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NONNEGATIVELY CURVED MANIFOLDS WITH FINITE FUNDAMENTAL GROUPS ADMIT METRICS WITH POSITIVE RICCI CURVATURE

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In this paper we address the question whether a complete Riemannian metric of nonnegative sectional curvature can be deformed to a metric of positive Ricci curvature. This problem came up implicitly in various recent new constructions for metrics with positive Ricci curvature. Grove and Ziller [GZ] showed that any compact cohomogeneity one manifold with finite fundamental group admits invariant metrics with positive Ricci curvature. The case that both non-regular orbits have codimension two is especially resilient. By earlier work of Grove and Ziller it has been known that these manifolds admit invariant nonnegatively curved metrics. However, in certain cases the Ricci curvature of these metrics is not positive at any point and hence they cannot apply the deformation theorem of Aubin [A] and Ehrlich [E]: a metric of nonnegative Ricci curvature is conformally equivalent to a metric with positive Ricci curvature if and only if the Ricci curvature is positive at some point. Similar problems arise in the work of Schwachhöfer and Tuschmann on quotient spaces [ST].

Our main result is:

Theorem A. *Let (M^n, g) be a compact Riemannian manifold with finite fundamental group and nonnegative sectional curvature. Then M^n admits a metric with positive Ricci curvature.*

Notice that by the theorem of Myers a closed manifold with infinite fundamental group cannot admit a metric with positive Ricci curvature. Similarly to Aubin's and Ehrlich's theorem, our deformation can be chosen to be invariant under the isometry group as well.

We emphasize that Theorem A does not assert the existence of metrics which have both nonnegative sectional as well as positive Ricci curvature.

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This remains open. We should also mention that the assumption of nonnegative sectional curvature in the theorem cannot be replaced by nonnegative Ricci curvature. In fact there are Ricci flat manifolds which do not admit positive scalar curvature, e.g. K3 surface. Since these manifolds have special holonomy, one might ask whether compact manifolds with nonnegative Ricci curvature and generic holonomy admit a metric with positive Ricci curvature.

By the work of Nash [N] and Bérard-Bergery [Be1] any vector bundle of rank ≥ 2 over a compact Riemannian manifold with positive Ricci curvature admits a metric of positive Ricci curvature. By combining Theorem A with the soul theorem of Cheeger and Gromoll we obtain

Corollary B. *An open nonnegatively curved complete manifold with finite fundamental group admits a complete metric with positive Ricci curvature if and only if* $H_{n-1}(M^n, \mathbb{Z}_2) = 0$.

Unlike in the compact case, there are open manifolds with metrics of positive Ricci curvature and infinite fundamental group [Be2], [We], [BW].

To prove Theorem A we deform the given nonnegatively curved metric by a family of Riemannian metrics g_t satisfying Hamilton's evolution equation $\frac{\partial}{\partial t} g_t = -2\,\mathrm{Ric}(g_t)$. We consider a time dependent family of curvature conditions which lie strictly between nonnegative Ricci curvature and nonnegative sectional curvature. As a consequence of an extended version of Hamilton's maximum principle (see [CL]), the Ricci flow preserves this family of curvature conditions. It follows that for any nonnegatively curved initial metric g_0 on a closed manifold M^n there exists $\varepsilon = \varepsilon(n, g_0) > 0$ such that the Ricci curvature of the evolved metrics g_t is nonnegative for $t \in [0, \varepsilon)$. The strong maximum principle implies that the Ricci curvature becomes positive unless M contains a flat factor and consequently has infinite fundamental group. For the convenience of the reader we include a simplified proof of the extended maximum principle in section 1.

The proof of Theorem A, given in section 2, raises the question whether any of the above curvature conditions is itself invariant under the Ricci flow. Recall that the Ricci flow preserves several curvature conditions such as positive scalar curvature, positive curvature operator and several further conditions on Kähler manifolds. However, other natural curvature conditions are not preserved. Recently, Ni [Ni] described complete non-compact manifolds with bounded nonnegative sectional curvature such that the Ricci flow does not preserve nonnegative sectional curvature. Furthermore, Knopf [K] provided complete non-compact Kähler manifolds of bounded curvature and nonnegative Ricci curvature, which immediately develop mixed Ricci curvature when evolved by the Kähler Ricci flow (cf. [PhS]).

In the next theorem we present a closed manifold of this kind.

Theorem C. *On the compact manifold* $M^{12} := Sp(3)/Sp(1)Sp(1)Sp(1)$ *the Ricci flow evolves certain positively curved metrics into metrics with mixed Ricci curvature.*

Taking products with spheres yields similar nonnegatively curved examples of arbitrary large dimension. Hence in dimensions above 12 there is no curvature condition between nonnegative sectional curvature and nonnegative Ricci curvature which is invariant under the Ricci flow.

Let us describe the above example more precisely (cf. section 3). The homogeneous space M^{12} has a two-dimensional family of homogeneous unit volume metrics. The biinvariant metric of $Sp(3)$ induces on M^{12} a homogeneous unit volume Einstein metric g_E of nonnegative sectional curvature. By shrinking the round fibers $\mathbb{S}^4$ of the fibration $\mathbb{S}^4 \to M^{12} \to \mathbb{H}\mathrm{P}^2$ and rescaling, we obtain a curve g_t, $t > 1$, of unit volume submersion metrics with positive sectional curvature emanating from g_E. The curve g_t is up to reparameterization a solution to the normalized Ricci flow. By analyzing the asymptotic behavior of solutions of the Ricci flow, we prove that for any homogeneous non-submersion initial metric, being close enough to g_2, the normalized Ricci flow evolves mixed Ricci curvature.

