*

$$\mathbb{B}_{\mathrm{st}}(L) := \mathbb{B}_{\mathrm{crys}}(L)[u_L]$$

and endow it with an extension of the map φ on $\mathbb{B}_{\mathrm{crys}}(L)$ given by

$$\varphi\colon \mathbb{B}_{\mathrm{st}}(L) \to \mathbb{B}_{\mathrm{st}}(L), u_L \mapsto p \cdot u_L.$$

There is also a natural action of G_K on the element u_L inherited from $\mathbb{B}_{\mathrm{dR}}^+(L)$ given by

$$g.u_L = \log_L^\flat(g.p_L^\flat) = \log_L^\flat(\varepsilon^{c_g} p_L^\flat) = c_g \cdot t + u_L$$

for some $c_g \in \mathbb{Z}_p^\times$. Furthermore let

$$N\colon \mathbb{B}_{\mathrm{st}}(L) \to \mathbb{B}_{\mathrm{st}}(L) \text{ given by } \sum_{n=0}^r b_n u_L^n \mapsto -\sum_{n=0}^r n b_n u_L^{n-1}$$

denote the Monodromy operator on $\mathbb{B}_{\mathrm{st}}(L)$. As usual we set $B_{\mathrm{st}} := \mathbb{B}_{\mathrm{st}}(\mathbb{C}_p)$.

Proposition 2.34 *1.* $B_{\mathrm{st}}^{G_{\check{L}}} = \mathbb{B}_{\mathrm{st}}(L)$.

2. u_L *is transcendental over* $\mathrm{Frac}(\mathbb{B}_{\mathrm{crys}}(L))$.

Proof: The fraction $\frac{\tilde{p}}{p_L^\flat}$ is contained in $\mathcal{O}_{\mathbb{C}_p^\flat}^\times$ since it has absolute value

$$|\frac{\tilde{p}}{p_L^\flat}|_p^\flat = |(\frac{\tilde{p}}{p_L^\flat})^\sharp|_p = 1.$$

Therefore we find $\lambda \in \overline{\mathbb{F}}_p^\times = \bigcup_n \mu_{p^n-1} \subseteq \mathcal{O}_{\mathbb{C}_p^\flat}^\times$ and $x \in 1 + \mathfrak{m}_{\mathbb{C}_p^\flat}$ such that $\frac{\tilde{p}}{p_L^\flat} = \lambda \cdot x$. We obtain

$$u_L = \log_L^\flat(p_L^\flat) = \underbrace{\log_p^\flat(\tilde{p})}_{=u} + \underbrace{\log_p^\flat(\lambda)}_{=0} + \underbrace{\log_p^\flat(x)}_{=:b\in B_{\mathrm{crys}}}.$$

This shows that the images of $B_{\mathrm{crys}}[u]$ and $B_{\mathrm{crys}}[u_L]$ inside B_{dR} are equal and since $G_{\check{L}}$ acts trivially on u_L we see that

$$B_{\mathrm{st}}^{G_{\check{L}}} = (B_{\mathrm{crys}}[u_L])^{G_{\check{L}}} = B_{\mathrm{crys}}^{G_{\check{L}}}[u_L] = \mathbb{B}_{\mathrm{st}}(L).$$

The existence of the isomorphism $B_{\mathrm{crys}}[u] \cong B_{\mathrm{crys}}[u_L]$ also implies that u_L is transcendental over $\mathrm{Frac}(B_{\mathrm{crys}})$ and hence transcendental over $\mathrm{Frac}(\mathbb{B}_{\mathrm{crys}}(L))$. $\square$

Now we are able to give a list of properties of $\mathbb{B}_{\mathrm{st}}(L)$ that will be exploited later on.

Lemma 2.35

- $\varphi(g.b) = g.\varphi(b)$ *holds for all* $b \in \mathbb{B}_{\mathrm{st}}(L)$ *and* $g \in \mathrm{Aut}_K(\check{L})$.
- $N(g.b) = g.N(b)$ *holds for all* $b \in \mathbb{B}_{\mathrm{st}}(L)$ *and* $g \in \mathrm{Aut}_K(\check{L})$.
- $(N \circ \varphi)(b) = p \cdot (\varphi \circ N)(b)$ *holds for all* $b \in \mathbb{B}_{\mathrm{st}}(L)$.
- *The map* $K \otimes_{K_0} \mathbb{B}_{\mathrm{st}}(L) \to \mathbb{B}_{\mathrm{dR}}(L)$ *given by* $\lambda \otimes b \mapsto \lambda \cdot b$ *is an injective* G_K*-equivariant map.*
- *The map* $(K \cdot P_0) \otimes_{P_0} \mathbb{B}_{\mathrm{st}}(L) \to \mathbb{B}_{\mathrm{dR}}(L)$ *given by* $\lambda \otimes b \mapsto \lambda \cdot b$ *is an injective* I_K*-equivariant map.*
- $\mathbb{B}_{\mathrm{st}}(L)^{G_K} = K_0$ *and* $\mathbb{B}_{\mathrm{st}}(L)^{I_K} = P_0$.

Proof: The first three properties are standard calculations. The map

$$K \otimes_{K_0} \mathbb{B}_{\mathrm{st}}(L) \to \mathbb{B}_{\mathrm{dR}}(L), \lambda \otimes b \mapsto \lambda \cdot b$$

$$(\text{resp. } (K \cdot P_0) \otimes_{P_0} \mathbb{B}_{\mathrm{st}}(L) \to \mathbb{B}_{\mathrm{dR}}(L), \lambda \otimes b \mapsto \lambda \cdot b)$$

is injective since the statement is true for $L = \mathbb{C}_p$ (compare [Fon94a, Théorème 4.2.4.]) and taking $G_{\check{L}}$-invariants preserves the injectivity. Thus $\mathbb{B}_{\mathrm{st}}(L)^{G_K}$ (resp. $\mathbb{B}_{\mathrm{st}}(L)^{I_K}$) is a one-dimensional vector space over K_0 (resp. P_0) and contains $K_0 = \mathbb{B}_{\mathrm{crys}}(L)^{G_K}$ (resp. $P_0 = \mathbb{B}_{\mathrm{crys}}(L)^{I_K}$) by Proposition 2.31. We conclude $\mathbb{B}_{\mathrm{st}}(L)^{G_K} = K_0$ (resp. $\mathbb{B}_{\mathrm{st}}(L)^{I_K} = P_0$). □

2.9 A Two-Dimensional Representation of G_K

We now discuss [BC09, Example 9.2.8] in detail since it gives a tangible example of a non-trivial p-adic Galois representations obtained from the period rings defined above.

Lemma 2.36 *Let V be a (finite dimensional) representation of G_K over $\mathbb{Q}_p$ and B be a (not necessarily finite dimensional) vector space over $\mathbb{Q}_p$ with G_K acting on it. Then the usual isomorphism*

$$B \otimes_{\mathbb{Q}_p} V^* \cong \mathrm{Hom}_{\mathbb{Q}_p}(V, B)$$

restricts to

$$(B \otimes_{\mathbb{Q}_p} V^\wedge)^{G_K} \cong \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V, B)$$

where $V^\wedge$ is the vector space V^* with the G_K-action given by

$$g.f(v) := f(g^{-1}v) \text{ for all } g \in G_K, v \in V.$$

For the remainder of this section we assume that $\mu_p(\overline{\mathbb{Q}}_p) \not\subseteq K$.

Lemma 2.37 *Fix the notations*

$$K_n := K(\mu_{p^n}) \text{ and } \Delta_n := \mathrm{Gal}(K_n/K).$$

Then an element $b \in K^\times \setminus (K^\times)^p$ has order p^n in the group $K_n^\times/(K_n^\times)^{p^n}$.

Proof: For every algebraic extension L/K and $k \geq 0$ we know by Kummer's theory that

$$H^1(G_L, \mu_k) \cong L^\times/(L^\times)^k.$$

Consider the inflation-restriction sequence associated to the normal subgroup G_{K_n} in G_K and the module $\mu_{p^l} := \mu_{p^l}(\overline{\mathbb{Q}}_p)$. We obtain exactness of

$$1 \to H^1(\Delta_n, (\mu_{p^l})^{G_{K_n}}) \to H^1(G_K, \mu_{p^l}) \to H^1(G_{K_n}, \mu_{p^l})^{\Delta_n} \to H^2(\Delta_n, (\mu_{p^l})^{G_{K_n}}).$$

But $H^2(\Delta_n, (\mu_{p^l})^{G_{K_n}}) \cong \hat{H}^0(\Delta_n, (\mu_{p^l})^{G_{K_n}}) = 1$ and since $(\mu_{p^l})^{G_K}$ is a finite module the Herbrand quotient $h(\Delta_n, (\mu_{p^l})^{G_{K_n}}) = 1$. By using the argument from Kummer's theory above we obtain

$$K^\times/(K^\times)^{p^l} \cong (K_n^\times/(K_n^\times)^{p^l})^{\Delta_n} \text{ for all } l \leq n.$$

By [Neu99, Chapter II, Proposition 5.7] we obtain the following isomorphism for any finite extension $L/\mathbb{Q}_p$ of degree d with residue field $\mathbb{F}_q$.

$$L^\times \cong \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}/p^m\mathbb{Z} \times \mathbb{Z}_p^d$$

where $m := \max\{k \geq 0 \mid \mu_{p^k} \subseteq L\}$. In our special case we have

$$K_n^\times \cong \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}_p^d$$

and therefore $K_n^\times/(K_n^\times)^{p^n}$ is isomorphic to a finite direct sum of copies of $\mathbb{Z}/p^n\mathbb{Z}$. Being no p^l-th power in $K_n^\times$ therefore implies that the residue class of b has order p^n in $K_n^\times/(K_n^\times)^{p^n}$. □

Lemma 2.38 *Let $a \in 1 + \mathfrak{m}_K$ be no root of unity. Then the Galois group of $K_\infty(\{a^{\frac{1}{p^n}} \mid n \geq 1\})/K$ is not abelian.*

Proof: It is enough to show that the $G_n := \mathrm{Gal}(K_n(a^{1/p^n})/K)$ is not abelian for some n. Without loss of generality we may assume that $a \notin (K^\times)^p$, otherwise replace a by $a^{1/p}$ which is still contained in $1 + \mathfrak{m}_K$ and no root of unity. We abbreviate

$$N_n := \mathrm{Gal}(K_n(a^{1/p^n})/K_n)$$

and

$$H_n := \mathrm{Gal}(K_n(a^{1/p^n})/K(a^{1/p^n})).$$

By Lemma 2.37 we know that $\mathrm{ord}(a(K_n^\times)^{p^n}) = p^n$, i.e. $a^l \notin (K_n^\times)^{p^n}$ for all $1 \leq l < p^n$. This leads to $[K_n(a^{1/p^n}) : K_n] = p^n$ in the following way. Assume that $X^{p^n} - a$ is divided by some f in $K_n[X]$. There exists $k < p^n$ and $i_1, \ldots, i_k \in \mathbb{Z}$ such that

$$f = \prod_{j=1}^{k} X - \xi_{p^n}^{i_j} a^{\frac{k}{p^n}} \text{ with } \xi_{p^n} \in \mu_{p^n} \text{ primitive .}$$

Examining the constant term of f we conclude $a^{k/p^n} \in K_n$, i.e. $a^k \in (K_n^\times)^{p^n}$ which contradicts the preceding argument. This means that N_n is a cyclic group of order p^n. The usual theory of p-power unit roots (e.g. [Neu99, Chapter II, (7.13)]) tells us that H_n is cyclic of a degree d dividing $p^{n-1}(p-1)$. G_n is a semidirect product of H_n and N_n which is not direct. In order to see this we are left to show that the map

$$H_n \to \mathrm{Aut}(N_n), h \mapsto (n \mapsto hnh^{-1})$$

is not the identity. Assume that n is chosen such that $|H_n| \neq 1 \neq |N_n|$ and take generators $\tau \in H_n$ and $\eta \in N_n$ as well as a primitive p^n-the root of unity $\xi_{p^n} \in K_n$. Then

$$\tau\eta\tau^{-1}(a^{1/p^n}) = \tau\eta(a^{1/p^n}) = \tau(\xi_{p^n} a^{1/p^n}) = \tau(\xi_{p^n})a^{1/p^n} \neq \xi_{p^n} a^{1/p^n} = \eta(a^{1/p^n})$$

shows the claim for suitable n. Therefore G_n is not abelian for some n. □

Corollary 2.39 *Let $a \in 1 + \mathfrak{m}_{K^{ab}}$ such that $a^{1/p^i} \in K^{ab}$ for all $i \geq 1$. Then a is a p-power root of unity.*

Example 2.40 *Let $a \in 1 + \mathfrak{m}_K$ be no root of unity and denote by α an element of $\mathcal{O}_{\mathbb{C}_p^\flat}$ such that $\alpha^\sharp = a$. Set*

$$v_1 := \log_{\mathrm{crys}}(\tau(\alpha)) = \sum_{n\geq 1} (-1)^{n+1} \frac{(\tau(\alpha)-1)^n}{n} \in B_{\mathrm{crys}}^+ \text{ and } v_2 := t \in B_{\mathrm{crys}}^+.$$

Investigating the G_K-action on v_1 delivers the following:

$$g.v_1 = \log_{\mathrm{crys}}(\tau(g\alpha)) = \log_{\mathrm{crys}}(\tau(\frac{g\alpha}{\alpha})) + \log_{\mathrm{crys}}(\tau(\alpha)).$$

But $(\frac{g\alpha}{\alpha})^\sharp = \frac{ga}{a} = 1$ and therefore we have

$$\frac{g\alpha}{\alpha} = \epsilon^{c(g)} \text{ for some (unique) } c(g) \in \mathbb{Z}_p.$$

Claim: $c(g)$ satisfies $c(gh) = c(g) + \chi(g)c(h)$ for all $g, h \in G_K$. This is due to:

$$\epsilon^{c(gh)} = \frac{(gh)\alpha}{\alpha} = \frac{g(\epsilon^{c(h)}\alpha)}{\alpha} = \epsilon^{c(h)\chi(g)}\frac{g\alpha}{\alpha} = \epsilon^{c(g)+\chi(g)c(h)}.$$

Therefore we get

$$g.v_1 = \log_{\mathrm{crys}}(\tau(\alpha)) + \log_{\mathrm{crys}}(\tau(\epsilon^{c(g)})) = v_1 + c(g)\cdot v_2$$

and

$$g.v_2 = g \cdot t = \chi(g) \cdot t$$

which implies that $V_a := \mathbb{Q}_p v_1 + \mathbb{Q}_p v_2$ *is a representation of* G_K*. We claim that* V_a *is two-dimensional. Assume that there is an* $\lambda \in \mathbb{Q}_p^\times$ *such that* $v_1 = \lambda v_2$ *and obtain for all* $g \in G_K$

$$\lambda \cdot \chi(g) v_2 = g(\lambda v_2) = g v_1 = v_1 + c(g) v_2 = \lambda v_2 + c(g) v_2.$$

Thus $c(g) = \lambda(\chi(g) - 1) = 0$ *for all* $g \in G_{K_\infty}$ *and* $g\alpha = \alpha$*, i.e.* $a^{\frac{1}{p^n}} \in K_\infty$ *which contradicts Lemma 2.38. We wish to show that* V_a *is crystalline. Therefore we make use of Lemma 2.36 and obtain*

$$\mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V_a, B_{\mathrm{crys}}) \cong (B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V_a^\wedge)^{G_K} = \mathbb{D}_{\mathrm{crys}}(V_a^\wedge).$$

Since $\dim_{K_0} \mathbb{D}_{\mathrm{crys}}(V_a) = \dim_{K_0} \mathbb{D}_{\mathrm{crys}}(V_a^\wedge) \leq \dim_{\mathbb{Q}_p}(V_a)$ *it is enough to show that there exists two* $\mathbb{Q}_p[G_K]$*-linear maps* $V_a \to B_{\mathrm{crys}}$ *which are linear independent. But*

$$\iota \colon V_a \to B_{\mathrm{crys}}$$

the canonical inclusion and

$$\pi \colon V_a \to V_a/\mathbb{Q}_p v_2 \cong \mathbb{Q}_p$$

the canonical projection are such maps. Using that V_a *is crystalline we can determine the Hodge polygon and Newton polygon associated to* D_a*. Let* x_π *and* x_ι *denote the elements of* $(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V_a^\wedge)^{G_K}$ *corresponding to* π *and* ι*. Then we obtain:*

$$x_\pi = 1 \otimes v_1^* \text{ and thus } \varphi(x_\pi) = \varphi(1) \otimes v_1^* = 1 \otimes v_1^* = x_\pi$$

$$x_\iota = v_1 \otimes v_1^* + v_2 \otimes v_2^* \text{ and thus } \varphi x_\iota = p \cdot x_\iota$$

since $\varphi(\log_{\mathrm{crys}}(\tau(x))) = \log_{\mathrm{crys}}(\tau(x^p)) = p \cdot \log_{\mathrm{crys}}(\tau(x))$ *for all* $x \in \mathcal{O}_{\mathbb{C}_p^\flat}$*. Therefore* $D_a^\wedge$ *decomposes as a* φ*-module and the Newton polygon is given by*

$$P_N(D_a^\wedge) = \{(0,0), (1,0), (2,1)\}.$$

The admissibility of $D_a^\wedge$ *implies that* $t_H(D_a^\wedge) = 1$ *and since* $x_\pi \notin \mathrm{Fil}^1(D_a^\wedge)_K$ *we conclude*

$$P_H(D_a^\wedge) = P_N(D_a^\wedge).$$

Chapter 3

(B-)Admissible Representations

Notation: We will use the following way to denote a pair $(\mathcal{F}, \mathcal{G})$ of quasi-inverse functors between two categories $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{F}\colon \mathcal{A} \rightleftharpoons \mathcal{B}\colon \mathcal{G},$$

where $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ and $\mathcal{G}\colon \mathcal{B} \to \mathcal{A}$ satisfy $\mathcal{G} \circ \mathcal{F} \cong \mathrm{id}_{\mathcal{A}}$ and $\mathcal{F} \circ \mathcal{G} \cong \mathrm{id}_{\mathcal{B}}$.

3.1 Fontaine's Equivalences of Categories

In order to state Fontaine's Theorems we need to define the relevant categories initially.

Definition 3.1 *The category of B_{st}-admissible (resp. B_{crys}-admissible) representations of G_K is denoted by $\mathrm{Rep}^{\mathrm{st}}(G_K)$ (resp. $\mathrm{Rep}^{\mathrm{crys}}(G_K)$) and we call it the category of (p-adic) log-crystalline (resp. crystalline) representations of G_K. Similarly we denote the category of B_{st}-admissible (resp. B_{crys}-admissible) representations of I_K by $\mathrm{Rep}^{\mathrm{st}}(I_K)$ (resp. $\mathrm{Rep}^{\mathrm{crys}}(I_K)$) and call it the category of (p-adic) log-crystalline (resp. crystalline) representations of I_K.*

Definition 3.2 *Let F be a field that contains K_0. A vector space V over F is called K-filtered if the scalar extension $V_K := K \otimes_{K_0} V$ is a filtered vector space over K, i.e. V_K carries a decreasing exhaustive and separated filtration $\mathrm{Fil}^\bullet$ (for details, see [BC09, Definition 4.1.1.]). A morphism $f\colon (V_1, \mathrm{Fil}_1^\bullet) \to (V_2, \mathrm{Fil}_2^\bullet)$ of K-filtered vector spaces over F is a F-linear map $f\colon V_1 \to V_2$ such that the induced map $f_K := f \otimes_{K_0} K$ satisfies $f_K(\mathrm{Fil}_1^i(V_K)) \subseteq \mathrm{Fil}_2^i(V_K)$ for all $i \in \mathbb{Z}$.*

Warning: Let F be a field that contains K_0. The category of K-filtered vector spaces over F is not abelian.

Definition 3.3 *Let $(W, \mathrm{Fil}^\bullet)$ be a filtered vector space over K. We define the Hodge number*

$$t_H(W) := t_H(W, \mathrm{Fil}^\bullet) := \sum_{i \in \mathbb{Z}} i \cdot \dim_K(\mathrm{Fil}^i(W)/\mathrm{Fil}^{i+1}(W)).$$

Definition 3.4 *We define the category of K-filtered (φ, N)-modules over K_0 (resp. P_0) as follows:*

- *The objects are tuples $D = (D, \mathrm{Fil}^\bullet, \varphi, N)$, where*
  - *$(D, \mathrm{Fil}^\bullet)$ is a K-filtered vector space over K_0 (resp. P_0),*
  - *(D, φ) is a φ-module over K_0 (resp. P_0) in the sense of section 1.4,*
  - *$N\colon D \to D$ is a K_0-linear (resp. P_0-linear) endomorphism,*
  - *$N\varphi = p\varphi N$ holds,*
- *A morphism $f\colon (D_1, \mathrm{Fil}_1^\bullet, \varphi_1, N_1) \to (D_2, \mathrm{Fil}_2^\bullet, \varphi_2, N_2)$ of K-filtered (φ, N)-modules over K_0 (resp. P_0) is a K_0-linear (resp. P_0-linear) map $f\colon D_1 \to D_2$ such that f is a morphism of K-filtered vector spaces over K_0 (resp. P_0) and $f \circ \varphi_1 = \varphi_2 \circ f$ as well as $f \circ N_1 = N_2 \circ f$ holds.*
- *The composition is the usual composition of maps.*

Definition 3.5 *We call a K-filtered (φ, N)-module $D = (D, \mathrm{Fil}^\bullet, \varphi, N)$ over K_0 (resp. P_0) admissible if $t_N(D) = t_H(D_K)$ and $t_N(D') \geq t_H(D'_K)$ holds for all subobjects $D' \subseteq D$. (The Newton Number $t_N(D)$ of a φ-module is explained in Definition B.12.)*

Remark 3.6 *The full subcategory consisting of admissible objects in the category of K-filtered (φ, N)-modules over K_0 (resp. P_0) is an abelian tensor category. This is a Theorem, see [BC09, Theorem 8.2.11.].*

Due to the work of Fontaine [Fon94a, Theorem 5.3.5.] we have the following:

Theorem 3.7 *There are two pairs of quasi-inverse functors:*

$$\mathbb{D}_{st}\colon \mathrm{Rep}^{st}(G_K) \rightleftharpoons (\text{admissible } K\text{-filtered } (\varphi, N)\text{-modules over } K_0) : \mathbb{V}_{st}$$

$$V \mapsto (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

$$\mathrm{Fil}^0(B_{\mathrm{st}} \otimes_{K_0} D)^{\varphi=\mathrm{id},N=0} \hookleftarrow D$$

and

$$\tilde{\mathbb{D}}_{st}\colon \mathrm{Rep}^{st}(I_K) \rightleftharpoons (\text{admissible } K\text{-filtered } (\varphi,N)\text{-modules over } P_0) : \tilde{\mathbb{V}}_{st}$$

$$V \mapsto (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{I_K}$$
$$\mathrm{Fil}^0(B_{\mathrm{st}} \otimes_{P_0} D)^{\varphi=\mathrm{id},N=0} \hookleftarrow D.$$

In particular we have the following *comparison isomorphisms*

$$\alpha_V\colon B_{\mathrm{st}} \otimes_{K_0} \mathbb{D}_{st}(V) \cong B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V, \tag{3.1}$$
$$\sum_{i,j} b_i \otimes b_j \otimes d_j \mapsto \sum_{i,j} (b_i \cdot b_j) \otimes d_j$$
$$\beta_D\colon B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} \mathbb{V}_{st}(D) \cong B_{\mathrm{st}} \otimes_{K_0} D, \tag{3.2}$$
$$\sum_{i,j} b_i \otimes b_j \otimes v_j \mapsto \sum_{i,j} (b_i \cdot b_j) \otimes v_j$$

in the first case and

$$\tilde{\alpha}_V\colon B_{\mathrm{st}} \otimes_{P_0} \tilde{\mathbb{D}}_{st}(V) \cong B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V, \tag{3.3}$$
$$\tilde{\beta}_D\colon B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} \tilde{\mathbb{V}}_{st}(D) \cong B_{\mathrm{st}} \otimes_{P_0} D \tag{3.4}$$

in the second case with maps in the same flavor as in the first case. Let V be a log-crystalline (p-adic) representation of G_K. Then

$$\begin{aligned}\tilde{\mathbb{D}}_{st}(V) &= (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{I_K}\\ &\overset{\alpha_V}{\cong} (B_{\mathrm{st}} \otimes_{K_0} (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K})^{I_K}\\ &= P_0 \otimes_{K_0} \mathbb{D}_{st}(V)\end{aligned}$$

shows that the diagram

$$\begin{array}{ccc} \mathrm{Rep}^{st}(G_K) & \xrightarrow{\mathbb{D}_{st}} & (\text{adm. } K\text{-filt. } (\varphi,N)\text{-mod.}/K_0) \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \cdot\otimes_{K_0} P_0} \\ \mathrm{Rep}^{st}(I_K) & \xrightarrow{\tilde{\mathbb{D}}_{st}} & (\text{adm. } K\text{-filt. } (\varphi,N)\text{-mod.}/P_0) \end{array} \tag{CD1}$$

is commutative, where $\mathcal{F}$ denotes the forgetful functor. In particular every log-crystalline representation of G_K is automatically log-crystalline as a representation of I_K. The converse is also true:

Lemma 3.8 *Let V be a representation of G_K. V is log-crystalline as a representation of G_K if and only if it is log-crystalline as a representation of I_K.*

Proof: Assume that V is log-crystalline as a representation of I_K and let $D := \tilde{\mathbb{D}}_{st}(V)$ denote the module corresponding to V. Then

$$\mathbb{D}_{st}(V) = ((B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{I_K})^{G_k} = D^{G_k}.$$

But since $H^1_{\mathrm{cont}}(G_k, \mathrm{GL}_n(P_0))$ is trivial for any $n \geq 1$ [BC09, Proof of Theorem 2.4.6.] we obtain an isomorphism $P_0 \otimes_{K_0} D^{G_k} \cong D$ of vector spaces over P_0 for any P_0-representation of G_k. This implies that $\dim_{K_0}(D^{G_k}) = \dim_{P_0}(D) = \dim_{\mathbb{Q}_p}(V)$. □

3.2 Log-crystalline Weil Group Representations

Definition 3.9 *A (p-adic) representation V of W_K is called log-crystalline (resp. de Rham, crystalline) if its restriction $V|_{I_K}$ is log-crystalline (resp. de Rham, crystalline). We denote the full subcategory of* $\mathrm{Rep}(W_K)$ *consisting of the log-crystalline (resp. de Rham, crystalline) representations by* $\mathrm{Rep}^{st}(W_K)$ *(resp.* $\mathrm{Rep}^{dR}(W_K)$*,* $\mathrm{Rep}^{crys}(W_K)$*).*

Remark 3.10 *By Lemma 3.8 and Remark 1.4 the category* $\mathrm{Rep}^{st}(G_K)$ *is a full subcategory of* $\mathrm{Rep}^{st}(W_K)$*.*

Let V denote a log-crystalline representation of W_K and $D := \tilde{\mathbb{D}}_{st}(V)$ the corresponding filtered (φ, N)-module. We define the following bijective self-map on $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$:

$$F_V = F\colon \sum_i b_i \otimes v_i \mapsto \sum_i \sigma_K.b_i \otimes \sigma_K.v_i$$

Since $I_K \trianglelefteq G_K$ is a normal subgroup the linear maps F restricts to a σ^f-semilinear (over P_0) bijective self-map of D. In particular F is independent of the choice of σ_K.

