J. reine angew. Math. **738** (2018), 313–328
DOI 10.1515/crelle-2015-0053

Journal für die reine und angewandte Mathematik

Revisiting homogeneous spaces with positive curvature

By *Burkhard Wilking* at Münster and *Wolfgang Ziller* at Philadelphia

Abstract. It was recently observed by M. Xu and J. Wolf that there is a gap in Berard Bergery's classification of odd-dimensional positively curved homogeneous spaces. Since this classification has been used in other papers as well, we give a modern, complete and self-contained proof (in odd as well as even dimensions), confirming that there are indeed no new examples.

The classification of compact simply connected homogeneous spaces of positive curvature is now almost 40 years old. It has been accomplished in a series of papers by M. Berger, N. Wallach, S. Aloff–N. Wallach, and L. Bérard Bergery [1–3, 11], with an omission in [3] as observed in [12]. As was recently observed by J. Wolf and M. Xu [15], there is a gap in Bérard Bergery's classification of odd-dimensional positively curved homogeneous spaces in the case of the Stiefel manifold $\mathsf{Sp}(2)/\mathsf{U}(1) = \mathrm{SO}(5)/\mathrm{SO}(2)$. Since this classification has been used in several other papers, for example, in the classification of positively curved cohomogeneity one manifolds in [6] and positively curved polar manifolds in [4], it seems desirable to correct this situation. We thus present here a modern complete and self-contained proof of the classification, confirming that there are indeed no new examples. To be more precise we will reprove the following:

Theorem. *Suppose a compact connected Lie group* $\bar{\mathsf{K}}$ *acts isometrically, effectively and transitively on a simply connected manifold of positive sectional curvature with stabilizer group* $\bar{\mathsf{H}}$. *Then the pair* $(\bar{\mathsf{K}}, \bar{\mathsf{H}})$ *is isomorphic to* $(\mathsf{K}/\mathsf{C}, \mathsf{H}/\mathsf{C})$ *for one of the triples* $(\mathsf{K}, \mathsf{H}, \mathsf{C})$ *in Tables* 1 *or* 2.

As far as the embeddings of H in K are concerned, $\mathsf{Sp}(2)\mathsf{S}^1$ is the normalizer of the subgroup $\mathsf{Sp}(2) \subset \mathsf{SU}(5)$ embedded by the four-dimensional representation, $\mathsf{Sp}(1)_{\max}$ is the unique three-dimensional maximal subgroup of $\mathsf{Sp}(2)$, and in the third example $\mathsf{U}(2)$ is the normalizer of $\Delta\mathsf{SU}(2) \subset \mathsf{SU}(2) \times \mathsf{SO}(3) \subset \mathsf{K}$. The last example is just an S^1-extension of the previous one and they are the Aloff–Wallach spaces [1], with

$$\mathsf{T}(p,q) := \{\mathrm{diag}(z^p\zeta, z^q, \bar{z}^{p+q}) : z, \zeta \in \mathsf{S}^1\} \quad \text{for } p \geq q \geq 1, \gcd(p,q) = 1.$$

The authors were supported by grants from the DFG (567049) and NSF (1112913).

K	H	K/H	Kernel C	N(H)/H
SU(5)	$\mathsf{Sp}(2)\cdot\mathsf{S}^1$	B^{13}	$\mathbb{Z}_5$	$\{e\}$
Sp(2)	$\mathsf{Sp}(1)_{\max}$	B^7	$\mathbb{Z}_2$	$\{e\}$
SU(3)× SO(3)	U(2)	$W^7_{1,1}$	$\mathbb{Z}_3$	$\{e\}$
SU(3)	T^2	W^6	$\mathbb{Z}_3$	S_3
Sp(3)	$\mathsf{Sp}(1)^3$	W^{12}	$\mathbb{Z}_2$	S_3
F_4	Spin(8)	W^{24}	$\{e\}$	S_3
SU(3)	$\mathsf{S}^1=\mathrm{diag}(z^p,z^q,\bar{z}^{p+q})$	$W^7_{p,q}$	$\mathbb{Z}_3$ if $p\equiv q \bmod 3$	S^1 if $p\neq q$
	$p\geq q\geq 1$, $\gcd(p,q)=1$		$\{e\}$ if $p\not\equiv q \bmod 3$	SO(3) if $p=q$
U(3)	$\mathsf{T}^2(p,q)$	$W^7_{p,q}$	$\mathbb{Z}_{p+2q}$	S^1

Table 1. Homogeneous spaces $M^n=\mathsf{K}/\mathsf{H}$ with positive sectional curvature, which are not diffeomorphic to rank 1 symmetric spaces.

K	H	K/H	Kernel C	N(H)/H
$\mathsf{SO}(n+1)$	$\mathsf{SO}(n)$	$\mathbb{S}^n$	$\{e\}$	$\mathbb{Z}_2$ (for $n\geq 2$)
$\mathsf{SU}(n+1)$	$\mathsf{SU}(n)$	$\mathbb{S}^{2n+1}$	$\{e\}$	S^1 (for $n\geq 2$)
$\mathsf{U}(n+1)$	$\mathsf{U}(n)$	$\mathbb{S}^{2n+1}$	$\{e\}$	S^1
$\mathsf{Sp}(n+1)$	$\mathsf{Sp}(n)$	$\mathbb{S}^{4n+3}$	$\{e\}$	S^3
$\mathsf{Sp}(n+1)\mathsf{Sp}(1)$	$\mathsf{Sp}(n)\Delta\mathsf{Sp}(1)$	$\mathbb{S}^{4n+3}$	$\Delta\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathsf{Sp}(n+1)\mathsf{U}(1)$	$\mathsf{Sp}(n)\Delta\mathsf{U}(1)$	$\mathbb{S}^{4n+3}$	$\Delta\mathbb{Z}_2$	S^1
Spin(9)	Spin(7)	$\mathbb{S}^{15}$	$\{e\}$	$\mathbb{Z}_2$
Spin(7)	G_2	$\mathbb{S}^7$	$\{e\}$	$\mathbb{Z}_2$
G_2	SU(3)	$\mathbb{S}^6$	$\{e\}$	$\mathbb{Z}_2$
$\mathsf{SU}(n+1)$	$\mathsf{U}(n)$	$\mathbb{CP}^n$	$\mathbb{Z}_{n+1}$	$\{e\}$ (for $n\geq 2$)
$\mathsf{Sp}(n+1)$	$\mathsf{Sp}(n)\mathsf{Sp}(1)$	$\mathbb{HP}^n$	$\mathbb{Z}_2$	$\{e\}$ (for $n\geq 2$)
$\mathsf{Sp}(n+1)$	$\mathsf{Sp}(n)\mathsf{U}(1)$	$\mathbb{CP}^{2n+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
F_4	Spin(9)	$\mathrm{Ca}\mathbb{P}^2$	$\{e\}$	$\{e\}$

Table 2. Transitive actions on rank 1 symmetric spaces.

The first three examples are the only ones in the list which admit a normal homogeneous metric of positive sectional curvature. The first two of those were discovered by Berger [3], while the third was added in [12] and is diffeomorphic to the Aloff–Wallach space $W_{1,1}$. The Wallach flag manifolds W^6, W^{12} and W^{24} are the only even-dimensional positively curved simply connected homogeneous spaces apart from the rank 1 symmetric space, [11]. We will not reprove that these spaces have positive curvature (see e.g. [16] for details). The group C is defined as the intersection of H with the center of K and corresponds to the kernel of the action of K on K/H. The normalizer N(H) of H in K can be determined by a standard computation and we keep track of the isomorphism type of N(H)/H in the last column, where S_3 stands for the permutation group with six elements.

Here we should add that Spin(8) acts in three different ways transitively on $\mathbb{S}^7$, but up to outer automorphisms of Spin(8) the actions are isomorphic.

A non-simply connected homogeneous space arises from a simply connected space K/H by replacing H by a finite extension $\hat{\mathsf{H}} \subset N(\mathsf{H})$. Thus the following corollary can be viewed as the classification of non-simply connected homogeneous spaces of positive sectional curvature.

Corollary. *Let* $(\mathsf{K}, \mathsf{H}, N(\mathsf{H})/\mathsf{H})$ *be one of the triples in Tables* 1 *or* 2*. If* K/H *is not a Wallach flag manifold, then all finite subgroups of* $N(\mathsf{H})/\mathsf{H}$ *give rise to a finite extension* $\hat{\mathsf{H}}$ *of* H *such that some positively curved metric descends to a* K*-invariant metric of* $\mathsf{K}/\hat{\mathsf{H}}$*. If* K/H *is a Wallach flag manifold, then only the* $\mathbb{Z}_2$*-extensions of* H *allow for some positively curved metric to descend.*

Using the description in [12] of the $\mathsf{SU}(3)$-equivariant principal bundle

$$\mathrm{SO}(3) \to W^7_{1,1} \to \mathbb{C}\mathbb{P}^2,$$

Shankar [9] was the first to observe that any finite subgroup $\mathsf{F} \subset \mathrm{SO}(3)$ can be realized as fundamental group of a positively curved homogeneous space $W^7_{1,1}/\mathsf{F}$.

Similarly, one can use the $\mathsf{Sp}(n+1)$ equivariant S^3-principal bundle

$$\mathsf{S}^3 \to \mathbb{S}^{4n+3} \to \mathbb{H}\mathbb{P}^n$$

to realize any finite subgroup $\mathsf{F} \subset \mathsf{S}^3$ as fundamental groups of a homogeneous space form $\mathbb{S}^{4n+3}/\mathsf{F}$.

By the corollary it is clear that any positively curved homogeneous spaces with a non-cyclic fundamental group must be equivariantly diffeomorphic to one of these. We should mention that the full isometry group of the examples in Table 1 was determined in [10] and various fundamental groups of *locally* homogeneous quotients have been exhibited [5] although a classification is open.