1 Hamilton's Maximum Principle

Let $\pi\colon V \to M$ be a vector bundle over a compact smooth manifold M of dimension n with a fixed metric k on the fibers $V_p = \pi^{-1}(p)$, $p \in M$. Let g_t be a time dependent metric on M and let ∇_t^L denote the corresponding Levi–Civita connection on (M, g_t). Furthermore, let ∇_t denote a time dependent metric connection on V. For a section $R\colon M \to V$ of the vector bundle one can define a new section $\Delta_t R\colon M \to V$ as follows. For $p \in M$ choose an orthonormal basis of V_p and extend it along radial geodesics in (M, g_t) emanating from p by parallel transport of ∇_t to an orthonormal basis $X_1(q), \dots, X_d(q)$ of V_q for all q in a small neighborhood of p. If we put $f_i := k(X_i, R)$, then

$$(\Delta_t R)(p) = \sum_{i=1}^{d} (\Delta_t f_i) \cdot X_i(p)$$

where Δ_t on the right-hand side denotes the Laplace Beltrami operator of (M, g_t). Notice that the Laplace operator $\Delta_t R$ can also be defined invariantly from the connections ∇_t^L and ∇_t (cf. [H2]).

Suppose that a time dependent section $R(\cdot, t) \in \Gamma(V)$ satisfies the parabolic equation

$$\tfrac{\partial}{\partial t} R(p,t) = (\Delta_t R)(p,t) + f\big(R(p,t)\big)\,, \tag{1.1}$$

where $f : V \to V$ is a local Lipschitz map mapping each fiber V_q to itself. Roughly speaking Hamilton's maximum principle asserts that the dynamics of the partial differential equation (1.1) is controlled by dynamics of the ordinary differential equations

$$\tfrac{d}{dt} R_p(t) = f\big(R_p(t)\big) \tag{1.2}$$

in the fibers V_p, $p \in M$. In all applications f is invariant under parallel transport in the following sense: If c is a curve in M and the section R is parallel along c with respect to ∇_t then $f(R)$ is parallel along c, too. Notice that in this case it suffices to consider (1.2) on one fiber.

Next, we describe Hamilton's maximum principle more precisely. For $t \geq 0$ let $C(t)$ denote a closed subset of V, invariant under parallel transport by the connection ∇_t, such that for every $p \in M$ the set

$$C_p(t) = C(t) \cap V_p \subseteq V_p$$

is convex. We assume that the sets $C(t)$ depend continuously on t (that is $\lim_{t \to t_0} C(t) = C(t_0)$ in the pointed Hausdorff topology) and moreover that the family $\{C(t)\}$ is invariant under the ordinary differential equations (1.2): This means that for all $t_0 \geq 0$, $p \in M$ and $R_0 \in C_p(t_0)$ the solution $R_p(t)$ of (1.2) with $R_p(t_0) = R_0$ satisfies $R_p(t) \in C_p(t)$ for all $t \geq t_0$ (for which the solution $R_p(t)$ exists).

Now we can state a special case of a maximum principle [CL] for the parabolic equation (1.1), which generalizes Hamilton's maximum principle [H2]. For the convenience of the reader we include a short self-contained proof. Let us mentioned that in contrast to Hamilton's original approach we use the converse Dini derivatives for functions which are not differentiable. This seems to simplify the proof in particular for the more general situation considered in [CL].

Theorem 1.1. *For $t \in [0, \delta]$ let $C(t) \subseteq V$ be a closed subset, depending continuously on t. Suppose that each of the sets $C(t)$ is invariant under parallel transport, fiberwise convex and that the family $\{C(t)\}_{0 \leq t \leq \delta}$ is invariant under the ordinary differential equations* (1.2). *Then, for any* *solution* $R(p,t) \in \Gamma(V)$ *on* $M \times [0,\delta]$ *of the parabolic equation* (1.1) *with* $R(p,0) \in C(0)$, *we have* $R(p,t) \in C(t)$ *for all* $t \in [0,\delta]$.

Proof. For any $S \in V_q$ we let

$$r_t(S) = d_k\big(S, C_t(q)\big)$$

denote the distance between S and the convex set $C_q(t)$ in the fiber V_q. For a solution $R(p,t)$ to the parabolic equation (1.1), defined on $[0,\delta]$, we consider the maximal distance to $C(t)$,

$$s(t) := \sup_{p \in M} r_t\big(R(p,t)\big)\,.$$

Even though s is not differentiable we define

$$s'(t_0) := \limsup_{h \searrow 0} \frac{s(t_0) - s(t_0 - h)}{h}\,.$$

Let r_0 denote the maximum of s on $[0,\delta]$. By assumption, we can find a constant $L > 0$ such that the restriction of f to the ball $B_{2r_0}(R(t,q))$ is $L/2$-Lipschitz continuous for all $q \in M$ and $t \in [0,\delta]$. We will show below that $s'(t) \le L \cdot s(t)$ for all $t \in [0,\delta]$. Thus, for $g(t) := s(t) \cdot e^{-Lt}$ we get $g'(t) \le 0$ for all $t \in [0,\delta]$. If $s(0) = 0$, then $g(0) = 0$, which implies $g(t) \le 0$ for all $t \in [0,\delta]$, and the theorem is proved.

It remains to compute $s'(t)$. For $t_0 \in [0,\delta]$ there exists $p_o \in M$ with $s(t_0) = r_{t_0}(R(p_o,t_0))$. We may assume $s(t_0) > 0$. Clearly, for $h > 0$ we have $s(t_0 - h) \ge r_{t_0 - h}(R(p_o, t_0 - h))$, consequently

$$\begin{aligned}
s'(t_0) &\le \limsup_{h \searrow 0} \frac{r_{t_0}(R(p_o,t_0)) - r_{t_0-h}(R(p_o,t_0-h))}{h} \\
&\overset{(1.1)}{=} \limsup_{h \searrow 0} \frac{r_{t_0}(R(p_o,t_0)) - r_{t_0-h}(R(p_o,t_0) - h\Delta_{t_0}R(p_o,t_0) - hf(R(p_o,t_0))}{h} \\
&\le \limsup_{h \searrow 0} \frac{r_{t_0}(R(p_o,t_0)) - r_{t_0-h}(R(p_o,t_0) - hf(R(p_o,t_0))}{h}\,.
\end{aligned}$$