We now want to use the additional datum F to construct a category of linear algebra data that is equivalent to $\mathrm{Rep}^{st}(W_K)$ in the flavor of Theorem 1.17. Hence we need to check the assumptions made in Section 1.4 for $B = B_{\mathrm{st}}$, $E = \mathbb{Q}_p$ (i.e. $r = 1$), $G = G_K$, $I = I_K$ and $\varsigma = \sigma_K$.

The category $\mathcal{C}_{W_K}$ from Section 1.4 is then the following.

Definition 3.11 *We define the category of admissible K-filtered (φ, N, F)-modules over P_0 as follows:*

- *The objects are pairs (D, F), where D is an admissible K-filtered (φ, N)-module over P_0 and $F\colon D \to D$ is a bijective, σ^f-semilinear map, that is strictly compatible with the filtration on D_K and commutes with φ and N.*
- *As morphisms we take the morphisms in the category of K-filtered (φ, N)-modules over P_0 that commute with F.*
- *The composition of morphisms is the usual composition of maps.*

We need to check the axioms 1.1-1.4 from Section 1.4:

- φ commutes with the action of G_K on B_{st} and $B_{\mathrm{st}}^{G_K} = K_0$ resp. $B_{\mathrm{st}}^{I_K} = P_0$ (compare Lemma 2.35) are fields. Furthermore B_{st} is $(G_K, \mathbb{Q}_p)$- and $(I_K, \mathbb{Q}_p)$-regular [BC09, Proposition 9.2.11].
- We need to check Axiom 1.1. The forgetful functor

  $$T\colon (\text{adm. } K\text{-filtered } (\varphi, N, F)\text{-mod. over } P_0\ ) \to (\varphi\text{-mod. over } P_0).$$

  The natural transformations $\xi_\bullet$ and $\eta_\bullet$ are given by

  $$\xi_V\colon (T \circ \tilde{\mathbb{D}}_{st})(V) \xrightarrow{=} (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^I$$

  by the identity for any log-crystalline representation V and

  $$\eta_D\colon \tilde{\mathbb{V}}_{st}(D) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} T(D))^{\varphi=\mathrm{id}, N=0, \mathrm{Fil}^0} \hookrightarrow (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} T(D))^{\varphi=\mathrm{id}}$$

  is given by the canonical inclusion for any admissible K-filtered (φ, N)-module D over P_0. Take the restrictions of the comparison isomorphisms in (3.3) resp. (3.4) for $\tilde{\alpha}_\bullet$ resp. $\tilde{\beta}_\bullet$. Then Axiom 1.1 is satisfied by Theorem 3.7.
- $t_N(\sigma_K^*(D)) = \nu(\det(\sigma_K^*(\varphi))) = \nu(\det(\varphi)) = t_N(D)$ and $t_H(\sigma_K^*(D)) = t_H(D)$ holds for all K-filtered φ-modules $D = (D, \varphi)$. Therefore $\sigma_K^*(D)$ is admissible and Axiom 1.2 is satisfied.
- The *monodromy operator* $N\colon B_{\mathrm{st}} \to B_{\mathrm{st}}$ is G_K-equivariant (see Lemma 2.35), therefore $F \circ N = N \circ F$ holds.

- The injective map $\iota\colon K \otimes_{K_0} B_{\rm st} \hookrightarrow B_{\rm dR}$ is G_K-equivariant (see Lemma 2.35), hence $F_K := K \otimes_{K_0} F$ is *strictly compatible* with the filtration on D_K, i.e. $F_K(\mathrm{Fil}^i(D_K)) = \mathrm{Fil}^i(D_K)$.
- Axiom 1.3 is satisfied due to the two preceding points: The self-map $F_V^{\mathrm{lin},\varphi}$ restricts to $\tilde{\mathbb{V}}_{st}(D)$ since F is compatible with the Monodromy operator N and strictly compatible with the filtration. We set $F_V^{\mathrm{lin}} := F_V^{\mathrm{lin},\varphi}|_{\tilde{\mathbb{V}}_{st}(D)}$ and obtain $T(F_V^{\mathrm{lin}}) = F_V^{\mathrm{lin},\varphi}$.
- Using the G_K-equivariance of ι and N and the calculation from Remark 1.14 we also receive that the map $\tilde{\mathbb{V}}_{st}(D) \to \tilde{\mathbb{V}}_{st}(D)$ given by $\sum_i b_i \otimes d_i \mapsto \sum_i \sigma_K.b_i \otimes d_i$ is well-defined, i.e. Axiom 1.4 holds.

Let (D, F) be an object of the category we just defined. Then the map $\tilde{\mathbb{V}}_{st}(D) \to \tilde{\mathbb{V}}_{st}(D)$ given by $\sum_i b_i \otimes d_i \mapsto \sum_i b_i \otimes F(d_i)$ is well-defined since F is strictly compatible with the filtration and commutes with φ and N. By a similar calculation as in Remark 1.15 this is an isomorphism. Therefore Axiom 1.5 is satisfied.

Theorem 3.12 *There is an equivalence of categories given as follows:*

$$\tilde{\mathbb{D}}_{st}\colon \mathrm{Rep}^{st}(W_K) \rightleftharpoons (\text{admissible } K\text{-filtered } (\varphi, N, F)\text{-modules over } P_0\ ) : \tilde{\mathbb{V}}_{st}$$

$$V \mapsto ((B_{\rm st} \otimes_{\mathbb{Q}_p} V)^{I_K}, F_V)$$
$$\mathrm{Fil}^0(B_{\rm st} \otimes_{P_0} D)^{\varphi=\mathrm{id}, N=0} \leftarrow\!\shortmid (D, F).$$

Proof: This is literally a corollary from Theorem 1.17. □

3.3 De Rham Weil Group Representations

At first we remark that any log-crystalline representation of G_K (resp. I_K) is also de Rham. This comes from the fact (see Lemma 2.35) that there is an injective morphism of K-algebras (which is G_K-equivariant)

$$K \otimes_{K_0} B_{\rm st} \to B_{\rm dR}$$

and the following calculation.

$$\begin{aligned} K \otimes_{K_0} \mathbb{D}_{st}(V) &= K \otimes_{K_0} (B_{\rm st} \otimes_{\mathbb{Q}_p} V)^{G_K} \\ &= (K \otimes_{K_0} B_{\rm st} \otimes_{\mathbb{Q}_p} V)^{G_K} \\ &\hookrightarrow (B_{\rm dR} \otimes_{\mathbb{Q}_p} V)^{G_K} = \mathbb{D}_{dR}(V). \end{aligned} \tag{3.5}$$

Thus $\dim_{\mathbb{Q}_p}(V) = \dim_K(K \otimes_{K_0} \mathbb{D}_{st}(V)) \leq \dim_K(\mathbb{D}_{dR}(V)) \leq \dim_{\mathbb{Q}_p}(V)$ (for the latter inequality, see [BC09, Theoreom 5.2.1.]) and V is de Rham.

Definition 3.13 *Consider a finite extension L/K and a representation V of I_K (resp. G_K).*

- *We define a relative version of the functor $\tilde{\mathbb{D}}_{\mathrm{crys}}$ (resp. $\mathbb{D}_{\mathrm{crys}}$) by*

$$\tilde{\mathbb{D}}_{\mathrm{crys},L}(V) := (B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)^{I_L}$$

$$(\text{resp. } \mathbb{D}_{\mathrm{crys},L}(V) := (B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)^{G_L}).$$

- *We define a relative version of the functor $\tilde{\mathbb{D}}_{st}$ (resp. $\mathbb{D}_{st}$) by*

$$\tilde{\mathbb{D}}_{st,L}(V) := (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{I_L}$$

$$(\text{resp. } \mathbb{D}_{st,L}(V) := (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_L}).$$

- *In the same fashion we set*

$$\tilde{\mathbb{D}}_{dR,L}(V) := (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{I_L}.$$

- *We call V potentially crystalline if $V|_{I_L}$ (resp. $V|_{G_L}$) is crystalline for some finite extension L/K. In the same manner we define V to be potentially log-crystalline if $V|_{I_L}$ (resp. $V|_{G_L}$) is log-crystalline for some finite extension L/K*

Remark 3.14 *Any potentially log-crystalline representation V of G_K (resp. I_K, resp. W_K) is de Rham. Take a representation V of G_K and assume that* $\dim_{L_0}(\mathbb{D}_{st,L}(V)) = \dim_{\mathbb{Q}_p}(V)$. *Without loss of generality we may enlarge L by its Galois envelope and assume that L/K is Galois. By Galois descent one obtains an isomorphism*

$$L \otimes_K (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} \cong (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_L}.$$

Combine this and (3.5) *to see that there exists an isomorphism*

$$L \otimes_{L_0} \mathbb{D}_{st,L}(V) \cong \mathbb{D}_{dR,L}(V) \cong L \otimes_K \mathbb{D}_{dR}(V),$$

in particular $\dim_{\mathbb{Q}_p}(V) = \dim_{L_0}(\mathbb{D}_{st,L}(V)) = \dim_K(\mathbb{D}_{dR}(V))$ *and V is de Rham.*

The following theorem due to Berger [Ber02, Cor. 5.22.] is called p-adic Monodromy theorem. Another proof, not using p-adic differential equations can be found in [Fon00, Theo. A].

Theorem 3.15 *Let V be representation of G_K (resp. I_K, resp. W_K). Then V is potentially log-crystalline if and only if V is de Rham.*

This theorem allows us to construct a category of (semi-)linear algebra data which is equivalent $\mathrm{Rep}^{dR}(G_K)$ (resp. $\mathrm{Rep}^{dR}(I_K)$, $\mathrm{Rep}^{dR}(W_K)$).

For a potentially log-crystalline representation of I_K we set

$$D := \tilde{\mathbb{D}}_{pst}(V) := \varinjlim_{L/K \text{ finite}} \tilde{\mathbb{D}}_{st,L}(V)$$

and remark that this is a vector space over P_0 of dimension $\dim_{\mathbb{Q}_p}(V)$. Then

$$\tilde{\mathbb{D}}_{pst}(V) = \tilde{\mathbb{D}}_{st,L}(V) = \tilde{\mathbb{D}}_{pst}(V)^{I_L} \tag{3.6}$$

for a finite extension L/K. Hence there is a discrete action of I_K on D, i.e. the action factors through a finite quotient. This allows us to endow $\tilde{\mathbb{D}}_{pst}(V)$ with the usual structure of a (φ, N)-module. Furthermore we endow this object with a $\overline{K}$-filtration:

$$\mathrm{Fil}^i(D_{\overline{K}}) := \overline{K} \otimes_L \mathrm{Fil}^i(\tilde{\mathbb{D}}_{st,L}(V)_L).$$

If we assume L/K to be Galois we obtain

$$\mathrm{Fil}^i(D_K)^{I_L} = L \otimes_{K_0} \mathrm{Fil}^i(D^{I_L})$$

by Galois descent. The above justifies the following definition.

Definition 3.16 *We define the category of admissible $\overline{K}/K$-filtered (φ, N, I_K)-modules (resp. admissible $\overline{K}/K$-filtered (φ, N, G_K)-modules) over P_0 (resp. K_0) as follows:*

- *The objects are tuples $(D, \mathrm{Fil}^\bullet, \varphi, N)$, where*
  - *$(D, \mathrm{Fil}^\bullet)$ is a $\overline{K}$-filtered vector space over P_0 (resp. K_0).*
  - *(D, φ) is a φ-module over P_0 (resp. K_0).*
  - *$N \colon D \to D$ is a P_0-linear (resp. K_0-linear) endomorphism.*
  - *I_K (resp. G_K) acts on D discretely.*
  - *$N \circ \varphi = p(\varphi \circ N)$ holds.*
  - *$\varphi \circ g = g \circ \varphi$ and $N \circ g = g \circ N$ for all $g \in I_K$ (resp. $g \in G_K$).*
- *A morphism $f \colon (D_1, \mathrm{Fil}_1^\bullet, \varphi_1, N_1) \to (D_2, \mathrm{Fil}_2^\bullet, \varphi_2, N_2)$ is a I_K-equivariant (resp. G_K-equivariant) P_0-linear (resp. K_0-linear) map $f \colon D_1 \to D_2$ such that f is a morphism of $\overline{K}$-filtered vector spaces and $f \circ \varphi_1 = \varphi_2 \circ f$ as well as $f \circ N_1 = N_2 \circ f$ holds.*

- *The composition is the usual composition of maps.*

This leads to the equivalence of categories stated in [Fon94b, §5.6.7.]:

Theorem 3.17 *There exists an equivalence of categories*

$$\tilde{\mathbb{D}}_{pst}\colon \mathrm{Rep}^{pst}(I_K) \rightleftharpoons (\text{adm. } \overline{K}/K\text{-filt. } (\varphi, N, I_K)\text{-mod.}/P_0) : \tilde{\mathbb{V}}_{pst}$$

given by

$$V \mapsto \varinjlim_{L/K \text{ finite}} \tilde{\mathbb{D}}_{st,L}(V)$$

$$\tilde{\mathbb{V}}_{pst}(D) \leftarrow\!\shortmid D$$

where $\tilde{\mathbb{V}}_{pst}(D) := \{x \in B_{\mathrm{st}} \otimes_{P_0} D \mid Nx = 0, \varphi(x) = x, 1 \otimes x \in \mathrm{Fil}^0(B_{\mathrm{st}} \otimes D)_{\overline{K}}\}$.

Remark 3.18 *Let D be an admissible $\overline{K}/K$-filtered (φ, N, I_K)-module over P_0. By choosing a sufficiently large extension L/K we obtain*

$$D = D^{I_L} \text{ and } \tilde{\mathbb{V}}_{pst}(D) = \tilde{\mathbb{V}}_{st,L}(D).$$

In the next step we will generalize this to the case of Weil group representations. Let V be a potentially log-crystalline representation of W_K and $D := \tilde{\mathbb{D}}_{pst}(V)$. Define

$$F_V\colon D \to D \text{ by } \sum_i b_i \otimes v_i \mapsto \sum_i \sigma_K b_i \otimes \sigma_K v_i.$$

Since we may assume that all L/K are Galois we obtain that $I_L \trianglelefteq G_K$ is a normal subgroup and therefore F is well-defined, bijective and σ^f-semilinear. One has to pay attention to the relation between F_V and the I_K-action. For all $u \in I_K$ and $d = \sum_i b_i \otimes v_i \in D$ we have

$$\begin{aligned} F_V(u.d) &= F_V(\sum_i u.b_i \otimes u.v_i) \\ &= \sum_i \sigma_K.(u.b_i) \otimes \sigma_K.(u.v_i) \\ &= \sum_i (\sigma_K u \sigma_K^{-1}).(\sigma_K.b_i) \otimes (\sigma_K u \sigma_K^{-1}).(\sigma_K.v_i) \\ &= (\sigma_K u \sigma_K^{-1}).F_V(d). \end{aligned} \tag{3.7}$$

This justifies to define the following category.

Definition 3.19 *We define the category of admissible $\overline{K}/K$-filtered (φ, N, I_K, F)-modules over P_0 as follows:*

- *The objects are pairs (D, F), where D is a $\overline{K}/K$-filtered (φ, N, I_K)-module over P_0 and $F\colon D \to D$ is a bijective, σ^f-semilinear map that is strictly compatible with the filtration on $D_{\overline{K}}$, commutes with φ and N and satisfies $F(u.d) = (\sigma_K u \sigma_K^{-1}).F(d)$ for all $u \in I_K$.*
- *A morphism $f\colon (D_1, F_1) \to (D_2, F_2)$ is a morphism $f\colon D_1 \to D_2$ in the category of admissible $\overline{K}/K$-filtered (φ, N, I_K)-modules over P_0 such that $f \circ F_1 = F_2 \circ f$ holds.*
- *The composition of morphisms is the usual composition of maps.*

Theorem 3.12 leads to:

Theorem 3.20 *There exist an equivalence of categories*

$$\tilde{\mathbb{D}}_{pst}\colon \mathrm{Rep}^{pst}(W_K) \rightleftharpoons (\text{adm. } \overline{K}/K\text{-filt. } (\varphi, N, I_K, F)\text{-mod.}/P_0) : \tilde{\mathbb{V}}_{pst}$$

given by

$$V \mapsto (\tilde{\mathbb{D}}_{pst}(V), F_V)$$
$$\tilde{\mathbb{V}}_{pst}(D) \leftarrow\!\shortmid D$$

Proof: The only significant difference in the proofs of this theorem and Theorem 3.12 is the fact that the group I_K acts on $\tilde{\mathbb{V}}_{pst}(D)$ diagonally. Hence we have to check that

$$\hat{\rho}\colon W_K \times V \to V \text{ given by } (u\sigma_K^n, v) \mapsto (u\sigma_K^n).v := \sum_i (u\sigma_K^n).b_i \otimes u.F^n(d_i)$$

for all $v = \sum_i b_i \otimes d_i \in \tilde{\mathbb{V}}_{pst}(D), u \in I_K$ and $n \in \mathbb{N}$ defines a representation of W_K. Take $g_1 = u_1\sigma_K^{n_1}, g_2 = u_2\sigma_K^{n_2} \in W_K$ and $v = \sum_i b_i \otimes d_i \in \tilde{\mathbb{V}}_{pst}(D)$ and see that

$$\begin{aligned}(g_1g_2).v &= (u_1\sigma_K^{n_1}u_2\sigma_K^{-n_1}).v \\ &= \sum_i (g_1g_2).b_i \otimes (u_1\sigma_K^{n_1}u_2\sigma_K^{-n_1}).F^{n_1+n_2}(d_i) \\ &= \sum_i g_1.(g_2.b_i) \otimes u_1.F^{n_1}(u_2.F^{n_2}(d_i)) \\ &= g_1.(g_2.v)\end{aligned}$$

□

Another way to interpret [Fon94a, §5.6.7.] is:

Theorem 3.21 *There exists an equivalence of categories*

$$\mathbb{D}_{pst}\colon \mathrm{Rep}^{pst}(G_K) \rightleftharpoons (\text{adm. } \overline{K}/K\text{-filt. } (\varphi, N, G_K)\text{-mod.}/\mathbb{Q}_p^{nr}) : \mathbb{V}_{pst}$$

given by

$$V \mapsto \varinjlim_{L/K \text{ finite}} \mathbb{D}_{st,L}(V)$$

$$\mathbb{V}_{pst}(D) \leftarrow\!\shortmid D$$

where

$$\mathbb{V}_{pst}(D) := \{x \in B_{\mathrm{st}} \otimes_{\mathbb{Q}_p^{nr}} D \mid Nx = 0, \varphi(x) = x, 1 \otimes x \in \mathrm{Fil}^0(B_{\mathrm{st}} \otimes D)_{\overline{K}}\}.$$

This leads to the following commutative diagram of functors:

$$\begin{array}{ccc} \mathrm{Rep}^{pst}(G_K) & \xrightarrow{\mathbb{D}_{pst}} & \{\text{adm. } \overline{K}/K\text{-filt. } (\varphi, N, G_K)\text{-mod. over } \mathbb{Q}_p^{nr}\} \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \cdot \otimes_{\mathbb{Q}_p^{nr}} P_0} \\ \mathrm{Rep}^{pst}(W_K) & \xrightarrow{\tilde{\mathbb{D}}_{pst}} & \{\text{adm. } \overline{K}/K\text{-filt. } (\varphi, N, I_K, F)\text{-mod. over } P_0\} \end{array} \qquad \text{(CD2)}$$

By Lemma 3.8 the forgetful functor is well-defined. Let D be an admissible $\overline{K}/K$-filtered (φ, N, G_K)-module over $\mathbb{Q}_p^{nr}$. Then the commutativity follows from

$$\begin{aligned} \tilde{\mathbb{D}}_{pst}(\mathbb{V}_{pst}(D)) &= \tilde{\mathbb{D}}_{st,L}(\mathbb{V}_{pst}(D)^{G_L}) \\ &= \tilde{\mathbb{D}}_{st,L}(\mathbb{V}_{st,L}(D^{G_L})) \\ &= P_0 \otimes_{L_0} D^{G_L} \\ &= P_0 \otimes_{\mathbb{Q}_p^{nr}} \underbrace{\mathbb{Q}_p^{nr} \otimes_{L_0} D^{G_L}}_{\cong D} \cong P_0 \otimes_{\mathbb{Q}_p^{nr}} D. \end{aligned}$$

F is given by

$$F(\lambda \otimes d) = \sigma_K(\lambda) \otimes \sigma_K.d$$

for all $u \in I_K, \lambda \in P_0$ and $d \in D$.

Remark 3.22 *Let D be as above. The action of I_K on $P_0 \otimes_{\mathbb{Q}_p^{nr}} D$ is given by*

$$u.(\lambda \otimes d) = u(\lambda) \otimes u.d = \lambda \otimes u.d$$

for all $u \in I_K, \lambda \in P_0$ and $d \in D$. Hence the semilinear action of G_K on D becomes a linear action of I_K on $P_0 \otimes_{\mathbb{Q}_p^{nr}} D$ since I_K acts trivially on P_0.

Chapter 4

Weil vs Galois group representations

The first aim is to characterize the admissible filtered (φ, N, F)-modules over P_0 that correspond to representations of the absolute Galois group G_K.

4.1 Lifting Maps from $\mathbb{Z}$ to $\hat{\mathbb{Z}}$

Our intermediate goal is to show that any group homomorphism $\mathbb{Z} \to \mathrm{GL}_r(\mathcal{O}_{P_0})$ has a continuous extension $\hat{\mathbb{Z}} \to \mathrm{GL}_r(\mathcal{O}_{P_0})$.

Lemma 4.1 *Let $\{G_i\}_{i\in I}$ be a projective system of groups such that each element $g \in G_i$ has finite order for all $i \in I$. For any homomorphism*

$$\varphi\colon \mathbb{Z} \to \varprojlim G_i$$

there exists a unique continuous extension

$$\hat{\varphi}\colon \hat{\mathbb{Z}} \to \varprojlim G_i$$

with respect to the projective limit topologies on both sides.

Proof: Set $G := \varprojlim G_i$, denote by $\pi_i\colon G \to G_i$ the projection maps and by e_{G_i} the neutral element in G_i. We define $m_i \in \mathbb{N}$ to be the order of $\pi_i(\varphi(1))$ for all $i \in I$. For $i \in I$ denote

$$N_i := \{n \in \mathbb{N} \mid m_i \text{ divides } n\} = \mathbb{N} \cap m_i\mathbb{Z}$$

and obtain a group homomorphism

$$\varphi_{n,i}\colon \mathbb{Z}/n\mathbb{Z} \to G_i, \text{ given by } \bar{1} \mapsto \pi_i(\varphi(1))$$

for all $n \in N_i$. If $t_{ij}\colon G_i \to G_j$ is the transition map for $i, j \in I$ such that $i \geq j$, we know that $t_{ij}(\pi_i(\varphi(1))) = \pi_j(\varphi(1))$ by the definition of a projective system. In particular

$$\pi_j(\varphi(1))^{m_i} = t_{ij}(\pi_i(\varphi(1)))^{m_i} = e_{G_j}.$$

Hence $m_j = \mathrm{ord}(\pi_j(\varphi(1)))$ divides m_i and we receive $N_i \subseteq N_j$. Therefore the map φ_n is compatible with the transition maps, i.e. $t_{ij} \circ \varphi_{n,i} = \varphi_{n,j}$ for all $i \geq j$ and $n \in N_i \subseteq N_j$. By the universal property of the projective limit we receive a continuous group homomorphism

$$\hat{\varphi}\colon \varprojlim_{n \in N} (\mathbb{Z}/n\mathbb{Z}) \to G \text{ for } N := \bigcup_{i \in I} N_i.$$

But $N \subseteq \mathbb{N}$ is a cofinal system and hence we get a continuous group homomorphism

$$\hat{\varphi}\colon \hat{\mathbb{Z}} \to G$$

that extends φ.