Except for the Wallach flag manifolds, all the examples have positively curved metrics which are $\mathrm{Ad}_{N(\mathsf{H})}$-invariant and thus the corollary is immediate for these examples. In the remaining cases one just has to use in addition that the fundamental group of an even-dimensional positively curved manifold has at most two elements by the Synge Lemma. The three $\mathbb{Z}_2$ subgroups of S_3 are conjugate and thus up to conjugation there is only one $\mathbb{Z}_2$-extension of H for the Wallach flag manifolds.

The rest of the paper is devoted to the proof of the theorem. The even-dimensional case is treated in Section 2 and the odd-dimensional case in Sections 3, 4 and 5. We explain the strategy in more detail at the end of the following section.

1. Obstructions to positive curvature

We will classify compact simply connected Riemannian homogeneous spaces K/H with positive sectional curvature. We can thus assume that K and H are compact and connected, and that the normal subgroup common to both is at most finite. We fix a biinvariant metric Q on the Lie algebra $\mathfrak{k}$ of K and let $\mathfrak{p}$ denote the orthogonal complement of the subalgebra $\mathfrak{h}$ in $\mathfrak{k}$. The K-invariant metrics of K/H are in one-to-one correspondence with positive definite selfadjoint endomorphisms $G\colon \mathfrak{p} \to \mathfrak{p}$ which commute with Ad_{H}. Indeed, one can use G to define a scalar product on $\mathfrak{p}$ by putting $\langle x, y\rangle = Q(Gx, y)$ for $x, y \in \mathfrak{p}$ and extend it equivariantly to K/H. We will implicitly assume that some G has been chosen. The following criteria for finding planes with zero or non-positive curvature is used as an obstruction in our classification.

Lemma 1.1. *The following statements hold.*

(a) *If $x, y \in \mathfrak{p}$ are linearly independent eigenvectors of G with $[x, y] = 0$, then they generate a zero curvature plane in K/H.*

(b) *Let $x \in \mathfrak{p}$ be an eigenvector to the smallest eigenvalue λ of G and assume we can find a linearly independent vector $z \in \mathfrak{p}$ with $[x, z] = 0$. If we put $y = G^{-1}z$, then x, y generate a plane of non-positive curvature in K/H.*

Proof. One can express the formula for the sectional curvature of the homogeneous metric in terms of the biinvariant metric (see e.g. [8] or [7]) as follows:

$$\begin{aligned}\langle R(x, y)y, x\rangle = {}& Q(B_-(x, y), [x, y]) - \frac{3}{4}Q(G[x, y]_{\mathfrak{p}}, [x, y]_{\mathfrak{p}})\\ &+ Q(B_+(x, y), G^{-1}B_+(x, y)) - Q(B_+(x, x), G^{-1}B_+(y, y)),\end{aligned}$$

where $B_\pm(x, y) = \frac{1}{2}([x, Gy] \mp [Gx, y])$ and $[x, y]_{\mathfrak{p}}$ is the Q-orthogonal projection of $[x, y]$ to $\mathfrak{p}$. This clearly implies (a).

For part (b), observe that $B_+(x, y) \in \mathfrak{p}$ for all $x, y \in \mathfrak{p}$. Indeed, since ad_v commutes with G for $v \in \mathfrak{h}$, this well-known fact (see e.g. [7, p. 624] or [2, p. 62]) follows from

$$\begin{aligned}Q([x, Gy], v) &= -Q(x, \mathrm{ad}_v\, Gy)\\ &= -Q(x, G\, \mathrm{ad}_v\, y)\\ &= -Q(Gx, [v, y])\\ &= Q([Gx, y], v).\end{aligned}$$

If x, y, z are as specified, then $[x, Gy] = 0$ and hence

$$B_-(x, y) = \frac{1}{2}\lambda[x, y], \quad B_+(x, y) = -\frac{1}{2}\lambda[x, y], \quad B_+(x, x) = 0.$$

Thus we also have $[x, y] \in \mathfrak{p}$, and x, y are linearly independent since $Gx = \lambda x$ and $Gy = z$ are. Altogether

$$\begin{aligned}\langle R(x, y)y, x\rangle &= \frac{1}{2}\lambda Q([x, y], [x, y]) + \frac{1}{4}\lambda^2 Q([x, y], G^{-1}[x, y]) - \frac{3}{4}Q(G[x, y], [x, y])\\ &\le \left(\frac{1}{2} + \frac{1}{4} - \frac{3}{4}\right)\lambda\|[x, y]\|_Q^2 = 0,\end{aligned}$$

where we used the inequalities

$$Q(Gu, u) \ge \lambda\|u\|_Q^2 \quad \text{and} \quad Q(G^{-1}u, u) \le \frac{1}{\lambda\|u\|_Q^2}$$

for all $u \in \mathfrak{p}$. □

The lemma can also be used to give a new proof for the following essential obstruction.

Lemma 1.2 (Berger). *If* K/H *is a positively curved* n*-dimensional homogeneous space, then* $\operatorname{rank}\mathsf{K} = \operatorname{rank}\mathsf{H}$ *if* n *is even, and* $\operatorname{rank}\mathsf{K} = \operatorname{rank}\mathsf{H} + 1$ *if* n *is odd.*

Proof. Consider first the special case of a trivial group H. Then K is endowed with a left invariant positively curved metric. In our above notation, if $x \in \mathfrak{p} = \mathfrak{k}$ is an eigenvector to the minimal eigenvalue of G, then by Lemma 1.1 (b) every vector $z \in \mathfrak{k}$ that commutes with x is linearly dependent to x. Thus $\operatorname{rank}(\mathsf{K}) \leq 1$.

We now use the following well-known fact, which will also be a crucial tool for us later on. Let $\mathsf{L} \subset \mathsf{H} \subset \mathsf{K}$ and $C(\mathsf{L})_0$ the identity component of the centralizer of L in K. Then $C(\mathsf{L})_0$ acts transitively on the component of the fixed point set $\operatorname{Fix}(L)_0 \subset \mathsf{K}/\mathsf{H}$ through $e\mathsf{H}$, as one easily sees by computing the tangent space of the orbit $C(\mathsf{L})_0 \cdot e\mathsf{H}$. Thus

$$\operatorname{Fix}(\mathsf{L})_0 = C(\mathsf{L})_0 / C(\mathsf{L})_0 \cap \mathsf{H}$$

is a totally geodesic submanifold of K/H and hence has positive curvature.

We now apply this to a maximal torus $\mathsf{T} \subset \mathsf{H}$. Then $C(\mathsf{T})_0 \cap \mathsf{H} = \mathsf{T}$ and hence $C(\mathsf{T})_0/\mathsf{T}$ acts transitively and freely on $\operatorname{Fix}(\mathsf{T})_0$. Thus $C(\mathsf{T})_0/\mathsf{T}$ admits a positively curved left invariant metric and by the above special case $\operatorname{rank}(C(\mathsf{T})_0/\mathsf{T}) \leq 1$ or equivalently $\operatorname{rank}(\mathsf{K}) \leq \operatorname{rank}(\mathsf{H})+1$. Since $\dim(\mathsf{L})-\operatorname{rank}(\mathsf{L})$ is an even number for any compact Lie group L, the lemma follows. □

By [13], if a group K of the form $\mathrm{SO}(n), \mathsf{SU}(n), \mathsf{Sp}(n)$ acts isometrically on a positively curved manifold in such a way that the principal isotropy group contains a $k \times k$-block of K with $k \geq 3$, then the underlying manifold is covered by a manifold which is homotopy equivalent to a rank 1 symmetric space. In the homogeneous case one can strengthen it as follows, which will be our main tool in the classification since it allows one to proceed by induction on the dimension of the Lie group. Although this result follows from [13] and the classification of homogeneous spaces homotopy equivalent to a rank 1 symmetric space, we give here a simple proof in the homogeneous case. In the following we will use the terminology lower $k \times k$-block to denote the $k \times k$ submatrix contained in the last k rows and last k columns, and similarly for the upper $k \times k$-block.

Lemma 1.3 (Block Lemma). *Let* $\mathsf{K} \in \{\mathrm{SO}(n), \mathsf{SU}(n), \mathsf{Sp}(n)\}$ *and assume that a connected proper subgroup* H *contains the lower* $k\times k$*-block of* K *with* $k \geq 3$ *if* $\mathsf{K} \in \{\mathrm{SO}(n), \mathsf{SU}(n)\}$ *and* $k \geq 2$ *if* $\mathsf{K} = \mathsf{Sp}(n)$. *If* K/H *admits an* K*-invariant positively curved metric, then* H *contains a group conjugate to a lower* $(n-1) \times (n-1)$*-block and* (K, H) *is one of the pairs listed in Table* 2.

Proof. We may assume that k is chosen maximal among all groups which are conjugate to H. We then let L_k denote the lower $k \times k$-block, $N(\mathsf{L}_k)$ its normalizer and let $\mathfrak{q} \subset \mathfrak{k}$ denote the orthogonal complement of the Lie algebra of $N(\mathsf{L}_k)$. An element $v \in \mathfrak{q}$ is determined by its entries in the upper right $(n-k) \times k$ corner and we say v is in the i-th row if the entries in the other rows are zero, $i = 1, \ldots, n-k$. Under the action of L_k the space $\mathfrak{q}$ is decomposed by the rows into $(n-k)$ pairwise equivalent sub-representations of L_k. If $\mathsf{K} = \mathrm{SO}(n)$ or $\mathsf{K} = \mathsf{SU}(n)$ with $k \geq 3$, these irreducible representations are orthogonal respectively unitary, and if $\mathsf{K} = \mathsf{Sp}(n)$ with $k \geq 2$ they are quaternionic. This in turn implies that the action of the upper $(n-k) \times (n-k)$-block U_{n-k} on $\mathfrak{q}$ induces a transitive action on the irreducible L_k-sub-representations.

We claim that L_k is normal in H. Otherwise, the isotropy representation of H/L_k contains a non-trivial irreducible sub-representation of L_k which can be seen as a sub-representation of $\mathfrak{q}$. As explained, we may assume that it is given by the last row in $\mathfrak{q}$. But now it is easy to see that H contains the lower $(k+1)\times(k+1)$-block in contradiction to our choice of k.