The equality holds since $R(p_o, t_0 - h) = R(p_o,t_0) - h \cdot \frac{d}{dt}|_{t=t_0} R(p_o,t) + o(h)$ and since the functions r_{t_0-h} are distance functions, hence uniformly Lipschitz continuous. To justify the last inequality we argue as follows: Using the convexity of $C(t)$ it is easy to see that for each t the function r_t is of class C^1 on $V \setminus C(t)$. We observe that the closed $s(t_0)$-tubular neighborhood $r_{t_0}^{-1}([0, s(t_0)])$ of C_{t_0} is convex. By construction the section R_{t_0} is contained in this neighborhood and we deduce from Lemma 1.2

$$k\big(\Delta_{t_0} R(t_0,p_o), \operatorname{grad}(r_{t_0})(R(t_0,p_o))\big) \le 0\,.$$

Since $C(t)$ is continuous with respect to the pointed Hausdorff topology it is easy to see that $\mathrm{grad}(r_t)$ is also continuous with respect to t. Thus for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$k\big(\Delta_{t_0} R(t_0, p_o), \mathrm{grad}(r_{t-h})(S))\big) \le \varepsilon$$

for all (S,h) with $|R(t_0,p_o) - S|_k + |h| \le \delta$. From the convexity of the function r_{t_0-h} it follows that

$$r_{t_0-h}\big(R(t_0,p_o) - hf(R(p_o,t_0) - h\Delta_{t_0} R(t_0,p_o))\big) \\ \ge -\varepsilon h + r_{t_0-h}\big(R(t_0,p_o) - hf(R(p_o,t_0)\big)$$

for small $h > 0$ which gives the asserted inequality, as $\varepsilon > 0$ was arbitrary.

Next, for $h > 0$ we choose the unique $S_h \in C_{p_o}(t_0 - h)$ with

$$r_{t_0-h}\big(R(p_o,t_0) - hf(R(p_o,t_0)\big) = \big|S_h + hf(R(p_o,t_0)) - R(p_o,t_0)\big|_k .$$

Applying triangle inequality and using that f is $L/2$-Lipschitz continuous on $B_{2s(t_0)}(R(p_o,t_0))$ we obtain

$$\begin{aligned} & r_{t_0-h}\big(R(p_o,t_0) - hf(R(p_o,t_0)\big) - r_{t_0}\big(R(p_o,t_0)\big) \\ & \ge \big|S_h + hf(S_h) - R(p_o,t_0)\big|_k - r_{t_0}\big(R(p_o,t_0)\big) - h\big|f(S_h) - f(R(p_o,t_0))\big|_k \\ & \ge d_k\big(S_h + hf(S_h), R(p_o,t_0)\big) - d_k\big(R(p_o,t_0), C(t_0)\big) - h\cdot L\cdot s(t_0) \\ & \ge -d_k\big(S_h + hf(S_h), C(t_0)\big) - h\cdot L\cdot s(t_0) . \end{aligned}$$

The term $S_h + hf(S_h)$ approximates the solution γ_{S_h} of (1.2) with $\gamma_{S_h}(t_0-h) = S_h \in C_{p_o}(t_0-h)$ up to first order. Since the family $C(t)$ is invariant under (1.2) we know $\gamma_{S_h}(t) \in C_{p_o}(t)$ for $t \ge t_0 - h$. Hence $d_k(C(t_0), S_h + hf(S_h)) = o(h)$ and we conclude $s'(t_0) \le L\cdot s(t_0)$ as claimed. □

The above maximum principle relies on the following elementary observation.

LEMMA 1.2. *Let $C \subset V$ be a closed subset such that $C_p = C \cap V_p$ is convex for all $p \in M$ and suppose that C is invariant under parallel transport with respect to a metric connection ∇_t. If $R \in \Gamma(V)$ is a section with $R_q \in C_q$ for all $q \in M$, then $\Delta_t R$ is a section with $(\Delta_t R)(q) \in T_{R(q)} C_q$, where $T_{R(q)} C_q$ denotes the tangent cone of the convex set $C_q \subset V_q$ at $R(q)$.*

Proof. Choose orthonormal vectorfields $X_1, \dots, X_d$ of V in a neighborhood of p. We may assume that each X_i is parallel along geodesics in M starting at p. Choose also an orthonormal basis $e_1, \dots, e_n$ of vectorfields in (M, g_t) in a neighborhood of p which are radially parallel. If we write R in coordinates

$R = \sum_{i=1}^{d} f_i \cdot X_i$ for suitable functions f_i then

$$(\Delta_t R)(p) = \sum_{i=1}^{d} (\Delta_t f_i)(p) \cdot X_i(p) = \sum_{j=1}^{n} \sum_{i=1}^{d} \big(e_j(p)(e_j f_i)\big) \cdot X_i(p)\,.$$

We let $\bar{C} \subset \mathbb{R}^d$ denote the convex set corresponding to C_p under the identification $V_q \cong \mathbb{R}^d$ induced by the choice of the above basis. Since $C_{\exp(se_j)}$ is parallel with respect to ∇_t along the geodesic $\exp(se_j)$ in (M, g_t), the curve $c_j(s) = (f_1, \dots, f_d)(\exp(se_j))$ is contained in $\bar{C} \subset \mathbb{R}^d$. Clearly this implies $c_j''(s) \in T_{h(s)}\bar{C}$, $\sum_j c_j''(0) \in T_{h(0)}\bar{C}$ and the result follows. □

Remark 1.3. (a) The maximum principle remains valid if f also depends explicitly on t. What is needed in the proof is that $f = f(t, R)$ is locally Lipschitz continuous.

(b) The condition that the family $C(t)$ is continuous with respect to the pointed Hausdorff topology can be removed. If R is a solution of (1.1) we can choose a compact convex subset $\tilde{C}(0) \subset C(0)$ containing the image of $R(\,\cdot\,, 0)$ which is invariant under parallel transport. Now one can consider $\bar{C}_t = \bigcap \tilde{C}(t)$, where the intersection is taken over all bounded closed convex families $\tilde{C}(t)_{0 \le t \le \delta'}$ extending the given set $\tilde{C}(0)$ which are invariant under parallel transport and invariant under (1.2). The *minimal* family $\bar{C}(t)$ is continuous with respect to the Hausdorff topology and from the theorem it follows that $R(q, t) \in \bar{C}_t \subset C_t$ for $t \in [0, \delta']$.