□

Remark 4.2 *Let R be a ring, $r \in \mathbb{N}$ and I be a totally ordered set such that $R = \bigcup_i R_i$ for a family of finite rings $\{R_i\}_{i \in I}$ such that $R_i \subseteq R_j$ for all $i \geq j$. Then each element in $\mathrm{GL}_r(R)$ has finite order.*

Combining Lemma 4.1 and Remark 4.2 has the following immediate consequence. We can modify the proof of Corollary 1.6 and obtain:

Corollary 4.3 *Let E be a local field with finite residue field. The forgetful functor*

$$\mathcal{F}\colon \mathrm{Rep}_{\overline{\mathbb{F}}_p}(G_E) \to \mathrm{Rep}_{\overline{\mathbb{F}}_p}(W_E)$$

is an equivalence of categories.

Lemma 4.4 *Each element of $\mathrm{GL}_r(\mathcal{O}_{P_0}/p^n\mathcal{O}_{P_0})$ has finite order.*

Proof: Choose a tower of finite sub-extensions $\mathbb{Q}_p \subseteq L_0 \subseteq L_1 \subseteq \cdots \subseteq \mathbb{Q}_p^{nr}$ such that $\mathbb{Q}_p^{nr} = \bigcup_i L_i$. Then $\mathcal{O}_{L_i}/p^n\mathcal{O}_{L_i}$ is a finite ring for each i and we can apply remark 4.2 to

$$\bigcup_i \mathcal{O}_{L_i}/p^n\mathcal{O}_{L_i} = \mathcal{O}_{\mathbb{Q}_p^{nr}}/p^n\mathcal{O}_{\mathbb{Q}_p^{nr}}.$$

Hence the elements of $\mathrm{GL}_r(\mathcal{O}_{\mathbb{Q}_p^{nr}}/p^n\mathcal{O}_{\mathbb{Q}_p^{nr}}) \cong \mathrm{GL}_r(\mathcal{O}_{P_0}/p^n\mathcal{O}_{P_0})$ are of finite order. □

Corollary 4.5 *For each group homomorphism $\phi\colon \mathbb{Z} \to \mathrm{GL}_r(\mathcal{O}_{P_0})$ there exists a unique continuous group homomorphism $\hat{\phi}\colon \hat{\mathbb{Z}} \to \mathrm{GL}_r(\mathcal{O}_{P_0})$ with $\hat{\phi}|_{\mathbb{Z}} = \phi$.*

Proof: Apply Lemma 4.1 to $\mathrm{GL}_r(\mathcal{O}_{P_0}) = \varprojlim_n \mathrm{GL}_r(\mathcal{O}_{P_0}/p^n\mathcal{O}_{P_0})$. □

Example 4.6 *Let R be a discrete valuation ring with uniformizer ϖ and E its field of fractions, e.g. $E = \mathbb{Q}_p$ or $E = P_0$ and $\varpi = p$.*

1. *Denote by $W := E^2$ be the representation of W_K where I_K acts trivially on W and σ_K has representing matrix*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}(R).$$

We denote all prime numbers with $p_1, p_2, \ldots$ starting with $p_1 = 3$. Then set $c_n := p_1^n p_2^n \cdots p_n^n \in \mathbb{Z}$ and by Bezout's Lemma there exist sequences $(a_n)_n$ and $(b_n)_n$ in $\mathbb{Z}$ such that $1 = a_n c_n + b_n 2^n$. Therefore $z := \lim_{n\to\infty} -a_n c_n$ is an element of $\hat{\mathbb{Z}}$ since

$$\hat{\mathbb{Z}} \cong \prod_{p\in\mathbb{P}} \mathbb{Z}_p$$

and $z_n := -a_n c_n$ converges to 1 in $\mathbb{Z}_2$ and to 0 in $\mathbb{Z}_p$ for $p \neq 2$. Now we see that the element $g := \sigma_K^z$ acts on W via

$$\rho(\sigma_K^z) = \lim_{n\to\infty} \rho(\sigma_K^{1+b_n 2^n}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2. *On the other hand let $W := E^2$ be the representation of W_K where I_K acts trivially on W and σ_K has representing matrix*

$$\begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \in \mathrm{GL}(R).$$

Then we can not extend the action of W_K to an action of G_K via continuity since

$$\rho(\sigma_K^z) = \lim_{n\to\infty} \rho(\sigma_K^{1+b_n 2^n}) = \lim_{n\to\infty} \varpi^{nb_n} \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}$$

does not exist in $\mathrm{GL}_2(E)$.

4.2 Identifying the Galois Group Representations

The following theorem gives a precise description of the log-crystalline Weil group representations that can be lifted to a Galois group representation.

Theorem 4.7 *Let V be a log-crystalline representation of W_K of dimension r and $\tilde{\mathbb{D}}_{st}(V) = (D, F)$ the corresponding module. The following statements are equivalent:*

1. *V is a log-crystalline representation of G_K.*
2. *There exists a basis $\mathcal{B}$ of D such that $F_{\mathcal{B}} \in \mathrm{GL}_r(\mathcal{O}_{P_0})$.*
3. *There exists a $\mathcal{O}_{P_0}$-lattice $M \subseteq D$ such that $F(M) = M$.*

We state the following lemma of topological nature in order to prove the theorem.

Lemma 4.8 *Let (D, F) be an admissible K-filtered (φ, N, F)-module over P_0 such there exists a $\mathcal{O}_{P_0}$-lattice $M \subseteq D$ satisfying $F(M) = M$. Take a basis $\mathcal{B}$ of D such that $F_{\mathcal{B}} \in \mathrm{GL}_r(\mathcal{O}_{P_0})$ and define $\hat{\phi}\colon \hat{\mathbb{Z}} \to \mathrm{GL}_r(\mathcal{O}_{P_0})$ to be the continuous map uniquely determined by $1 \mapsto F_{\mathcal{B}}$ by Corollary 4.5. Let F^z denote the map represented by $\hat{\phi}(z)$ for all $z \in \hat{\mathbb{Z}}$. Then:*

1. *$F^z \circ \varphi = \varphi \circ F^z$ for all $z \in \hat{\mathbb{Z}}$.*
2. *$F^z \circ N = N \circ F^z$ for all $z \in \hat{\mathbb{Z}}$.*
3. *$F^z_K(\mathrm{Fil}^i(D_K)) = \mathrm{Fil}^i(D_K)$ for all $z \in \hat{\mathbb{Z}}$.*

Proof: $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ is metrizable by [Que76, Korollar 10.18] as a countable product of metric spaces. Therefore we can choose for any $z \in \hat{\mathbb{Z}}$ a sequence $(z_n)_{n\in\mathbb{N}}$ in $\hat{\mathbb{Z}}$ converging to z. We obtain

$$F^z \circ \varphi = \lim_{n\to\infty} (F^{z_n} \circ \varphi) = \lim_{n\to\infty} (\varphi \circ F^{z_n}) = \varphi \circ F^z$$

for all $z \in \hat{\mathbb{Z}}$. If one replaces φ with N we also see that N commutes with F^z for any $z \in \hat{\mathbb{Z}}$. Furthermore $F^z_K(d) = \lim_{n\to\infty} F^{z_n}_K(d) \in \mathrm{Fil}^i(D_K)$ for all $d \in \mathrm{Fil}^i(D_K)$, $z \in \hat{\mathbb{Z}}$ and $i \in \mathbb{Z}$ since $\mathrm{Fil}^i(D_K) \subseteq D_K$ is closed. □

Now we prove Theorem 4.7.

Proof: 2. and 3. are equivalent by definition. Assume 2. holds. Take a basis $\mathcal{B}$ of D such that $F_{\mathcal{B}} \in \mathrm{GL}_r(\mathcal{O}_{P_0})$ and define $\hat{\phi}\colon \hat{\mathbb{Z}} \to \mathrm{GL}_r(\mathcal{O}_{P_0})$ to be the continuous map uniquely determined by $1 \mapsto F_{\mathcal{B}}$ by Corollary 4.5. Let F^z denote the map represented by $\hat{\phi}(z)$ for all $z \in \hat{\mathbb{Z}}$. Define the map

$$\hat{\rho}\colon G_K \times V \to V \text{ via } g.v := \sum_i g.b_i \otimes F^{\deg_K(g)}(d_i)$$

for $v = \sum_i b_i \otimes d_i$. We need to check that this is well-defined, i.e. that the image of the map is contained in $V = \mathrm{Fil}^0(B_{\mathrm{st}} \otimes_{P_0} D)^{\varphi=\mathrm{id}, N=\mathrm{id}} \subseteq B_{\mathrm{st}} \otimes_{P_0} D$. But this is covered by the formulas in Lemma 4.8 as we see in the following. Take $v = \sum_i b_i \otimes d_i \in V$ and $g \in G_K$ and calculate

$$\begin{aligned}
\varphi(g.v) &= \varphi(\sum_i g.b_i \otimes F^{\deg_K(g)}(d_i)) \\
&= \sum_i \varphi(g.b_i) \otimes (\varphi \circ F^{\deg_K(g)})(d_i) \\
&= \sum_i g.\varphi(b_i) \otimes F^{\deg_K(g)}(\varphi(d_i)) \\
&= g.\varphi(v) = g.v
\end{aligned}$$

as well as

$$\begin{aligned}
N(g.v) &= N(\sum_i g.b_i \otimes F^{\deg_K(g)}(d_i)) \\
&= \sum_i N(g.b_i) \otimes (N \circ F^{\deg_K(g)})(d_i) \\
&= \sum_i g.N(b_i) \otimes F^{\deg_K(g)}(N(d_i)) \\
&= g.N(v) = 0
\end{aligned}$$

For any i we have $1 \otimes b_i \otimes d_i \in \mathrm{Fil}^j(K \otimes_{K_0} B_{\mathrm{st}}) \otimes_K \mathrm{Fil}^{-j}(D_K)$ for some $j \in \mathbb{Z}$.

$$\begin{aligned}
g.(b_i \otimes d_i) &= \sum_i g.b_i \otimes F^{\deg_K(g)}(d_i) \\
&\in \mathrm{Fil}^j(K \otimes_{K_0} B_{\mathrm{st}}) \otimes F_K^{\deg_K(g)}(\mathrm{Fil}^{-j}(D_K)) \\
&= \mathrm{Fil}^j(K \otimes_{K_0} B_{\mathrm{st}}) \otimes \mathrm{Fil}^{-j}(D_K)
\end{aligned}$$

Therefore $g.(b_i \otimes d_i) \in \mathrm{Fil}^0(B_{\mathrm{st}} \otimes D)$ for all i, hence $g.v \in \mathrm{Fil}^0(B_{\mathrm{st}} \otimes D)$ and overall we have $g.v \in V$. By the same calculation as in (1.2) $\hat{\rho}$ defines a group action of G_K on V. In order to show that this action is continuous we would like to apply Proposition 1.5. Hence it is necessary to show that the map

$$I_K \times \hat{\mathbb{Z}} \to I_K \text{ given by } (u, z) \mapsto \sigma^z u \sigma^{-z}$$

is continuous. Take an open subset of $U \subseteq I_K$ and we may assume without loss of generality that U is a normal open subgroup of I_K, in particular U is a normal subgroup of G_K. Then the preimage of U under the map above is

$$\bigcup_{z\in\hat{\mathbb{Z}}} \sigma^{-z} U \sigma^{z} \times \{z\} = \bigcup_{z\in\hat{\mathbb{Z}}} U \times \{z\} = U \times \hat{\mathbb{Z}} \subseteq I_K \times \hat{\mathbb{Z}}.$$

We conclude that we are allowed to apply Proposition 1.5 and V is a continuous representation of G_K. The representation obtained this way is log-crystalline since it is log-crystalline as a representation of I_K by Lemma 3.8. Now assume 1. holds. Then $G_k \cong G_K/I_K$ acts continuously (and diagonally) on $\tilde{\mathbb{D}}_{st}(V) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{I_K}$. By Lemma 1.2 there exists an $\mathcal{O}_{P_0}$-lattice M which is invariant under the action of G_k. Now F acts in the same way on M as the topological generator $\overline{\sigma}_K$ of G_K/I_K and therefore the image of M under F is contained in M and F is of slope 0, i.e. $F(M) = M$. □

An easy reformulation can be given in terms of slopes in the spirit of section B.3. We stress out that there are two Newton slopes on an admissible K-filtered (φ, N, F)-module D over P_0. On the one hand we have the usual $t_N(D, \varphi) := \nu(\det(\varphi))$, which is the Newton number with respect to the map φ. But we also have a Newton slope with respect to F which is characterized by the Newton number $t_N(D, F) := \nu(\det(F))$.

Corollary 4.9 *Let V be a log-crystalline representation of W_K of dimension r and $\tilde{\mathbb{D}}_{pst}(V) = (D, F)$ the corresponding module. The following statements are equivalent:*

- *V is a log-crystalline representation of G_K.*
- *(D, F) is isoclinic of Newton slope 0 (with respect to F).*

We obtain the same result for potentially log-crystalline representations from the following "division with remainder" in $\hat{\mathbb{Z}}$.

Lemma 4.10 *Take two elements $z \in \hat{\mathbb{Z}}$ and $f \in \mathbb{Z}$. Then there exist two unique elements $\beta \in \{0, \dots, f-1\} \subseteq \mathbb{Z}$ and $\alpha \in \hat{\mathbb{Z}}$ such that*

$$z = f \cdot \alpha + \beta.$$

Proof: First of all we remark that $\hat{\mathbb{Z}}/f\hat{\mathbb{Z}} \cong \mathbb{Z}/f\mathbb{Z}$ holds. Let

$$\mathrm{pr}_f \colon \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/f\mathbb{Z}$$

denote the projection to $\mathbb{Z}/f\mathbb{Z}$. Hence we find an unique element $\beta \in \{0, \ldots, f-1\} \subseteq \mathbb{Z}$ such that $\mathrm{pr}_f(z) = \beta + f\mathbb{Z}$ holds. We see that $z - \beta \equiv 0 \mod f\hat{\mathbb{Z}}$ and hence find an element $\alpha \in \hat{\mathbb{Z}}$ such that $z - \beta = f \cdot \alpha$. If we have two elements $\alpha_1, \alpha_2 \in \hat{\mathbb{Z}}$ such that $z - \beta = f \cdot \alpha_i$, we receive $f \cdot (\alpha_1 - \alpha_2) = 0$. But $\hat{\mathbb{Z}}$ is $\mathbb{Z}$-torsion-free and hence $\alpha_1 = \alpha_2$. □

Corollary 4.11 *Let V be a potentially log-crystalline representation of W_K of dimension r and $\tilde{\mathbb{D}}_{pst}(V) = (D, F)$ the corresponding module. The following statements are equivalent:*

- *V is a potentially log-crystalline representation of G_K.*
- *(D, F) is isoclinic of Newton slope 0 (with respect to F).*

Proof: We only need to show that the second point implies the first. Assume that L/K is a finite Galois extension such that $V|_L$ is a log-crystalline representation of I_L, i.e. $D^{I_L} = D$ holds. Hence, by the previous Corollary 4.9, V is a log-crystalline representation of G_L. It is left to show that G_K acts on V. By Theorem 3.20 we know that V is already a representation of I_K. Let $f := f(L/K)$ be the inertia index, take an element $g \in G_K$ and set $z := \deg_K(g) \in \hat{\mathbb{Z}}$. By Lemma 4.10 we may write $z = \alpha \cdot f + \beta$ for some $\alpha \in \hat{\mathbb{Z}}$ and $\beta \in \mathbb{Z}$. We define the map

$$G_K \times V \to V, (\sigma_K^z u, v) \mapsto (\sigma_K^f)^\alpha.(\sigma_K^\beta.(u.v)) = (\sigma_L)^\alpha.(\sigma_K^\beta.(u.v))$$

for $z \in \hat{\mathbb{Z}}$ and $u \in I_K$. This map is continuous since the G_L-action on V is continuous and $G_L \subseteq G_K$ is an open subgroup. It is left to show that this defines an action of G_K on V. Take $g_1 = \sigma_K^{z_1} u_1, g_2 = \sigma_K^{z_2} u_2 \in G_K$ and set $\tilde{u}_1 := \sigma_K^{-z_2} u_1 \sigma_K^{z_2} \in I_K$. Write $z_1 = \alpha_1 \cdot f + \beta_1$ and $z_2 = \alpha_2 \cdot f + \beta_2$ such that $\alpha_1, \alpha_2 \in \hat{\mathbb{Z}}$ and $\beta_1, \beta_2 \in \{0, \ldots, f-1\}$ as in Lemma 4.10. Take sequences $(\alpha_{1,n})_n$ resp. $(\alpha_{2,n})_n$ in $\mathbb{Z}$ converging to α_1 resp. α_2 to obtain:

$$\begin{aligned}
(g_1 g_2).v &= \sigma_L^{\alpha_1+\alpha_2}.(\sigma_K^{\beta_1+\beta_2}.((\tilde{u}_1 u_2).v)) \\
&= \lim_{n\to\infty} \sigma_L^{\alpha_{1,n}+\alpha_{2,n}}.(\sigma_K^{\beta_1+\beta_2}.((\tilde{u}_1 u_2).v)) \\
&= \lim_{n\to\infty} (\sigma_L^{\alpha_{1,n}} \sigma_K^{\beta_1} \sigma_K^{f\alpha_{2,n}+\beta_2} \tilde{u}_1 \sigma_K^{-(f\alpha_{2,n}+\beta_2)}).((\sigma_L^{\alpha_{2,n}} \sigma_K^{\beta_2} u_2).v)) \\
&= (\sigma_L^{\alpha_1} \sigma_K^{\beta_1} (\underbrace{\lim_{n\to\infty} \sigma_K^{f\alpha_{2,n}+\beta_2} \tilde{u}_1 \sigma_K^{-(f\alpha_{2,n}+\beta_2)}}_{=u_1})).((\sigma_L^{\alpha_2} \sigma_K^{\beta_2} u_2).v)) = g_1.(g_2.v)
\end{aligned}$$

for all $v \in V$. □

The operator F allows us to decompose the objects in the category of linear algebra data corresponding to the category of representations of the Weil group. This will be done in the next chapter.

4.3 Decomposition of Weil Group Representations

We need the following preparations that can be found in [Bou81, chapter V, §10.4]. Let E be a field, $\Gamma \subseteq \mathrm{Aut}(E)$ a subgroup and $E_0 := E^\Gamma$ the Γ-invariants of E. As usual [Bou74, chapter II, §8] an E_0-structure on V is an E_0 subspace $V_0 \subseteq V$ such that the map

$$m\colon E \otimes_{E_0} V_0 \to V, \text{ given by } \lambda \otimes x \mapsto \lambda \cdot x$$

is bijective. Let $V_0 \subseteq V$ be such an E_0-structure. For any $\gamma \in \Gamma$ define V^γ to be the vector space over E with the underlying additive group $(E, +)$ and scalar multiplication given by

$$E \times V^\gamma \to V^\gamma, (\lambda, v) \mapsto \gamma(\lambda) \cdot v.$$

Set $X := \bigoplus_{\gamma \in \Gamma} V^\gamma$ and remark that the scalar multiplication on this E-vector space is given by

$$E \times X \to X, (\lambda, x) \mapsto (\gamma(\lambda) \cdot x_\gamma)_{\gamma \in \Gamma}.$$

We finally define the map

$$\psi\colon E \otimes_{E_0} V \to X, \text{ given by } \lambda \otimes x \mapsto (\gamma(\lambda) \cdot x)_{\gamma \in \Gamma}.$$

Then [Bou81, chapter V, §10.4, Proposition 8] tells us:

Proposition 4.12 *ψ is injective and it is bijective if Γ is finite.*

Now we will make use of the following refinement of FONTAINE's equivalences of categories which can be found in [Fon00, §4]. Let $\mathbb{Q}_{p^r} := W(\mathbb{F}_{p^r})[\frac{1}{p}]$ be the unique unramified extension of $\mathbb{Q}_p$ of degree r. Then we have

$$\mathrm{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p) = \langle \sigma \rangle \cong \mathbb{Z}/r\mathbb{Z},$$

where σ denotes a generator and restricts to the p-th power Frobenius map on $\mathbb{F}_{p^r}$. Take two vector spaces V and W over $\mathbb{Q}_{p^r}$ and decompose its tensor product over $\mathbb{Q}_p$ via Proposition 4.12:

$$\begin{aligned} V \otimes_{\mathbb{Q}_p} W &\cong V \otimes_{\mathbb{Q}_{p^r}} (\mathbb{Q}_{p^r} \otimes_{\mathbb{Q}_p} W) \\ &\cong V \otimes_{\mathbb{Q}_{p^r}} (\bigoplus_{0\le m<r} W^{\sigma^m}) \\ &\cong \bigoplus_{0\le m<r} V \otimes_{\mathbb{Q}_{p^r},m} W, \end{aligned}$$

where

$$V \otimes_{\mathbb{Q}_{p^r},m} W := \{x \in V \otimes_{\mathbb{Q}_p} W \mid (1 \otimes \lambda)x = (\sigma^m(\lambda) \otimes 1)x \text{ for all } \lambda \in \mathbb{Q}_{p^r}\}.$$

Assume that $\mathbb{Q}_{p^r} \subseteq K$ and let W be a log-crystalline $\mathbb{Q}_{p^r}$-representation of I_K, i.e. W is log-crystalline as a $\mathbb{Q}_p$-representation of I_K. Then

$$\tilde{\mathbb{D}}_{st}(W) = \bigoplus_{0\le m<r} (B_{\mathrm{st}} \otimes_{\mathbb{Q}_{p^r},m} W)^{I_K}$$

holds and we set $\tilde{\mathbb{D}}_{st,m}(W) := (B_{\mathrm{st}} \otimes_{\mathbb{Q}_{p^r},m} W)^{I_K}$. By [Fon00, §4] the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Rep}^{st}_{\mathbb{Q}_{p^r}}(I_K) & \xrightarrow{\tilde{\mathbb{D}}_{st,0}} & (\text{adm. } K\text{-filt. } (\varphi^r, N)\text{-mod.}/P_0) \\ \downarrow \mathcal{F} & & \downarrow \cdot\otimes_{\mathbb{Q}_p[\varphi^r]}\mathbb{Q}_p[\varphi] \\ \mathrm{Rep}^{st}_{\mathbb{Q}_p}(I_K) & \xrightarrow{\tilde{\mathbb{D}}_{st}} & (\text{adm. } K\text{-filt. } (\varphi, N)\text{-mod.}/P_0) \end{array} \qquad \text{(CD3)}$$

and the horizontal arrows are equivalences of abelian tensor categories. Furthermore let W be a log-crystalline $\mathbb{Q}_{p^r}$-representation of W_K, i.e. W is log-crystalline as a $\mathbb{Q}_p$-representation of I_K. We claim that $\tilde{\mathbb{D}}_{st,m}(W)$ is stable under the map F_V we defined on $\tilde{\mathbb{D}}_{st}(V)$ to establish the equivalence of categories in Theorem 3.12. Take $x = \sum b_i \otimes v_i \in \tilde{\mathbb{D}}_{st,m}(V)$ and apply F_V to obtain:

$$\begin{aligned} \sum_i \sigma_K(b_i) \otimes \lambda\sigma_K.v_i &= \sum_i \sigma_K(b_i) \otimes \sigma_K.\lambda v_i \\ &= F(\sum_i b_i \otimes \lambda v_i) \\ &= F(\sum_i \sigma^m(\lambda) b_i \otimes v_i) \\ &= \sum_i \sigma^m(\lambda)\sigma_K(b_i) \otimes \sigma_K.v_i. \end{aligned}$$

for all $\lambda \in \mathbb{Q}_{p^r}$. Hence $F_V(x) \in \tilde{\mathbb{D}}_{st,m}(V)$ and we have verified the claim. We do the usual business in order to apply the results of section 1.4.

Definition 4.13 *We define the category of K-filtered (φ^r, N, F)-modules over P_0 as follows:*

- *An object is a pair (D, F), where D is an K-filtered (φ^r, N)-module over P_0 and*
$$F\colon D \to D$$
is a bijective σ^f-semilinear map such that
  - $\varphi^r \circ F = F \circ \varphi^r$,
  - $N \circ F = F \circ N$ *and*
  - $F_K(\mathrm{Fil}^i(D_K)) = \mathrm{Fil}^i(D_K)$ *holds for all $i \in \mathbb{Z}$.*
- *A morphism $f\colon (D_1, F_1) \to (D_2, F_2)$ is a morphism $f\colon D_1 \to D_2$ in the category of K-filtered (φ^r, N)-modules over P_0 such that $F_2 \circ f = f \circ F_1$ holds.*
- *The composition of morphisms is the usual composition of maps.*

We call an object (D, F) of this category admissible if D is admissible in the category of K-filtered (φ^r, N)-modules over P_0 and denote the full subcategory consisting of admissible objects by

$$(\text{adm. } K\text{-filt. } (\varphi^r, N, F)\text{-mod.}/P_0).$$

In the notation of section 1.4 we are in the following situation:

$$E = \mathbb{Q}_{p^r},\ \varsigma = \sigma_K \text{ and } B = B_{\mathrm{st}}.$$

As consequence of Theorem 1.17 we receive:

Theorem 4.14 *There is an equivalence of categories given as follows:*

$$\tilde{\mathbb{D}}_{st,0}\colon \mathrm{Rep}^{st}_{\mathbb{Q}_{p^r}}(W_K) \sim (\text{adm. } K\text{-filt. } (\varphi^r, N, F)\text{-mod.}/P_0) : \tilde{\mathbb{V}}_{st,0}$$

$$V \mapsto (\tilde{\mathbb{D}}_{st,0}(V), F_V|_{\tilde{\mathbb{D}}_{st,0}(V)})$$

$$\mathrm{Hom}_{(KP_0\text{-filt. } (\varphi^r, N)\text{-modules over } P_0)}(P_0, B_{\mathrm{st}} \otimes_{P_0} D) \hookleftarrow D.$$

Proof: The verification of the axioms in section 1.4 is the same as in section 3.2, taking into account that we already checked that F_V is well-defined. □

Now we make use of Theorem B.18, i.e. the classification theorem of Dieudonne and Manin that allows us to decompose the modules (D, F) as follows.