Thus L_k is normal in H. If $k = n-1$, then clearly (K,H) is one of the pairs listed in Table 2. Suppose on the contrary $k \le n-2$. We choose an irreducible sub-representation $V \subset \mathfrak{q}$ consisting of eigenvectors of the selfadjoint endomorphism G. As before $\mathrm{Ad}_g V$ is given by the last row in $\mathfrak{q}$ for some $g \in \mathsf{U}_{n-k}$. After conjugating G and H with Ad_g we can therefore assume that the last row in $\mathfrak{q}$ consists of eigenvectors. Clearly, we can iterate this argument and we may assume without loss of generality that each row of $\mathfrak{q}$ consists of eigenvectors of G. But now it is obvious that we can find commuting eigenvectors and by Lemma 1.1 (a) this is a contradiction. □

Remark. The subgroups $\mathsf{U}(2) \subset \mathrm{SO}(4)$ and $\mathsf{Sp}(2) \subset \mathsf{SU}(4)$ contain a 2×2 but no 3×3-block. Thus we cannot allow $k = 2$ for $\mathsf{K} \in \{\mathsf{SU}(n), \mathrm{SO}(n)\}$. The proof breaks down, since for a 2×2-block $\mathsf{L}_2 \subset \mathsf{K}$ the irreducible sub-representation in $\mathfrak{q}$ are complex if $\mathsf{K} = \mathrm{SO}(n)$ and quaternionic for $\mathsf{K} = \mathsf{SU}(n)$ and the upper $(n-2)\times(n-2)$-block no longer acts transitively on them.

The proof of the theorem in the next few sections will go by induction on the dimension of the Lie group, that is, at all times we will assume that the main theorem holds for all Lie groups with dimension strictly below $\dim(\mathsf{K})$. For any element $\iota \in \mathsf{H}$ the fixed point set $\mathrm{Fix}(\iota)$ is totally geodesic and hence positively curved. If $\mathrm{Fix}(\iota)_0$ is the component of $\mathrm{Fix}(\iota)$ containing the base point, then the id component of the centralizer $C(\iota)_0$ acts transitively on it, with stabilizer group $\mathsf{H}^\iota := C(\iota)_0 \cap \mathsf{H}$ and thus $C(\iota)_0/\mathsf{H}^\iota$ is positively curved. Although $\mathrm{Fix}(\iota)_0$ may not be simply connected, we obtain a contradiction if its universal cover is not listed in Table 1 or 2.

For simplicity of notation we let $\mathrm{Fix}(\iota)$ stand for $\mathrm{Fix}(\iota)_0$ and $C(\iota)$ for $C(\iota)_0$. Since ι is contained in a maximal torus of H which in turn can be extended to a maximal torus of K, we have

$$\operatorname{rank} C(\iota) = \operatorname{rank} \mathsf{K}, \quad \operatorname{rank}(\mathsf{H}^\iota) = \operatorname{rank}(\mathsf{H})$$

and ι is a central element in the identity component of H^ι. Hence the codimension of these fixed point sets $C(\iota)/\mathsf{H}^\iota$ is always even, and we can do the induction in even and odd dimensions separately.

In all of the cases we will consider, Ad_ι is an involution, and hence $\mathsf{K}/C(\iota)$ is a symmetric space with $\operatorname{rank}(C(\iota)) = \operatorname{rank}(\mathsf{K})$. If K is an exceptional Lie group, then the classification of symmetric spaces only allows the following possibilities for the pair $(\mathsf{K}, C(\iota))$:

$$\begin{aligned}
&\mathsf{G}_2\colon\ \mathrm{SO}(4),\\
&\mathsf{F}_4\colon\ (\mathsf{Sp}(3)\times\mathsf{Sp}(1))/\Delta\mathbb{Z}_2,\ \mathrm{Spin}(9),\\
&\mathsf{E}_6\colon\ (\mathrm{Spin}(10)\times\mathsf{S}^1)/\Delta\mathbb{Z}_4,\ (\mathsf{SU}(6)\times\mathsf{SU}(2))/\Delta\mathbb{Z}_2,\\
&\mathsf{E}_7\colon\ (\mathrm{Spin}(12)\times\mathsf{SU}(2))/\Delta\mathbb{Z}_2,\ (\mathsf{E}_6\times\mathsf{S}^1)/\Delta\mathbb{Z}_3,\ \mathsf{SU}(8)/\mathbb{Z}_2,\\
&\mathsf{E}_8\colon\ (\mathsf{E}_7\times\mathsf{SU}(2))/\Delta\mathbb{Z}_2,\ \mathrm{Spin}(16)/\mathbb{Z}_2 =: \mathrm{SO}'(16).
\end{aligned}$$

We can assume, by making the action ineffective if necessary, that the semisimple part of K is simply connected. But when $\mathsf{K} = \mathrm{Spin}(n)$, we will usually replace it by $\mathrm{SO}(n)$, at the expense of possibly making K/H not simply connected. We will use the following notational conventions:

- If we write a Lie group L as $\mathsf{L} = \mathsf{L}_1 \cdot \mathsf{L}_2$, then L_1 and L_2 are normal subgroups generating L and $\mathsf{L}_1 \cap \mathsf{L}_2$ is finite.
- If a group is identified as SO(4), then $\mathsf{SU}(2)_\pm$ will stand for the two simple normal subgroups.
- $G \colon \mathfrak{p} \to \mathfrak{p} = \mathfrak{h}^\perp$ will always stand for a positive definite selfadjoint endomorphism inducing a positively curved metric on the homogeneous space K/H, see first paragraph of this section. Both H and K are connected and the kernel of the action of K on K/H is finite.
- Sometimes we will just say that we found commuting eigenvectors of G. This is the same as saying that the case under consideration cannot occur, as it contradicts Lemma 1.1 (a). Similarly, if a situation arises where our induction hypothesis can be used to show that some fixed point component cannot have positive sectional curvature, it should be understood that we can move on to the next case.

2. The even-dimensional case

Since $\operatorname{rank}\mathsf{K} = \operatorname{rank}\mathsf{H}$, all irreducible sub-representations of H in $\mathfrak{p}$ are inequivalent, and hence the metric is diagonal, that is, every vector in an irreducible sub-representation of H is an eigenvector of G. In addition, we can assume that K is simple since otherwise K/H is a product homogeneous space and every K-invariant metric is a product metric, contradicting positive curvature. Furthermore, all elements in K are, up to conjugacy, also contained in H and hence for each symmetric pair (K, L) with $\operatorname{rank}(\mathsf{K}) = \operatorname{rank}(\mathsf{L})$ we can find an element ι in H such that $C(\iota) \cong \mathsf{L}$ and Ad_ι is an involution. In case of the classical Lie groups we always assume that H contains the classical diagonal torus and then $\iota \in \mathsf{H}$ will be diagonal as well.

Since $C(\iota)/\mathsf{H}^\iota$ is again an even-dimensional space of positive curvature, only one factor in $C(\iota)$ can act effectively on $\mathrm{Fix}(\iota)$, and the others must lie in H^ι and hence in H. Furthermore, if $C(\iota)/\mathsf{H}^\iota$ is not a point, then the action of the remaining factor is listed in Table 1 or 2, since we always assume that the main theorem holds for Lie groups of dimension strictly smaller than $\dim(\mathsf{K})$. We point out though that low-dimensional isomorphisms of Lie algebras sometimes give rise to less obvious presentations, e.g. $\mathbb{C}\mathbb{P}^3 = \mathrm{SO}(6)/\mathsf{U}(3) = \mathrm{SO}(5)/\mathsf{U}(2)$.

We now discuss each simple Lie group separately. Due to the low-dimensional isomorphisms

$$\mathrm{Spin}(5) \cong \mathsf{Sp}(2) \quad \text{and} \quad \mathrm{Spin}(6) \cong \mathsf{SU}(4)$$

we only need to consider $\mathrm{Spin}(n)$ for $n \geq 7$, and in that case replace it by $\mathrm{SO}(n)$ for simplicity. The aim is to confirm that (K, H) is listed in Table 1 or 2.

2.1. The case $\mathsf{K} = \mathsf{SU}(3)$. The only rank 2 subgroups of $\mathsf{SU}(3)$ are $\mathsf{U}(2)$ and T^2, and both pairs correspond to listed examples.

2.2. The case $\mathsf{K} = \mathsf{SU}(4)$. In this case, the fixed point set $\mathrm{Fix}(\iota) = \mathsf{S}(\mathsf{U}(2)\mathsf{U}(2))/\mathsf{H}^\iota$ has positive curvature, for $\iota = \mathrm{diag}(-1, -1, 1, 1)$, and thus H^ι, and hence also H, contains an $\mathsf{SU}(2)$-block, say the upper 2×2-block.

We next look at the involution $\iota_2 = \mathrm{diag}(1, -1, -1, 1)$ and repeat the argument. It shows that the group H also contains either the middle $\mathsf{SU}(2)$-block or the $(1,4)$-$\mathsf{SU}(2)$-block. Neither of these blocks commutes with the upper $\mathsf{SU}(2)$-block and in fact the two blocks generate an $\mathsf{SU}(3)$-block and we are done by the Block Lemma.

2.3. The case $\mathsf{K} = \mathsf{SU}(k)$ $(k \geq 5)$. In this case we look at the fixed point set of $\iota = \mathrm{diag}(\zeta, \dots, \zeta, -\zeta) \in \mathsf{H}$, where $\zeta \in \mathsf{S}^1$ is a primitive $2k$-th root of unity. Thus $C(\iota) = \mathsf{U}(k-1)$ and the fixed point component $\mathsf{U}(k-1)/\mathsf{H}^\iota$ admits positive curvature. Since $k - 1 \geq 4$, it follows that H contains a $(k-2) \geq 3$ block, and we are done by the Block Lemma.

