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Rigidity of group actions on solvable Lie groups

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Abstract. We establish analogs of the three Bieberbach theorems for a lattice Γ in a semidirect product $\mathsf{S} \rtimes \mathsf{K}$ where S is a connected, simply connected solvable Lie group and K is a compact subgroup of its automorphism group. We first prove that the action of Γ on S is metrically equivalent to an action of Γ on a supersolvable Lie group. The latter is shown to be determined by Γ itself up to an affine diffeomorphism. Then we characterize these lattices algebraically as polycrystallographic groups. Furthermore, we realize any polycrystallographic group Γ as a lattice in a semidirect product $\mathsf{S} \rtimes \mathsf{F}$ with F being a finite group whose order is bounded by a constant only depending on the dimension of S. This generalization of the first Bieberbach theorem is used to obtain a partial generalization of the third one as well. Finally we show for any torsion free closed subgroup $\Upsilon \subset \mathsf{S} \rtimes \mathsf{K}$ that the quotient S/Υ is the total space of a vector bundle over a compact manifold B, where B is the quotient of a solvable Lie group by a torsion free polycrystallographic group.

1 Introduction and main results

The classical Bieberbach theorems investigate the structure of crystallographic groups, i.e. of discrete cocompact subgroups of the isometry group of the Euclidean space $\mathrm{Iso}(\mathbb{R}^d) = \mathbb{R}^d \rtimes \mathrm{O}(d)$.

Bieberbach's First Theorem. *Let $\Gamma \subset \mathbb{R}^d \rtimes \mathrm{O}(d)$ be a crystallographic group. Then $\Gamma \cap \mathbb{R}^d$ has finite index in Γ.*

Bieberbach's Second Theorem. *Let $\Gamma_1, \Gamma_2 \subset \mathbb{R}^d \rtimes \mathrm{O}(d)$ be two crystallographic groups. Suppose there exists an isomorphism $\iota\colon \Gamma_1 \to \Gamma_2$ of abstract groups. Then ι is given by conjugation with an element in the group of affine motions $\mathbb{R}^d \rtimes \mathrm{GL}(d)$.*

Bieberbach's Third Theorem. *In each dimension there are only finitely many isomorphism classes of crystallographic groups.*

We will study discrete, cocompact subgroups of semidirect products $\mathsf{S} \rtimes \mathsf{K}$ where S is a connected, simply connected solvable Lie group and K is a compact

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subgroup of its automorphism group $\mathrm{Aut}(\mathsf{S})$. Recall that a connected solvable Lie group S contains closed subgroups

$$\{e\} = \mathsf{N}_1 \subset \cdots \subset \mathsf{N}_k = \mathsf{S}$$

such that N_i is normal in N_{i+1} and $\mathsf{N}_{i+1}/\mathsf{N}_i \cong \mathbb{R}$ or $\mathsf{N}_{i+1}/\mathsf{N}_i \cong \mathsf{S}^1$. If the groups $\mathsf{N}_1, \ldots, \mathsf{N}_k$ are normal in S, the group S is called supersolvable. A connected nilpotent Lie group is supersolvable; the converse however is not true.

The automorphism group $\mathrm{Aut}(\mathsf{S})$ of a simply connected Lie group S is a Lie group with finitely many connected components. Consequently, any compact subgroup of $\mathrm{Aut}(\mathsf{S})$ is contained in a maximal compact subgroup and all maximal compact subgroups are conjugate, compare Remark 3.1. Notice that a semidirect product $\mathsf{S} \rtimes \mathsf{K}$ acts on S by $(\tau, A) \star v = \tau \cdot A(v)$ for $(\tau, A) \in \mathsf{S} \rtimes \mathsf{K}$, $v \in \mathsf{S}$.

Theorem 1. *Let S be a connected, simply connected solvable Lie group, and let $\mathsf{K} \subset \mathrm{Aut}(\mathsf{S})$ be a maximal compact subgroup. Then there is*

a) a unique maximal connected, simply connected supersolvable normal subgroup R of $\mathsf{S} \rtimes \mathsf{K}$ such that K also can be viewed as a subgroup of $\mathrm{Aut}(\mathsf{R})$,
b) an isomorphism $\iota\colon \mathsf{R} \rtimes \mathsf{K} \to \mathsf{S} \rtimes \mathsf{K}$ of Lie groups and
c) an equivariant isometry $f\colon (\mathsf{R}, g_1) \to (\mathsf{S}, g)$ for suitable left invariant metrics g_1, g on R and S, i.e. $f(h \star v) = \iota(h) \star f(v)$ for $v \in \mathsf{R}$ and $h \in \mathsf{R} \rtimes \mathsf{K}$. More precisely, if g is a left invariant metric on S such that $g_{|e}$ is invariant under the natural representation of K in the Lie algebra $\mathfrak{s}$ of S, then the pull back metric $g_1 := f^*g$ is left invariant on R.