Lemma 4.15 *Let (D, F) be an admissible K-filtered (φ^f, N, F)-module over P_0. We denote the decomposition of D into isoclinic components along F (via Theorem B.18) by*

$$D = \bigoplus_{q \in \mathbb{Q}} D_q.$$

Then $\varphi^f(D_q) = D_q$ and $N(D_q) \subseteq D_q$ for all $q \in \mathbb{Q}$.

Proof: $F^{-1} \circ \varphi^f$ is a P_0-linear automorphism of D, that commutes with φ, and such maps from D_{q_1} to D_{q_2} are the zero if $q_1 \neq q_2$ by Lemma B.14. Thus $F^{-1} \circ \varphi^f(D_q) = D_q$ for all $q \in \mathbb{Q}$, i.e. $\varphi^f(D_q) = F(D_q) = D_q$. For the same reason the P_0-linear operator N has to map D_q into itself. □

Lemma 4.16 *We make the same assumptions as in the previous lemma and assume that K/K_0 is a finite Galois extension. Let*

$$D \cong \bigoplus_{q \in \mathbb{Q}} S_q^{n_q}$$

denote the decomposition of D along F into standard isocrystals (via Theorem B.22). Then

$$\mathrm{Fil}^i(D_K) = K \otimes_{K_0} \left(\bigoplus_{q \in J_i} S_q^{n_{q,i}} \right)$$

for a finite subset $J_i \subseteq \mathbb{Q}$, $1 \leq n_{q,i} \leq n_q$ for all $q \in J_i$ and $i \in \mathbb{Z}$.

Proof: By the definiton of (D, F) we have $F_K(\mathrm{Fil}^i(D_K)) = \mathrm{Fil}^i(D_K)$ and set $W := \mathrm{Fil}^i(D_K)^{\mathrm{Gal}(K/K_0)}$. Then the diagram

$$\begin{array}{ccc} K \otimes_{K_0} W & \xrightarrow{\cong} & \mathrm{Fil}^i(D_K) \\ \downarrow \subseteq & & \downarrow \subseteq \\ K \otimes_{K_0} D & \xrightarrow{=} & D_K \end{array}$$

is commutative by Hilbert 90. In particular $D = (D_K)^{\mathrm{Gal}(K/K_0)}$ where $D \hookrightarrow D_K$ via $d \mapsto 1 \otimes d$. Thus

$$\begin{aligned} F(W) &= D \cap F_K(K \otimes_{K_0} W) \\ &= D \cap F_K(\mathrm{Fil}^i(D_K)) \\ &= D \cap \mathrm{Fil}^i(D_K) \\ &= D_K^{\mathrm{Gal}(K/K_0)} \cap \mathrm{Fil}^i(D_K) = W. \end{aligned}$$

Therefore $W = \bigoplus_{q \in J_i} S_q^{n_{q,i}}$ and $\mathrm{Fil}^i(D_K) = K \otimes_{K_0} (\bigoplus_{q \in J_i} S_q^{n_{q,i}})$. □

Remark 4.17 *Let $\mathbb{K}$ be a field, (D, Fil) a $\mathbb{K}$-filtered vector space and*

$$D_1, \ldots, D_n \subseteq D$$

sub-objects. Then $D \cong \bigoplus_{i=1}^n D_i$ in the cateogry of $\mathbb{K}$-filtered vector spaces if and only if $\mathrm{Fil}^j(D) = \bigoplus_{i=1}^n \mathrm{Fil}^j(D_i)$ for all $j \in \mathbb{Z}$. Warning: This condition may easily fail, even if the D_i are endowed with the subspace filtration of D. For instance take $D = \mathbb{K}^2$,

$$\mathrm{Fil}^0(D) = \mathbb{K}^2 \supset \mathrm{Fil}^1(D) = \mathbb{K}(e_1 + e_2) \supset \mathrm{Fil}^2 = 0$$

and $D_1 = \mathbb{K} \cdot e_1$ and $D_2 = \mathbb{K} e_2$.

With Lemma 4.16 we excluded this situation. Now we need [CF00, Theorem 4.3.] to proceed.

Theorem 4.18 *Let D be a K-filtered (φ, N)-module over P_0 of dimension $h \geq 1$. Then $\tilde{\mathbb{V}}_{st}(D)$ has finite dimension over $\mathbb{Q}_p$ if and only if $t_H(D') \leq t_N(D')$ for all sub-objects $D' \subseteq D$ (in the category of K-filtered (φ, N)-modules over P_0). In this case we have $\dim_{\mathbb{Q}_p}(\tilde{\mathbb{V}}_{st}(D)) \leq h$.*

Theorem 4.19 *Let (D, F) be an admissible K-filtered (φ^f, N)-module of dimension $h \geq 1$ over P_0 and let $D = \bigoplus_{q \in \mathbb{Q}} D_q$ denote its decomposition relative to F into isoclinic components (via Theorem B.18). Then the summands D_q are admissible.*

Proof: By Lemma 4.16 this decomposition is a decomposition of K-filtered vector spaces and by Lemma 4.15 it is a decomposition of (φ^f, N)-modules. D is admissible if and only if $\mathbb{Q}_p[\varphi] \otimes_{\mathbb{Q}_p[\varphi^f]} D$ is admissible as K-filtered (φ, N)-module over P_0 by definition. Apply Theorem 4.18 to

$$\tilde{\mathbb{V}}_{st}(\mathbb{Q}_p[\varphi] \otimes_{\mathbb{Q}_p[\varphi^f]} D) \cong \bigoplus_{q \in \mathbb{Q}} \tilde{\mathbb{V}}_{st}(\mathbb{Q}_p[\varphi] \otimes_{\mathbb{Q}_p[\varphi^f]} D_q)$$

and obtain

$$\begin{aligned} h &= \dim_{\mathbb{Q}_p} \tilde{\mathbb{V}}_{st}(\mathbb{Q}_p[\varphi] \otimes_{\mathbb{Q}_p[\varphi^f]} D) \\ &= \sum_{q \in \mathbb{Q}} \underbrace{\dim_{\mathbb{Q}_p} \tilde{\mathbb{V}}_{st}(\mathbb{Q}_p[\varphi] \otimes_{\mathbb{Q}_p[\varphi^f]} D_q)}_{\leq \dim_{P_0}(\mathbb{Q}_p[\varphi] \otimes_{\mathbb{Q}_p[\varphi^f]} D_q)} \\ &\leq \dim_{P_0}(\mathbb{Q}_p[\varphi] \otimes_{\mathbb{Q}_p[\varphi^f]} D) = h. \end{aligned}$$

Hence $\dim_{\mathbb{Q}_p} \tilde{\mathbb{V}}_{st}(\mathbb{Q}_p[\varphi] \otimes_{\mathbb{Q}_p[\varphi^f]} D_q) = \dim_{P_0}(\mathbb{Q}_p[\varphi] \otimes_{\mathbb{Q}_p[\varphi^f]} D_q)$ for all $q \in \mathbb{Q}$, i.e. all the $\mathbb{Q}_p[\varphi] \otimes_{\mathbb{Q}_p[\varphi^f]} D_q$ are admissible. Therefore all the D_q are admissible in the category of K-filtered (φ^f, N, F)-modules over P_0. □

From the proof we extract the following.

Corollary 4.20 *Let V be a log-crystalline (p-adic) representation of W_K that is coming from a $\mathbb{Q}_{p^f}$-representation of W_K by scalar restriction. Then $\tilde{\mathbb{D}}_{st}(V)$ admits a decomposition into isoclinic components along F in the category of admissible K-filtered (φ, N, F)-modules over P_0.*

Remark that a combination of (CD2) and (CD3) gives us the following commutative diagram of functors for any $r \leq f$:

$$\begin{array}{ccc} \mathrm{Rep}^{pst}_{\mathbb{Q}_{p^r}}(G_K) & \xrightarrow{\mathbb{D}_{pst,0}} & (\text{adm. } \overline{K}/K\text{-filt. } (\varphi^r, N, G_K)\text{-mod. over } \mathbb{Q}_p^{nr}) \\ \downarrow \mathcal{F} & & \downarrow \cdot \otimes_{\mathbb{Q}_p^{nr}} P_0 \\ \mathrm{Rep}^{pst}_{\mathbb{Q}_{p^r}}(W_K) & \xrightarrow{\tilde{\mathbb{D}}_{pst,0}} & (\text{adm. } \overline{K}/K\text{-filt. } (\varphi^r, N, I_K, F)\text{-mod. over } P_0) \end{array} \qquad \text{(CD4)}$$

Corollary 4.21 *Let V be a potentially log-crystalline representation of W_K that is coming from a $\mathbb{Q}_{p^f}$-representation of W_K by scalar restriction. Then $\tilde{\mathbb{D}}_{pst}(V)$ admits a decomposition along F in the category of admissible $\overline{K}/K$-filtered (φ, N, I_K, F)-modules over P_0.*

Proof: This can be proven the same way Theorem 4.19 was proven with the additional information that any $u \in I_K$ can be understood as a linear operator $u \colon D \to D$ and therefore must respect the decomposition along F into isoclinic components. □

4.4 Generators of Abelian Tensor Categories

Initially we need to give the word "generating" a meaning in our context. Since we are dealing with abelian tensor categories (even Tannakian categories) most of the time, it seems reasonable to adopt the usual definition of a *tensor generating family* from [DM82, §1, Tensor subcategories]. Nevertheless we refine the definition to state the results more precisely.

Definition 4.22 *Let $\mathcal{C}$ be an abelian tensor category, $\mathcal{U}$ be a strictly full subcategory and $(X_i)_{i\in I}$ a collection of objects in $\mathcal{C}$. We say that:*

- *$\mathcal{U}$ is a tensor subcategory if it is closed under the formation of finite tensor products.*
- *$(X_i)_{i\in I}$ is a tensor generating family of $\mathcal{C}$ if every object of $\mathcal{C}$ is isomorphic to a subquotient of $P(X_i)$ for some $P \in \mathbb{N}[(t_i)_{i\in I}]$. (Interpret multiplication as $\otimes$ and addtion as $\oplus$.)*
- *$(X_i)_{i\in I}$ is a tensor integrally generating family of $\mathcal{C}$ if every object of $\mathcal{C}$ is isomorphic to $P(X_i)$ for some $P \in \mathbb{N}[(t_i)_{i\in I}]$.*
- *$(X_i)_{i\in I}$ is a tensor rationally generating family of $\mathcal{C}$ if every object X of $\mathcal{C}$ satisfies $l \cdot X = X^{\oplus l} \cong P(X_i)$ for some $l \in \mathbb{N}$ and $P \in \mathbb{N}[(t_i)_{i\in I}]$.*

Remark 4.23 *Clearly any tensor integrally generating family is a tensor rationally generating family. Take a tensor rationally generating family $(X_i)_i$ in some abelian tensor category $\mathcal{C}$ and an arbitrary object X. Consider a projection $\mathrm{pr}\colon X^{\oplus l} \twoheadrightarrow X$ such that $X^{\oplus l} \cong P(X_1, \ldots, X_n)$ for some objects $X_1, \ldots, X_n$ in the tensor integrally generating family. We obtain an isomorphism $X \cong P(X_1, \ldots, X_n)/\ker(\mathrm{pr})$ and therefore $(X_i)_i$ is a tensor generating family in $\mathcal{C}$.*

4.5 Generators of the category of Weil group representations

In this section we will construct a family of Weil group representations that can't be lifted to Galois group representations. Later on we will see that this family and the family of Galois group representations are a tensor rationally generating family of the category $\mathrm{Rep}^{st}(W_K)$.

Let $r \in \mathbb{N}$ and K_r/K be the unramified extension of degree r and set

$$\sigma_{K_r} := \sigma_K^r.$$

Remark 4.24 *Consider the one-dimensional representation* $\mathbb{Q}_p(|\cdot|_{K_r})$ *of* W_{K_r} *given in Example 1.3. We recall that* I_{K_r} *acts trivially on* $\mathbb{Q}_p(|\cdot|_{K_r})$*. Then the induction* $\mathrm{Ind}_{W_{K_r}}^{W_K}(\mathbb{Q}_p(|\cdot|_{K_r})$ *is crystalline since*

$$\begin{aligned}(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} (\mathrm{Ind}_{W_{K_r}}^{W_K}(\mathbb{Q}_p(|\cdot|_{K_r})))^{I_K} &= (B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} (\bigoplus_{i=0}^{r-1} \sigma_K^i * \mathbb{Q}_p(|\cdot|_{K_r}))^{I_K} \\ &= P_0 \otimes_{\mathbb{Q}_p} (\bigoplus_{i=0}^{r-1} \sigma_K^i * \mathbb{Q}_p(|\cdot|_{K_r})) \\ &\cong S_{\frac{1}{r}},\end{aligned}$$

where $S_{\frac{1}{r}}$ *denotes the standard isocrystal over* P_0 *with respect to* F *(compare Definition B.20). This justifies to define*

$$V_{\frac{1}{r}} := \mathrm{Ind}_{W_{K_r}}^{W_K}(\mathbb{Q}_p(|\cdot|_{K_r})) \text{ and } V_{-\frac{1}{r}} := \mathrm{Ind}_{W_{K_r}}^{W_K}(\mathbb{Q}_p(|\cdot|_{K_r})^{-1})$$

for $r \in \mathbb{N}$*, where* $\mathbb{Q}_p(|\cdot|_{K_r})^{-1}$ *denotes the character given by the composition*

$$W_{K_r} \twoheadrightarrow W_{K_r}^{ab} \xrightarrow{r_{K_r}} {K_r}^\times \xrightarrow{|\cdot|_{K_r}} p^{\mathbb{Z}} \subseteq \mathbb{Q}_p^\times \xrightarrow{x \mapsto x^{-1}} \mathbb{Q}_p^\times .$$

Warning: The induction of a (log-)crystalline representation will not be (log-)crystalline in general.

Now we can state the main theorem as follows.

Theorem 4.25 $\mathrm{Rep}^{st}(W_K)$ *(resp.* $\mathrm{Rep}^{\mathrm{crys}}(W_K)$*) is rationally generated as an abelian tensor category by* $\mathrm{Rep}^{st}(G_K)$ *(resp.* $\mathrm{Rep}^{\mathrm{crys}}(G_K)$*) and the family* $\{V_{\frac{1}{r}}\}_{r\in\mathbb{Z}\setminus\{0\}}$.

Proof: Let V be a (p-adic) log-crystalline representation of W_K. Without loss of generality we assume that V is coming from a log-crystalline $\mathbb{Q}_{p^f}$-representation of W_K by considering $\mathbb{Q}_{p^f} \otimes_{\mathbb{Q}_p} V \cong V^{\oplus f}$ instead of V. Use Corollary 4.20 to decompose

$$D := \tilde{\mathbb{D}}_{st}(V) = \bigoplus_{i=1}^{n} D_{\frac{r_i}{s_i}}$$

such that $D_{\frac{r_i}{s_i}}$ is isoclinic of slope $\frac{s_i}{r_i}$ (reduced fraction such that $r_i \geq 1$) with respect of F. Then

$$\tilde{D}_i := D_{\frac{s_i}{r_i}} \otimes S_{\frac{r_i - s_i}{r_i}} \otimes S_{-1}$$

is isoclinic of slope 0 (with respect to F) by Lemma B.21. By Theorem 4.7 $\tilde{D}_i$ corresponds to an object of $\mathrm{Rep}^{st}(G_K)$. Multiply the equation with $S_{\frac{s_i-r_i}{r_i}}$ and rearrange to obtain

$$S_{\frac{s_i-r_i}{r_i}} \otimes S_1 \otimes \tilde{D}_i \cong D_{\frac{s_i}{r_i}} \otimes \underbrace{S_{\frac{0}{r_i^2}}}_{\cong P_0^{r_i^2}} \cong D_{\frac{s_i}{r_i}}^{\oplus r_i^2}.$$

If $r_i = s_i$ holds we are done. In the case that $s_i > r_i$ holds, take the $r_i^{s_i-r_i-1}$-fold sum on both sides. By Corollary B.23 this leads to

$$S_{\frac{1}{r_i}}^{\otimes(s_i-r_i)} \otimes S_1 \otimes \tilde{D}_i \cong D_{\frac{s_i}{r_i}}^{\oplus r_i^{s_i-r_i+1}}.$$

If $s_i < r_i$ holds, take the $r_i^{r_i-s_i-1}$-fold sum on both sides and receive

$$S_{-\frac{1}{r_i}}^{\otimes(r_i-s_i)} \otimes S_1 \otimes \tilde{D}_i \cong D_{\frac{s_i}{r_i}}^{\oplus r_i^{r_i-s_i+1}}.$$

Set $l := \prod_{i=1}^n r_i^{\pm(s_i-r_i)+1}$ (where $\pm$ is the appropriate sign from above depending on the index i) and see that

$$D^{\oplus l} \cong \bigoplus_{i=1}^n (S_{\pm\frac{1}{r_i}}^{\otimes\pm(s_i-r_i)} \otimes S_1 \otimes \tilde{D}_i)^{\oplus l/r_i^{\pm(s_i-r_i)+1}}.$$

Translating everything back to the category of representations gives the result. The same proof works for the crystalline case since the representations $V_{\frac{1}{r}}$ are crystalline for all $r \in \mathbb{Z} \setminus \{0\}$ by Remark 4.24. □

We obtain the analogous result for potentially log-crystalline representations.

Corollary 4.26 $\mathrm{Rep}^{pst}(W_K)$ *is rationally generated as an abelian tensor category by* $\mathrm{Rep}^{pst}(G_K)$ *and the family* $\{V_{\frac{1}{r}}\}_{r\in\mathbb{Z}\setminus\{0\}}$.

Proof: Follow the proof above and use Corollary 4.11 as well as Corollary 4.21 in the appropriate places. □

Example 4.27 *The following categories are equivalent:*

$$\mathrm{Rep}^{st}(W_K/I_K) \sim \mathrm{Rep}(W_K/I_K) \sim \mathrm{Rep}(\mathbb{Z})$$

(all the representations are assumed to be vector spaces over $\mathbb{Q}_p$*). Consider*

$$\rho\colon \mathbb{Z} \to \mathrm{GL}(V) \cong \mathrm{GL}_2(\mathbb{Q}_p)$$

given by $1 = \overline{\sigma}_K \mapsto \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$ *as an object of those categories. Then* V *is neither decomposable nor irreducible and lifts to a representation of* $\hat{\mathbb{Z}}$ *since* $\rho(1) \in \mathrm{GL}_2(\mathbb{Z}_p)$. *Then* $D := \tilde{\mathbb{D}}_{st}(V) = P_0 e_1 + P_0 e_2$ *with* $F(e_1) = e_1 + pe_2$ *and* $F(e_2) = e_2$. *By Theorem B.22* D *decomposes as an* F*-module into standard isocrystals. Therefore we find* $0 \neq a, b, \lambda \in P_0$ *such that*

$$F(ae_1 + be_2) = \lambda(ae_1 + be_2).$$

Hence $\lambda = \frac{\sigma_K(a)}{a}$ *and* $\sigma_K(b) = \lambda b - \sigma_K(a) \cdot p$. *One might now think that* D *is decomposable which is false:*

$$\varphi(ae_1 + be_2) = \sigma_K(a)e_1 + \sigma_K(b)e_2 = \sigma_K(a)e_1 + (\lambda b - \sigma_K(a)p)e_2$$

implies (by assuming decomposabilty) that there exists $\mu \in P_0$ *such that*

$$\sigma_K(a)e_1 + \sigma_K(b)e_2 = \sigma_K(a)e_1 + (\lambda b - \sigma_K(a) \cdot p)e_2 = \mu ae_1 + \mu be_2$$

and therefore $\mu = \frac{\sigma(a)}{a} = \lambda$ *and hence* $\sigma_K(a) \cdot p = \lambda b - \mu b = 0$ *which contradicts* $a \neq 0$. *In particular* φ *does not respect the decomposition along* F *into standard isocrystals. Nevertheless it always respects the decomposition into isoclinic components.*

Chapter 5

(φ, Γ, F)-Modules

Inspired by the results in the case of (potentially log-)crystalline representations we try to obtain similar results for general (p-adic) representations. As we will see later this approach unfortunately is only successful to a limited (and minor) extend.

Notations: We adopt the notations from [Sch17] and [BC09] as follows:

- $\mathbb{E}_K^+$ denotes the image of $k[[X]]$ in $\mathcal{O}_{K_\infty^\flat}$ via $X \mapsto \varpi$ (for the definition of ϖ, see [Sch17, Lemma 1.4.14 and below]), which is a complete discrete valuation ring with residue field k and fraction field $\mathbb{E}_K$ isomorphic to $k((X))$.

- $\mathbb{E}_K^{sep}$ denotes the separable closure of $\mathbb{E}_K$ in $\mathbb{C}_p^\flat$.

- $\mathbb{A}_K$ denotes the image of the complete discrete valuation ring

  $$\{\sum_{i\in\mathbb{Z}} a_i X^i \mid a_i \in W(k) \text{ and } \lim_{i\to-\infty} a_i = 0\}$$

  with residue field $k((X))$ in $W(\mathbb{E}_K)$ via a lift of the isomorphism $k((X)) \cong \mathbb{E}_K$ (see [Sch17, Section 2.1]). Therefore $\mathbb{A}_K$ is a complete discrete valuation ring with residue field $\mathbb{E}_K$ and we denote its fraction field by $\mathbb{B}_K$, which is isomorphic to

  $$\{\sum_{i\in\mathbb{Z}} a_i X^i \mid \{a_i\}_{i\in\mathbb{Z}} \subseteq W(k)[\frac{1}{p}] \text{ bounded and } \lim_{i\to-\infty} a_i = 0\}.$$

  $\mathbb{A}_K$ is a Cohen ring of $\mathbb{E}_K$ in the sense of [GD64, Théorème 19.8.6.].

- $\mathbb{A}_K^{nr}$ denotes the union of all unramified ring extensions of $\mathbb{A}_K$ with respect to the residue field $\mathbb{E}_K$. $\mathbb{A}_K^{nr}$ embeds into $W(\mathbb{E}_K^{sep})$ and the G_K-action on $W(\mathbb{E}_K^{sep})$ preserves $\mathbb{A}_K^{nr}$ (see discussion before [Sch17, Remark 3.1.4.]).

- $\mathbb{B}_K^{nr}$ the fraction field of $\mathbb{A}_K^{nr}$. $\mathbb{B}_K^{nr}$ embeds into $W(\mathbb{E}_K^{sep})[\frac{1}{p}]$ and the G_K-action on $W(\mathbb{E}_K^{sep})[\frac{1}{p}]$ preserves $\mathbb{B}_K^{nr}$.

- $\mathbb{A}$ denotes the p-adic completion of $\mathbb{A}_K^{nr}$. $\mathbb{A}$ is a complete discrete valuation ring with prime element p and residue field $\mathbb{E}_K^{sep}$. $\mathbb{A}$ embeds into $W(\mathbb{E}_K^{sep})$ and the Frobenius map σ as well as the G_K-action on $W(\mathbb{E}_K^{sep})$ preserve $\mathbb{A}$. Furthermore $\mathbb{A}^{\mathrm{Gal}(\mathbb{E}_K^{sep}/\mathbb{E}_K)} = \mathbb{A}_K$ (see [Sch17, Lemma 3.1.6.]).

- $\mathbb{B}$ denotes the fraction field of $\mathbb{A}$. From the point above we see that $\mathbb{B}^{\sigma=\mathrm{id}} = \mathbb{Q}_p$ as well as $\mathbb{B}^{H_K} = \mathbb{B}_K$.

Note that we slightly differ from the notation in [Sch17] here. In SCHNEIDER's notation the ring $\mathbb{A}_K$ above is the ring $\mathbb{A}_{W(k)[\frac{1}{p}]}$.

The following two sections will seem redundant since we have already seen that mod-p-representations of G_K and W_K coincide [Corollary 1.7]. Anyway we will use the basic calculations later on. In addition this section should serve as a reality check to see that our rather abstract arguments work out correctly in a concrete situation.

5.1 (φ, F)-Modules and Mod-p-Representations

We use the following abbreviation:

$$\tilde{\mathbb{E}}_K := \mathbb{E}_{K^{nr}} := (\mathbb{E}_K^{sep})^{I_{\mathbb{E}_K}} \cong \overline{\mathbb{F}}_p((X)).$$

Definition 5.1 *A φ-module D (semilinear w.r.t. σ) over $\mathbb{E}_K$ (resp. $\tilde{\mathbb{E}}_K$) is called etale if the linearization*

$$\sigma^*(D) \to D \text{ given by } \lambda \otimes d \mapsto \lambda\varphi(d)$$

is an isomorphism.