2.4. The case $\mathsf{K} = \mathsf{Sp}(2)$. We may assume that the involution $\iota = \mathrm{diag}(-1, 1)$ is in H. Since $\mathsf{Sp}(1) \times \mathsf{Sp}(1)/\mathsf{H}^\iota$ admits positive curvature, H contains an $\mathsf{Sp}(1)$-block. Thus either $\mathsf{H} = \mathsf{Sp}(1) \cdot \mathsf{Sp}(1)$ or $\mathsf{H} = \mathsf{Sp}(1) \cdot \mathsf{S}^1$ and both quotients are listed in Table 2.

2.5. The case $\mathsf{K} = \mathsf{Sp}(3)$. We may assume that the involutions $\iota_1 = \mathrm{diag}(-1, 1, 1)$, $\iota_2 = \mathrm{diag}(1, -1, 1)$ and $\iota_3 = \mathrm{diag}(1, 1, -1)$ are in H and that H contains no 2×2-block. Then we have $\mathrm{Fix}(\iota_h) = \mathsf{Sp}(2)\mathsf{Sp}(1)/\mathsf{H}^{\iota_h}$ and since H does not contain an $\mathsf{Sp}(2)$, it must contain the $\mathsf{Sp}(1)$-block – given by the S^3 in the h-th diagonal entry. Hence $\{\mathrm{diag}(a, b, c) : a, b, c \in \mathsf{S}^3\}$ is a subgroup of H. Since H contains no 2×2-block equality must hold and we are left with the twelve-dimensional Wallach flag manifold.

2.6. The case $\mathsf{K} = \mathsf{Sp}(k)$ $(k \geq 4)$. We may assume $\iota = \mathrm{diag}(-1, -1, 1, \dots, 1) \in \mathsf{H}$ and hence $\mathrm{Fix}(\iota) = \mathsf{Sp}(2) \cdot \mathsf{Sp}(k-2)/\mathsf{H}^\iota$. Thus H contains either the upper 2×2-block or the lower $(k-2) \times (k-2)$-block. In either case the result follows from the Block Lemma.

2.7. The case $\mathsf{K} = \mathrm{SO}(k)$ $(k \geq 7)$. We may assume $\iota = \mathrm{diag}(-1, -1, -1, -1, 1 \dots, 1)$ is in H and hence $\mathrm{Fix}(\iota) = \mathrm{SO}(4) \cdot \mathrm{SO}(k-4)/\mathsf{H}^\iota$ admits positive curvature. The group H^ι must contain all but one of the connected simple normal subgroups of $\mathrm{SO}(4) \cdot \mathrm{SO}(k-4)$. This implies that H contains either the upper 4×4-block or the lower $(k-4) \times (k-4)$-block and we are done by the Block Lemma.

2.8. The case $\mathsf{K} = \mathsf{G}_2$. For any involution $\iota \in \mathsf{H}$, we have $\mathrm{Fix}(\iota) = \mathrm{SO}(4)/\mathsf{H}^\iota$, which implies that H^ι contains at least a group isomorphic to $\mathsf{U}(2) \subset \mathrm{SO}(4)$. We claim that $\mathsf{H} = \mathsf{U}(2)$ cannot hold. In fact, otherwise we could choose ι as a non-central involution in $\mathsf{U}(2)$ and would get $\mathsf{H}^\iota = \mathsf{T}^2$, but $\mathrm{SO}(4)/\mathsf{T}^2$ does not admit positive curvature. Thus H is strictly bigger than $\mathsf{U}(2)$. The only connected proper subgroups of G_2 satisfying this are $\mathsf{SU}(3)$ and $\mathrm{SO}(4)$. In the former case, $\mathsf{G}_2/\mathsf{SU}(3) \cong \mathbb{S}^6$ is listed in Table 2, while the latter case is not possible as G_2/H would be isometric to the rank 2 symmetric space $\mathsf{G}_2/\mathrm{SO}(4)$.

2.9. The case $\mathsf{K} = \mathsf{F}_4$. Choose an involution $\iota \in \mathsf{H}$ whose centralizer in F_4 is given by $\mathrm{Spin}(9)$. Since $\mathrm{Fix}(\iota) = \mathrm{Spin}(9)/\mathsf{H}^\iota$ has positive curvature, H contains $\mathrm{Spin}(8)$. The only proper connected subgroups in F_4 satisfying this are $\mathrm{Spin}(8)$ and $\mathrm{Spin}(9)$ and both correspond to listed quotients.

2.10. The case $\mathsf{K} = \mathsf{E}_i$, $i = 6, 7, 8$. We choose an element $\iota \in \mathsf{H}$ whose centralizer $C(\iota)$ is given by $\mathsf{S}^1 \cdot \mathrm{Spin}(10)$ if $i = 6$, $\mathsf{S}^1 \cdot \mathsf{E}_6$ if $i = 7$ and $\mathrm{SO}'(16)$ if $i = 8$. By induction we can use Tables 1 and 2 in the introduction to see that $C(\iota)$ cannot act transitively by isometries on a positively curved manifold of positive even dimension. Hence $C(\iota) = \mathsf{H}^\iota \subset \mathsf{H}$ and equality must hold since $C(\iota)$ is maximal – a contradiction as $\mathsf{K}/C(\iota)$ is a higher rank symmetric space.

3. K not semisimple

In the remaining three sections we assume that K/H is an almost effective representation of an odd-dimensional homogeneous space of positive sectional curvature and in this section we treat the case of a non-semisimple compact group K. Since $\mathrm{rank}(\mathsf{K}) = \mathrm{rank}(\mathsf{H}) + 1$, the center of K can be at most one-dimensional. After passing to a finite cover we can assume $\mathsf{K} = \mathsf{S}^1 \times \mathsf{K}_2$ with K_2 semisimple. We let $\mathrm{pr}_2(\mathsf{H})$ denote the projection of H to the second factor. Since K/H has finite fundamental group, the projection of H to the first factor is surjective.

If we put $\mathsf{H}_2 = \mathsf{K}_2 \cap \mathsf{H}$, then $\mathsf{H} = \Delta\mathsf{S}^1 \cdot \mathsf{H}_2$. Since the projection to the first factor is surjective, K_2 acts transitively on K/H with stabilizer H_2 and by induction on the dimension of the Lie group $(\mathsf{K}_2, \mathsf{H}_2)$ is up to a finite covering one of the pairs listed in Table 1 or 2. The group $\mathrm{pr}_2(\mathsf{H})$ is contained in the normalizer of H_2 in K_2 and thus $N(\mathsf{H}_2)/\mathsf{H}_2$ is at least one-dimensional. Combining this with the fact that K_2 is semisimple we deduce that $(\mathsf{K}_2, \mathsf{H}_2)$ is given by $(\mathsf{SU}(n), \mathsf{SU}(n-1))$, $(\mathsf{Sp}(n), \mathsf{Sp}(n-1))$ or $(\mathsf{SU}(3), \mathrm{diag}(z^p, z^q, \bar{z}^{p+q}))$ (with $p \geq q \geq 1$ and $\gcd(p,q) = 1$). In either case the corresponding S^1-extension is also listed in Tables 1 and 2 and thus we are done.

4. K semisimple but not simple

We assume in this section that $\mathsf{K} = \mathsf{K}_1 \times \mathsf{K}_2$ is a simply connected product group with semisimple factors of positive rank. Notice that $\mathrm{rank}(\mathsf{K}_i) - \mathrm{rank}(\mathsf{H}_i) \leq \mathrm{rank}(\mathsf{K}) - \mathrm{rank}(\mathsf{H}) = 1$ holds for $\mathsf{H}_i = \mathsf{K}_i \cap \mathsf{H}$, $i = 1, 2$. We distinguish among three cases.

4.1. H_1 and H_2 are finite. Then $\mathrm{rank}(\mathsf{K}_i) \leq \mathrm{rank}(\mathsf{H}_i) + 1 = 1$ and thus $\mathsf{K} = \mathsf{S}^3 \times \mathsf{S}^3$. If H is three-dimensional, then it is necessarily given by $\Delta\mathsf{S}^3$ and $\mathsf{K}/\mathsf{H} \cong \mathbb{S}^3$ is in our list. Otherwise H is a circle and we can assume $\mathsf{H} = \{(z^p, z^q) : z \in \mathsf{S}^1\}$ with $p \geq q \geq 1$, $\gcd(p,q) = 1$. We want to rule out these potential examples by finding commuting eigenvectors. The tangent space $\mathfrak{p}$ splits into a trivial one-dimensional module and a four-dimensional module spanned by $(j,0), (k,0), (0,j), (0,k)$ on which H acts as a rotation on the span of the first two and the last two vectors. If $p \neq q$, the sub-representations are inequivalent and thus G-invariant. Therefore $(j,0)$ and $(0,j)$ are commuting eigenvectors. If $p = q$, we can assume that one eigenvector is given by $e_1 = (\alpha j, \beta j e^{i\psi})$ for some $\alpha, \beta, \psi \in \mathbb{R}$ with $\alpha^2 + \beta^2 = 1$. A second eigenvector to the same eigenvalue is then obtained by the action of H to be $e_2 = (\alpha k, \beta k e^{i\psi})$. Any vector in the four-dimensional module which is Q-orthogonal to both must thus be an eigenvector as well. Thus $e_3 = (-\beta j, \alpha j e^{i\psi})$ is an eigenvector and it clearly commutes with e_1. Here we used indirectly that we can choose the biinvariant metric Q such that both factors are weighted equally.

4.2. H_1 is finite but H_2 is not. Then $\mathrm{rank}(\mathsf{K}_1) = 1$ and thus $\mathsf{K} = \mathsf{S}^3 \times \mathsf{K}_2$.