So we may restrict attention to actions on supersolvable Lie groups, and thereby we can view the following theorem as an analogue of the second Bieberbach theorem:

Theorem 2. *Let S_i be a connected, simply connected supersolvable Lie group, $\mathsf{K}_i \subset \mathrm{Aut}(\mathsf{S}_i)$ a compact subgroup, and let $\Gamma_i \subset \mathsf{S}_i \rtimes \mathsf{K}_i$ be a discrete cocompact subgroup, $i = 1, 2$. Suppose there exists an isomorphism $\iota\colon \Gamma_1 \to \Gamma_2$ of abstract groups. Then there is an isomorphism $\varphi\colon \mathsf{S}_1 \to \mathsf{S}_2$ and an element $\tau \in \mathsf{S}_2$ such that the affine diffeomorphism*

$$f\colon \mathsf{S}_1 \to \mathsf{S}_2, \quad v \mapsto \varphi(v) \cdot \tau$$

is equivariant. In particular, $\iota(\gamma) \star w = f(\gamma \star f^{-1}(w))$ for all $w \in \mathsf{S}_2$, $\gamma \in \Gamma_1$.

In the special case of a nilpotent Lie group S the theorem is due to Auslander (1961a), and the group Γ_1 is then called an *almost crystallographic group*.

Theorem 2 is also a partial generalization of the main result of Farrell and Jones (1997). They proved for a pair of torsion free, closed, cocompact subgroups $\Upsilon_i \subset \mathsf{S}_i \rtimes \mathsf{K}_i$ for which the identity components are contained in the nilradicals of S_i ($i = 1, 2$) that the following holds: Any isomorphism $\pi_1(\mathsf{S}_1/\Upsilon_1) \to$

$\pi_1(\mathsf{S}_2/\Upsilon_2)$ between the fundamental groups is induced by a diffeomorphism $\mathsf{S}_1/\Upsilon_1 \to \mathsf{S}_2/\Upsilon_2$ provided that $\dim(\mathsf{S}_i/\Upsilon_i) \neq 4$. In the special case of discrete groups Υ_1 and Υ_2 this is of course equivalent to saying that for any isomorphism $\Upsilon_1 \to \Upsilon_2$ there exists a corresponding equivariant diffeomorphism $\mathsf{S}_1 \to \mathsf{S}_2$. Farrell and Jones (1997) employ strong topological theorems to obtain their result. Our proof in contrast is elementary although it is quite long. Notice also that Theorem 2 is stronger to some extent, since we do not require that the considered groups are torsion free and since we prove in a "normalized category" the existence of an affine diffeomorphism.

In order to generalize the other Bieberbach theorems we first characterize the above lattices algebraically. Therefore we recall that a group Λ is called polycyclic if there are subgroups

$$\{e\} = \mathsf{N}_1 \subset \cdots \subset \mathsf{N}_k = \Lambda$$

such that N_i is a normal subgroup of N_{i+1} and the factor group $\mathsf{N}_{i+1}/\mathsf{N}_i$ is cyclic. The number of factor groups satisfying $\mathsf{N}_{i+1}/\mathsf{N}_i \cong \mathbb{Z}$ does not depend on the choice of the subgroups and is called the Hirsch-rank or for short the rank of Λ. The group Λ is called strongly polycyclic if for a suitable choice $\mathsf{N}_{i+1}/\mathsf{N}_i \cong \mathbb{Z}$ for all i. If Π is a group containing a polycyclic subgroup Λ of finite index, we define $\operatorname{rank}(\Pi) := \operatorname{rank}(\Lambda)$. This definition is easily seen to be independent of the choice of Λ. Finally the nilradical $\operatorname{nil}(\Pi)$ of Π is then defined as the maximal nilpotent normal subgroup of Π.