Recall the equivalence of abelian tensor categories initially established by FONTAINE [BC09, Theo. 3.1.8.]:

Theorem 5.2 *There are equivalences of abelian tensor categories given by*

$$\mathbb{D}_{\mathrm{mod}}\colon \mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{E}_K}) \rightleftharpoons (\text{etale } \varphi\text{-modules over } \mathbb{E}_K) : \mathbb{V}_{\mathrm{mod}}$$

$$V \mapsto (\mathbb{E}_K^{sep} \otimes_{\mathbb{F}_p} V)^{G_{\mathbb{E}_K}}$$
$$(\mathbb{E}_K^{sep} \otimes_{\mathbb{E}_K} D)^{\varphi=\mathrm{id}} \leftarrow D$$

and

$$\tilde{\mathbb{D}}_{\mathrm{mod}}\colon \mathrm{Rep}_{\mathbb{F}_p}(I_{\mathbb{E}_K}) \rightleftharpoons (\text{etale } \varphi\text{-modules over } \tilde{\mathbb{E}}_K) : \tilde{\mathbb{V}}_{\mathrm{mod}}$$

$$V \mapsto (\mathbb{E}_K^{sep} \otimes_{\mathbb{F}_p} V)^{I_{\mathbb{E}_K}}$$
$$(\mathbb{E}_K^{sep} \otimes_{\tilde{\mathbb{E}}_K} D)^{\varphi=\mathrm{id}} \leftarrow D.$$

In particular we have the following comparison isomorphisms

$$\alpha_V\colon \mathbb{E}_K^{sep} \otimes_{\mathbb{E}_K} \mathbb{D}_{\mathrm{mod}}(V) \cong \mathbb{E}_K^{sep} \otimes_{\mathbb{F}_p} V, \tag{5.1}$$
$$\sum_{i,j} b_i \otimes b_j \otimes d_j \mapsto \sum_{i,j} (b_i \cdot b_j) \otimes d_j$$
$$\beta_D\colon \mathbb{E}_K^{sep} \otimes_{\mathbb{F}_p} \mathbb{V}_{\mathrm{mod}}(D) \cong \mathbb{E}_K^{sep} \otimes_{\mathbb{E}_K} D, \tag{5.2}$$
$$\sum_{i,j} b_i \otimes b_j \otimes v_j \mapsto \sum_{i,j} (b_i \cdot b_j) \otimes v_j$$

in the first case and

$$\tilde{\alpha}_V\colon \mathbb{E}_K^{sep} \otimes_{\tilde{\mathbb{E}}_K} \tilde{\mathbb{D}}_{\mathrm{mod}}(V) \cong \mathbb{E}_K^{sep} \otimes_{\mathbb{F}_p} V, \tag{5.3}$$
$$\tilde{\beta}_D\colon \mathbb{E}_K^{sep} \otimes_{\mathbb{F}_p} \tilde{\mathbb{V}}_{\mathrm{mod}}(D) \cong \mathbb{E}_K^{sep} \otimes_{\tilde{\mathbb{E}}_K} D \tag{5.4}$$

in the second case with maps in the same flavor as in the first case. These restrict to natural isomorphisms $\mathbb{V}_{\mathrm{mod}} \circ \mathbb{D}_{\mathrm{mod}} \cong \mathrm{id}$ *(resp.* $\tilde{\mathbb{V}}_{\mathrm{mod}} \circ \tilde{\mathbb{D}}_{\mathrm{mod}} \cong \mathrm{id}$*) and* $\mathbb{D}_{\mathrm{mod}} \circ \mathbb{V}_{\mathrm{mod}} \cong \mathrm{id}$ *(resp.* $\tilde{\mathbb{D}}_{\mathrm{mod}} \circ \tilde{\mathbb{V}}_{\mathrm{mod}} \cong \mathrm{id}$*).*

We obtain the following commutative diagram of functors:

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{E}_K}) & \xrightarrow{\mathbb{D}_{\mathrm{mod}}} & (\text{etale } \varphi\text{-modules over } \mathbb{E}_K) \\ \downarrow \mathcal{F} & & \downarrow \tilde{\mathbb{E}}_K \otimes_{\mathbb{E}_K} \cdot \\ \mathrm{Rep}_{\mathbb{F}_p}(I_{\mathbb{E}_K}) & \xrightarrow{\tilde{\mathbb{D}}_{\mathrm{mod}}} & (\text{etale } \varphi\text{-modules over } \tilde{\mathbb{E}}_K). \end{array} \tag{CD5}$$

(The φ-module structure on the scalar extension $\tilde{\mathbb{E}}_K \otimes_{\mathbb{E}_K} D$ is given by

$$\varphi(\lambda \otimes d) := \sigma(\lambda) \otimes \varphi(d)$$

for $\lambda \in \tilde{\mathbb{E}}_K$ and $d \in D$.)

Now take $V \in \mathrm{Rep}_{\mathbb{F}_p}(W_{\mathbb{E}_K})$ and denote $D := (\mathbb{E}_K^{sep} \otimes_{\mathbb{F}_p} V)^{I_{\mathbb{E}_K}}$. We fix an element $\sigma_K \in G_{\mathbb{E}_K}$ such that $\deg_{\mathbb{E}_K}(\sigma_K) = 1$ and obtain a σ_K-semilinear bijective map by

$$F_V \colon D \to D, \sum_i \lambda_i \otimes v_i \mapsto \sum_i \sigma_K(\lambda_i) \otimes \sigma_K.v_i.$$

(This map is well-defined since $I_{\mathbb{E}_K}$ is a normal subgroup of $G_{\mathbb{E}_K}$.) φ is the p-th power map on $\tilde{\mathbb{E}}_K$ and σ_K is a ring homomorphism, therefore they commute on $\tilde{\mathbb{E}}_K$. This implies

$$F_V \circ \varphi = \varphi \circ F_V$$

and motivates the following definition.

Definition 5.3 *We define the category of etale (φ, F)-modules over $\tilde{\mathbb{E}}_K$ as follows:*

- *The objects are pairs (D, F), where D is an etale φ-module over $\tilde{\mathbb{E}}_K$ and $F \colon D \to D$ is a bijective σ^f-semilinear map that commutes with φ.*
- *A morphism $f \colon (D_1, F_1) \to (D_2, F_2)$ consists of a morphism $f \colon D_1 \to D_2$ in the category of etale φ-modules such that $f \circ F_1 = F_2 \circ f$ holds.*
- *The composition is the usual composition of maps.*

In order to apply the results of section 1.4 we need to check the axioms. The setup is the following:

$$B = \mathbb{E}_K^{sep}, E = \mathbb{F}_p, G = G_{\mathbb{E}_K}, I = I_{\mathbb{E}_K} = G_{\tilde{\mathbb{E}}_K} \text{ and } \varsigma = \sigma_K.$$

Take the forgetful functor

$$(\text{etale } \varphi\text{-modules over } \tilde{\mathbb{E}}_K) \to (\varphi\text{-modules over } \mathbb{E}_K)$$

for T (this functor is fully faithful), consider the natural isomoprhisms in (5.3) and (5.4) for $\tilde{\alpha}_\bullet$ and $\tilde{\beta}_\bullet$. Insert the trivial (identity) natural transformations for $\eta_\bullet$ and $\xi_\bullet$. Then Axiom 1.1 is satisfied by Theorem 5.2.

Let (D, φ) be an etale φ-module over $\tilde{\mathbb{E}}_K$. Using the functoriality of σ_K^* on the isomorphism $\sigma^*(D) \cong D$ we get

$$\sigma^*(\sigma_K^*(D)) \cong \sigma_K^*(\sigma^*(D)) \cong \sigma_K^*(D)$$

by Lemma 1.12, i.e. $\sigma_K^*(D)$ is an etale $\sigma_K^*(\varphi)$-module. Hence, the functor σ_K^* extends to a self-equivalence of the category of etale φ-modules over $\tilde{\mathbb{E}}_K$, i.e. Axiom 1.2 is satisfied. Axiom 1.3 holds since T is a fully faithful functor. Remark 1.14 (resp. Remark 1.15) shows that Axiom 1.4 (resp. Axiom 1.5) is satisfied.

From Theorem 1.17 one receives:

Theorem 5.4 *There exists an equivalence of abelian tensor categories*

$$\tilde{\mathbb{D}}_{\mathrm{mod}} : \mathrm{Rep}_{\mathbb{F}_p}(W_{\mathbb{E}_K}) \rightleftharpoons (\text{etale } \varphi\text{-modules over } \tilde{\mathbb{E}}_K) : \tilde{\mathbb{V}}_{\mathrm{mod}}$$

given by

$$V \mapsto (\mathbb{E}_K^{sep} \otimes_{\mathbb{F}_p} V)^{I_{\mathbb{E}_K}}$$
$$(\mathbb{E}_K^{sep} \otimes_{\tilde{\mathbb{E}}_K} D)^{\varphi=\mathrm{id}} \leftarrow D.$$

5.2 (φ, Γ, F)-Modules and Mod-p Representations

Now we want to transition to mod-p representations of W_K. This can be realized, loosely speaking, by "adding" the action of

$$\Gamma := \Gamma_K := I_K/\tilde{H}_K = \mathrm{Gal}(K_\infty K^{nr}/K^{nr}) \cong \mathrm{Gal}(K_\infty/K)$$

on both sides. We use the following abbreviations:

$$H_K := \mathrm{Gal}(\overline{\mathbb{Q}}_p/K_\infty) \text{ and } \tilde{H}_K := \mathrm{Gal}(\overline{\mathbb{Q}}_p/K^{nr}K_\infty).$$

By the main theorem of the theory of norm fields [BC09, Theorem 13.4.3.] we have

$$H_K \cong \mathrm{Gal}(\mathbb{E}_K^{sep}/\mathbb{E}_K) \text{ and } \tilde{H}_K \cong \mathrm{Gal}(\mathbb{E}_K^{sep}/\tilde{\mathbb{E}}_K) \tag{5.5}$$

as topological groups. Hence we may interpret the element σ_K from the previous section as an element $\sigma_K \in H_K$ such that $\deg_{K_\infty}(\sigma_K) = 1$.

Lemma 5.5 *Let T/K be a Galois extension contained in $\overline{K}$.*

- $W_K/W_T \hookrightarrow \mathrm{Gal}(T/K)$ *is an injective continuous dense group homomorphism.*
- $W_K/W_T \cong \mathrm{Gal}(T/K)$ *if T/K is totally ramified.*

Proof: Consider the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & W_T & \xrightarrow{\subseteq} & W_K & \longrightarrow & W_K/W_T & \longrightarrow & 1 \\
 & & \downarrow \subseteq & & \downarrow \subseteq & & \vdots\ \iota & & \\
1 & \longrightarrow & G_T & \xrightarrow{\subseteq} & G_K & \longrightarrow & \mathrm{Gal}(T/K) & \longrightarrow & 1
\end{array}$$

with exact rows and remark that the left square is cartesian, i.e. $W_T = G_T \cap W_K$. By an elementary diagram chase the dotted arrow ι exists and has dense image since the middle vertical arrow has dense image. If T/K is totally ramified we have $I_K/I_T = \mathrm{Gal}(T \cdot K^{nr}/K^{nr}) = \mathrm{Gal}(T/K)$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & I_T & \xrightarrow{\subseteq} & I_K & \longrightarrow & \mathrm{Gal}(T/K) & \longrightarrow & 1 \\
 & & \downarrow \subseteq & & \downarrow \subseteq & & \vdots\ \eta & & \\
1 & \longrightarrow & W_T & \xrightarrow{\subseteq} & W_K & \longrightarrow & W_K/W_T & \longrightarrow & 1
\end{array}$$

with exact rows. Again the left square is cartesian and hence η exists. ι and η are inverse to each other. □

Now take $V \in \mathrm{Rep}_{\mathbb{F}_p}(W_K)$ and denote $D := (\mathbb{E}_K^{sep} \otimes_{\mathbb{F}_p} V)^{\tilde{H}_K}$. Consider the σ_K-semilinear bijective map given by

$$F_V \colon D \to D, \sum_i \lambda_i \otimes v_i \mapsto \sum_i \sigma_K(\lambda_i) \otimes \sigma_K.v_i$$

again. We check the compatibility with the action of Γ. Γ acts on D via the residual action of I_K on $\mathbb{E}_K^{sep} \otimes_{\mathbb{F}_p} V$ after taking $\tilde{H}_K$-invariants. For $u \in I_K$ and $d = \sum \lambda_i \otimes v_i \in D$ we have

$$\begin{aligned} F_V(u.d) &= \sum_i (\sigma_K \circ u)(\lambda_i) \otimes (\sigma_K \circ u).v_i \\ &= (\sigma_K \circ u \circ \sigma_K^{-1} \circ u^{-1}).(u.F_V(d)). \end{aligned}$$

But $\sigma_K \circ u \circ \sigma_K^{-1} \circ u^{-1} \in \tilde{H}_K$ and therefore

$$F_V(\gamma.d) = \gamma.F_V(d)$$

for all $\gamma \in \Gamma$ and $d \in D$. φ is the p-th power map on $\tilde{\mathbb{E}}_K$ and σ_K is a ring homomorphism, therefore they commute on $\tilde{\mathbb{E}}_K$. This implies

$$F_V \circ \varphi = \varphi \circ F_V.$$

This motivates the following definition.

Definition 5.6 *We define the category of etale (φ, Γ, F)-modules over $\tilde{\mathbb{E}}_K$ as follows:*

- *The objects are pairs (D, F), where D is an etale (φ, Γ)-module over $\tilde{\mathbb{E}}_K$ and $F: D \to D$ is a bijective σ_K-semilinear map that commutes with φ and the action of Γ.*
- *A morphism $f: (D_1, F_1) \to (D_2, F_2)$ consists of a morphism $f: D_1 \to D_2$ in the category of etale (φ, Γ)-modules such that $f \circ F_1 = F_2 \circ f$ holds.*
- *The composition is the usual composition of maps.*

From Theorem 5.4 one receives:

Theorem 5.7 *There exists an equivalence of abelian tensor categories*

$$\tilde{\mathbb{D}}_{\mathrm{mod}}: \mathrm{Rep}_{\mathbb{F}_p}(W_K) \rightleftharpoons (\text{etale } (\varphi, \Gamma, F)\text{-modules over } \tilde{\mathbb{E}}_K) : \tilde{\mathbb{V}}_{\mathrm{mod}}$$

given by

$$V \mapsto (\mathbb{E}_K^{sep} \otimes_{\mathbb{F}_p} V)^{\tilde{H}_K}$$
$$(\mathbb{E}_K^{sep} \otimes_{\tilde{\mathbb{E}}_K} D)^{\varphi=\mathrm{id}} \leftarrow D.$$

Proof: Denote the canonical projection

$$W_K \twoheadrightarrow W_K/\tilde{H}_K \cong W_K/I_{K_\infty} \cong \Gamma \times \mathbb{Z}$$

by $g \mapsto (\gamma_g, z_g)$ for all $g \in W_K$. Take a module (D, F) from the right hand side and define the map

$$W_K \times \tilde{\mathbb{V}}_{\mathrm{mod}}(D) \to \tilde{\mathbb{V}}_{\mathrm{mod}}(D), (g, v) \mapsto g.v := \sum_i g.b_i \otimes \gamma_g.F^{z_g}(d_i)$$

for $v = \sum_i b_i \otimes d_i \in \tilde{\mathbb{V}}_{\mathrm{mod}}(D)$ and $g \in W_K$. This map is well-defined since

$$\begin{aligned}
\varphi(g.v) &= \varphi(\sum_i g.b_i \otimes \gamma_g.F^{z_g}(d_i)) \\
&= \sum_i \varphi(g.b_i) \otimes \varphi(\gamma_g.F^{z_g}(d_i)) \\
&= \sum_i g.\varphi(b_i) \otimes \gamma_g.(\varphi \circ F^{z_g})(d_i) \\
&= \sum_i g.\varphi(b_i) \otimes \gamma_g.F^{z_g}(\varphi(d_i)) \\
&= g.v
\end{aligned}$$

holds for all $g \in W_K$ and $v = \sum_i b_i \otimes d_i \in \tilde{\mathbb{V}}_{\mathrm{mod}}(D)$. Furthermore this defines a group action since

$$\begin{aligned}
(gh).v &= \sum_i (gh).b_i \otimes \gamma_{gh}.F^{z_{gh}}(d_i) \\
&= \sum_i g(h.b_i) \otimes (\gamma_g \gamma_h).F^{z_g + z_h}(d_i) \\
&= \sum_i g(h.b_i) \otimes \gamma_g.F^{z_g}(\gamma_h.F^{z_h}(d_i)) \\
&= g.(h.v)
\end{aligned}$$

holds for all $g, h \in W_K$ and $v = \sum_i b_i \otimes d_i \in \tilde{\mathbb{V}}_{\mathrm{mod}}(D)$. It remains to check that the map, that defines the W_K-action, is continuous. By assumption its restriction to $I_K \times V \to V$ is continuous and I_K is open in W_K. Hence $I_K \times V$ is open in $W_K \times V$ and $W_K \times V \to V$ is therefore continuous. In addition we need to check that $\tilde{\mathbb{V}}_{\mathrm{mod}}$ is still a functor after varying the source and target category. Take a morphism $f\colon (D_1, F_1) \to (D_2, F_2)$. For all $g \in W_K$ and $v = \sum_i b_i \otimes d_i \in \tilde{\mathbb{V}}_{\mathrm{mod}}(D)$ we have

$$\begin{aligned}
\tilde{\mathbb{V}}_{\mathrm{mod}}(f)(g.v) &= \tilde{\mathbb{V}}_{\mathrm{mod}}(f)(\sum_i g.b_i \otimes \gamma_g.F_1^{z_g}(d_i)) \\
&= \sum_i g.b_i \otimes f(\gamma_g.F_1^{z_g}(d_i)) \\
&= \sum_i g.b_i \otimes \gamma_g.F_2^{z_g}(f(d_i))) \\
&= \sum_i g.b_i \otimes f(\gamma_g.F_2^{z_g}(d_i)) \\
&= g.\tilde{\mathbb{V}}_{\mathrm{mod}}(f)(v).
\end{aligned}$$

For a morphism $f\colon V_1 \to V_2$ of W_K-representations

$$\begin{aligned}
\tilde{\mathbb{D}}_{\mathrm{mod}}(f)(\gamma.d) &= \tilde{\mathbb{D}}_{\mathrm{mod}}(f)(\gamma.\sum_i \lambda_i \otimes v_i) \\
&= \tilde{\mathbb{D}}_{\mathrm{mod}}(f)(\sum_i u_\gamma.\lambda_i \otimes u_\gamma.v_i) \\
&= \sum_i u_\gamma.\lambda_i \otimes f(u_\gamma.v_i) \\
&= \sum_i u_\gamma.\lambda_i \otimes u_\gamma.f(v_i) \\
&= \gamma.\tilde{\mathbb{D}}_{\mathrm{mod}}(f)(d)
\end{aligned}$$

holds for all $d = \sum_i \lambda_i \otimes v_i \in \tilde{\mathbb{D}}_{\mathrm{mod}}(V_1)$ and $\gamma = u_\gamma \tilde{H}_K \in \Gamma$. We verify that the functors above are quasi-inverse to each other. It is enough to show that the comparison maps

$$\tilde{\alpha}_V\colon \tilde{\mathbb{V}}_{\mathrm{mod}}(\tilde{\mathbb{D}}_{\mathrm{mod}}(V)) \to V \text{ and } \tilde{\beta}_D\colon \tilde{\mathbb{D}}_{\mathrm{mod}}(\tilde{\mathbb{V}}_{\mathrm{mod}}(D)) \to D$$

are isomorphisms in the stated categories. For $g \in W_K$, the image $\gamma_g \in \Gamma$ of g and $v = \sum_{i,j} b_i \otimes b_{ij} \otimes v_j \in \tilde{\mathbb{V}}_{\mathrm{mod}}(\tilde{\mathbb{D}}_{\mathrm{mod}}(V))$ we have

$$\begin{aligned}
\tilde{\alpha}_V(g.v) &= \tilde{\alpha}_V(g.(\sum_i b_i \otimes \underbrace{\sum_j b_{ij} \otimes v_j}_{\in \tilde{\mathbb{D}}_{\mathrm{mod}}(V)})) \\
&= \tilde{\alpha}_V(\sum_i g.b_i \otimes \gamma_g F_V^{z_g}(\sum_j b_{ij} \otimes v_j)) \\
&= \tilde{\alpha}_V(\sum_i g.b_i \otimes (\sum_j g.b_{ij} \otimes g.v_j)) \\
&= \sum_{i,j} g.(b_i b_{ij}) g.v_j \\
&= g.\tilde{\alpha}_V(v).
\end{aligned}$$

Hence $\tilde{\alpha}_V$ is a W_K-equivariant linear bijection. For $\gamma = u_\gamma \tilde{H}_K \in \Gamma$ and

$d=\sum_{i,j} b_i\otimes b_{ij}\otimes d_j \in \tilde{\mathbb{D}}_{\mathrm{mod}}(\tilde{\mathbb{V}}_{\mathrm{mod}}(D))$ we have:

$$\begin{aligned}
(\tilde{\beta}_D\circ\gamma)(d) &= \tilde{\beta}_D(\gamma(\sum_i b_i\otimes \underbrace{\sum_j b_{ij}\otimes d_j}_{\in\tilde{\mathbb{V}}_{\mathrm{mod}}(D)}))\\
&= \tilde{\beta}_D(\sum_i u_\gamma.b_i\otimes u_\gamma.(\sum_j b_{ij}\otimes d_j))\\
&= \tilde{\beta}_D(\sum_i u_\gamma.b_i\otimes \sum_j u_\gamma.b_{ij}\otimes \gamma F^{\deg(u_\gamma)}(d_j))\\
&= \sum_{i,j} u_\gamma.(b_ib_{ij})\otimes\gamma(d_j))\\
&= (\gamma\circ\tilde{\beta}_D)(d).
\end{aligned}$$

This shows that $\tilde{\beta}_D$ is an isomorphism in the category of etale (φ,Γ,F)-modules which completes the proof. □

5.3 Reality Check

We remind that the actions of φ and $\sigma:=\sigma_{\mathbb{Q}_p}$ on the field $k((X))$ are given as follows:

$$\varphi(f)=f^p \text{ and } \sigma(f)=\sum_{i\geq m} a_i^pX^i$$

for all $f=\sum_{i\geq m} a_iX^i\in k((X))$.

One-dimensional etale (φ,Γ)-modules over $\mathbb{E}_{\mathbb{Q}_p}$ and $\tilde{\mathbb{E}}_{\mathbb{Q}_p}$: We classify all one-dimensional mod-p-representations of $G_{\mathbb{Q}_p}$ and $I_{\mathbb{Q}_p}$ as follows. Take $k\in\{\mathbb{F}_p,\overline{\mathbb{F}}_p\}$ and let D denote a one-dimensional etale (φ,Γ)-module over $k((X))$. Fix a generator $0\neq e\in D$ and find $h\in k((X))^\times$ such that $\varphi(e)=h\cdot e$. We write $h=h_0T^aH$ with $h_0\in k^\times, a\in\mathbb{Z}$ and $H\in 1+Xk[[X]]$. For any $u\in k((X))^\times$ we have

$$\varphi(ue)=\varphi(u)he=\varphi(u)u^{-1}h(ue).$$

The map $k[[X]]^\times\to 1+Xk[[X]]$ given by $u\mapsto u\varphi(u)^{-1}$ is surjective (a preimage of an element b is given by $\prod_{j=0}^\infty\varphi^j(b)$). Thus we find $u\in k[[X]]$ such that $\varphi(ue)=h_0X^a(ue)$ and we may assume w.l.o.g. (by base change) that

$$\varphi(e)=h_0X^ae \text{ with } 0\leq a\leq p-1.$$

Take $g \in k((X))^\times$ and a generator $\gamma \in \Gamma$ such that $\gamma.e = ge$. Apply $\gamma \circ \varphi = \gamma \circ \varphi$ on e and obtain:

$$h_0((1+X)^{\chi(\gamma)} - 1)^a ge = \varphi(g) h_0 X^a e.$$

Set $z := \chi(\gamma) \in \mathbb{Z}_p^\times$ and compare the leading coefficients of the above equation. One receives

$$\overline{z}^a X^a g_m X^m = g_m^p X^{pm} X^a.$$

This implies $m = 0$, $g_0 \in \mathbb{F}_p^\times$ and therefore $\overline{z}^a = 1$. Then $a = 0$ since $\overline{z} \neq 1$ and $h = h_0 \in k^\times$. We see that the one-dimensional mod-p-representations of $G_{\mathbb{Q}_p}$ correspond to elements of $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$ and the one-dimensional mod-p-representations of $I_{\mathbb{Q}_p}$ correspond to elements of $\mathbb{F}_p^\times \times \overline{\mathbb{F}}_p^\times$.

One-dimensional etale (φ, Γ, F)-modules over $\tilde{\mathbb{E}}_{\mathbb{Q}_p}$: Now we calculate all one-dimensional mod-p-representations of $W_{\mathbb{Q}_p}$. We can immediately restrict to the situation $k = \overline{\mathbb{F}}_p$ above and assume now that D is a (φ, Γ, F)-module over $k((X))$. Take $f \in k((X))^\times$ such that $F(e) = fe$ and apply $F \circ \varphi = \varphi \circ F$ to e and receive

$$h_0^p fe = f^p h_0 e, \text{ i.e. } h_0 f^{-1} \in \mathbb{F}_p^\times.$$

This implies $f = f_0 \in k^\times$ and since taking the $(p-1)$-th power on $k^\times$ is surjective we find $u \in k^\times$ such that $\sigma(u)u^{-1} = f_0^{-1}$. Then we have

$$F(ue) = \sigma(u) f_0 e = ue.$$

and we change the base now by $e \mapsto e' := ue$. Hence $\varphi(e') = \varphi(u)u^{-1}h_0 \cdot e'$ and we set $h_0' = h' := \varphi(u)u^{-1}h_0 \in k^\times$. Apply $F \circ \varphi = \varphi \circ F$ to e' and obtain

$$(h_0')^p \cdot e' = h_0' \cdot e', \text{ in particular } h_0' \in \mathbb{F}_p^\times.$$

We see that the one-dimensional mod-p-representations of $W_{\mathbb{Q}_p}$ correspond to elements of $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$ as predicted by Corollary 1.6.