2 Ricci Flow Deformations

Let (M^n, g_0) be a compact Riemannian manifold with finite fundamental group and nonnegative sectional curvature. We consider an abstract vector bundle W isomorphic to TM and endow it with a fixed fiber metric k. Let g_t denote the solution to the unnormalized Ricci flow

$$\tfrac{\partial}{\partial t}\, g_t = -2\,\mathrm{Ric}(g_t)$$

with initial metric g_0. We choose an isometry $u : W \to TM$ at time $t = 0$ and let this isometry evolve by the equation

$$\tfrac{\partial}{\partial t}\, u_a^i = g^{ij}\,\mathrm{Ric}_{jk}\, u_a^k\,.$$

Then, the pull-back metric $k_{ab} = g_{ij} u_a^i u_b^j$ is constant in time [H2]. With the help of the isometries $u(t) : (W, k) \to (TM, g_t)$ we can pull back any vector bundle over M associated to the principal bundle P of orthonormal frames of (TM, g_t). In particular, the $(4, 0)$-curvature tensor R of the Riemannian manifold (M^n, g_t) can be considered a section in the associated bundle

$$V = P_W \times_{O(n)} S\big(\Lambda^2(\mathbb{R}^n)\big)\,,$$

where $S(\Lambda^2(\mathbb{R}^n))$ denotes the space of symmetric bilinear forms on $\Lambda^2(\mathbb{R}^n)= T_eO(n)$ and P_W is the principal bundle of orthonormal frames in (W,k). The fiber metric k on W induces a fiber metric on V again denoted by k. Notice that we use Hamilton's sign convention, that is on the round sphere we have $R_{xvxv} \geq 0$.

We can also pull back the Levi–Civita connection of the tangent bundle of $(M^n, g(t))$ and the induced connection of any associated vector bundle. As explained in the previous section the Laplacian of a section in such a bundle can be defined. One arrives at the following evolution equation for the curvature tensor R (see [H2]):

$$\tfrac{\partial}{\partial t} R_{abcd} = (\Delta R)_{abcd} + 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc})\,,$$

where

$$B_{abcd} := k^{ef} k^{gh} R_{eagb} R_{fchd}\,.$$

Notice that we are precisely in the situation described in the last section.

The first crucial step in the proof of Theorem A is the following

PROPOSITION 2.1. *For each $n \in \mathbb{N}$ and $\kappa > 0$ there is a constant $\varepsilon > 0$ such that the following holds. If (M, g_0) is a compact nonnegatively curved n-manifold with upper curvature bound κ, then the solution g_t of the Ricci flow with initial metric g_0 exists on $[0,\varepsilon]$ and all metrics g_t have nonnegative Ricci curvature.*

Proof. Let D_1, D_2, E_1, E_2 be positive constants to be fixed later on. For $t \geq 0$ we consider the subset $C(t)$ of curvature tensors R in V which satisfy the following constraint equations:

$$0 \leq \mathrm{Ric}(v,v) \quad \forall\, v \in W\,, \tag{2.1}$$

$$(R_{xvxw})^2 \leq (D_1 + tE_1)\cdot \mathrm{Ric}(v,v)\cdot \mathrm{Ric}(w,w) \quad \forall\, v,w,x \in W\,,\ |x| = 1\,, \tag{2.2}$$

$$\|R\| \leq D_2 + tE_2\,. \tag{2.3}$$

Here, $\|R\|$ denotes the 2-norm of the curvature tensor R.

Clearly, the sets $C(t)$ are closed and invariant under parallel transport. We show that for every $p \in M$ the sets $C_p(t) = C(t) \cap V_p$ are convex.

The set of curvature tensors satisfying (2.1) and (2.3) is obviously convex. Let $R, S \in V_p$ denote curvature tensors satisfying (2.1) and (2.2). For $v \in W_p$ we set $A = R_{xvxw}$, $a_1 = \mathrm{Ric}(v,v)$, $a_2 = \mathrm{Ric}(w,w)$ and for the tensor S we define B, b_1 and b_2 accordingly. Let $\lambda \in [0,1]$. We need to show that $\lambda R + (1-\lambda)S$ satisfies the equation (2.2) as well:

$$\begin{aligned}\big(\lambda A + (1-\lambda)B\big)^2 &\le \lambda^2 A^2 + 2\lambda(1-\lambda)|A||B| + (1-\lambda)^2 B^2\\ &\le (D_1 + tE_1)\cdot\big(\lambda^2 a_1 a_2 + 2\lambda(1-\lambda)\sqrt{a_1a_2b_1b_2} + (1-\lambda)^2 b_1 b_2\big)\\ &\le (D_1+tE_1)\cdot\big(\lambda a_1 + (1-\lambda)b_1\big)\cdot\big(\lambda a_2 + (1-\lambda)b_2\big)\,.\end{aligned}$$

The first inequality follows from Cauchy–Schwarz, the second one from (2.2). The last inequality is equivalent to $2\sqrt{a_1a_2b_1b_2} \le a_1b_2 + a_2b_1$ which follows from (2.1). This shows that the sets $C_p(t)$ are convex.

In order to apply Hamilton's maximum principle, stated in Theorem 1.1, we need to show that the family $\{C(t)\}_{0\le t\le\varepsilon}$ is invariant under the ordinary differential equation

$$\tfrac{d}{dt}R_{abcd} = 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}) \tag{2.4}$$

for suitable constants $\varepsilon, D_1, D_2, E_1, E_2$ depending on n and κ.

First notice that the curvature bounds for g_0 give a bound D_2 on the norm of R. Clearly we can find a constant $E_2 > 0$ and an $\varepsilon_1 > 0$ such that the family of curvature tensors satisfying (2.3) is invariant under the ordinary differential equation (2.4). The solution of the Ricci flow exists as long as the curvature tensor stays bounded. Combining with the maximum principle we deduce that there exists solution of the Ricci flow on the interval $[0,\varepsilon_1]$ and that its curvature tensor satisfies (2.3).