We start with the case where H projects surjectively to the first factor. In this case we have $\mathsf{H} = \Delta\mathsf{S}^3 \cdot \mathsf{H}_2$ and the factor K_2 acts transitively on the homogeneous space K/H with stabilizer H_2. By induction on the dimension of the Lie group, the pair $(\mathsf{K}_2, \mathsf{H}_2)$ is up to a finite covering listed in Table 1 or 2. Since $\mathrm{pr}_2(\mathsf{H})$ is contained in the normalizer of H_2, it follows that $N(\mathsf{H}_2)/\mathsf{H}_2$ is three-dimensional. Therefore the pair $(\mathsf{K}_2, \mathsf{H}_2)$ is given by $(\mathsf{SU}(3), \mathrm{diag}(z, z, \bar{z}^2))$ or $(\mathsf{Sp}(n), \mathsf{Sp}(n-1))$. But then (K, H) is (up to finite kernel) either $(\mathsf{SU}(3) \times \mathrm{SO}(3), \mathsf{U}(2))$ or $(\mathsf{Sp}(n) \times \mathsf{Sp}(1), \mathsf{Sp}(n)\Delta\mathsf{Sp}(1))$ and both are listed in Tables 1 and 2.

The projection of H to the S^3 factor cannot be trivial as otherwise $\mathsf{H} = 1 \times \mathsf{H}_2$ would be a product subgroup of $\mathsf{S}^3 \times \mathsf{K}_2$ and every invariant metric of K/H would be a product metric.

It remains to consider the subcase where the projection of H to the S^3 factor is given by an S^1. Let $N(\mathsf{H}_2)$ denote the normalizer of H_2 in K_2. If $N(\mathsf{H}_2)/\mathsf{H}_2$ is three-dimensional, then a fixed point component of H_2 is locally isometric to $(\mathsf{S}^3 \times \mathsf{S}^3)/\mathsf{S}^1$, which is impossible as we saw in Section 4.1.

Otherwise, we have $\dim(N(\mathsf{H}_2)/\mathsf{H}_2) \leq 1$. This implies that the two-dimensional irreducible representation of H in the first factor is not equivalent to any other sub-representation of H in $\mathfrak{p}$ since no other non-trivial sub-representation has H_2 in its kernel. Thus the two-dimensional sub-representation in the first factor consists of eigenvectors and it is now easy to find commuting eigenvectors of G.

4.3. Both H_1 and H_2 are infinite. In this case there is a non-trivial irreducible sub-representation which does not contain H_1 in its kernel. Since any such sub-representation is tangent to the first factor, there are eigenvectors in the first factor. Similarly there are also eigenvectors in the second factor and thus we found commuting eigenvectors.

5. The odd-dimensional case with K simple

The proof is again by induction on the dimension of the group. Again we frequently use that the centralizer $C(\iota)$ of an element $\iota \in \mathsf{H}$ acts transitively on an odd-dimensional fixed point component of ι and by induction this action is, up to possibly a larger kernel, (locally) given by one listed in Tables 1 and 2 in the introduction. We point out though that low-dimensional isomorphisms of Lie algebras sometimes give rise to less obvious presentations:

$$\begin{aligned} \mathbb{RP}^7 &= \mathrm{SO}(5)/\mathsf{SU}(2)_\pm = \mathrm{SO}(6)/\mathsf{SU}(3) = \mathrm{SO}(7)/\mathsf{G}_2 = \mathrm{SO}(8)/\mathrm{Spin}(7), \\ \mathbb{RP}^7 &= \mathrm{SO}(5)\mathsf{SU}(2)/\mathsf{SU}(2)_- \cdot \Delta\mathsf{SU}(2)_+, \quad \mathbb{S}^5 = \mathsf{SU}(4)/\mathsf{Sp}(2), \\ \mathbb{RP}^{15} &= \mathrm{SO}(9)/\mathrm{Spin}(7). \end{aligned}$$

As explained, for $\iota \in \mathsf{H}$, the group H is either equal to $\mathsf{H}^\iota = C(\iota) \cap \mathsf{H}$ or an equal rank enlargement thereof. The latter are rather rare, as follows from [14, table on p. 281]. For example, up to covers the only equal rank enlargements of simple Lie groups are

$$\begin{array}{lll} \mathrm{SO}(2n) \subset \mathrm{SO}(2n+1), & \mathsf{SU}(3) \subset \mathsf{G}_2, & \mathrm{Spin}(9) \subset \mathsf{F}_4, \\ \mathsf{SU}(8)/\mathbb{Z}_2 \subset \mathsf{E}_7, & \mathsf{SU}(9)/\mathbb{Z}_3 \subset \mathsf{E}_8, & \mathrm{Spin}(16)/\mathbb{Z}_2 \subset \mathsf{E}_8. \end{array}$$

The group $\mathsf{S}^3 \times \mathsf{S}^3$ has only $\mathsf{Sp}(2)$ as equal rank enlargement whereas $\mathrm{SO}(4)$ has $\mathrm{SO}(5)$ and G_2.

We go through the list of simple Lie groups. By passing to a $\mathbb{Z}_2$ quotient if necessary we again can deal with the group $\mathrm{SO}(k)$ rather than $\mathrm{Spin}(k)$ as long as we allow fundamental group $\mathbb{Z}_2$ for K/H. As in the even-dimensional case we only need to consider this case for $k \geq 7$.

5.1. The case $\mathsf{K} = \mathsf{SU}(3)$. If $\mathsf{H} = \mathrm{SO}(3)$, then K/H is isometric to a symmetric space of rank 2, and hence does not have positive curvature, and if $\mathsf{H} = \mathsf{SU}(2)$, it is a sphere. Otherwise, H is one-dimensional and we may assume $\mathsf{H} = \mathrm{diag}(z^p, z^q, \bar{z}^{p+q})$ with $p \geq q \geq 0$ and $\gcd(p,q) = 1$. Then K/H is an Aloff–Wallach space, which has positive curvature unless $(p,q) = (1,0)$. In the latter case we choose the involution $\iota = \mathrm{diag}(-1,1,-1) \in \mathsf{H}$ whose fixed point set $\mathrm{Fix}(\iota) = \mathsf{U}(2)/\mathsf{H} = (\mathbb{S}^2 \times \mathbb{S}^1)/\Delta\mathbb{Z}_2$ cannot have positive curvature.

5.2. The case $\mathsf{K} = \mathsf{SU}(4)$. There exists an involution $\iota \in \mathsf{H}$ which is not central since $\mathrm{rank}(\mathsf{H}) = 2$. In this case, $\mathrm{Fix}(\iota) = \mathsf{S}(\mathsf{U}(2)\mathsf{U}(2))/\mathsf{H}^\iota$, which can only have positive curvature if H contains $\Delta\mathsf{SU}(2)$ or an $\mathsf{SU}(2)$-block. In the first case, K/H is effectively a quotient of $\mathrm{SO}(6)$. The image of $\mathsf{S}(\mathsf{U}(2)\mathsf{U}(2))$ in $\mathrm{SO}(6)$ is $\mathrm{SO}(2)\,\mathrm{SO}(4)$, and hence the image of $\Delta\mathsf{SU}(2)$ is a 3×3-block in $\mathrm{SO}(6)$, and we are done by the Block Lemma.

If H contains a block (say lower) $\mathsf{SU}(2)$, there are four possible enlargements of rank 2. If H is simple, then it must be $\mathsf{SU}(3)$ or $\mathsf{Sp}(2)$, since $\mathsf{SU}(4)$ does not contain a G_2. But then $\mathsf{K}/\mathsf{H} = \mathbb{S}^7$ or $\mathbb{S}^5$ is in our list. A third possibility is that $\mathsf{H} = \mathsf{SU}(2) \cdot \mathsf{SU}(2)$ (lower and upper 2×2-block), but then K/H is effectively given by $\mathrm{SO}(6)/\mathrm{SO}(4)$ and we are done by the Block Lemma.

The final possibility is $\mathsf{H} = \mathsf{SU}(2) \cdot \mathrm{diag}(z^{2p}, z^{2q}, \bar{z}^{p+q}, \bar{z}^{p+q})$ for some $p, q \in \mathbb{Z}$ with $\gcd(p,q) = 1$ and this can be ruled out as follows. If $|p| \neq |q|$, then the two rows in the orthogonal complement $\mathfrak{q}$ of $\mathsf{S}(\mathsf{U}(2)\mathsf{U}(2))$ correspond to inequivalent representations and hence are contained in eigenspaces of G which clearly yields commuting eigenvectors. If $p = q$, the representation of H in $\mathfrak{q}$ decomposes into two equivalent *complex* representations and the normalizer of H contains the upper 2×2-block. Hence we can argue as in the proof of the Block Lemma to find commuting eigenvectors. If $p = -q$, then the involution $\mathrm{diag}(1,1,-1,-1) \in \mathsf{H}$ has a three-dimensional fixed point component $\mathsf{S}(\mathsf{U}(2)\mathsf{U}(2))/\mathsf{H} = (\mathbb{S}^2 \times \mathbb{S}^1)/\mathbb{Z}_2$ with infinite fundamental group.

5.3. The case $\mathsf{K} = \mathsf{SU}(5)$. Let $\mathsf{F} \cong \mathbb{Z}_2^4$ denote the group of diagonal matrices in $\mathsf{SU}(5)$ with eigenvalues ± 1. Since we can assume that the three-dimensional torus in H is diagonal, there exists an index 2 subgroup E of F contained in H. We claim that one element in E has an eigenvalue -1 with multiplicity 4. Suppose not. If $\iota_1, \iota_2 \in \mathsf{F}$ are two elements both of which have the eigenvalue -1 with multiplicity 4, then it would follow $\iota_1 \cdot \iota_2 \in \mathsf{E}$. But these products generate the whole group F – a contradiction.

Therefore without loss of generality $\iota = \mathrm{diag}(-1,-1,-1,-1,1) \in \mathsf{H}$. By induction, the fixed point set $\mathrm{Fix}(\iota) = \mathsf{U}(4)/\mathsf{H}^\iota$ must be one of $\mathsf{U}(4)/\mathsf{U}(3)$, $\mathsf{U}(4)/\mathsf{SU}(4)$ or $\mathsf{SU}(4)/\mathsf{Sp}(2)$. In the first two cases, H contains a 3×3-block, and we are done. In the last case, H contains $\mathsf{Sp}(2) \cdot \mathsf{S}^1$, which gives rise to the positively curved Berger space $\mathsf{SU}(5)/\mathsf{Sp}(2) \cdot \mathsf{S}^1$.