5.4 (φ, F)-Modules and p-adic Representations

As a start we remark that the theory of (φ, Γ)-modules developed by Fontaine (see [Fon90] or [BC09, §13]) does not require the residue field of the p-adic field K', which we begin with, to be finite, hence we are free to start with a p-adic field $K' = KP_0$ with residue field $\overline{\mathbb{F}}_p$. This would now lead to a category of (φ, Γ)-modules classifying the p-adic representations of $G_{K'}$ but by the Theorem of Ax-Sen-Tate [Tat67, §(3.3), Theorem 1] we obtain

$$G_{K'} = \mathrm{Gal}(\overline{K'}/K') \cong \mathrm{Aut}_{\mathrm{cont}}(\mathbb{C}_p/KP_0) \cong \mathrm{Gal}(\overline{K}/K^{nr}) = I_K.$$

In this situation the following rings matter.

- $\tilde{\mathbb{A}}_K$ denotes a Cohen ring for $\tilde{\mathbb{E}}_K$ which is isomorphic to

$$\{\sum_{i \in \mathbb{Z}} a_i X^i \mid a_i \in W(\overline{\mathbb{F}}_p) \text{ and } \lim_{i \to -\infty} a_i = 0\}$$

  via a lift of the isomorphism $\overline{\mathbb{F}}_p((X)) \cong \tilde{\mathbb{E}}_K$. For a precise construction see [BC09, §13.5].

- $\tilde{\mathbb{B}}_K$ denotes the quotient field of $\tilde{\mathbb{A}}_K$.

From [Sch17, Proposition 1.2.6.] and (5.5) we know that

$$G_{\mathbb{E}_K} \cong H_K \cong \mathrm{Gal}(\mathbb{B}_K^{nr}/\mathbb{B}_K) \text{ and } I_{\mathbb{E}_K} \cong \tilde{H}_K \cong \mathrm{Gal}(\mathbb{B}_K^{nr}/\tilde{\mathbb{B}}_K)$$

as topological groups.

Definition 5.8 *A φ-module D over $\mathbb{B}_K$ (resp. $\tilde{\mathbb{B}}_K$) is called etale if there exists an $\mathbb{A}_K$-lattice (resp. $\tilde{\mathbb{A}}_K$-lattice) $M \subseteq D$ such that the linearization*

$$(\sigma|_{\mathbb{A}_K})^*(M) \to M \text{ given by } a \otimes x \mapsto a\varphi(x)$$

$$(\text{resp. } (\sigma|_{\tilde{\mathbb{A}}_K})^*(M) \to M \text{ given by } a \otimes x \mapsto a\varphi(x))$$

is an isomorphism.

Warning: The meaning of being "etale" depends on the coefficient ring of the φ-modules. Maybe "φ-module containing an etale lattice" would be the better term but we stick to the literature here.

Recall the equivalence of abelian tensor categories initially established by Fontaine (see [BC09, Theo. 3.3.4.]):

Theorem 5.9 *There are equivalences of abelian tensor categories given by*

$$\mathbb{D}\colon \mathrm{Rep}(G_{\mathbb{E}_K}) \rightleftharpoons (\text{etale } \varphi\text{-modules over } \mathbb{B}_K) : \mathbb{V}$$

$$V \mapsto (\mathbb{B} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{E}_K}}$$

$$(\mathbb{B} \otimes_{\mathbb{B}_K} D)^{\varphi=\mathrm{id}} \leftarrow\!\shortmid D$$

and

$$\tilde{\mathbb{D}}\colon \mathrm{Rep}(I_{\mathbb{E}_K}) \rightleftharpoons (\text{etale } \varphi\text{-modules over } \tilde{\mathbb{B}}_K) : \tilde{\mathbb{V}}$$
$$V \mapsto (\mathbb{B} \otimes_{\mathbb{Q}_p} V)^{I_{\mathbb{E}_K}}$$
$$(\mathbb{B} \otimes_{\tilde{\mathbb{B}}_K} D)^{\varphi=\mathrm{id}} \leftarrow\!\shortmid D.$$

We immediately obtain the following commutative diagram of functors

$$\begin{array}{ccc} \mathrm{Rep}(G_{\mathbb{E}_K}) & \xrightarrow{\mathbb{D}} & (\text{etale } \varphi\text{-modules over } \mathbb{B}_K) \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \cdot \otimes_{\mathbb{B}_K} \tilde{\mathbb{B}}_K} \\ \mathrm{Rep}(I_{\mathbb{E}_K}) & \xrightarrow{\tilde{\mathbb{D}}} & (\text{etale } \varphi\text{-modules over } \tilde{\mathbb{B}}_K). \end{array} \tag{CD6}$$

since $\tilde{\mathbb{B}}_K/\mathbb{B}_K$ is a Galois extension with group G_k.

Definition 5.10 *We define the category of etale (φ, F)-modules over $\tilde{\mathbb{B}}_K$ as follows:*

- *The objects are pairs (D, F), where D is an etale φ-module over $\tilde{\mathbb{B}}_K$ and $F\colon D \to D$ is a bijective σ^f-semilinear map that commutes with φ.*
- *A morphism $f\colon (D_1, F_1) \to (D_2, F_2)$ consists of a morphism $f\colon D_1 \to D_2$ in the category of etale φ-modules such that $f \circ F_1 = F_2 \circ f$ holds.*
- *The composition is the usual composition of maps.*

In order to apply the results of section section 1.4 we need to check the axioms. The setup is the following:

$$B = \mathbb{B}, E = \mathbb{Q}_p, G = G_{\mathbb{E}_K}, I = I_{\mathbb{E}_K} = G_{\tilde{\mathbb{E}}_K} \text{ and } \varsigma = \sigma_K.$$

Take the forgetful functor for T (which forgets the property of being etale) and the trivial (identity) natural transformations for $\eta_\bullet$ and $\xi_\bullet$. Then Axiom 1.1 is satisfied by Theorem 5.9.

Let (D, φ) be an etale φ-module over $\tilde{\mathbb{B}}_K$. We set $\sigma_K^*(M) := A \otimes_{A,\sigma_K} M$ and verify that $\sigma_K^*(M)$ is an $\tilde{\mathbb{A}}_K$-lattice in $\sigma_K^*(D)$:

We claim that the canonical map $\sigma_K^*(M) \to \sigma_K^*(D)$ given by $a \otimes m \mapsto a \otimes m$ is injective. Take a linearly independent set of elements $x_1, \ldots, x_l \in \sigma_K^*(M)$ and $b_1, \ldots, b_l \in \tilde{\mathbb{B}}_K$ such that

$$\sum_i b_i x_i = 0.$$

Since $\tilde{\mathbb{B}}_K$ is a discretely valued field with ring of integers $\tilde{\mathbb{A}}_K$ and uniformizer p, we find $N \gg 0$ such that $p^N b_i \in \tilde{\mathbb{A}}_K$. Hence

$$\sum_i p^N b_i x_i = 0$$

and by linear independence over $\tilde{\mathbb{A}}_K$, we obtain $p^N b_i = 0$ for all $i = 1, \ldots, l$, therefore $b_i = 0$ for all $i = 1, \ldots, l$. Now we show that $(\sigma|_{\tilde{\mathbb{A}}_K})^*(\sigma_K^*(M)) \cong \sigma_K^* M$. Using the functoriality of σ_K^* on the isomorphism $(\sigma|_{\tilde{\mathbb{A}}_K})^*(M) \cong M$ we get

$$(\sigma|_{\tilde{\mathbb{A}}_K})^*(\sigma_K^*(M)) \cong \sigma_K^*((\sigma|_{\tilde{\mathbb{A}}_K})^*(M)) \cong \sigma_K^*(M),$$

by the same argument as in Lemma 1.12, i.e. $\sigma_K^*(D)$ is an etale $\sigma_K^*(\varphi)$-module. Hence, the functor σ_K^* extends to a self-equivalence of the category of etale φ-modules over $\tilde{\mathbb{B}}_K$, i.e. Axiom 1.2 is satisfied.

Axiom 1.3 holds since T is fully faithful. Remark 1.14 (resp. Remark 1.15) shows that Axiom 1.4 (resp. Axiom 1.5) is satisfied.

From Theorem 1.17 one receives:

Theorem 5.11 *There exists an equivalence of abelian tensor categories*

$$\tilde{\mathbb{D}}\colon \mathrm{Rep}(W_{\mathbb{E}_K}) \rightleftharpoons (\text{etale } (\varphi, F)\text{-modules over } \tilde{\mathbb{B}}_K) : \tilde{\mathbb{V}}$$

given by

$$V \mapsto (\mathbb{B} \otimes_{\mathbb{Q}_p} V)^{I_{\mathbb{E}_K}}$$
$$(\mathbb{B} \otimes_{\tilde{\mathbb{B}}_K} D)^{\varphi = \mathrm{id}} \leftarrow D.$$

5.5 (φ, Γ, F)-Modules and p-adic Representations

Now we extend to representations of W_K in the same way as in section 5.2

Definition 5.12 *We define the category of etale (φ, Γ, F)-modules over $\tilde{\mathbb{B}}_K$ as follows:*

- *The objects are pairs (D, F), where D is an etale (φ, Γ)-module over $\tilde{\mathbb{B}}_K$ and $F\colon D \to D$ is a bijective σ_K-semilinear map that commutes with φ and the action of Γ.*
- *A morphism $f\colon (D_1, F_1) \to (D_2, F_2)$ consists of a morphism $f\colon D_1 \to D_2$ in the category of etale (φ, Γ)-modules such that $f \circ F_1 = F_2 \circ f$ holds.*
- *The composition is the usual composition of maps.*

One replaces the field $\tilde{\mathbb{E}}_K$ by $\tilde{\mathbb{B}}_K$ and $\mathbb{E}_K^{sep}$ by $\mathbb{B}$ in section 5.2 and the does the same with the functors, i.e. we consider $\tilde{\mathbb{V}}$ instead of $\tilde{\mathbb{V}}_{\mathrm{mod}}$ and $\tilde{\mathbb{D}}$ instead of $\tilde{\mathbb{D}}_{\mathrm{mod}}$. Then all the modified calculations continue to be valid. Hence one can imitate the proof of Theorem 5.7 and receive:

Theorem 5.13 *There exists an equivalence of abelian tensor categories*

$$\tilde{\mathbb{D}}\colon \mathrm{Rep}(W_K) \rightleftharpoons (\text{etale } (\varphi, \Gamma, F)\text{-modules over } \tilde{\mathbb{B}}_K) : \tilde{\mathbb{V}}$$

given by

$$V \mapsto (\mathbb{B} \otimes_{\mathbb{Q}_p} V)^{\tilde{H}_K}$$

$$(\mathbb{B} \otimes_{\tilde{\mathbb{B}}_K} D)^{\varphi = \mathrm{id}} \leftarrow D.$$

Proof: Imitate the proof of Theorem 5.7 using the modifications explained above. □

Corollary 5.14 *Let V be an object of $\mathrm{Rep}(W_K)$. The W_K-action on V can be extended continuously to an action of G_K on V (i.e. V is an object of $\mathrm{Rep}(G_K)$) if and only if there exists an etale (φ, Γ)-module D over $\mathbb{B}_K$ such that $\tilde{\mathbb{D}}(V) \cong \tilde{\mathbb{B}}_K \otimes_{\mathbb{B}_K} D$. Here the bijective σ_K-semilinear selfmap F on $\tilde{\mathbb{B}}_K \otimes_{\mathbb{B}_K} D$ is given by $F(b \otimes d) = \sigma_K(b) \otimes d$ for all $b \in \tilde{\mathbb{B}}_K$ and $d \in D$.*

Proof: Assume that V is an object of $\mathrm{Rep}(G_K)$. Then

$$\tilde{\mathbb{B}}_K \otimes_{\mathbb{B}_K} \mathbb{D}(V) \cong \tilde{\mathbb{B}}_K \otimes_{\mathbb{B}_K} (\mathbb{B} \otimes_{\mathbb{Q}_p} V)^{I_K} \cong (\tilde{\mathbb{B}}_K \otimes_{\mathbb{Q}_p} V)^{I_K} = \tilde{\mathbb{D}}(V).$$

Conversely assume that $\tilde{\mathbb{D}}(V) \cong \tilde{\mathbb{B}}_K \otimes_{\mathbb{B}_K} D$ as above. This implies

$$\mathbb{D}(V) \cong (\mathbb{B} \otimes_{\mathbb{Q}_p} V)^{G_K} = ((\mathbb{B} \otimes_{\mathbb{Q}_p} V)^{I_K})^{G_k} \cong \tilde{\mathbb{D}}(V)^{G_k} \cong (\tilde{\mathbb{B}}_K \otimes_{\mathbb{B}_K} D)^{G_k} \cong D$$

and therefore V is a Galois representation. □

Corollary 5.15 *Let V be an object of* $\mathrm{Rep}(G_K) \subseteq \mathrm{Rep}(W_K)$ *and* $\tilde{\mathbb{D}}(V) = (D, F)$ *the associated (φ, Γ, F)-module over $\tilde{\mathbb{B}}_K$. Then (D, F) satisfies*

$$\det(F) \in \mathbb{A}_K^\times \text{ i.e. } \nu_{\mathbb{B}_K}(\det(F)) = 0,$$

where $\nu_{\mathbb{B}_K}$ denotes the discrete valuation on $\mathbb{B}_K$.

Now one would wish to apply the methods from chapter 4 to the current case in order to establish sufficient criteria that distinguish Weil and Galois group representations. The ring $\tilde{\mathbb{A}}_K$ is a discrete valuation ring with residue field $\overline{\mathbb{F}}_p((X))$. In general the elements of this residue field do not have finite order, hence we are not able to lift a morphism $\mathbb{Z} \to \mathrm{GL}_d(\tilde{\mathbb{A}}_K)$ to a morphism $\hat{\mathbb{Z}} \to \mathrm{GL}_d(\tilde{\mathbb{A}}_K)$ by using Lemma 4.1. Therefore this technique does not work out in the present context.

Appendices

Appendix A

Divided Powers

The following paragraph is extracted from [BO78].

Definition A.1 *Let A be a ring. We call (I, γ) a divided power structure on A if I is an ideal in A and γ collection of maps $\{\gamma_i\}_{i \geq 0}$ such that the following properties are satisfied:*

1. $\gamma_0(x) = 1, \gamma_1(x) = x, \gamma_i(x) \in I$ *for all* $x \in I, i \geq 1$.
2. $\gamma_k(x + y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(y)$ *for all* $x, y \in I$.
3. $\gamma_k(\lambda x) = \lambda^k \gamma_k(x)$ *for all* $\lambda \in A, x \in I$.
4. $\gamma_i(x)\gamma_j(x) = \frac{(i+j)!}{i! \cdot j!}\gamma_{i+j}(x)$ *for all* $x \in I$.
5. $\gamma_p(\gamma_q(x)) = \frac{(pq)!}{p!(q!)^p}\gamma_{pq}(x)$ *for all* $x \in I$.

We call the triplet (A, I, γ) a divided powers ring which will be abbreviated by "PD-ring". Similarly we call (I, γ) a divided power ideal, abbreviated by "PD-ideal", and γ a divided power structure, abbreviated by "PD-structure". Furthermore we call $J \subseteq I$ a sub PD-ideal if $J \subseteq A$ is an ideal and $\gamma_i(x) \in J$ for all $x \in J$ and $i \geq 1$.

Remark A.2 *All rational coefficients appearing in the definition above are integers.*

Definition A.3 *Let (A, I, γ) and (B, J, δ) be PD-rings and $f \colon A \to B$ a ring homomorphism. We call f a PD-morphism if $f(I) \subseteq J$ and $\delta_n(f(x)) = f(\gamma_n(x))$ for all $x \in I$.*

Example A.4 *A PD-ring of our interest will be $(\mathbb{Z}_p, (p), \gamma)$ where $\gamma_n(x) = \frac{x^n}{n!}$. Obviously $((0), \gamma)$ is a sub PD-ideal of $((p), \gamma)$.*

Lemma A.5 *Let (A, I, γ) be a PD-ring and $J \subseteq A$ an ideal. Denote the canonical projection by $\pi\colon A \to A/J$. There exists a unique PD-structure $\bar{\gamma}\colon \pi^{-1}(I) \to \pi^{-1}(I)$ such that $\pi\colon (A, I, \gamma) \to (A/J, \pi^{-1}(I), \bar{\gamma})$ is a PD-morphism if and only if $J \cap I \subseteq I$ is a sub PD-ideal.*

Proof: [BO78, Lemma 3.5]. □

A.1 Universal Enveloping Divided Power Ring

Let B be a ring and M be an B-module.

Definition A.6 *We call $(\mathcal{U}_B(M), \mathcal{U}_B^+(M), \mu)$ universal enveloping PD-ring of M if it is a PD-ring and there exists a B-module homomorphism*

$$\iota\colon M \to U_B^+(M)$$

satisfying the following universal property: For any PD-ring (C, J, δ) over B and B-module homomorphism $\Psi\colon M \to J$ there is a unique PD-morphism

$$\bar{\Psi}\colon (\mathcal{U}_B(M), \mathcal{U}_B^+(M), \mu) \to (C, J, \delta)$$

such that $\bar{\Psi} \circ \iota = \Psi$. If no confusion is possible we abbreviate

$$\mu_n(x) := \mu_n(\iota(x))$$

for $x \in M$.

Theorem A.7 *$(\mathcal{U}_B(M), \mathcal{U}_B^+(M), \mu)$ exists.*

Proof: Set $G_B(M) := B[\{T_{(x,n)} \mid x \in M, n \in \mathbb{N}\}]$ and consider the following subsets of $G_B(M)$:

$$E_1 := \{T_{(x,0)} - 1 \mid x \in M\}$$

$$E_2 := \{T_{(bx,n)} - b^n T_{(x,n)} \mid x \in M, b \in B, n \in \mathbb{N}\}$$

$$E_3 := \{T_{(x,n)} T_{(x,m)} - \frac{(n+m)!}{n!m!} \cdot T_{(x,n+m)} \mid x \in M, n, m \in \mathbb{N}\}$$

$$E_4 := \{T_{(x+y,n)} - \sum_{i+j=n} T_{x,i} T_{y,j} \mid x, y \in M, n \in \mathbb{N}\}$$

Let $I_B(M)$ denote the ideal generated by $E_1 \cup E_2 \cup E_3 \cup E_4$. There is an obvious grading on $G_B(M)$ given by

$$G_B(M) = \bigoplus_{n \geq 0} B[T_{(x,n)} \mid x \in M]$$

and $I_B(M)$ is a homogenous ideal with respect to this grading. Therefore $\mathcal{U}_B(M) := G_B(M)/I_B(M)$ is a graded ring, i.e.

$$\mathcal{U}_B(M) = \bigoplus_{i\geq 0} \mathcal{U}_B^i(M).$$

We define

$$\mathcal{U}_B^+(M) = \bigoplus_{i>0} \mathcal{U}_B^i(M).$$

We set $x^{[n]} := T_{(x,n)} + I_B(M)$ and

$$\varphi\colon M \to \mathcal{U}_B^+(M), x \mapsto x^{[1]}.$$

By [BO78, Theorem A9] there exists a unique PD-structure μ such that

$$\mu_i(x^{[1]}) = x^{[i]}$$

for all $i \geq 1$ and $x \in M$. Thus $(\mathcal{U}_B(M), \mathcal{U}_B(M)^+, \mu)$ satisfies the universal property. □

Corollary A.8 *The assignment $M \mapsto (\mathcal{U}_B(M), \mathcal{U}_B^+(M), \mu)$ defines a functor from the category of B-modules into the category of PD-rings.*

A.2 Divided Power Envelopes

Let (A, I, γ) be a PD-ring, B an A-algebra and J an ideal in B.

Definition A.9 *We say that γ extends to B if there is a PD-structure γ' on IB such that $\psi\colon (A, I, \gamma) \to (B, IB, \gamma')$ is a PD-morphism.*

Proposition A.10 *If I is principal, γ extends to B.*

Proof: [BO78, Proposition 3.15]. □

Proposition A.11 *Assume that (J, δ) is a PD-ideal in B. The following statements are equivalent:*

1. *γ extends to a PD-structure γ' on B and $\gamma'(x) = \delta(x)$ for all $x \in IB \cap J$.*

2. *$K := \psi(I)B + J$ has a unique PD-structure δ' such that $\psi\colon (A, I, \gamma) \to (B, K, \delta')$ and $\mathrm{id}_B\colon (B, J, \delta) \to (B, K, \delta')$ are PD-morphisms.*

3. *There is an ideal $K \subseteq B$ with $\psi(I)B + J \subseteq K$ with a PD-structure κ such that $\psi\colon (A, I, \gamma) \to (B, K, \kappa)$ and $\mathrm{id}_B\colon (B, J, \delta) \to (B, K, \kappa)$ are PD-morphisms.*

If the conditions hold we say that γ and δ are compatible.

Proof: [BO78, Proposition 3.16]. □

Definition A.12 *We call a PD-ring $(\mathcal{D}, \mathcal{J}, \eta)$ PD-envelope of (B, J) with respect to (A, I, γ) if $J\mathcal{D} \subseteq \mathcal{J}$, η is compatible with γ and the following universal property is satisfied: For any PD-ring (C, K, δ) such that C is a B-algebra, $JC \subseteq K$ and δ is compatible with γ, there exists a unique PD-morphism*

$$(\mathcal{D}, \mathcal{J}, \eta) \to (C, K, \delta)$$

making the obvious diagram commutative. We denote the PD-envelope of (B, J) with respect to (A, I, γ) by $\mathcal{D}_{(A,I,\gamma)}(B, J)$.

Theorem A.13 *$\mathcal{D}_{(A,I,\gamma)}(B, J)$ exists.*

Proof: If I is not contained in J replace J by $J + I$. Let $\iota\colon J \to \mathcal{U}_B^+(J)$ be the universal map from Theorem A.7. Since $J \subseteq B$ we can interpret the elements of J as elements of the B-algebra $\mathcal{U}_B(J)$. Set

$$F_1 := \{\iota(x) - x \mid x \in J\}$$

$$F_2 := \{\mu_n(\iota(y)) - \gamma_n(y) \mid y \in I\}$$

and let $\mathcal{I}_{B,(A,I,\gamma)}(J)$ denote the ideal in $\mathcal{U}_B(J)$ generated by F_1 and F_2. One can proof that the quotient $\mathcal{U}_B(J)/\mathcal{I}_{B,(A,I,\gamma)}(J)$ has the required properties, compare for example [BO78, Theorem 3.19]. □

A.3 Compatibility with Tensor Products

The following is partly extracted from [Rob63].

Let B be an A-algebra and as in the previous section let (A, I, γ) be a PD-ring, R an B-algebra and J an ideal in B such that the canonical (surjective) map

$$R \otimes_B J \to JR \text{ given by } r \otimes j \mapsto j \cdot r \tag{A.1}$$

is an isomorphism. This condition is for example satisfied if $B \subseteq R$ is a ring extension of integral domains and J is a principal ideal. It is our aim to show that

$$\mathcal{D}_{(A,I,\gamma)}(R, JR) \cong R \otimes_B \mathcal{D}_{(A,I,\gamma)}(B, J).$$

At first we need to understand the B-linear homomorphisms from $\mathcal{U}_B(M)$ into R. For this purpose we introduce

$$\exp(R) := \{f \in R[[T]] \mid f(0) = 1 \text{ and } f(T_1 + T_2) = f(T_1)f(T_2)\}$$

which is a subgroup of $R[[T]]^\times$ and becomes an R-module via $r.f(T) := f(rT)$.

Lemma A.14 *There is a bijection*

$$\mathrm{Map}(M, R[[T]]) \cong \mathrm{Hom}_{(B-alg)}(G_B(M), R)$$

given as follows: Let $f\colon M \to R[[T]]$ be a map. Then for each $x \in M$ we can write

$$f(x) = \sum_{n \geq 0} c_{x,n} T^n \text{ with } c_{x,n} \in R.$$

Define the image of f as the map φ which is uniquely determined by

$$\varphi(T_{(x,n)}) := c_{x,n}.$$

Proof: The inverse of the map given above is

$$\varphi \mapsto (x \mapsto \sum_{n \geq 0} \varphi(T_{(x,n)}) T^n).$$

□

Proposition A.15 *The map*

$$\mathrm{Hom}_{(B-alg)}(\mathcal{U}_B(M), R) \to \mathrm{Hom}_{(B-mod)}(M, \exp(R))$$

given by

$$\varphi \mapsto (x \mapsto \sum_{n \geq 0} \varphi(\mu_n(x)) T^n)$$

is a bijection.

Proof: The map defined in the assertion is a restriction of the map in Lemma A.14:

$$\mathrm{Map}(M, R[[T]]) \to \mathrm{Hom}_{(B-alg)}(G_B(M), R).$$

Take corresponding elements

$$f \mapsto \varphi$$

under this bijection. Calculations deliver the following:

- f is additive if and only if $\varphi(E_4) = 0$.
- We have $f(ax) = a\beta(x)$ for all $a \in A, x \in M$ if and only if $\varphi(E_2) = 0$.
- We have $f(x)(0) = 1$ if and only if $\varphi(E_1) = 0$.
- We have $f(x)(T_1 + T_2) = f(x)(T_1)f(x)(T_2)$ if and only if $\varphi(E_3) = 0$.

(E_i's as in the proof of Theorem *A.7.*) This shows the claim since $\mathcal{U}_B(M)$ is the quotient of $G_B(M)$ by the ideal generated by the E_i's. □

Proposition A.16 *We have* $\mathcal{U}_R(R \otimes_B M) \cong R \otimes_B \mathcal{U}_B(M)$.