Theorem 3. *For a group Γ the following statements are equivalent.*

a) *Γ is isomorphic to a discrete, cocompact subgroup of a semidirect product $\mathsf{S}_1 \rtimes \mathsf{K}$, where S_1 is a connected, simply connected solvable Lie group and K is a compact subgroup of the automorphism group* $\operatorname{Aut}(\mathsf{S}_1)$.
b) *There is an almost crystallographic group Γ_N, a crystallographic group Γ_A and an exact sequence*

$$\{1\} \to \Gamma_N \to \Gamma \to \Gamma_A \to \{1\}.$$

c) *There are subgroups*

$$\{e\} = \Gamma_0 \subset \cdots \subset \Gamma_n = \Gamma$$

such that Γ_i is a normal subgroup of Γ_{i+1} and the factor group Γ_{i+1}/Γ_i is isomorphic to a crystallographic group.
d) *Γ is polycyclic up to finite index, and Γ does not contain any nontrivial finite normal subgroup.*
e) *Γ contains a strongly polycyclic normal subgroup Λ of finite index such that the centralizer of Λ is contained in Λ.*

f) Γ *is isomorphic to a discrete, cocompact subgroup of a semidirect product* $\mathsf{S}_2 \rtimes \mathsf{F}$*, where* S_2 *is a connected, simply connected solvable Lie group and* F *is a finite subgroup of the automorphism group* $\mathrm{Aut}(\mathsf{S}_2)$.

The equivalence $d) \Leftrightarrow b)$ is due to Dekimpe. Moreover, he also proved $e) \Rightarrow d)$ and $a) \Rightarrow d)$, see (Dekimpe, 1996, Theorems 3.4.3 and 3.4.6).

Auslander and Johnson (1976) have verified a conjecture that is related to the implication $d) \Rightarrow f)$: Under the additional assumption that Γ is torsion free they have realized Γ as the fundamental group of a compact manifold that is finitely covered by a solvmanifold. The implication $a) \Rightarrow f)$ is due to Farrell and Jones (1997). Actually they proved it only under the additional assumption that Γ is torsion free but their proof carries over to the present situation.

Notice that the implication $a) \Rightarrow f)$ can be viewed as a partial generalization of the first Bieberbach theorem, since the actions of Γ on S_1 and S_2 arising from the conditions a) and f) are by the Theorems 1 and 2 equivalent.

Condition c) in the above theorem suggests the following notation: A group Γ is called polycrystallographic, if and only if it satisfies one of the conditions of Theorem 3. Using the above theorems it is easy to see:

Corollary 4. *Let* Γ *be a polycrystallographic group. Then there is a connected, simply connected supersolvable Lie group* S*, a compact subgroup* K *of its automorphism group and a homomorphism* $\iota\colon \Gamma \to \mathsf{S} \rtimes \mathsf{K}$ *mapping* Γ *isomorphically onto a discrete, cocompact subgroup, such that* $\iota(\Gamma)\cdot\big(\mathsf{S}\times\{e\}\big)$ *is dense in* $\mathsf{S}\rtimes\mathsf{K}$.

Moreover, if $\iota_2\colon \Gamma \to \mathsf{S}_2 \rtimes \mathsf{K}_2$ *is another embedding satisfying the above assumptions, there is a unique isomorphism* $\varphi\colon \mathsf{S}_2 \rtimes \mathsf{K}_2 \to \mathsf{S} \rtimes \mathsf{K}$ *for which* $\varphi \circ \iota_2 = \iota$*, and then* $\varphi(\mathsf{S}_2 \times \{e\}) = \mathsf{S} \times \{e\}$.

The embedding of Corollary 4 has nice algebraic properties, see Sect. 8.

If Γ is an almost crystallographic group, then the group K in Corollary 4 is finite, the group $\Gamma^* := \iota^{-1}(\mathsf{S} \times \{e\})$ can be viewed as the translational part of Γ, and it coincides with the nilradical of Γ. A theorem of Dekimpe et al. (1994) generalizing the third Bieberbach theorem states that there are only finitely many almost crystallographic groups containing a fixed group as its nilradical.