5.4. The case $\mathsf{K} = \mathsf{SU}(k)$, $k \geq 6$. We can assume that the maximal diagonal torus of H has at least a one-dimensional intersection with the maximal torus of $\Delta\mathsf{SU}(3) \subset \mathsf{SU}(3)^2$ contained in the upper 6×6-block of $\mathsf{SU}(k)$. Clearly, any involution ι in this intersection has a complex eigenvalue -1 with multiplicity 4. Then $\mathrm{Fix}(\iota) = \mathsf{S}(\mathsf{U}(4) \cdot \mathsf{U}(k-4))/\mathsf{H}^\iota$.

If $k \geq 7$, then H^ι either contains the upper 4×4 or the lower $(k-4) \times (k-4)$-block and we are done.

If $k = 6$, the stabilizer group of the action of $\mathsf{SU}(4)$ on $\mathrm{Fix}(\iota)$ is $\mathsf{Sp}(2)$ unless it contains an $\mathsf{SU}(3)$-block. Thus we may assume $\mathsf{Sp}(2)\mathsf{SU}(2) \cdot \mathsf{S}^1 \subset \mathsf{H}$ with $\mathsf{S}^1 = \mathrm{diag}(z,z,z,z,\bar{z}^2,\bar{z}^2)$.

This in turn implies that $\iota_2 = \mathrm{diag}(i, -i, i, -i, i, -i) \in \mathsf{Sp}(2)\mathsf{SU}(2) \subset \mathsf{H}$. The centralizer $C(\iota_2)$ is isomorphic to $\mathsf{S}(\mathsf{U}(3)\mathsf{U}(3))$ and hence $\mathrm{Fix}(\iota_2) = \mathsf{S}(\mathsf{U}(3)\mathsf{U}(3))/\mathsf{H}^{\iota_2}$ has positive curvature. By induction H contains up to conjugation a 3×3-block and we are done by the Block Lemma.

5.5. The case $\mathsf{K} = \mathsf{Sp}(2)$. There are three three-dimensional subgroups of $\mathsf{Sp}(2)$. One quotient is a sphere $\mathbb{S}^7 = \mathsf{Sp}(2)/\mathsf{Sp}(1)$, the second the Berger space $\mathsf{Sp}(2)/\mathsf{Sp}(1)_{\max}$ with positive curvature, and the third the Stiefel manifold $\mathsf{Sp}(2)/\Delta\mathsf{Sp}(1) = \mathrm{SO}(5)/\mathrm{SO}(3)$ which contains a 3×3-block, and thus is ruled out by the Block Lemma.

It remains to consider $\mathsf{H} = \mathrm{diag}(z^p, z^q)$ with $p \geq q \geq 0$ and $\gcd(p, q) = 1$. Then the weights of the adjoint action of H on $\mathfrak{p}$ are $0, 2p, 2q, p - q, p + q$. If they are all distinct, the metric G is diagonal, and there are two commuting eigenvectors. If $(p, q) = (1, 0)$, then $\mathrm{Fix}(\iota) = \mathsf{Sp}(1)\mathsf{Sp}(1)/\mathsf{S}^1 \times 1$ cannot have positive curvature, where ι is the involution in H.

This leaves us with two exceptional cases.

If $(p, q) = (1, 1)$, then $\mathsf{Sp}(2)/\Delta\mathsf{S}^1 = \mathrm{SO}(5)/\mathrm{SO}(2)$ is the Stiefel manifold where we can think of SO(2) as the lower 2×2-block. In this case, $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$ where H acts trivially on the three-dimensional module $\mathfrak{p}_0$ (upper 3×3-block), and as the direct sum of three equivalent two-dimensional representations on $\mathfrak{p}_1$. We use Lemma 1.1 (b) to find an obstruction. If an eigenvector corresponding to the smallest eigenvalue of G lies in $\mathfrak{p}_0$, then it has rank 2 in $\mathfrak{so}(5)$ and there is a vector in $\mathfrak{p}_1$ that commutes with it. If the eigenvector lies in $\mathfrak{p}_1$, we can use the fact that $\mathrm{SO}(5)/\mathrm{SO}(3)\,\mathrm{SO}(2)$ is a symmetric space of rank 2 to find a linearly independent vector in $\mathfrak{p}_1$ that commutes with it. In either case Lemma 1.1 (b) implies that $\mathsf{Sp}(2)/\mathsf{H}$ does not have positive curvature.

Ruling out the remaining case of $(p, q) = (3, 1)$ we postpone to the end since it is the only case that requires a more detailed argument, see Section 5.14.

5.6. The case $\mathsf{K} = \mathsf{Sp}(3)$. If $\iota \in \mathsf{H}$ is an involution which is not central, then

$$\mathrm{Fix}(\iota) = \mathsf{Sp}(2)\mathsf{Sp}(1)/\mathsf{H}^{\iota}$$

and by induction there are only four odd-dimensional quotients which have positive curvature, corresponding to $\mathsf{H}^{\iota} = \mathsf{Sp}(2), \mathsf{Sp}(1)\Delta\mathsf{Sp}(1), \mathsf{Sp}(1)\mathsf{Sp}(1)$ or $\mathsf{Sp}(1)_{\max}\mathsf{Sp}(1)$. By the Block Lemma we may assume that H does not contain a 2×2-block and thus $\mathsf{H} = \mathsf{H}^{\iota}$.

If the group H^{ι} is given by $\mathsf{Sp}(1)\Delta\mathsf{Sp}(1) = \mathrm{diag}(q, r, r)$ or $\mathsf{Sp}(1)\mathsf{Sp}(1) = \mathrm{diag}(1, q, r)$ with $q, r \in \mathsf{Sp}(1)$, then we can choose a second involution $\iota_2 = \mathrm{diag}(1, -1, -1) \in \mathsf{H}$ with fixed point set

$$\mathsf{Sp}(1)\mathsf{Sp}(2)/\mathsf{H} = \mathsf{Sp}(2)/\Delta\mathsf{Sp}(1)$$

in the first case, and

$$\mathsf{Sp}(1)\mathsf{Sp}(2)/\mathsf{H} = \mathsf{Sp}(1) \times (\mathsf{Sp}(2)/\mathsf{Sp}(1)\mathsf{Sp}(1))$$

in the second case. Neither one admits positive curvature.

Thus we are left with the case $\mathsf{H}^{\iota} = \mathsf{H} = \mathsf{Sp}(1)_{\max}\mathsf{Sp}(1) \subset \mathsf{Sp}(2)\mathsf{Sp}(1)$. Then we have $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, where $\mathfrak{p}_1$ is the irreducible sub-representation of H given by the orthogonal complement of $\mathsf{Sp}(1)_{\max}$ in the upper $\mathfrak{sp}(2)$-block and $\mathfrak{p}_2$ is the irreducible inequivalent sub-representation given by $(\mathfrak{sp}(2) \oplus \mathfrak{sp}(1))^{\perp}$. Clearly, $\mathfrak{p}_i$ is contained in an eigenspace of G. We may assume that a maximal torus of H is given by $\{\mathrm{diag}(z^3, z, \zeta) : z, \zeta \in \mathsf{S}^1\}$ (see e.g. [3, p. 237]). The circle $\mathrm{diag}(z^3, z, 1)$ acts on the Lie algebra of $\mathsf{Sp}(1)_{\max}$ with weight 2 and thus $x = \mathrm{diag}(j, 0, 0) \in \mathfrak{p}_1$, as x lies in a two-dimensional sub-representation of the circle with weight 6. Clearly, we can find a commuting eigenvector in $\mathfrak{p}_2$.

5.7. The case $\mathsf{K} = \mathsf{Sp}(k)$, $k \geq 4$. Fixed point groups of non-central involutions are of the form $\mathsf{Sp}(r)\mathsf{Sp}(s)$ with $r + s = k$ and $r \geq s$. Thus either $r \geq 3$ or $r = s = 2$. In either case, it follows from our induction hypothesis that $\mathsf{H}^\iota \subset \mathsf{H}$ contains a 2×2-block and we are done.

5.8. The case $\mathsf{K} = \mathrm{SO}(k)$, $k \geq 7$. We may assume that the maximal torus of H has at least a one-dimensional intersection with $\mathsf{SU}(3) \subset \mathsf{U}(3)$ contained in the upper the 6×6-block. Any involution ι in this intersection has the eigenvalue -1 with real multiplicity four. Hence without loss of generality $\iota = \mathrm{diag}(-1, -1, -1, -1, 1, \dots, 1) \in \mathsf{H}$.

A fixed point component is given by $\mathrm{SO}(4)\,\mathrm{SO}(k-4)/\mathsf{H}^\iota$. If $k \geq 10$, then by induction H^ι either contains the upper 4×4-block or the lower $(k-4) \times (k-4)$-block and we are done.

If $k = 7$, then H^ι contains a 3×3-block unless

$$\mathsf{H}^\iota_0 = \mathsf{SU}(2)_- \cdot \Delta\mathsf{SU}(2) \cong \mathrm{SO}(4) \quad \text{with} \quad \Delta\mathsf{SU}(2) \subset \mathsf{SU}(2)_+ \cdot \mathrm{SO}(3).$$

If $\mathsf{H} = \mathsf{H}^\iota$, then we consider another involution $\iota_2 = \mathrm{diag}(-1, -1, 1, 1, -1, -1, 1) \in \mathsf{H}$ coming from $(\mathrm{diag}(i, -i), \mathrm{diag}(i, -i)) \in \mathsf{SU}(2)_- \cdot \Delta\mathsf{SU}(2)$ with fixed point set $\mathrm{SO}(4)\,\mathrm{SO}(3)/\mathsf{H}^{\iota_2}$ which cannot have positive curvature since $(\mathsf{H}^{\iota_2})_0 = \mathsf{U}(1)\mathsf{U}(1)$. Thus $\mathsf{H} \neq \mathsf{H}^\iota$ and H is simple. Since $\mathrm{SO}(5)$ only embeds as a 5×5-block into $\mathrm{SO}(7)$, it necessarily follows that $\mathsf{H} = \mathsf{G}_2$ and $\mathrm{SO}(7)/\mathsf{G}_2 \cong \mathbb{RP}^7$ is two-fold covered by a listed example.