Proof: To prove this statement without confusion we fix the following notation. The PD-structures on

- $\mathcal{U}_B(M)$ is given by μ.
- $\mathcal{U}_B(R \otimes_B M)$ is given by η.
- $\mathcal{U}_R(R \otimes_B M)$ is given by $\hat{\eta}$.

We define the map

$$f \colon R \otimes_B M \to \mathcal{E} := \exp(R \otimes_B \mathcal{U}_B(M))$$

as the R-linear continuation of

$$r \otimes x \mapsto \sum_{n \geq 0} (r^n \otimes \mu_n(x)) T^n.$$

One needs to check that this is well-defined. Since we have

$$f(x)(0) = 1 \otimes 1 = 1$$

it is enough to show that

$$f(x)(T_1 + T_2) = f(x)(T_1)f(x)(T_2)$$

for all $x \in R \otimes_B M$. Take $x = \sum_i r_i \otimes m_i \in R \otimes_B M$:

$$\begin{aligned}
f(x)(T_1)f(x)(T_2) &= (\prod_i \sum_n (r_i^n \otimes \mu_n(m_i))T_1^n)(\prod_i \sum_n (r_i^n \otimes \mu_n(m_i))T_2^n) \\
&= \sum_{n\geq 0} \sum_{k+l=n} (\prod_i (r_i^k \otimes \mu_k(m_i))T_1^k)(\prod_j (r_i^l \otimes \mu_l(m_i))T_1^l) \\
&= \sum_{n\geq 0} \sum_{k+l=n} \prod_i (r_i^{k+l} \otimes \mu_k(m_i)\mu_l(m_i))T_1^k T_2^l \\
&= \sum_{n\geq 0} \sum_{k+l=n} \prod_i (\frac{(k+l)!}{(k!)(l!)} r_i^{k+l} \otimes \mu_{k+l}(m_i))T_1^k T_2^l \\
&= \sum_{n\geq 0} \sum_{k+l=n} \prod_i (\frac{n!}{(n-l)!(l!)} r_i^n \otimes \mu_n(m_i))T_1^{n-l} T_2^l \\
&= \sum_{n\geq 0} (\prod_i r_i^n \otimes \mu_n(m_i)) \sum_{l=0}^{n} \binom{n}{l} T_1^{n-l} T_2^l \\
&= \sum_{n\geq 0} (\prod_i r_i^n \otimes \mu_n(m_i))(T_1 + T_2)^n \\
&= f(x)(T_1 + T_2)
\end{aligned}$$

Now we use Proposition A.15 and obtain an R-algebra homomorphism

$$\varphi\colon \mathcal{U}_B(R \otimes_B M) \to R \otimes_B \mathcal{U}_B(M)$$

corresponding to f satisfying

$$\varphi([T_{(x,n)}]) = c_{x,n} \text{ for all } x \in R \otimes_B M, n \in \mathbb{N}$$

where $f(x) = \sum_{n\geq 0} c_{x,n}T^n$. In particular

$$\varphi([T_{(r\otimes y,n)}]) = r^n \otimes \mu_n(y) \text{ for all } r \in R, y \in M, n \in \mathbb{N}.$$

To finish the prove we construct an inverse ψ of φ as follows. Let N be an R-module. By [Rob63, Proposition III.4, p.261] there exists a unique homomorphism of B-algebras

$$\delta_N\colon \mathcal{U}_B(N) \to \mathcal{U}_R(N)$$

satisfying $\mu_{B,n}(x) \mapsto \mu_{R,n}(x)$ for all $x \in N$ and $n \in \mathbb{N}$ (where $\mathcal{U}_B(N) = (\mathcal{U}_B(N), \mathcal{U}_B^+(N), \mu_{B,n})$ and $\mathcal{U}_R(N) = (\mathcal{U}_R(N), \mathcal{U}_R^+(N), \mu_{R,n})$). Apply the functor $\mathcal{U}_B$ to the canonical B-module homomorphism $M \to R \otimes_B M$ given by $x \mapsto 1 \otimes x$ and denote the resulting B-algebra homomorphism by

$$\delta_0\colon \mathcal{U}_B(R \otimes_B M) \to \mathcal{U}_R(R \otimes_B M).$$

Then

$$\delta := \delta_{R\otimes_B M} \circ \delta_0 \colon \mathcal{U}_B(M) \to \mathcal{U}_R(R \otimes_B M)$$

satisfies $\delta(\mu_n(x)) = \hat{\eta}(1 \otimes x)$. We define the B-bilinear map

$$\mathcal{U}_B(M) \times R \to \mathcal{U}_R(R \otimes_B M), (u, r) \mapsto \delta(u) \cdot r$$

and by the universal property of the tensor product we finally obtain

$$\psi \colon \mathcal{U}_B(M) \otimes_B R \to \mathcal{U}_R(R \otimes_B M)$$

given by $u \otimes r \mapsto \delta(u) \cdot r$. For all $r \in R, x \in M$ and $n \in \mathbb{N}$ we have:

$$\begin{aligned}
\psi(\varphi([T_{(r\otimes x,n)}])) &= \psi(r^n \otimes \mu_n(x)) \\
&= \delta(\mu_n(x)) \cdot r^n \\
&= \hat{\eta}_n(1 \otimes x) \cdot r^n \\
&= \hat{\eta}_n(r \otimes x) \\
&= [T_{(r\otimes x,n)}]
\end{aligned}$$

and

$$\begin{aligned}
\varphi(\psi(r \otimes \mu_n(x))) &= \varphi(\delta(\mu_n(x)) \cdot r) \\
&= r \cdot \varphi(\delta(\mu_n(x))) \\
&= r \cdot \varphi(\hat{\eta}_n(1 \otimes x)) \\
&= r \cdot \varphi([T_{(1\otimes x)}]) \\
&= r \otimes \mu_n(x)
\end{aligned}$$

This shows that φ and ψ are inverse to each other since they are R-linear and $\mathcal{U}_R(R \otimes_B M)$ is as a R-algebra generated by elements of the form $[T_{(r\otimes x,n)}]$ as above, furthermore $\mathcal{U}_B(M)$ is as a B-algebra generated by $\mathcal{U}_B^+(M)$. □

Corollary A.17 *Let $\mathcal{H}$ denote the set of all $f \in \mathrm{Hom}_{(B-mod)}(J, \exp(R))$ that satisfy the conditions*

(a) $f(x) = 1 + xT+$ *"terms of higher exponent" for all* $x \in J$.

(b) $f(y) = \sum_{n\geq 0} \gamma_n(y)T^n$ *for all* $y \in I$.

Then for $M = J$ the bijection of Proposition A.15 restricts to a bijection

$$\mathcal{H} \to \mathrm{Hom}_{(B-alg)}(\mathcal{D}_{(A,I,\gamma)}(B, J), R).$$

Proof: It is sufficient to check that for $f \in \mathrm{Hom}_{(A-alg)}(\mathcal{U}_A(M), R)$ the condition (a) is equivalent to $\varphi(F_1) = 0$ and condition (b) is equivalent to $\varphi(F_2) = 0$ with F_1 and F_2 given as in the proof of Theorem A.13. But this is true since

$$\varphi(\mu_1(\iota(x))) = \varphi([T_{(x,n)}]) = c_{x,1} = x = \varphi(x) \text{ for all } x \in J$$

where $c_{x,1}$ is given by $f(x) = \sum_{n \geq 0} c_{x,n} T^n$ and

$$\varphi(\mu_n(\iota(y))) = \varphi([T_{(y,n)}]) = \gamma_n(y) = \varphi(\gamma_n(y)) \text{ for all } y \in I.$$

□

With these tools we are now able to verify the statement we were looking for.

Theorem A.18 *There is an isomorphism*

$$\mathcal{D}_{(A,I,\gamma)}(R, JR) \cong R \otimes_B \mathcal{D}_{(A,I,\gamma)}(B, J).$$

Proof: Set

$$\mathcal{E}' := \exp(R \otimes_B \mathcal{D}_{(A,I,\gamma)}(B, J)).$$

Define

$$f\colon JR \cong R \otimes_B J \to \mathcal{E}' \text{ given by } r \otimes j \mapsto \sum_{n \geq 0} (r \otimes [T_{(j,n)}])T^n$$

in the same manner as in Proposition A.16. We claim that f satisfies condition (a) and (b) from Corollary A.17, i.e. $f \in \mathcal{H}$. Indeed let without loss of generality $x = r \otimes j$ be an element of $R \otimes_B J$. Then

$$f(x) = 1 + (r \otimes \mu_1(j))T + \ldots = 1 + (r \otimes \iota(j))T + \ldots = 1 + (r \otimes j)T + \ldots$$

by the relations due to F_1. This implies (a). For $y \in I$ we have

$$f(y) = \sum_{n \geq 0} (1 \otimes \mu_n(y))T^n = \sum_{n \geq 0} (1 \otimes \gamma_n(y))T^n$$

by the relations due to F_2. This implies (b) and we fix a homomorphism of R-algebras

$$\varphi\colon \mathcal{D}_{(A,I,\gamma)}(R, JR) \to R \otimes_B \mathcal{D}_{(A,I,\gamma)}(B, J)$$

corresponding to f. Now consider the homomorphism of R-algebras

$$\overline{\psi}\colon \mathcal{U}_B(J) \otimes_B R \to \mathcal{U}_R(R \otimes_B J) \to \mathcal{D}_{(A,I,\gamma)}(R, JR)$$

where the first map is the map ψ of Proposition A.16 composed with the canonical projection. We claim that $\overline{\psi}$ factorizes over $R \otimes_B \mathcal{D}_{(A,I,\gamma)}(B, J)$, i.e. $\mathcal{I}_B(J) \otimes_B R \subseteq \ker \overline{\psi}$. For $x \in J$ and $r \in R$ we have

$$\begin{aligned} \overline{\psi}((\iota(x) - x) \otimes r) &= (\delta(\iota(x)) - \delta(x)) \cdot r \\ &= (\delta(\mu_1(x)) - x) \cdot r \\ &= (\hat{\eta}_1(1 \otimes x) - x) \cdot r \\ &= (\iota(1 \otimes x) - x) \cdot r \\ &= (x - x) \cdot r = 0 \end{aligned}$$

since the relation F_1 in $\mathcal{I}_R(R \otimes_B J)$ implies $\hat{\eta}_1(1 \otimes x) = \iota(1 \otimes x)$. For $y \in I$ we have

$$\begin{aligned} \overline{\psi}((\mu_n(\iota(y)) - \gamma_n(y)) \otimes 1) &= (\delta(\mu_n(y)) - \delta(\gamma_n(y))) \cdot r \\ &= (\hat{\eta}_n(1 \otimes y) - \gamma_n(y)) \cdot r \\ &= (\gamma_n(y) - \gamma_n(y)) \cdot r = 0 \end{aligned}$$

since the relation F_2 in $\mathcal{I}_R(R \otimes_B J)$ implies $\hat{\eta}_n(1 \otimes x) = \gamma_n(y)$. This shows that $\overline{\psi}$ can be interpreted as a homomorphism of R-algebras

$$\overline{\psi}\colon R \otimes_B D_{(A,I,\gamma)}(B, J) \to \mathcal{D}_{(A,I,\gamma)}(R, JR)$$

which is inverse to φ. $\square$

Appendix B

Slope filtrations

The formalism of slopes occurs in different areas in mathematics and often is treated adjusted to the associated situation. In contrast to the many incarnations of slope filtrations in the literature André introduced a purely category theoretical approach. Since there appear different slopes (more precisely slope functions) in the course of this thesis it is convenient to introduce the basic results about them in this chapter. As announced the following is extracted from [And09].

Let $\mathcal{C}$ denote an essentially small abelian category and let Γ be a totally ordered (abelian) group such that Γ is divisible. One may always assume $\Gamma = \mathbb{Z} \times \cdots \times \mathbb{Z}$ or $\Gamma = \mathbb{R}_{>0} \times \cdots \times \mathbb{R}_{>0}$ with the lexicographic order.

B.1 Slopes

We denote by $\mathrm{sk}(\mathcal{C})$ a skeleton of $\mathcal{C}$, i.e. a set of representatives for the isomorphism classes in $\mathcal{C}$. Furthermore we assume that there exists

- a rank function $\mathrm{rk}\colon \mathrm{sk}(\mathcal{C}) \to \mathbb{N}$ that maps the zero object to 0 and is additive on short exact sequences, i.e. $\mathrm{rk}(N) = \mathrm{rk}(M) + \mathrm{rk}(P)$ for any short exact sequence $0 \to M \to N \to P \to 0$.
- a slope function $\mu\colon \mathrm{sk}(\mathcal{C}) \setminus \{0\} \to \Gamma$, such that the degree function $\deg := \mu \cdot \mathrm{rk}$ is additive on short exact sequences.

Lemma B.1 *1. For any short exact sequence $0 \to M \to N \to P \to 0$ of non-zero objects in $\mathcal{C}$. Then*

$$\min(\mu(M), \mu(P)) \leq \mu(N) \leq \max(\mu(M), \mu(P))$$

holds and both inequalities are strict unless $\mu(M) = \mu(N) = \mu(P)$.

2. *Let $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$ denote a flag such that $M_i/M_{i-1} \neq 0$ for $1 \leq i \leq r$. Then*

   $$\min\{\mu(M_i/M_{i-1}) \mid 1 \leq i \leq r\} \leq \mu(M) \leq \max\{\mu(M_i/M_{i-1}) \mid 1 \leq i \leq r\}.$$

   Again, both inequalities are strict unless $\mu(M_i/M_{i-1}) = \mu(M)$ for all $1 \leq i \leq r$.

Proof: deg is additive on short exact sequences, hence we obtain

$$\mu(N) = \mu(M)\frac{\mathrm{rk}(M)}{\mathrm{rk}(N)} + \mu(P)\frac{\mathrm{rk}(P)}{\mathrm{rk}(N)}.$$

Set $\alpha := \frac{\mathrm{rk}(M)}{\mathrm{rk}(N)}$ and receive $\frac{\mathrm{rk}(P)}{\mathrm{rk}(N)} = 1 - \alpha$ since rk is additive on short exact sequences. But

$$\min(\mu(M), \mu(P)) \leq \mu(M)\alpha + \mu(P)(1-\alpha) \leq \max(\mu(M), \mu(P))$$

holds for any $\alpha \in [0, 1]$ and we have proven the first part. The second part is proven by induction on r. □

Definition B.2 $0 \neq N \in \mathcal{C}$ *is called (μ-)semistable (resp. (μ-)stable) if $\mu(M) \leq \mu(N)$ (resp. $\mu(M) < \mu(N)$) holds for any subobject $0 \neq M \subsetneq N$.*

Lemma B.3 *Let N be a non-zero object of $\mathcal{C}$.*

1. *N is semistable if and only if $\mu(P) \geq \mu(N)$ holds for any non-zero quotient P of N.*

2. *Let N be semistable and $0 \neq M \subseteq N$ denote a subobject such that $\mu(M) = \mu(N)$ holds. Then M is semistable.*

3. *Let N be semistable and $P \neq 0$ denote a quotient of N such that $\mu(P) = \mu(N)$ holds. Then M is semistable.*

4. *Let N be semistable and $M \neq 0$ be a direct summand of N. Then M is semistable of slope $\mu(M) = \mu(N)$.*

5. *Let $0 \neq M \subseteq N$ be a subobject of minimal rank, such that $\mu(M) \geq \mu(N)$ holds. Then M is semistable.*

6. *Let $P \neq 0$ be a quotient of N of minimal rank, such that $\mu(P) \leq \mu(N)$ hold. Then P is semistable.*

7. *Let $0 \to M \to N \to P \to 0$ be an exact sequence in $\mathcal{C}$ of non-zero objects. If two object in the sequence are semistable, the third is also semistable.*

Proof: Straightforward, see [And09, Lemma 1.3.7.]. □

The following lemma [And09, Lemma 1.3.8.] states that there are no non-trivial morphisms between semistable objects of decreasing slope.

Lemma B.4 *Let M and N be semistable objects of $\mathcal{C}$. $\mu(M) \leq \mu(N)$ holds if there exists a non-zero morphism $f\colon M \to N$.*

Proof: Consider the factorization $M \twoheadrightarrow M/\ker(f) \cong \mathrm{Im}(f) \hookrightarrow N$ of f. Then $\mu(M) \leq \mu(M/\ker(f)) = \mu(\mathrm{Im}(f)) \leq \mu(N)$ holds by the semistability of M and N. □

Definition B.5 *Let N be a non-zero object of $\mathcal{C}$. $0 \neq M \subseteq N$ is a universal destabilizing subobject of N (with respect to μ) if for any non-zero subobject $M' \subseteq N$ the following holds:*

- $\mu(M') \leq \mu(M)$
- *If $\mu(M') = \mu(M)$, then $M' \subseteq M \subseteq N$.*

A universal destabilizing subobject M of $N \in \mathcal{C}$ is semistable by definition and unique. We have the following Lemma ([And09, Lemma 1.3.12.]).

Lemma B.6 *Let N be a non-zero object of $\mathcal{C}$. Then there exists a universal destabilizing object of M.*

Proof: We prove this by induction on $\mathrm{rk}(N)$. If N is already semistable we are already done, in particular the statement is true for the case $\mathrm{rk}(N) = 1$. Assume that N is not semistable and consider all quotients $0 \neq P \neq N$ of N such that $\mu(P) < \mu(N)$. Choose such a P of minimal rank and set $N' := \ker(N \twoheadrightarrow P)$. P is semistable by item 6. of Lemma B.3 and we have $\mathrm{rk}(N') < \mathrm{rk}(N)$ since the rk is additive. We deduce $\mu(P) < \mu(N) < \mu(N')$ from

$$\min(\mu(N'), \mu(P)) < \mu(N) < \max(\mu(N'), \mu(P)).$$

By induction we know that N' has an universal destabilizing subobject M. We claim that M is also an universal destabilizing object for N. It is immediate that $\mu(N) < \mu(N') \leq \mu(M)$. Take $0 \neq M' \subseteq N$ such that $N' \subsetneq M'$. Thus $M' \subseteq N \twoheadrightarrow P$ is nonzero and we obtain $\mu(M') \leq \mu(P) < \mu(N) \leq \mu(M)$ by Lemma B.4. This verifies the claim. □

B.2 Filtrations

Let $\mathcal{C}$ be an abelian category endowed with a rank function, defined as in the previous paragraph. Furthermore let Γ be a totally ordered, uniquely divisible, abelian group. We consider Γ as a category by $\mathrm{Ob}(\Gamma) := \Gamma$ and

$$\mathrm{Hom}_\Gamma(\gamma, \delta) := \begin{cases} \{*\} & \text{if } \gamma \leq \delta \\ \emptyset & \text{if } \gamma > \delta. \end{cases}$$

Then composition of morphisms in Γ is already uniquely determined.

Definition B.7 *A (decreasing) filtration is a functor* $\mathrm{Fil}^\bullet(\cdot)\colon \Gamma^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ *that assigns to an object* (γ, M) *a subobject* $\mathrm{Fil}^\gamma(M)$ *of* M*. We call a filtration*

1. *separated if*
$$\varprojlim_{\gamma \in \Gamma} \mathrm{Fil}^\gamma(M) = 0$$
holds for any M *in* $\mathcal{C}$*.*

2. *exhaustive if*
$$\varinjlim_{\gamma \in \Gamma} \mathrm{Fil}^\gamma(M) = M$$
holds for any M *in* $\mathcal{C}$*.*

3. *left continuous if*
$$\mathrm{Fil}^\gamma(M) = \varinjlim_{\delta < \gamma} \mathrm{Fil}^\delta(M)$$
holds for any $\gamma \in \Gamma$ *and* M *in* $\mathcal{C}$*.*

For a separated, exhaustive and left continuous filtration $\mathrm{Fil}^\bullet(\cdot)$ *and any object* M*, we receive a partition of*

$$\Lambda = (-\infty, \lambda_r] \sqcup \cdots \sqcup (\lambda_2, \lambda_1] \sqcup (\lambda_1, \infty)$$

such that $\mathrm{Fil}^\bullet(\cdot)$ *is constant on each of the intervals above. The values*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_r$$

are called the breaks of $\mathrm{Fil}^\bullet(\cdot)$*. Set:*

$$\mathrm{gr}^i(M) := \mathrm{Fil}^{\lambda_i}(M)/\mathrm{Fil}^{\lambda_{i-1}}(M),$$

$$\deg_{\mathrm{Fil}}\colon \mathrm{sk}(\mathcal{C}) \to \Lambda, M \mapsto \sum_{i=1}^{r} \lambda_i \cdot \mathrm{rk}(\mathrm{gr}^{\lambda_i}(M)), \tag{B.1}$$

$$\mu_{\mathrm{Fil}} \colon \mathrm{sk}(\mathcal{C}) \setminus \{0\} \to \Lambda, M \mapsto \frac{\deg_{\mathrm{Fil}}(M)}{\mathrm{rk}(M)}. \tag{B.2}$$

We call $\mathrm{Fil}^\bullet(\cdot)$ *a slope filtration if*

1. *the filtration on* $\mathrm{Fil}^\lambda(M)$ *is induced by the filtration of* M*, i.e.*

$$\mathrm{Fil}^\eta(\mathrm{Fil}^\lambda(M)) = \begin{cases} \mathrm{Fil}^\eta(M) & \text{if } \eta \geq \lambda \\ \mathrm{Fil}^\lambda(M) & \text{if } \eta \leq \lambda \end{cases}.$$

2. *the filtration on* $M/\mathrm{Fil}^\lambda(M)$ *is induced by the filtration of* M*, i.e.*

$$\mathrm{Fil}^\eta(M/\mathrm{Fil}^\lambda(M)) = \begin{cases} \mathrm{Fil}^\eta(M)/\mathrm{Fil}^\lambda(M) & \text{if } \eta \leq \lambda \\ 0 & \text{if } \eta \geq \lambda \end{cases}.$$

3. μ_{Fil} *is a slope function.*

In the following we assume all filtrations to be exhaustive, separated and left continuous.

Lemma B.8 *Let* $\mathrm{Fil}^\bullet(\cdot)$ *denote a slope filtration with breaks*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_r$$

and let μ *be the corresponding slope function. An object* $0 \neq N$ *of* $\mathcal{C}$ *is semistable (with respect to* μ*) if and only if* $r = 1$.

Proof: We assume $r = 1$ and define $\lambda := \lambda_1$. Then $N = \mathrm{gr}^\lambda(N)$ and $\mu(N) = \lambda$. Now take a subobject $0 \neq M \subseteq N$. By functoriality of $\mathrm{Fil}^\bullet(\cdot)$ we obtain

$$\mathrm{Fil}^\eta(M) \subseteq \mathrm{Fil}^\eta(N) = 0 \text{ for all } \eta > \lambda.$$

Hence the breaks $\eta_1 > \eta_2 > \cdots > \eta_s$ of M satisfy $\lambda \geq \eta_1$. We obtain

$$\lambda \cdot \mathrm{rk}(M) \geq \lambda \cdot \sum_{i=1}^{s} \mathrm{rk}(\mathrm{gr}^{\eta_i}(M)) \geq \mathrm{rk}(M) \cdot \mu(M),$$

in particular $\mu(N) = \lambda \geq \mu(M)$, i.e. N is semistable. Now assume that $r \geq 2$ and prove by induction that N is not semistable. We abbreviate $N_i := \mathrm{Fil}^{\lambda_i}(N)$. We consider the case $r = 2$ and remark that

$$\mu(N_1) = \lambda_1 \text{ and } \mu(N/N_1) = \lambda_2$$

since $\mathrm{Fil}^\bullet(\cdot)$ is a slope filtration (N_1 and N/N_1 carry the induced filtrations). From $\mu(N_1) = \lambda_1 > \lambda_2 = \mu(N/N_1)$ and Lemma B.1 we see that

$$\mu(N/N_1) < \mu(N) < \mu(N_1),$$

in particular N is not semistable. Now assume that $r \geq 2$ and that N is semistable. For any subobject $0 \neq M \subseteq N_1$ we receive

$$\mu(M) = \lambda_1 \cdot \mathrm{rk}(M) \leq \lambda_1 \cdot \mathrm{rk}(N_1) = \mu(N_1).$$

Therefore N_1 as well as N/N_1 are semistable. Consider the induced slope filtration on N/N_1 given by

$$0 \subseteq N_2/N_1 \subseteq \cdots \subseteq N_r/N_1 = N/N_1$$

which has $r-1$ breaks and hence N/N_1 can not be semistable by induction. This is a contradiction and N is not semistable. □

From this proof we get the following additional information.

Corollary B.9 *Let* $\mathrm{Fil}^\bullet(\cdot)$ *denote a slope filtration with breaks*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_r$$

and let μ *be the corresponding slope function. Take an object* N *of* $\mathcal{C}$ *and abbreviate* $N_i := \mathrm{Fil}^{\lambda_i}(N)$ *for* $i = 1, \ldots, r$.

1. *All graded pieces* N_i/N_{i-1} *are semistable.*
2. $\mu(N_1) = \lambda_1 < \mu(N_2/N_1) = \lambda_2 < \cdots < \mu(N_r/N_{r-1}) = \lambda_r$.

Proposition B.10 *Let* $\mathrm{Fil}^\bullet(\cdot)$ *denote a slope filtration with breaks*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_r$$

and let μ *be the corresponding slope function. Take an object* N *of* $\mathcal{C}$ *and abbreviate* $N_i := \mathrm{Fil}^{\lambda_i}(N)$ *for* $i = 1, \ldots, r$. *Then*

$$\mathcal{F} : 0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_r = N$$

is the unique flag (up to unique isomorphism) satisfying:

1. *For all* $i = 1, \ldots, r$ *the quotient* N_i/N_{i-1} *is semistable.*
2. $\mu(N_1) > \mu(N_2/N_1) > \cdots > \mu(N/N_{r-1})$ *holds.*

Furthermore N_i *is the preimage of the universal destabilizing subobject of* N/N_{i-1} *under the canonical projection* $N \twoheadrightarrow N/N_{i-1}$.