However, in the general situation the group K is not finite and for that reason we also consider different embeddings:

Theorem 5. *For a polycrystallographic group* Γ *of rank* n *there is*

(i) *a connected, simply connected solvable Lie group* S*,*
(ii) *a finite subgroup* $\mathsf{F} \subset \mathrm{Aut}(\mathsf{S})$ *with* $\mathrm{ord}(\mathsf{F}) \le C_n$*, where* C_n *is a constant only depending on* n *and*
(iii) *a homomorphism* $\iota\colon \Gamma \to \mathsf{S} \rtimes \mathsf{F}$ *satisfying* $\mathsf{S} \rtimes \mathsf{F} = \iota(\Gamma)\cdot(\mathsf{S}\times\{e\})$ *and mapping* Γ *isomorphically onto a discrete, cocompact subgroup of* $\mathsf{S} \rtimes \mathsf{F}$

for which the following holds: Let $\Gamma' \subset \Gamma$ *be a subgroup of finite index, and let* $\mathsf{F}' \subset \mathsf{F}$ *be the unique group with* $\mathsf{S} \rtimes \mathsf{F}' = \iota(\Gamma')\cdot(\mathsf{S}\times\{e\})$*. Then any* *automorphism of* $\iota(\Gamma') \cong \Gamma'$ *can be extended uniquely to an automorphism of* $\mathsf{S} \rtimes \mathsf{F}'$.

The embedding ι in Theorem 5 is not unique, and the group $\Gamma^* := \iota^{-1}(\mathsf{S} \times \{e\})$ is not independent of the choice of ι. However, the index of the subgroup Γ^* in Γ is bounded by C_n, and a theorem of Segal (1978) states that there are only finitely many isomorphism classes of groups containing a fixed polycyclic group as a normal subgroup of a given index. Thus we can regard Segal's result in connection with Theorem 5 as a generalization of Bieberbach's third theorem.

Finally, we consider torsion free subgroups and the corresponding quotients.

Theorem 6. *Let* S *be a connected, simply connected solvable Lie group,* $\mathsf{K} \subset \mathrm{Aut}(\mathsf{S})$ *a compact subgroup, and let* $\Upsilon \subset \mathsf{S} \rtimes \mathsf{K}$ *be a torsion free closed subgroup. Then*

a) S/Υ *is the total space of a vector bundle over a compact manifold* B.
b) The fundamental group $\Gamma := \pi_1(B)$ *is a torsion free polycrystallographic group, and for an embedding* $\iota\colon \Gamma \to \hat{\mathsf{S}} \rtimes \hat{\mathsf{K}}$ *satisfying the assumption of Corollary 4 the quotient* $\hat{\mathsf{S}}/\Gamma$ *is diffeomorphic to* B.

If in addition the identity component of Υ *is contained in a normal supersolvable subgroup of* S*, then a finite cover of* S/Υ *is diffeomorphic to a product of a compact manifold and a vector space.*

In the special case $\mathsf{K} = \{e\}$ the statement a) of the theorem was conjectured by Mostow (1951) and proved by Auslander and Tolimieri (1970). Furthermore, Mostow (1951) has shown that a finite cover of a noncompact solvmanifold is homeomorphic to a product of a compact solvmanifold and a vectorspace.

The action of $\mathsf{S} \rtimes \mathsf{K}$ on S is isometric with respect to a suitable left invariant metric g on S. At first view it seems to be more general to consider a subgroup $\Upsilon \subset \mathsf{S} \rtimes \mathsf{K}$ for which the quotient $(\mathsf{S}, g)/\Upsilon$ is a Riemannian manifold. However, by applying (Eschenburg, 1984, Satz 12,13) one can show that such a manifold is isometric to a quotient $(\mathsf{S}', g')/\Upsilon'$ with Υ' being torsion free.

Remarks. 1. The manifolds occurring in Theorem 6 are called infrasolvmanifolds. Compact infrasolvmanifolds have a nice geometric characterization. According to Tuschmann (1997) a compact topological manifold M is homeomorphic to an infrasolvmanifolds if and only if M admits a sequence of Riemannian structures g_μ with uniformly bounded sectional curvature such that $(M, g_\mu)_{\mu\in\mathbb{N}}$ collapses in the Gromov Hausdorff sense to a flat orbifold.

2. After finishing this paper the author realized that compact solvmanifolds can also be used to construct compact Riemannian manifolds with noncompact holonomy groups, see Wilking (1999).

3. Discrete subgroups of supersolvable Lie groups resemble in many respects discrete subgroups of nilpotent Lie groups, compare Subsect. 8.2. It would be

interesting to know whether compact supersolvmanifolds, i.e. quotients of supersolvable Lie groups by lattices, have any special geometric properties.