Next, observe that at time $t=0$ the constraint equation (2.2) is fulfilled with $D_1 = 1$ since the metric g_0 has nonnegative sectional curvature. This is seen as follows: The symmetric bilinear form $R_{x,\cdot,x,\cdot}$ has nonnegative eigenvalues $0 \le \lambda_1^2,\dots,\lambda_n^2$ corresponding to eigenvectors $e_1,\dots,e_n$ of norm one. Let $v = \sum_{i=1}^n v_i\cdot e_i$ and $w = \sum_{j=1}^n w_j \cdot e_j$. Then

$$\begin{aligned}(R_{xvxw})^2 &= \Big(\sum_{i=1}^n v_i w_i \lambda_i^2\Big)^2\\ &\le \Big(\sum_{i=1}^n (v_i\lambda_i)^2\Big)\cdot\Big(\sum_{j=1}^n (w_j\lambda_j)^2\Big)\\ &= R_{xvxv}\cdot R_{xwxw}\\ &\le \mathrm{Ric}(v,v)\cdot\mathrm{Ric}(w,w)\,.\end{aligned}$$

Put $D_1 := 1$ and choose constants E_2, D_2 with $D_2 + \varepsilon_1 E_2 \le 2D_2$. We claim that for suitable large $E_1 \ge 1/\varepsilon_1$ we can set $\varepsilon := 1/E_1$ and then the family $C(t)_{0\le t\le\varepsilon}$ is invariant under (2.4). Notice that $\varepsilon \le \varepsilon_1$.

As seen above the constraint equation (2.3) is invariant by itself. Next, let $R = R(t)$ be a solution of (2.4) and assume that $R(t) \in C(t)$ for some $t \in [0, \varepsilon]$. At a point p we choose an orthonormal basis $e_1, \dots, e_n$ with $\mathrm{Ric}_{ab} = \mathrm{Ric}(e_a, e_b) = 0$ for $a \neq b$. Since $\mathrm{Ric}_{ac} = k^{bd} R_{abcd}$ and $k'_{ab} = 0$, we deduce from (2.4) and [H1, Lem. 7.4]

$$\tfrac{d}{dt} \mathrm{Ric}_{ac} = 2k^{be} k^{df} R_{abcd} \mathrm{Ric}_{ef} = \sum_{b=1}^{n} 2R_{abcb} \mathrm{Ric}_{bb} \,.$$

Thus we get for unit vectors $v, w, x \in W_p$:

$$\begin{aligned}
&\tfrac{d}{dt}\big((1+tE_1)\cdot \mathrm{Ric}(v,v)\cdot \mathrm{Ric}(w,w) - (R_{vxwx})^2\big) \\
&\quad = E_1 \cdot \big|\mathrm{Ric}(v,v)\big| \cdot \big|\mathrm{Ric}(w,w)\big| + 2(1+tE_1) \sum_{a=1}^{n} R_{vava} \cdot \mathrm{Ric}_{aa} \cdot \mathrm{Ric}(w,w) \\
&\qquad + 2(1+tE_1) \sum_{a=1}^{n} R_{wawa} \cdot \mathrm{Ric}_{aa} \cdot \mathrm{Ric}(v,v) - 2R_{vxwx} \cdot \tfrac{\partial}{\partial t} R_{vxwx} \\
&\quad \geq E_1 \cdot \big|\mathrm{Ric}(v,v)\big| \cdot \big|\mathrm{Ric}(w,w)\big| - 4\sum_{a=1}^{n} |R_{vava}| \cdot \mathrm{Ric}_{aa} \cdot \mathrm{Ric}(w,w) \\
&\qquad - 4\sum_{a=1}^{n} |R_{wawa}| \cdot \mathrm{Ric}_{aa} \cdot \mathrm{Ric}(v,v) - 4\sqrt{\mathrm{Ric}(v,v)\,\mathrm{Ric}(w,w)} \cdot \big|\tfrac{d}{dt} R_{vxwx}\big| \,,
\end{aligned}$$

where we used $t \leq 1/E_1$ and the constraint equation (2.2) for the last inequality. We will show that this expression is nonnegative. The first term dominates the second and the third term by (2.2) and (2.3) if we choose E_1 large enough. By (2.4) we get

$$\begin{aligned}
\tfrac{d}{dt} R_{vxwx} = 2 \sum_{a,b=1}^{n} & R_{vaxb} R_{waxb} - R_{vaxb} R_{xawb} \\
& + R_{vawb} R_{xaxb} - R_{vaxb} R_{xawb} \,.
\end{aligned} \tag{2.5}$$

This equation remains valid if we replace $e_1, \dots, e_n$ by another orthonormal basis. Therefore without loss of generality $R_{xaxb} = 0$ for $a \neq b$. Thus

$$\begin{aligned}
\sum_{a,b=1}^{n} R_{vawb} R_{xaxb} &= \sum_{a=1}^{n} R_{xaxa} R_{vawa} \\
&\leq \mathrm{Ric}(x,x) \cdot \sqrt{(D_1 + tE_1) \cdot \mathrm{Ric}(v,v) \cdot \mathrm{Ric}(w,w)} \\
&\leq 2nD_2 \cdot \sqrt{\mathrm{Ric}(v,v) \cdot \mathrm{Ric}(w,w)} \,.
\end{aligned}$$

In order to get control on the other three summands in (2.5) it is sufficient to show that the norm $\|R_w\|$ of the tensor $R_{w,\cdot,\cdot,\cdot}$ is uniformly bounded by

$C\sqrt{|\operatorname{Ric}(w,w)|}$ for all unit vectors $w \in W_p$, where C is independent of E_1. To this end, let $x, \tilde{x}, u \in W_p$ with $\|x\| = \|\tilde{x}\| = \|u\| = 1$. Then, by (2.2) we know

$$(R_{u,(x+\tilde{x}),w,(x+\tilde{x})})^2 \leq 16(D_1 + tE_1) \cdot \operatorname{Ric}(u,u) \cdot \operatorname{Ric}(w,w)\,,$$

hence

$$\begin{aligned} |R_{uxw\tilde{x}} + R_{u\tilde{x}wx}| &\leq 6\sqrt{1+tE_1} \cdot \sqrt{\operatorname{Ric}(u,u)\operatorname{Ric}(w,w)} \\ &\leq \underbrace{12n \cdot \sqrt{D_2} \cdot \sqrt{\operatorname{Ric}(w,w)}}_{:=c_w}\,. \end{aligned} \tag{2.6}$$

By the first Bianchi identity

$$\begin{aligned} 0 &= R_{uxw\tilde{x}} + R_{u\tilde{x}xw} + R_{uw\tilde{x}x} \\ &= -R_{u\tilde{x}wx} - R_{u\tilde{x}wx} - R_{\tilde{x}xwu} + r_{u,w} \\ &= -3R_{u\tilde{x}wx} + r_{u,w} + r_{\tilde{x},w} \end{aligned}$$

with $|r_{\tilde{x},w}| \leq c_w$ and $|r_{u,w}| \leq c_w$. Therefore we can find a constant C which only depends on D_2 and n with $\|R_w\| \leq C\sqrt{|\operatorname{Ric}(w,w)|}$.