Proof: Let M be a universal destabilizing subobject of N and set

$$j := \min\{i = 1, \ldots, r \mid M \subseteq N_i\}.$$

Consider the nonzero morphism

$$M \twoheadrightarrow M/N_{j-1} \hookrightarrow N_j/N_{j-1}$$

between semistable objects, given by the composition of the canonical projection and the natural inclusion. We deduce $\mu(M) \leq \mu(N_j/N_{j-1})$ by Lemma B.4. But since M is a universal destabilizing subobject of N we also have $\mu(N_j/N_{j-1}) \leq \mu(N_1) \leq \mu(M)$ and this implies $j = 1$, hence $M = N_1$. Now proceed inductively. □

Theorem B.11 (Harder-Narasimhan) *The map* $\mathrm{Fil}^{\bullet}(\cdot) \mapsto \mu_{\mathrm{Fil}}$ *given as in* (B.2) *establishes a bijection between slope filtrations and slope function on* $\mathcal{C}$.

Proof: Injectivity immediately follows from Proposition B.10. Let μ be an arbitrary slope function on $\mathcal{C}$ and let N denote an object of N. As indicated in Proposition B.10 we define a flag on N inductively by defining N_i to be the preimage of the universal destabilizing subobject of N under the canonical projection $N \twoheadrightarrow N/N_{i-1}$ (and $N_0 := 0$). Let $M_i \subseteq N/N_{i-1}$ denote an universal destabilizing subobject. Then we receive an exact sequence

$$0 \to N_{i-1} \to N_i \to M_i \to 0.$$

M_i and N_{i-1} are semistable by definition resp. induction and hence all N_i are semistable. The existence of the rank function implies that there are only finitely many N_i, assume $N_r = N$. Set $\lambda_i := \mu(N_i/N_{i-1})$ and

$$\mathrm{Fil}^{\lambda}(N) := N_i \text{ for all } \lambda \in (\lambda_{i+1}, \lambda_i].$$

as well as $\mathrm{Fil}^{\lambda}(N) = N$ for $\lambda \geq \lambda_r$ and $\mathrm{Fil}^{\lambda}(N) = 0$ for $\lambda > \lambda_1$. This filtration is a slope filtration and we obtain surjectivity. □

B.3 Dieudonné-Manin Classification

Since we already introduced the concept of a Harder-Narasimhan filtration the smoothest way to proof the Classification Theorem is the one executed by Y. W. Ding and Y. Ouyang in their short article [DO12]. For all technical details we likewise refer to [DO12].

Let k be a perfect field of characteristic $p > 0$ and denote by F the field of fractions of the ring of Witt vectors $W(k)$. We denote by ν the discrete valuation on F induced by $W(k)$. As usual a φ-module D over F is a finite-dimensional vector space over F together with at σ-semilinear map φ where $\sigma := W(\overline{\sigma})$ and $\overline{\sigma}\colon k \to k$ is the Frobenius map given by $x \mapsto x^p$.

In the following definition we introduce the *Newton slope* μ_N of a φ-module D over F. The rank function is the given by the dimension of D and the degree function is the so called *Newton number* t_N (compare [BC09, Definition 8.1.7.]).

Definition B.12 *Let $D \neq 0$ denote a φ-module over F. Choose a basis of D and denote by A the matrix representing the map φ with respect to this basis. Set*

$$t_N(D) := \nu(\det(A))$$

and

$$\mu_N(D) := \frac{t_N(D)}{\dim_F(D)}.$$

Furthermore D is called isoclinic of slope $\lambda \in \mathbb{Q}$ if there exists a $W(k)$-lattice $M \subseteq D$ such that

$$\varphi^h(M) = p^d M,$$

where $\lambda = \frac{d}{h}$ and $d, h \in \mathbb{Z}$ and $h \geq 1$. (The slope $\lambda \in \mathbb{Q}$ does not depend on the choice of the lattice, which follows from the third point of the subsequent Remark B.13.)

Remark B.13

1. *The Newton number t_N is independent of the choice of a basis. Indeed a change of the basis results in σ-conjugation of the matrix A, i.e. A is replaced by $\sigma(B)AB^{-1}$ for some $B \in \mathrm{GL}(D)$. But*

$$\begin{aligned}\nu(\det(\sigma(B)AB^{-1})) &= \nu(\det(\sigma(B)) + \nu(\det(A)) + \nu(\det(B^{-1})) \\ &= \nu(\sigma(\det(B)) + \nu(\det(A)) - \nu(\det(B)) \\ &= \nu(\det(A)).\end{aligned}$$

2. *t_N is additive on exact sequences, i.e. for any exact sequence*

$$0 \to D_1 \to D \to D_2 \to 0$$

of φ-modules over F the equation $t_N(D) = t_N(D_1) + t_N(D_2)$ holds. This is obvious since det *is multiplicative on such exact sequences.*

3. *Any φ-module D over F, that is isoclinic of slope $\lambda = \frac{d}{h} \in \mathbb{Q}$, has Newton slope $\mu_N(D) = \lambda$. Let M be a $W(k)$-lattice in D such that $\varphi^h(M) = p^d M$ holds and let A denote a matrix representing φ with respect to a basis of M. Then*

$$\begin{aligned}\mu_N(D) &= \frac{1}{h} \cdot t_N(A^h) \cdot \dim_F(D)^{-1} \\ &= \frac{1}{h} \cdot \nu(\det(p^d \mathbb{1})) \cdot \dim_F(D)^{-1} \\ &= \frac{d}{h} = \lambda\end{aligned}$$

4. *The subsequent Lemma B.14 shows that any isoclinic φ-module over F is semistable.*

5. *It is not at all clear yet that a φ-module D over F, which is semistable (with respect to μ_N), is indeed isoclinic of Newton slope $\mu_N(D)$. This fact is a crucial part of the proof of the Classification Theorem of Dieudonné-Manin.*

From the definition we may also draw the following conclusion.

Lemma B.14 1. *Let $0 \to (D_1, \varphi_1) \overset{\alpha}{\to} (D, \varphi) \overset{\beta}{\to} (D_2, \varphi_2) \to 0$ be an exact sequence of φ-modules over F and assume that (D, φ) is isoclinic. Then (D_1, φ_1) and (D_2, φ_2) are isoclinic and $\mu_N(D, \varphi) = \mu_N(D_1, \varphi_1) = \mu_N(D_2, \varphi_2)$.*

2. *Let (D_1, φ_1) and (D_2, φ_2) be isoclinic φ-modules over F of distinct slopes, i.e. $\mu_N(D_1, \varphi_1) \neq \mu_N(D_2, \varphi_2)$ and let $\gamma\colon (D_1, \varphi_1) \to (D_2, \varphi_2)$ be a morphism of φ-modules over F. Then γ is the zero morphism.*

Proof: Let $M \subseteq D$ denote a lattice such that $\varphi^h(M) = p^d M$ holds for suitable $h, d \in \mathbb{Z}$. We see that $\beta(M) \subseteq D_2$ is a lattice and $\varphi_2^h(\beta(M)) = \beta(\varphi^h(M)) = \beta(p^d M) = p^d \beta(M)$, hence (D_2, φ_2) is isoclinic of slope $\frac{d}{h}$. By the second part of Remark B.13 we obtain

$$\begin{aligned}\mu_N(D_1) &= \frac{t_N(D) - t_N(D_2)}{\dim_F(D) - \dim_F(D_2)} \\ &= \frac{t_N(D) - \mu_N(D)\dim_F(D_2)}{\dim_F(D) - \dim_F(D_2)} \\ &= \frac{t_N(D)}{\dim_F(D)} = \mu_N(D)\end{aligned}$$

This proves the first point. Now assume that $\gamma\colon (D_1, \varphi_1) \to (D_2, \varphi_2)$ is non-zero. Then $\mu_N(D_2, \varphi_2) = \mu_N(\mathrm{Im}(\gamma)) = \mu_N(D_1/\ker(\gamma)) = \mu_N(D_1, \varphi_1)$ holds by the first part and contradicts the assumption. □

Now Theorem B.11 automatically provides a filtration for any φ-module D over F. We need to show that this filtration splits and propose that the graded pieces, i.e. the direct summands, are isoclinic of the appropriate slope.

Definition B.15 *Let D be a φ-module over F. Choose a $W(k)$-lattice $M \subseteq D$ and set*

$$M_{h,d} := \bigcap_{n \geq 0} \varphi_{h,d}^{-n}(M)$$

and we call $\mathrm{Fil}_N^\bullet$ given by

$$\mathrm{Fil}_N^\lambda(D) := M_{h,d}[\frac{1}{p}]$$

for $\lambda = \frac{d}{h} \in \mathbb{Q}$ the Newton filtration on D.

Remark B.16 $\mathrm{Fil}_N^\lambda(D)$ *is independent of the choices of the lattice M and the choice of the pair (h, d) and $\mathrm{Fil}^\bullet$ is indeed a filtration. See [DO12, Propostion 2.1.].*

There are two facts that are crucial for the proof of the Classification Theorem but also rather technical. We will source the details out by only stating them and providing a (well-written) reference.

Proposition B.17 *Let D be a φ-module over F.*

1. $\mathrm{Fil}_N^\bullet$ *is a slope filtration and $\mu_{\mathrm{Fil}_N} = \mu_N$.*

2. *Assume that $0 \to D_1 \to D \to D_2 \to 0$ is a short exact sequence of φ-modules. Then*

$$0 \to \mathrm{Fil}_N^\lambda(D_1) \to \mathrm{Fil}_N^\lambda(D) \to \mathrm{Fil}_N^\lambda(D_2) \to 0$$

   is also exact for any $\lambda \in \mathbb{Q}$.

Proof: Let $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ denote the break points of $\mathrm{Fil}_N^\bullet$ and set $D_i := \mathrm{Fil}_N^{\lambda_i}(D)$ for all $i = 1, \ldots, r$. Then D_i/D_{i-1} is isoclinic of slope λ_i by

[DO12, Proposition 2.6.] and hence $\mu_N(D_i/D_{i-1}) = \lambda_i$ by the third part of Remark B.13. Since t_N is additive on exact sequences we obtain

$$\begin{aligned}\mu_N(D) &= (\sum_{i=1}^{r} t_N(D_i/D_{i-1}))/\dim_F(D)\\ &= (\sum_{i=1}^{r} \lambda_i \cdot \dim_F(D_i/D_{i-1}))/\dim_F(D)\\ &= \mu_{\mathrm{Fil}_N}(D).\end{aligned}$$

In particular μ_{Fil_N} is a slope function and hence $\mathrm{Fil}_N^\bullet$ is a slope filtration. The second part is subject of [DO12, Proposition 2.8.]. □

Theorem B.18 (Dieudonné-Manin) *Let D be a φ-module over F. Then*

1. $$D = \bigoplus_{i=1}^{r} D_{\lambda_i},$$
   where $D_{\lambda_i} := \mathrm{Fil}_N^{\lambda_i}(D)/\mathrm{Fil}_N^{\lambda_{i-1}}(D)$, $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ are the break points of $\mathrm{Fil}_N^\bullet$ and $\lambda_0 > \lambda_1$ is arbitrary.

2. *There exists a $W(k)$-lattice $M \subseteq D_{\lambda_i}$ such that $\varphi^{h_i}(M) = p^{d_i}M$ where $\lambda_i = \frac{d_i}{h_i} \in \mathbb{Q}$.*

Proof: The second part of the theorem follows directly from Proposition B.17. Set $D_i := \mathrm{Fil}_N^{\lambda_i}(D)$ for all $i = 1, \ldots, r$. Since φ is bijective we may consider D as φ^{-1}-module over F (where φ^{-1} is semilinear with respect to σ^{-1}). We denote the Newton slope with respect to φ^{-1} by $\mu_N^{'}$ and remark the following. A φ-module is isoclinic of slope λ if and only if it is isoclinic of slope $-\lambda$ considered as an φ^{-1}-module. Denote the Harder-Narasimhan filtration corresponding to $\mu_N^{'}$ by $\mathrm{Fil}_N^{'\bullet}$ and set $D_i^{'} := \mathrm{Fil}_N^{'i}(D)$. Then the chain

$$0 = D_0^{'} \subseteq D_1^{'} \subseteq \cdots \subseteq D_s^{'} = D$$

has isoclinic quotients $D_i^{'}/D_{i-1}^{'}$ of slope $\mu_N(D_i^{'}/D_{i-1}^{'}) = -\lambda_i^{'}$ for all $i = 1, \ldots, s$. Now we prove the following statements by induction on s:

- $\mathrm{Fil}_N^\lambda(D) = D_{s-i+1}$ for all $\lambda \in (-\lambda_{i-1}^{'}, -\lambda_i^{'}]$ and $i = 1, \ldots, s$.
- $\mathrm{Fil}_N^\lambda(D) = 0$ for all $\lambda > -\lambda_s$.
- $-\lambda_i^{'} = \lambda_{r-i+1}$ for all $i = 1, \ldots, s$.

- $D \cong \bigoplus_{i=1}^{s} D_i^{'}/D_{i-1}^{'}$.

In the case that $s = 1$ all the conditions are satisfied since D is isoclinic of slope $\lambda_1 = -\lambda_1^{'}$. By induction we know that $\mathrm{Fil}^{-\lambda_s^{'}}(D_{s-1}^{'}) = 0$ and deduce

$$\mathrm{Fil}_N^{-\lambda_s^{'}}(D) = \mathrm{Fil}_N^{-\lambda_s^{'}}(D)/\mathrm{Fil}_N^{-\lambda_s^{'}}(D_{s-1}^{'}) = \mathrm{Fil}_N^{-\lambda_s^{'}}(D/D_{s-1}^{'}) = D/D_{s-1}^{'} \neq 0$$

by applying Proposition B.17 and the fact that $D/D_{s-1}^{'}$ is isoclinic of slope $-\lambda_s^{'}$. Similarly

$$\mathrm{Fil}_N^{\lambda}(D) = \mathrm{Fil}_N^{\lambda}(D)/\mathrm{Fil}_N^{\lambda}(D_{s-1}^{'}) = \mathrm{Fil}_N^{\lambda}(D/D_{s-1}^{'}) = 0$$

holds for all $\lambda > -\lambda_s^{'}$. This provides $\lambda_1 = -\lambda_s^{'}$ and therefore $\mathrm{Fil}_N^{-\lambda_s^{'}}(D) = D_1$. In particular we see that $D_1 \cong D/D_{s-1}^{'}$ and $0 \to D_{s-1}^{'} \to D \to D/D_{s-1}^{'} \to 0$ splits. We finish the proof with the remark that for all $i = 1, \dots, r$ we have an isomorphism

$$D_i^{'}/D_{i-1}^{'} \cong D_{r-i+1}/D_{r-i}.$$

□

Corollary B.19 *Any φ-module D over F, which is semistable (with respect to μ_N), is indeed isoclinic of Newton slope $\mu_N(D)$.*

This corollary reveals a 'down to earth' meaning of the Classification Theorem. Take a φ-module D of dimension d over F and pick a basis to form the representing matrix A of φ. If the determinant of A is contained in $W(k)^\times$, then we may find a basis such that the representing matrix $\sigma(B)AB^{-1}$ (for some $B \in \mathrm{GL}_d(W(k))$) is contained in $\mathrm{GL}_d(W(k))$.

Definition B.20 *Let $\lambda = \frac{s}{r} \in \mathbb{Q}$ be a reduced fraction with $r \geq 1$. We call the vector space F^r endowed with the semilinear map given by*

$$\varphi(e_i) := \begin{cases} e_{i+1} \text{ for all } 1 \leq i \leq r-1 \\ p^s \cdot e_1 \text{ for } i = r \end{cases}$$

the standard isocrystal $S_{\frac{s}{r}}$ of slope $\lambda = \frac{s}{r}$, which is an isoclinic φ-module over F of slope λ, that does not contain non-trivial sub objects in the category of φ-modules.

Lemma B.21 *Let D_1 and D_2 denote isoclinic φ-modules over F of slope $\frac{s_1}{r_1}$ and $\frac{s_2}{r_2}$. The tensor product $D_1 \otimes_F D_2$ is endowed with the structure of a φ-module over F given by*

$$\varphi := \varphi_1 \otimes \varphi_2 \colon d_1 \otimes d_2 \mapsto \varphi_1(d_1) \otimes \varphi_2(d_2).$$

Then $D := D_1 \otimes_F D_2$ is isoclinic of Newton slope $\frac{r_1 s_2 + r_2 s_1}{r_1 r_2}$.

Proof: There exist $W(k)$-lattices $M_1 \subseteq D_1$ and $M_2 \subseteq D_2$ such that

$$\varphi_!^{r_1}(M_1) = p^{s_1} M_1 \text{ and } \varphi_2^{r_2}(M_2) = p^{s_2} M_2.$$

Then $M := M_1 \otimes_{W(k)} M_2$ is a $W(k)$-lattice in D and we obtain

$$\begin{aligned}\varphi^{r_1 r_2}(M) &= \varphi_1^{r_1 r_2}(M_1) \otimes_{W(k)} \varphi_2^{r_1 r_2}(M_2) \\ &= p^{r_2 s_1} M_1 \otimes_{W(k)} p^{r_1 s_2} M_2 \\ &= p^{r_2 s_1 + r_1 s_2} M.\end{aligned}$$

□

Theorem B.22 *Assume that the residue field of F is algebraically closed. Let D be an isoclinic φ-module over F of slope $\lambda \in \mathbb{Q}$, where $\lambda = \frac{s}{r}$ is a reduced fraction. Then D is isomorphic to a direct sum of (finitely many) copies of the standard isocrystal $S_{\frac{s}{r}}$.*

Proof: See [Ked10, Remark 14.6.5.]. □

Applying this and a comparison of dimensions delivers:

Corollary B.23 *Assume that the residue field of F is algebraically closed and abbreviate $\otimes := \otimes_F$. Let $r_1, r_2, s_1, s_2 \in \mathbb{Z}$ such that $r_1, r_2 \geq 1$ and $(r_1, s_1) = 1 = (r_2, s_2)$ holds. We denote the reduced fraction representing $\frac{s_1}{r_1} + \frac{s_2}{r_2}$ by $\frac{s}{r}$. Then there exists an isomorphism*

$$S_{\frac{s_1}{r_1}} \otimes S_{\frac{s_2}{r_2}} \cong S_{\frac{s}{r}}^{\oplus \frac{r_1 r_2}{r}}$$

of φ-modules over F. In particular we have the following isomorphisms of φ-modules over F:

1. $S_{\frac{s}{r}} \otimes S_{\frac{r-s}{r}} \cong S_1^{\oplus r^2}$ *for $r, s \in \mathbb{Z}$ such that $r \geq 1$ and $(r, s) = 1$.*
2. $S_{\frac{1}{r}}^{\otimes s} \cong S_{\frac{s}{r}}^{\oplus r^{s-1}}$ *for $r, s \in \mathbb{Z}$ such that $r, s \geq 1$ and $(r, s) = 1$.*
3. $S_{-\frac{1}{r}}^{\otimes -s} \cong S_{\frac{s}{r}}^{\oplus r^{-s-1}}$ *for $r, s \in \mathbb{Z}$ such that $r \geq 1, s \leq -1$ and $(r, s) = 1$.*

Proof: The φ-module $S_{\frac{s_1}{r_1}} \otimes S_{\frac{s_2}{r_2}}$ has dimension $r_1 \cdot r_2$, Newton slope $\frac{s}{r}$ and is isoclinic by Lemma B.21. By Theorem B.22 it is isomorphic to a direct sum of copies of $S_{\frac{s}{r}}$, which has dimension r. We compare the dimensions and see that the number of copies is $\frac{r_1 r_2}{r}$. The isomorphism stated in 1. is a special case of this. We prove the isomorphism in 2. by induction. For $s = 1$ there is nothing to show. Assume that the statement holds for $s - 1$ and see that

$$\begin{aligned} S_{\frac{1}{r}}^{\otimes s} &\cong S_{\frac{1}{r}}^{\otimes(s-1)} \otimes S_{\frac{1}{r}} \\ &\cong S_{\frac{s-1}{r}}^{\oplus r^{s-2}} \otimes S_{\frac{1}{r}} \\ &\cong (S_{\frac{s-1}{r}} \otimes S_{\frac{1}{r}})^{\oplus r^{s-2}} \\ &\cong S_{\frac{s}{r}}^{\oplus r^{s-1}} \end{aligned}$$

The isomorphism in 3. is proved the same way. □

Bibliography

[And09] Y. Andre. Slope Filtrations. In *Confluentes Mathematici*, volume 1, chapter Slope Filtrations, pages 1–85. World Scientific Publishing Company, 2009.

[BC09] O. Brinon and B. Conrad. CMI Summer School Notes on p-adic Hodge Theory, 2009.

[Ber02] L. Berger. Représentations p-adiques et équations différentielles. In *Inventiones mathematicae*, volume 148, pages 219–284. Springer-Verlag Berlin Heidelberg 2002, May 2002.

[BO78] P. Berthelot and A. Ogus. *Notes on Crystalline Cohomology*. Princeton University Press and University of Tokio Press, 1978.

[Bou71] N. Bourbaki. *General Topology, Chapters 1-4*. Elements of Mathematics. Springer-Verlag Berlin Heidelberg, 1971.

[Bou74] N. Bourbaki. *Algebra I, Chapters 1-3*. Elements of Mathematics. Springer-Verlag Berlin Heidelberg, 1974.

[Bou81] N. Bourbaki. *Algebra II, Chapters 4-7*. Elements of Mathematics. Springer-Verlag Berlin Heidelberg, 1981.

[CF00] P. Colmez and J.-M. Fontaine. Constructions des representations p-adiques semi-stables. *Inventiones mathematicae 140*, 2000.

[DM82] P. Deligne and J.S. Milne. Lecture Notes in Mathematics. In *Hodge Cycles, Motives, and Shimura Varieties*, volume 900, chapter Tannakian Categories, pages 101–228. Springer-Verlag Berlin Heidelberg, 1 edition, 1982.

[DO12] Y.W. Ding and Y. Ouyang. A Simple Proof of Dieudonné-Manin Classification Theorem. In *Acta Mathematica Sinica, English Series*, volume 28, pages 1553–1558. August 2012.

[FF11] L. Fargues and J.-M. Fontaine. *AMS/IP Studies in Advanced Mathematics*, volume 51, chapter Vector bundles and p-adic Galois representations. American Mathematical Society and International Press, 2011.

[Fon90] J.-M. Fontaine. *The Grothendieck Festschrift*, volume II, chapter Represenstations p-adiques des corps locaux, pages 249–310. Birkhäuser, 1990.

[Fon94a] J.-M. Fontaine. Le corps des périodes p-adiques. In *Periodes p-adiques*, volume 223 of *Astérisque*, pages 59–111. Scoiété Mathématique de France, 1994.

[Fon94b] J.-M. Fontaine. Représentations p-adiques semi-stables. In *Periodes p-adiques*, volume 223 of *Astérisque*, pages 113–184. Scoiété Mathématique de France, 1994.

[Fon00] J.-M. Fontaine. Representations de de Rham et representations semi-stables, 2000. Orsay preprint number 2004-12, avaiable online at https://www.math.u-psud.fr/~biblio/ppo/2004/fic/ppo_2004_12.pdf.

[GD64] A. Grothendieck and J. A. Dieudonné. Eléments de Géométrie Algébrique IV. *lHÉS*, 20(24):5–259, 1964.

[Hen00] G. Henniart. Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique. In *Inventiones mathematicae*, volume 139, pages 439–455. Springer-Verlag Berlin Heidelberg, 2000.

[HT01] M. Harris and R. Talyor. The geometry and cohomology of some simple Shimura varieties. In *Annals of Mathematics Studies*, volume 151. Princeton University Press, 2001.

[Ked10] K. S. Kedlaya. *p-adic Differential Equations*, volume 125 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2010.

[ML78] S. Mac Lane. *Categories for the Working Mathematician*. Number 5 in Graduate Texts in Mathematics. Springer-Verlag New York, 1978.

[Neu86] J. Neukirch. *Class Field Thoery*, volume 280 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg, 1986.

[Neu99] J. Neukirch. *Algebraic Number Theory*, volume 322 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg, 1999.

[Que76] B. v. Querenburg. *Mengentheoretische Topologie*. Springer-Lehrbuch. Springer-Verlag Berlin Heidelberg, 1976.

[Rob63] N. Roby. *Lois polynomes et lois formelles en theorie des modules*. Annales Scientifiques de l'E.N.S. 1963.

[RZ00] L. Ribes and P. Zaleski. Profinite Groups. In *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics*, volume 40 of *3*. Springer-Verlag Berlin Heidelberg, 2000.

[Sch11] P. Scholze. Perfectoid Spaces. In *Publications mathématiques de l'IHÉS*, volume 116, pages 245–313. IHES and Springer-Verlag 2012, 2011.

[Sch17] P. Schneider. *Galois Representations and* (φ, Γ)*-Modules*. Cambridge Studies in Advanced Mathematics, 2017.

[Ser79] J. P. Serre. *Local Fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag New York, 1979.

[Tat67] J.T. Tate. p-Divisible Groups. In *Proceedings of a Conference on Local Fields*, pages 158–183. Springer, Berlin, Heidelberg, 1967.

[Tat79] J. Tate. Number Theoretic Background. *Proceedings of Symposia in Pure Mathematics*, 33, 1979.

[Was82] L. C. Washington. *Introduction to Cyclotomic Fields*, volume 83 of *Graduate Texts in Mathematics*. Springer-Verlag New York, 1982.