If $k = 8$, then at least two of the four normal connected simple subgroups of $\mathrm{SO}(4)\,\mathrm{SO}(4)$ are in H^ι. If they form a 4×4-block, then we are done by the Block Lemma. Otherwise, we can choose a suitable complex structure such that H^ι contains the subgroup L given as the upper and lower 2×2-block of $\mathsf{SU}(4) \subset \mathrm{SO}(8)$. If we choose an automorphism of the Lie algebra $\mathfrak{so}(8)$ that moves the subalgebra $\mathfrak{su}(4)$ into a 6×6-block $\mathfrak{so}(6)$, then the image of the Lie algebra of L will be a 4×4-block (see first paragraph of Section 5.2). By the Block Lemma, $\mathsf{H} \cong \mathrm{Spin}(7)$ and $\mathrm{SO}(8)/\mathrm{Spin}(7) \cong \mathbb{RP}^7$ is two-fold covered by a listed example.

It remains to consider $k = 9$. Then $\mathrm{Fix}(\iota) = \mathrm{SO}(5)\,\mathrm{SO}(4)/\mathsf{H}^\iota$. By the Block Lemma, we can assume that both $\mathrm{SO}(5)$ and one of the simple factors in $\mathrm{SO}(4)$ must act non-trivially on $\mathrm{Fix}(\iota)$. This leaves only the possibility that effectively

$$\mathrm{Fix}(\iota) = \mathbb{RP}^7 = \mathrm{SO}(5)\mathsf{SU}(2)_+/\mathsf{SU}(2)_- \cdot \Delta\mathsf{SU}(2)_+.$$

Thus $\Delta\mathsf{SU}(2)_+ \subset \mathsf{H}$ which implies that H contains another involution ι_2 with eigenvalue -1 of multiplicity 8 and hence has fixed point component $\mathrm{SO}(8)/\mathsf{H}^{\iota_2}$. Thus either H contains a 7×7-block and we are done, or $\mathrm{Fix}(\iota_2) = \mathbb{RP}^7 = \mathrm{SO}(8)/\mathrm{Spin}(7)$. But then $\mathsf{H} = \mathrm{Spin}(7)$ and $\mathsf{K}/\mathsf{H} = \mathbb{RP}^{15}$ is two-fold covered by a listed example.

5.9. The case $\mathsf{K} = \mathsf{G}_2$. For every involution $\iota \in \mathsf{H}$, we have $\mathrm{Fix}(\iota) = \mathrm{SO}(4)/\mathsf{H}^\iota$. Since ι is contained in the center of H^ι, we deduce that $\mathsf{H}^\iota \not\cong \mathrm{SO}(3)$ and so $\mathsf{H}^\iota = \mathsf{H}$ is a normal subgroup of $\mathrm{SO}(4)$. We let $\mathfrak{p}_0 \subset \mathfrak{p}$ denote the three-dimensional trivial sub-representation of H corresponding to the dual normal subgroup of $\mathrm{SO}(4)$. The orthogonal complement $\mathfrak{q} := (\mathfrak{p}_0)^\perp \cap \mathfrak{p}$ corresponds to the tangent space of the rank 2 symmetric space $\mathsf{G}_2/\mathrm{SO}(4)$. Thus $G_{|\mathfrak{q}}$ cannot be a multiple of the identity because otherwise we could find commuting eigenvectors. This implies that the representation of $\mathsf{H} \cong \mathsf{SU}(2)$ on the eight-dimensional space $\mathfrak{q}$ is reducible. Since $\mathrm{Ad}_{\iota|\mathfrak{q}} = -\mathrm{id}$, there are no trivial or three-dimensional sub-representations and it must be given as the sum of two four-dimensional sub-representations.

Altogether, $G_{|\mathfrak{q}}$ has two four-dimensional eigenspaces $\mathfrak{p}_1$ and $\mathfrak{p}_2$ corresponding to distinct eigenvalues λ_1 and λ_2. If $X \in \mathfrak{p}_1 \setminus \{0\}$, then ad_X moves $\mathfrak{p}_2$ to $\mathfrak{p}_0 \oplus \mathfrak{h}$ as $\mathsf{G}_2/\mathrm{SO}(4)$ is a symmetric space. Since $\mathfrak{p}_0$ is three-dimensional, we have $[X, Y] \in \mathfrak{h}$ for some $Y \in \mathfrak{p}_2 \setminus \{0\}$. Now $B_+(X, Y) = [GX, Y] - [X, GY] = (\lambda_1 - \lambda_2)[X, Y] \in \mathfrak{p}$ (see proof of Lemma 1.1) gives $[X, Y] = 0$ and we found two commuting eigenvectors.

5.10. The case $\mathsf{K} = \mathsf{F}_4$. In this case, $C(\iota)$ is either $\mathsf{Sp}(3) \cdot \mathsf{Sp}(1)$ or Spin(9). Since $\mathrm{rank}(\mathsf{H}) = 3$, one of its involutions lies in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, the center of $\mathrm{Spin}(8) \subset \mathrm{Spin}(9)$. Since $\mathsf{Sp}(3)\mathsf{Sp}(1)$ does not contain a Spin(8), the fixed point set is $\mathrm{Fix}(\iota) = \mathrm{Spin}(9)/\mathsf{H}^\iota$. Thus we have $\mathsf{H}^\iota = \mathrm{Spin}(7)$ since H^ι has rank 3, which in turn implies that $\mathsf{H} = \mathrm{Spin}(7)$ since it has no equal rank enlargement.

Choose an involution $\iota_2 \in \mathrm{Spin}(7)$ with $\mathsf{H}^{\iota_2} = \mathrm{Spin}(4)\,\mathrm{Spin}(3)$. Again $C(\iota_2)$ is either $\mathsf{Sp}(3) \cdot \mathsf{Sp}(1)$ or Spin(9) and by induction $C(\iota_2)/\mathsf{H}^\iota$ does not have positive curvature.

5.11. The case $\mathsf{K} = \mathsf{E}_6$. Here $C(\iota)$ is either $\mathsf{SU}(6) \cdot \mathsf{Sp}(1)$ or $(\mathrm{Spin}(10) \times \mathsf{S}^1)/\Delta\mathbb{Z}_4$. As in the previous case, we can choose an involution $\iota \in Z(\mathrm{Spin}(8)) \cap \mathsf{H}$ with fixed point set $\mathrm{Fix}(\iota) = \mathrm{Spin}(10) \cdot \mathsf{S}^1/\mathsf{H}^\iota$. Thus either $\mathrm{Spin}(9) \cdot \mathsf{S}^1 \subset \mathsf{H}$ or $\mathrm{Spin}(10) \subset \mathsf{H}$. If H is a strict equal rank enlargement of H^ι, then it is isomorphic to $\mathsf{Sp}(1) \cdot \mathrm{Spin}(9)$ or Spin(11) and the central element $\iota \in \mathrm{Spin}(9)$ of H^ι would remain central in H, which is impossible as these groups are not in $C(\iota)$.

If $\mathsf{H} = \mathrm{Spin}(9) \cdot \mathsf{S}^1$, then we can choose another involution $\iota_2 \in Z(\mathrm{Spin}(8)) \cap \mathsf{H}$ with $\mathsf{H}^{\iota_2} = \mathrm{Spin}(8) \cdot \mathsf{S}^1$. But then $\mathrm{Fix}(\iota_2) = \mathrm{Spin}(10) \cdot \mathsf{S}^1/\mathrm{Spin}(8) \cdot \mathsf{S}^1$ which does not admit positive curvature.

If $\mathsf{H} = \mathrm{Spin}(10)$, then we can choose an involution $\iota_2 \in \mathsf{SU}(2) \subset \mathrm{Spin}(4) \subset \mathrm{Spin}(10)$ with $\mathsf{H}^{\iota_2} = \mathrm{Spin}(4)\,\mathrm{Spin}(6)$ and as before $C(\iota_2) \cong \mathrm{Spin}(10) \cdot \mathsf{S}^1$ or $\mathsf{SU}(6) \cdot \mathsf{Sp}(1)$ but in either case $C(\iota_2)/\mathsf{H}^{\iota_2}$ cannot have positive curvature by our induction hypothesis.

5.12. The case $\mathsf{K} = \mathsf{E}_7$. We let ι denote an involution in H which is not central in E_7. The potential candidates for $C(\iota)$ are $\mathsf{E}_6 \cdot \mathsf{S}^1$, $(\mathrm{Spin}(12) \times \mathsf{Sp}(1))/\Delta\mathbb{Z}_2$ or $\mathsf{SU}(8)/\mathbb{Z}_2$. Notice that ι must be contained in the center of $C(\iota)$. The center of $\mathsf{E}_6 \cdot \mathsf{S}^1$ and $\mathsf{SU}(8)/\mathbb{Z}_2$ only contains one involution and this must be contained in the center of E_7 which is $\mathbb{Z}_2$ – a contradiction.

Thus we have $C(\iota) = (\mathrm{Spin}(12) \times \mathsf{Sp}(1))/\Delta\mathbb{Z}_2$. Then H^ι is either $\mathrm{Spin}(11) \cdot \mathsf{Sp}(1) = \mathsf{H}$ or $\mathrm{Spin}(12) = \mathsf{H}$. But then we can choose another involution $\iota_2 \in Z(\mathrm{Spin}(4)) \subset \mathrm{Spin}(11) \cap \mathsf{H}$ with $\mathsf{H}^{\iota_2} = \mathrm{Spin}(4)\,\mathrm{Spin}(7)\mathsf{Sp}(1)$ or $\mathrm{Spin}(4)\,\mathrm{Spin}(8)$. As before $C(\iota_2) \cong \mathrm{Spin}(12) \cdot \mathsf{Sp}(1)$ and the fixed point set does not have positive curvature.