4. Suppose that Π is a group that is polycyclic up to finite index. It is elementary to show that Π contains a maximal finite normal subgroup $\mathsf{E} \subset \Pi$. By condition d) of Theorem 3 the factor group Π/E is polycrystallographic. So polycrystallographic groups might be of algebraic interest as well.

5. A subgroup Γ of the group of affine motions $\mathbb{R}^d \rtimes \mathrm{GL}(d)$ is called an affine crystallographic group if the corresponding action of Γ on $\mathbb{R}^d$ is discontinuous and cocompact. It is conjectured that affine crystallographic groups are virtually polycyclic. Evidently, a virtually polycyclic affine crystallographic group Γ does not contain any finite normal subgroup, and hence it is polycrystallographic. Moreover, it is not hard to see that the affine action of Γ on $\mathbb{R}^d$ is smoothly equivalent to the action of Γ on the supersolvable Lie group arising from the embedding in Corollary 4; in fact the supersolvable Lie group acts then simply transitive on $\mathbb{R}^d$ by affine diffeomorphisms. Of course one can use this to obtain structure results for virtually polycyclic affine crystallographic groups. However, there is already a nice structure theory for these groups, see Grunewald and Segal (1994). Also notice that not any polycrystallographic group is affine crystallographic, see (Benoist, 1995) for a nilpotent counterexample. Hence the results in (Grunewald and Segal, 1994) do not imply structure results for polycrystallographic groups.

Contents

The main aim of this paper is, of course, to prove the main results. However, some of the results stated below might be of some interest in itself: In Sect. 2 we give two counterexamples. They will uncover some mistakes occurring in this context in the literature. Lemma 4.1 introduces a different characterization of supersolvable Lie groups that is needed subsequently. Section 6 is the heart of the proofs of the Theorems 2 and 6. We study there subgroups of a semidirect product $\mathsf{S} \rtimes_\beta \mathsf{K}$ with S being supersolvable. Our results depend on a good understanding of the exponential map of $\mathsf{S} \rtimes_\beta \mathsf{K}$. Here Theorem 6.5 is of its own right.

The proof of Theorem 5 needs some additional preparations which we have placed in Subsect. 7.3. There a sufficient condition on a polycrystallographic group Γ is given which ensures that for the embedding $\iota\colon \Gamma \to \mathsf{S} \rtimes \mathsf{K}$ of Corollary 4 the group $\mathsf{S} \rtimes \mathsf{K}$ is connected. Actually this result can be interpreted as a correction of a similar theorem of Auslander from which we show in Example 2.1 that the original version is not correct. Furthermore, we prove without additional assumptions that the number of connected components of $\mathsf{S} \rtimes \mathsf{K}$ is bounded by a constant only depending on the rank of Γ.

Finally, we investigate in Sect. 8 the algebraic properties of the embedding of Corollary 4. To some extent this embedding is the natural generalization of the Malcev completion of a torsion free nilpotent group.

The author would like to thank the referees for bringing several references to his attention.

2. Counterexamples

Let Λ be a torsion free polycyclic group. Then the nilradical $\mathrm{nil}(\Lambda)$ of Λ is finitely generated. Hence there is a connected, simply connected nilpotent Lie group N, called the Malcev completion of $\mathrm{nil}(\Lambda)$, such that $\mathrm{nil}(\Lambda)$ is a lattice in N, see (Raghunathan, 1972, Theorem 2.18). The action of Λ on $\mathrm{nil}(\Lambda)$ by conjugation induces an action of Λ on the Malcev completion N of $\mathrm{nil}(\Lambda)$, see (Raghunathan, 1972, Theorem 2.11). Let $\rho\colon \Lambda \to \mathrm{GL}(\mathfrak{n})$ be the corresponding representation in the Lie algebra of N. The group Λ is called predivisable if and only if $\Lambda/\mathrm{nil}(\Lambda)$ is free abelian and for all $g \in \Lambda$ the following is true: any eigenvalue λ of $\rho(g)$ is either real and positive or the number λ^n is not real for all positive integers n.