The first constraint equation $\operatorname{Ric} \geq 0$ is obviously fulfilled at time $t = 0$. To conclude that nonnegative Ricci curvature is preserved as well, for solutions which fulfill (2.2), we argue as follows. Because of (2.2) at each time t the Ricci curvature is either nonnegative or nonpositive. Thus if the Ricci curvature would change sign at some time $t_0 > 0$, then we would conclude $\operatorname{Ric}(t_0) = 0$ and, by equation (2.2), $R(t_0) = 0$. Using that $R(t)$ is a solution of the ordinary differential equation (2.4) we see $R(t) = 0$ for all t.

We conclude that the family $C(t)_{0\leq t\leq\varepsilon}$ is invariant under the ordinary differential equation (2.4). By Hamilton's maximum principle the same is true for the solution g_t of the Ricci flow, hence $\operatorname{Ric}(g_t) \geq 0$ for all $t \in [0,\varepsilon]$. □

One may ask whether the curvature conditions described by (2.1), (2.2), (2.3) are themselves invariant under the Ricci flow. The example described in Theorem C shows that in dimension $n = 12$ and higher this is not the case.

Proof of Theorem A. To establish Theorem A we have to show that the Ricci tensor becomes positive definite if we assume in addition that M has finite fundamental group. From the evolution equation

$$\frac{\partial}{\partial t}\operatorname{Ric}_{aa} = (\Delta \operatorname{Ric})_{aa} + 2R_{abad} \cdot \operatorname{Ric}_{bd}$$

for the Ricci curvature and from the above constraint equations, which are fulfilled by the solution g_t of the Ricci flow with initial metric g_0, we deduce the existence of a constant $H > 0$ such that the modified Ricci tensor

$$\widetilde{\mathrm{Ric}}(t) := e^{tH} \cdot \mathrm{Ric}(g(t))$$

satisfies the inequality

$$\tfrac{\partial}{\partial t}\, \widetilde{\mathrm{Ric}}_{aa} \geq (\Delta \widetilde{\mathrm{Ric}})_{aa} \tag{2.7}$$

for $t \in [0, \varepsilon]$.

Below, we will show that the rank of $\widetilde{\mathrm{Ric}}$ is constant on the interval $(0, \varepsilon]$. If $\widetilde{\mathrm{Ric}}$ has maximal rank then we are done. Thus, let v denote a smooth vector field on M, depending smoothly on $t \in I$, with $\widetilde{\mathrm{Ric}}(v, v) = 0$. Since $\widetilde{\mathrm{Ric}} \geq 0$, we deduce from (2.7)

$$0 = \left(\tfrac{\partial}{\partial t}\, \widetilde{\mathrm{Ric}}\right)(v, v) \geq 2 \sum_{a=1}^{n} \widetilde{\mathrm{Ric}}(\nabla_a v, \nabla_a v)\,.$$

This shows that the kernel of the Ricci tensor $\mathrm{Ric}(g(t))$ is invariant under parallel transport. Consequently, the Riemannian metrics $g(t)$ are Riemannian product metrics with one flat factor, a contradiction to the assumption that M^n has finite fundamental group.

It remains to show that the modified Ricci tensor $\widetilde{\mathrm{Ric}}(g(t))$ has constant rank for $t > 0$. Let $0 \leq \mu_1 \leq \cdots \leq \mu_n$ denote the eigenvalues of $\widetilde{\mathrm{Ric}}(g(t))$ and let

$$\begin{aligned} \sigma_l &:= \mu_1 + \cdots + \mu_l \\ &= \min\{\mathrm{tr}\, \widetilde{\mathrm{Ric}}|_X \mid X \subset \mathbb{R}^n \text{ subspace of dimension } l\}\,. \end{aligned}$$

Now fix $p \in M$ and let $(e_1(t_0), \ldots, e_l(t_0))$ denote an orthonormal basis of (W_p, k) such that $\sigma_l(t_0) = \sum_{i=1}^{l} \widetilde{\mathrm{Ric}}_{t_0}(e_i(t_0), e_i(t_0))$. Then we have

$$\begin{aligned} \sigma_l'(t_0) &:= \liminf_{t \nearrow t_0} \frac{\sigma_l(t_0) - \sigma_l(t)}{t_0 - t} \\ &\geq \tfrac{d}{dt}\Big|_{t=t_0} \sum_{i=1}^{l} \widetilde{\mathrm{Ric}}_t\big(e_i(t_0), e_i(t_0)\big) \\ &\geq \sum_{i=1}^{l} (\Delta \widetilde{\mathrm{Ric}})_{t_0}\big(e_i(t_0), e_i(t_0)\big)\,. \end{aligned}$$

As above we may extend the orthonormal basis $(e_1(t_0), \ldots, e_l(t_0))$ by radial parallel transport to a local orthonormal basis. The locally defined function $\sum_{i=1}^{l} \widetilde{\mathrm{Ric}}_{t_0}(e_i(t_0), e_i(t_0)) - \sigma_l$ attains a local minimum at p, hence in the sense of support functions we obtain

$$\tfrac{\partial}{\partial t}\, \sigma_l \geq \Delta \sigma_l\,.$$

From the strong maximum principle for functions it follows that for some $\varepsilon > 0$ we either have $\sigma_l \equiv 0$ on $(0, \varepsilon)$ or $\sigma_l > 0$ on $(0, \varepsilon]$. Observe that the strong maximum principle for functions satisfying the differential inequality can be easily deduced from the maximum principle for solutions by showing that the minimum of the function satisfying inequality growth faster than the minimum of the function satisfying the equality. Therefore the rank of $\widetilde{\mathrm{Ric}}$ is constant on $(0, \varepsilon]$. □

3 Ricci Flow and Positive Curvature

The Ricci flow does not preserve nonnegative sectional curvature on complete Riemannian manifolds with bounded curvature in general. Recently, Ni [Ni] has described such manifolds for the first time: The tangent bundle $T\mathbb{S}^n$, $n \geq 2$, of the sphere provides a concrete example.