5.13. The case $\mathsf{K} = \mathsf{E}_8$. Here $C(\iota)$ is either $\mathsf{E}_7 \cdot \mathsf{Sp}(1)$ or $\mathrm{SO}'(16)$. In the former case, we would get $\mathsf{H} = \mathsf{E}_7$ and there is a Riemannian submersion from $\mathsf{E}_8/\mathsf{E}_7$ to the higher rank symmetric space $\mathsf{E}_8/\mathsf{E}_7 \cdot \mathsf{Sp}(1)$ as the isotropy representation of $\mathsf{E}_8/\mathsf{E}_7\mathsf{Sp}(1)$ remains irreducible when restricted to E_7 – a contradiction.

Thus we have $\mathrm{Fix}(\iota) = \mathrm{SO}'(16)/\mathsf{H}^\iota$ and hence $\mathsf{H}^\iota = \mathrm{Spin}(15) = \mathsf{H}$. We can now choose another involution $\iota_2 \in Z(\mathrm{Spin}(12)) \subset \mathsf{H}$ with $\mathsf{H}^{\iota_2} = \mathrm{Spin}(12)\,\mathrm{Spin}(3)$. As before we must have $C(\iota_2) \cong \mathrm{SO}'(16)$ but then $C(\iota_2)/\mathsf{H}^{\iota_2}$ does not have positive curvature.

5.14. The case $\mathsf{K} = \mathsf{Sp}(2)$ and $\mathsf{H} = \mathrm{diag}(e^{i\theta}, e^{3i\theta})$. Finally, we discuss the example left out in Section 5.5.

Claim. *Without loss of generality G commutes with* Ad_a*, where* $a = \mathrm{diag}(j, j)$.

We first want to explain why it is enough to prove the claim. By the claim it suffices to consider $\mathrm{Ad}_{\hat{\mathsf{H}}}$-invariant metrics with

$$\hat{\mathsf{H}} := \mathsf{H} \cup a\mathsf{H} \cong \mathsf{Pin}(2).$$

Of course, any such metric descends via the two-fold cover $\mathsf{K}/\mathsf{H} \to \mathsf{K}/\hat{\mathsf{H}}$. Notice that $\mathrm{Ad}_{a|\mathfrak{p}}$ has a negative determinant and thus right multiplication with a induces an orientation reversing isometry of $\mathsf{Sp}(2)/\mathsf{H}$ and $\mathsf{Sp}(2)/\hat{\mathsf{H}}$ is a non-orientable manifold. On the other hand, we know that a positively curved odd-dimensional manifold is orientable by the Synge Lemma.

It remains to verify the claim. The element $b = \mathrm{diag}(e^{i\psi}, e^{i\psi})$ is in the normalizer of H. Therefore the isometry type does not change if we replace G by $G_b = \mathrm{Ad}_b\, G\, \mathrm{Ad}_{b^{-1}}$ to define an induced metric on $\mathsf{Sp}(2)/\mathsf{H}$. We plan to show that for a suitable choice of b the endomorphisms G_b and Ad_a commute.

Consider the isotropy decomposition $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_4 \oplus \mathfrak{p}_6$ preserved by G, where H acts trivially on the one-dimensional space $\mathfrak{p}_0$ and with weight n on $\mathfrak{p}_n$, $n = 2, 4, 6$. These subspaces can be described explicitly as follows: $\mathfrak{p}_0 = \mathbb{R}\,\mathrm{diag}(-3i, i)$, $\mathfrak{p}_6 = \mathbb{C} \cdot \mathrm{diag}(0, j)$,

$$\mathfrak{p}_2 = \left\{ \begin{pmatrix} w & -\bar{z} \\ z & 0 \end{pmatrix} : w \in \mathbb{C} \cdot j,\ z \in \mathbb{C} \right\} \quad \text{and} \quad \mathfrak{p}_4 = \left\{ \begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix} : w \in \mathbb{C} \cdot j \right\}.$$

In particular, $\mathfrak{p}_2$ is four-dimensional, whereas $\mathfrak{p}_4$ and $\mathfrak{p}_6$ are two-dimensional. Notice that each of these spaces is also Ad_a-invariant. Since G_b restricts to a multiple of the identity on $\mathfrak{p}_i$ for $i = 0, 4, 6$, it remains to show that $G_{b|\mathfrak{p}_2}$ commutes with $\mathrm{Ad}_{a|\mathfrak{p}_2}$ for a suitable choice of b.

The action of H induces a natural complex structure on $\mathfrak{p}_2$ and we can view $G_{|\mathfrak{p}_2}$ as hermitian endomorphism with respect to this complex structure and the scalar product Q. The element $\mathrm{Ad}_{a|\mathfrak{p}_2}$ corresponds to complex conjugation if we identify the real vector subspace

$$W = \left\{ \begin{pmatrix} \beta j & -\delta \\ \delta & 0 \end{pmatrix} \delta, \beta \in \mathbb{R} \right\}$$

of $\mathfrak{p}_2$ with $\mathbb{R}^2$. Hence if we consider the hermitian 2×2 matrix representing $G_{|\mathfrak{p}_2}$ with respect to an orthonormal basis of W, then G commutes with Ad_a if this matrix is real. If we replace G by G_b, the corresponding 2×2 matrix changes by conjugating it with $\mathrm{diag}(e^{i2\psi}, 1)$ and clearly we can turn the given hermitian matrix into a symmetric real matrix for a suitable choice of ψ.

Final remarks

An analysis of the proof shows: If a simply connected compact homogeneous space K/H satisfies the conclusion of the Berger Lemma, but does not admit an invariant metric of positive sectional curvature, then either one can find commuting eigenvectors or K/H is given by $\mathsf{Sp}(2)/\mathrm{diag}(z^3, z)$, $\mathsf{Sp}(2)/\mathrm{diag}(z, z)$ or $\mathsf{Sp}(3)/\{\mathrm{diag}(z, z, g) : z \in \mathsf{S}^1, g \in \mathsf{S}^3\}$. The first space is ruled out in Section 5.14. It is also the most difficult case in [2] where one finds another proof that it does not admit positive curvature, by exhibiting two commuting vectors (not necessarily eigenvectors) with zero curvature. The third space contains the second one as a totally geodesic submanifold. It was pointed out by M. Xu and J. A. Wolf that Bérard Bergery did not consider the most general class of metrics on the second space when he tried to rule out positive curvature. They also show that one can find metrics on it where all planes spanned by commuting vectors have positive curvature. We recall that we rule out this potential example in Section 5.5 using part (b) of Lemma 1.1.

References

[1] *S. Aloff* and *N. Wallach*, An infinite family of 7-manifolds admitting positively curved Riemannian structures, Bull. Amer. Math. Soc. **81** (1975), 93–97.

[2] *L. Bérard Bergery*, Les variétés´riemanniennes homogènes simplement connexes de dimension impaire à courbure strictement positive, J. Math. Pure Appl. (9) **55** (1976), 47–68.

[3] *M. Berger*, Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive, Ann. Sc. Norm. Super. Pisa Cl. Sci. (3) **15** (1961), 191–240.

[4] *F. Fang*, *K. Grove* and *G. Thorbergsson*, Tits geometry and positive curvature, preprint 2012, http://arxiv.org/abs/1205.6222v1.

[5] *K. Grove* and *K. Shankar*, Rank 2 fundamental groups of positively curved manifolds, J. Geom. Anal. **10** (2000), 679–682.

[6] *K. Grove*, *B. Wilking* and *W. Ziller*, Positively curved cohomogeneity one manifolds and 3-Sasakian geometry, J. Differential Geom. **78** (2008), 33–111.

[7] *K. Grove* and *W. Ziller*, Cohomogeneity one manifolds with positive Ricci curvature, Invent. Math. **149** (2002), 619–646.

[8] *T. Püttmann*, Optimal pinching constants of odd-dimensional homogeneous spaces, Invent. Math. **138** (1999), 631–684.

[9] *K. Shankar*, On the fundamental groups of positively curved manifolds, J. Differential Geom. **49** (1998), 179–182.

[10] *K. Shankar*, Isometry groups of homogeneous, positively curved manifolds, Differential Geom. Appl. **14** (2001), 57–78.

[11] *N. Wallach*, Compact homogeneous Riemannian manifolds with strictly positive curvature, Ann. of Math. (2) **96** (1972), 277–295.

[12] *B. Wilking*, The normal homogeneous space $\mathsf{SU}(3) \times \mathsf{SO}(3)/\mathsf{U}(2)$ has positive sectional curvature, Proc. Amer. Math. Soc. **127** (1999), 1191–1994.

[13] *B. Wilking*, Positively curved manifolds with symmetry, Ann. of Math. (2) **163** (2006), 607–668.

[14] *J. A. Wolf*, Spaces of constant curvature, 5th ed., Publish or Perish, Boston 1984.

[15] *J. A. Wolf* and *M. Xu*, $\mathsf{Sp}(2)/\mathsf{U}(1)$ and a positive curvature problem, preprint 2015, http://arxiv.org/abs/1502.02755.

[16] *W. Ziller*, Examples of Riemannian manifolds with nonnegative sectional curvature, in: Metric and comparison geometry, Surv. Differ. Geom. **11**, International Press, Somerville (2007), 63–102.

Burkhard Wilking, Mathematisches Institut, University of Münster,
Einsteinstraße 62, 48149 Münster, Germany
e-mail: wilking@math.uni-muenster.de

Wolfgang Ziller, Department of Mathematics, University of Pennsylvania,
Philadelphia, PA 19104, USA
e-mail: wziller@math.upenn.edu

Eingegangen 15. April 2015, in revidierter Fassung 13. Juni 2015