A theorem of Auslander (1961b) states that a predivisable polycyclic group Λ is isomorphic to a lattice in a connected, simply connected solvable Lie group. Another theorem of Auslander (1969) asserts that there is a connected solvable Lie group D containing Λ as a uniform lattice such that any automorphism of Λ can be extended uniquely to an automorphism of D. Both theorems are not correct:

Example 2.1. *There is a torsion free, predivisable polycyclic group Λ which can not be realized as a discrete, cocompact subgroup of a connected solvable Lie group.*

The construction of Λ needs some preparations. Set

$$p(X) := X^4 + 4X^3 + 3X^2 - 2X + 1 = (X+1)^4 - 3(X+1)^2 + 3.$$

Clearly, p is an irreducible polynomial, and the zeros of p are the numbers:

$$z_1 = -1 + \sqrt[4]{3}\, e^{i\pi/12}, \quad z_2 = -1 + \sqrt[4]{3}\, e^{-i\pi/12},$$
$$z_3 = -1 - \sqrt[4]{3}\, e^{i\pi/12}, \quad z_4 = -1 - \sqrt[4]{3}\, e^{-i\pi/12}.$$

We claim that z_i^n is not real for all positive integers n, $i = 1, \dots, 4$. In fact, otherwise the number $z_i/\bar{z}_i$ would be a root of unity. It is easy to see that the roots of unity in $\mathbb{Q}(z_1, z_2, z_3, z_4)$ form a cyclic group of order 12. Thus it is sufficient to verify that $(z_i/\bar{z}_i)^{12} \neq 1$ which is trivial.

Let $\widetilde{\mathbb{Z}}$ be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(z_1)$, i.e. $\widetilde{\mathbb{Z}}$ consists of those numbers in $\mathbb{Q}(z_1)$ for which the corresponding (normalized) minimal polynomial is in $\mathbb{Z}[X] \subset \mathbb{Q}[X]$. Evidently, $z_1 \in \widetilde{\mathbb{Z}}$ and as additive group $\widetilde{\mathbb{Z}}$ is isomorphic to $\mathbb{Z}^4$. It is a well-known and elementary fact that the characteristic polynomial of the $\mathbb{Z}$-linear map $\mathrm{L}_{z_1} : \widetilde{\mathbb{Z}} \to \widetilde{\mathbb{Z}},\ x \mapsto z_1 x$ is given by the minimal polynomial $p(X)$. In particular, $\mathrm{L}_{z_1} \in \mathrm{GL}(\widetilde{\mathbb{Z}}) \cong \mathrm{GL}(4, \mathbb{Z})$, and z_1, z_2, z_3, z_4 are the eigenvalues of L_{z_1}. Put

$$\mathsf{L} := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in \widetilde{\mathbb{Z}} \subset \mathbb{C} \right\}.$$

Clearly, L is a finitely generated, torsion free nilpotent group of rank 12. We define an automorphism of L as follows.

$$\sigma : \mathsf{L} \to \mathsf{L}, \quad A \mapsto \mathrm{diag}(z_1, z_1^2, 1) \cdot A \cdot \mathrm{diag}(z_1^{-1}, z_1^{-2}, 1).$$

As usual σ induces an automorphism $\bar{\sigma}$ on the Malcev completion N of L. It is straightforward to check that the corresponding automorphism $\bar{\sigma}_{*e} : \mathfrak{n} \to \mathfrak{n}$ of the Lie algebra of N has the eigenvalues $z_i, z_i^{-1}, z_i^2, i = 1, \ldots, 4$.

Next we consider the product group $\mathsf{L}^2 := \mathsf{L} \times \mathsf{L}$ with the automorphism

$$\psi : \mathsf{L}^2 \to \mathsf{L}^2, \quad (g, h) \mapsto \big(\sigma(h), \sigma(g)\big).$$

Similarly to above ψ induces an automorphism $\bar{\psi}$ on N^2, and the eigenvalues of $\bar{\psi}_{*e} : \mathfrak{n} \oplus \mathfrak{n} \to \mathfrak{n} \oplus \mathfrak{n}$ are given by $\pm z_i, \pm z_i^{-1}, \pm z_i^2, i = 1, \ldots, 4$. Observe that, for an eigenvalue λ of $\bar{\psi}_{*e}$ the number λ^n is not real for all positive integers n. Therefore the semidirect product

$$\Lambda := (\mathsf{L}^2) \rtimes \mathbb{Z}, \quad (a, m) \cdot (b, n) := \big(a \cdot \psi^m(b), m + n\big)$$

is a torsion free, predivisable polycyclic group.