We will describe compact manifolds for which the Ricci flow evolves an initial metric of positive sectional curvature to metrics with mixed Ricci curvature. Consequently the Ricci flow does neither preserve positive sectional curvature, nor nonnegative Ricci curvature, nor any condition in between on compact Riemannian manifolds.

Theorem 3.1. *On the flag manifold $Sp(3)/Sp(1)Sp(1)Sp(1)$ the Ricci flow evolves certain positively curved metrics into metrics with mixed Ricci curvature.*

Proof. Let $G/H = Sp(3)/Sp(1)Sp(1)Sp(1)$. We consider the symplectic group $G = Sp(3)$ as a subgroup of $Gl(3, \mathbb{H})$. On the Lie algebra $\mathfrak{g} = \mathfrak{sp}(3)$ of $G = Sp(3)$ we consider the $Ad(G)$-invariant scalar product

$$Q(V, W) = -\tfrac{1}{2}\mathrm{Re}\,\mathrm{tr}(V \cdot W)\,.$$

The orthogonal complement $\mathfrak{m}$ of the Lie subalgebra $\mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ of $\mathfrak{g}$ can be decomposed into three irreducible (and pairwise inequivalent) $Ad(H)$-invariant subspaces:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3\,.$$

We have $\mathfrak{sp}(2)_i \oplus \mathfrak{sp}(1)_i = \mathfrak{h} \oplus \mathfrak{m}_i$, where $\mathfrak{sp}(2)_i \oplus \mathfrak{sp}(1)_i$ denotes the Lie algebra of one of the three intermediate subgroups $H \subsetneq K_1, K_2, K_3 \subsetneq G$ which are all isomorphic to $Sp(2) \times Sp(1)$.

It follows from Schur's lemma, that every $Ad(H)$-invariant scalar product on $\mathfrak{m}$ can be described by

$$g = x_1 \cdot Q|_{\mathfrak{m}_1} \perp x_2 \cdot Q|_{\mathfrak{m}_2} \perp x_3 \cdot Q|_{\mathfrak{m}_3}$$

where $x_1, x_2, x_3 > 0$. As a consequence, the space of G-invariant metrics on G/H is 3-dimensional, parameterized by the positive real numbers (x_1, x_2, x_3). By the same reasoning we have

$$\mathrm{Ric}(g) = (x_1 \cdot r_{11}) \cdot Q|_{\mathfrak{m}_1} \perp (x_2 \cdot r_{22}) \cdot Q|_{\mathfrak{m}_2} \perp (x_3 \cdot r_{33}) \cdot Q|_{\mathfrak{m}_3}$$

for $r_{11}, r_{22}, r_{33} \in \mathbb{R}$.

In order to compute the Ricci tensor of a G-invariant metric on G/H we apply the following formula (cf. [WaZ], [PS]):

$$r_{ii} \;=\; \frac{b_i}{2x_i} - \frac{1}{4d_i} \sum_{j,k} [ijk] \frac{2x_k^2 - x_i^2}{x_i x_j x_k} \,.$$

In this formula, for each i, $-B|_{\mathfrak{m}_i} = b_i Q|_{\mathfrak{m}_i}$, where B denotes the Killing form of G, and $d_i = \dim \mathfrak{m}_i$; the triple $[ijk] = \sum Q([X_\alpha, X_\beta], X_\gamma)^2$ is summed over $\{X_\alpha\}$, $\{X_\beta\}$, and $\{X_\gamma\}$, Q-orthonormal bases for $\mathfrak{m}_i$, $\mathfrak{m}_j$, and $\mathfrak{m}_k$, respectively. Notice that $[ijk]$ is totally symmetric in i, j, k.

In our case we have $d_1 = d_2 = d_3 = 4$, $b_1 = b_2 = b_3 = 32$, $[123] = 16$ and all other structure constants vanish. We conclude

$$r_{11} = 2 \cdot \left(\frac{8}{x_1} + \frac{x_1}{x_2 x_3} - \frac{x_2}{x_1 x_3} - \frac{x_3}{x_1 x_2} \right),$$
$$r_{22} = 2 \cdot \left(\frac{8}{x_2} + \frac{x_2}{x_1 x_3} - \frac{x_3}{x_1 x_2} - \frac{x_1}{x_2 x_3} \right),$$
$$r_{33} = 2 \cdot \left(\frac{8}{x_3} + \frac{x_3}{x_1 x_2} - \frac{x_1}{x_2 x_3} - \frac{x_2}{x_1 x_3} \right).$$

Hence, the scalar curvature $\mathrm{scal}(g)$ of g is given by

$$\mathrm{scal}(g) = 8 \cdot \left(\frac{8}{x_1} + \frac{8}{x_2} + \frac{8}{x_3} - \frac{x_1}{x_2 x_3} - \frac{x_2}{x_1 x_3} - \frac{x_3}{x_1 x_2} \right).$$

As a consequence, the normalized Ricci flow $\frac{\partial}{\partial t} g_t = -2\,\mathrm{Ric}(g_t) + \frac{2}{n} \cdot \mathrm{scal}(g_t) \cdot g_t$ for homogeneous initial metrics is given by

$$x_1' = \frac{8}{3} \cdot \left(-8 - 2 \cdot \frac{x_1^2}{x_2 x_3} + \left(\frac{x_2}{x_3} + \frac{x_3}{x_2} \right) + 4 \cdot \left(\frac{x_1}{x_2} + \frac{x_1}{x_3} \right) \right)$$
$$x_2' = \frac{8}{3} \cdot \left(-8 - 2 \cdot \frac{x_2^2}{x_1 x_3} + \left(\frac{x_1}{x_3} + \frac{x_3}{x_1} \right) + 4 \cdot \left(\frac{x_2}{x_1} + \frac{x_2}{x_3} \right) \right)$$
$$x_3' = \frac{8}{3} \cdot \left(-8 - 2 \cdot \frac{x_3^2}{x_1 x_2} + \left(\frac{x_1}{x_2} + \frac{x_2}{x_1} \right) + 4 \cdot \left(\frac{x_3}{x_1} + \frac{x_3}{x_2} \right) \right).$$

Since the normalized Ricci flow keeps the volume constant we may assume

$$x_1 x_2 \equiv \tfrac{1}{x_3} \,.$$

Plugging x_3 into the first two equations and rescaling the vector field by the factor 3/8 we get

$$x_1' = -8 - 2 \cdot x_1^3 + \left(x_1 x_2^2 + \frac{1}{x_1 x_2^2} \right) + 4 \cdot \left(\frac{x_1}{x_2} + x_1^2 x_2 \right)$$
$$x_2' = -8 - 2 \cdot x_2^3 + \left(x_1^2 x_2 + \frac{1}{x_1^2 x_2} \right) + 4 \cdot \left(\frac{x_2}{x_1} + x_2^2 x_1 \right).$$

Notice the symmetry of both these equation with respect to exchanging x_1 with x_2. Based on this observation we introduce the new coordinates

$$\varphi = x_1 + x_2 \quad \text{and} \quad \psi = x_1 - x_2\,.$$

A computation shows that the above system is equivalent to

$$\varphi' = -16 + \frac{3}{4} \cdot \varphi^3 - \frac{11}{4} \cdot \varphi\psi^2 + 16 \cdot \frac{\varphi}{(\varphi^2 - \psi^2)^2} + 8 \cdot \frac{\varphi^2 + \psi^2}{\varphi^2 - \psi^2}$$
$$\psi' = \psi \cdot \left(-\frac{3}{4} \cdot \varphi^2 - \frac{5}{4} \cdot \psi^2 + 16 \cdot \frac{\varphi}{\varphi^2 - \psi^2} + 16 \cdot \frac{1}{(\varphi^2 - \psi^2)^2} \right).$$

The set $\{\psi \equiv 0\}$ is invariant under the above ordinary differential equation.

We are interested in initial values $(\varphi(0), \psi(0))$ with $\varphi(0) = N$ very large (but fixed) and $\psi(0) < 0$ with $|\psi(0)|$ very small to be fixed later on. The above system shows clearly that for such initial values we have $\varphi' > 1$ and $\psi' > 0$. That is, the φ-axis is (asymptotically) a local attractor. We change the parameterization of solutions again and consider the system

$$\varphi' = 1$$
$$\psi' = \psi \cdot \frac{-\frac{3}{4} \cdot \varphi^2 - \frac{5}{4} \cdot \psi^2 + 16 \cdot \frac{\varphi}{\varphi^2 - \psi^2} + 16 \cdot \frac{1}{(\varphi^2 - \psi^2)^2}}{-16 + \frac{3}{4} \cdot \varphi^3 - \frac{11}{4} \cdot \varphi\psi^2 + 16 \cdot \frac{\varphi}{(\varphi^2 - \psi^2)^2} + 8 \cdot \frac{\varphi^2 + \psi^2}{\varphi^2 - \psi^2}}\,.$$

In these new coordinates the Ricci curvature r_{11} can be written as follows:

$$r_{11} = 2 \cdot \left(\frac{8}{x_1} + x_1^2 - x_2^2 - \frac{1}{x_1^2 x_2^2} \right) = 2 \cdot \left(\frac{16}{\varphi - \psi} + \varphi \cdot \psi - \frac{16}{(\varphi^2 - \psi^2)^2} \right). \quad (3.1)$$

In order to conclude that r_{11} turns negative for large t, we will show that ψ does not converge to fast to zero. We have

$$\varphi(t) = N + t$$

and

$$\psi' \le -\eta \cdot \psi \cdot \tfrac{1}{\varphi} \quad (3.2)$$

where the second inequality holds with $\eta > 1$ for all large t. In particular, we may assume $\eta < 2$. Since $\psi(t) < 0$ for all $t \ge 0$ we conclude

$$\psi(t) \le -\psi(0) \cdot \frac{\varphi(0)^\eta}{\varphi(t)^\eta}\,.$$

It follows

$$\lim_{t\to+\infty} \psi(t)\cdot\varphi^2(t) = -\infty\,,$$

which in view of (3.1) shows that r_{11} is negative for large t.

Now it is well known that any metric with $x_1 = x_2 > x_3$ has positive sectional curvature [W]. By continuity we can find positively curved metrics with $x_1 + x_2 = N = \varphi(0)$ and $x_1 - x_2 = \psi(0) < 0$ as small as we like. □

Let us mention that the above local analysis cannot be applied to the complex flag manifold $SU(3)/T^2$. For this homogeneous space the structure constants are different, so that in (3.2) the best possible η must be bigger than two. An even more careful computation shows that in this case almost submersion metrics do not develop mixed Ricci curvature. Notice however, this does not exclude the above theorem to be true in this case, too.

Remark 3.2. (a) It is not hard to show that one can find initial metrics of positive sectional curvature which are arbitrarily close to the normal homogeneous Einstein metric $(1,1,1)$ and which evolve mixed Ricci curvature under the Ricci flow.

(b) Using similar methods as above one can show that on $SU(3)/T^2$ the Ricci flow does not preserve positive sectional curvature. In this case we consider submersion metrics with slightly blown up fibers $\mathbb{S}^2$. Again, the corresponding curve of unit volume homogeneous metrics is a solution to the normalized Ricci flow. It starts at the normal homogeneous Einstein metric and approaches the homogeneous Kähler Einstein metric on $SU(3)/T^2$, which does not have nonnegative sectional curvature but positive Ricci curvature of course. More generally, on $U(3p)/U(p)U(p)U(p)$, $p \geq 1$, and also on real flag manifolds $SO(3p)/SO(p)SO(p)SO(p)$, $p \geq 3$, the Ricci flow does not preserve nonnegative sectional curvature.

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