In order to show that Λ has the claimed properties, it is important to verify that the automorphism $\bar{\psi} : \mathsf{N}^2 \to \mathsf{N}^2$ is not contained in the identity component of $\mathrm{Aut}(\mathsf{N}^2)$. Notice that N is not abelian, so there is an element $g \in \mathsf{N}$ for which the connected group $[g, \mathsf{N}] := \langle\{ghg^{-1}h^{-1} \mid h \in \mathsf{N}\}\rangle$ is not trivial. Choose the minimal positive integer k for which the set $S := \{g \in \mathsf{N} \mid \dim([g, \mathsf{N}]) = k\}$ is not empty. The set $M := \{g \in \mathsf{N}^2 \mid \dim([g, \mathsf{N}^2]) = k\}$ is invariant under $\mathrm{Aut}(\mathsf{N}^2)$, and it is trivial to compute that $M = (S \times \mathsf{C}) \cup (\mathsf{C} \times S)$, where C is the center of N. Evidently, $\bar{\psi}$ swaps the subsets $S \times \mathsf{C}$ and $\mathsf{C} \times S$ of M. Taking into account that these two subsets are open and closed in M, we see that $\bar{\psi}$ is not contained in the identity component of $\mathrm{Aut}(\mathsf{N}^2)$.

Suppose, on the contrary, that Λ can be realized as a discrete, cocompact subgroup of a connected solvable Lie group S. It is convenient to assume that $\dim(\mathsf{S})$ is minimal. Let $\tilde{\mathsf{N}}$ be the maximal connected, nilpotent normal subgroup of S. If $\tilde{\mathsf{N}}$ is not simply connected, there is a nontrivial, connected, maximal compact central subgroup T of $\tilde{\mathsf{N}}$, compare Lemma 4.2 a) below. Then T is a characteristic subgroup of the normal subgroup $\mathsf{N} \subset \mathsf{S}$ and accordingly normal in S. The factor group S/T is again a connected solvable group. Since Λ is torsion free and discrete in S, the projection $\mathsf{S} \to \mathsf{S}/\mathsf{T}$ maps Λ isomorphically onto a discrete, cocompact subgroup of S/T which is impossible because $\dim(\mathsf{S})$ is minimal. Thus $\tilde{\mathsf{N}}$ is simply connected.

By a theorem of Mostow the group $\mathsf{H} := \Lambda \cap \tilde{\mathsf{N}}$ is a lattice in $\tilde{\mathsf{N}}$, see (Raghunathan, 1972, Theorem 3.3). Furthermore, H is contained in $\mathsf{L}^2 = \mathsf{L}^2 \times \{e\}$, the nilradical of Λ. The factor group $\mathsf{S}/\tilde{\mathsf{N}}$ is abelian, and hence $[\Lambda, \Lambda] \subset \mathsf{H}$. It is easy to see that the commutator group $[\Lambda, \Lambda] \subset \mathsf{H}$ is of finite index in $\mathsf{L}^2 \subset \Lambda$. Therefore $\mathsf{H} \subset \mathsf{L}^2$ is lattice in $\tilde{\mathsf{N}}$ and also a lattice in $\mathsf{N}^2 = \mathsf{N} \times \mathsf{N}$. Since both groups are connected, simply connected nilpotent Lie groups, there exists an isomorphism $\iota\colon \tilde{\mathsf{N}} \to \mathsf{N}^2$ with $\iota_{|\mathsf{H}} = \mathrm{id}$, see (Raghunathan, 1972, Theorem 2.11). The connected group S acts on $\tilde{\mathsf{N}}$ by conjugation, and the image of the induced homomorphism $\mathsf{S} \to \mathrm{Aut}(\tilde{\mathsf{N}})$ is contained in the identity component of $\mathrm{Aut}(\tilde{\mathsf{N}})$.

Set $g := (e, 1) \in (\mathsf{L}^2) \rtimes \mathbb{Z} = \Lambda$. The automorphism $c_g\colon \tilde{\mathsf{N}} \to \tilde{\mathsf{N}}$, $h \mapsto ghg^{-1}$ leaves the subgroup $\mathsf{H} = \Lambda \cap \tilde{\mathsf{N}}$ invariant and clearly $c_{g|\mathsf{H}} = \psi$. Hence $\iota \circ c_g \circ \iota^{-1}$ and $\bar{\psi}$ coincide on the cocompact subgroup H of N^2, so they must be equal. But this implies that $\bar{\psi}$ is contained in the identity component of $\mathrm{Aut}(\mathsf{N}^2)$, a contradiction.

Remarks 2.2. 1. Actually the two theorems of Auslander are only wrong in detail but true in spirit. In fact, one just has to replace the assumption that the group Γ is predivisable by the assumption that Γ is generated by elements which are absolutely net, see Subsect. 7.3 for the definition and details.

2. If one is willing to disregard the minor mistake in (Auslander, 1969, Theorem 1), its statement implies nearly directly the existence part of Corollary 4. However, we will not make use of this fact. Notice also that the uniqueness part of Corollary 4 was not known before, even not in the special case of a predivisable polycyclic group Γ.

Example 2.3. *There is a nontrivial, connected, simply connected solvable Lie group* S*, a compact group* $\mathsf{K} \subset \mathrm{Aut}(\mathsf{S})$ *and a discrete, cocompact subgroup* $\Gamma \subset \mathsf{S} \rtimes \mathsf{K}$ *such that* Γ *and* $\mathsf{S} = \mathsf{S} \times \{e\}$ *have only the trivial element in common.*

Choose a two-dimensional real subspace $\mathsf{V} \subset \mathbb{R}^3$ with $\mathbb{Z}^3 \cap \mathsf{V} = \{0\}$. There is a one-dimensional connected subgroup $\mathrm{SO}(2) \subset \mathrm{SO}(3)$ leaving V invariant. Let G denote the semidirect product $\mathbb{R}^3 \rtimes \mathrm{SO}(2)$. Then $\mathsf{V} \times \{e\}$ is a normal subgroup of G, and the factor group $\mathsf{Q} := \mathsf{G}/\mathsf{V} \times \{e\}$ is isomorphic to a direct product $\mathbb{R} \times \mathrm{SO}(2)$. The projection $\pi : \mathsf{G} \to \mathsf{Q}$ maps the group $\Gamma := \mathbb{Z}^3 \times \{e\}$ monomorphically onto a subgroup of Q. Clearly, we can find a connected, one-dimensional, cocompact, closed subgroup A of Q that has trivial intersection with $\pi(\Gamma)$. Let S be the preimage $\pi^{-1}(\mathsf{A})$. By construction S is a connected, simply connected, closed, solvable normal subgroup that has trivial intersection with Γ. Moreover, $\mathsf{K} = \{e\} \times \mathrm{SO}(2)$ is a group complement of S in G, that is $\mathsf{K} \cdot \mathsf{S} = \mathsf{G}$ and $\mathsf{K} \cap \mathsf{S} = \{e\}$. Finally, it is easy to see that K acts on $\mathsf{S} \supset \mathsf{V} \times \{e\}$ effectively by conjugation. Thus we can view K as a subgroup $\mathrm{Aut}(\mathsf{S})$, and under this identification $\mathsf{S} \rtimes \mathsf{K}$ is isomorphic to G, compare Subsect. 3.2. Furthermore, $\Gamma \cap \mathsf{S} = \{e\}$.

Remarks 2.4. 1. Example 2.3 contradicts Corollary 8.25 in (Raghunathan, 1972), which in the above situation states that for the maximal connected nilpotent normal subgroup N of S the group $\mathsf{N} \cap \Gamma$ is a lattice in N.

2. The error occurs in the proof of (Corollary 8.25, Raghunathan, 1972) where is claimed that the maximal connected nilpotent normal subgroup of $\mathsf{S} \rtimes \mathsf{K}$ is contained in S. In the above example this is not true. In fact, there the maximal connected nilpotent normal subgroup of $\mathsf{S} \rtimes \mathsf{K}$ is isomorphic to $\mathbb{R}^3$; whereas the maximal connected nilpotent normal subgroup of S is isomorphic to $\mathbb{R}^2$. Actually the mistake and the corollary itself is related to a similar assertion that is stated in the proof of (Theorem 2, Auslander, 1961a).

3. In general for a lattice Λ in a connected, simply connected solvable Lie group S not any automorphism of Λ can be extended to an automorphism of S: Consider the homomorphism $\beta\colon \mathbb{R} \to \mathbb{C}^* \subset \mathrm{Aut}(\mathbb{C})$, $t \mapsto \exp(2\pi i t)$ and the semidirect product $\mathsf{S} = \mathbb{C} \rtimes_\beta \mathbb{R}$. The group $\Lambda := (\mathbb{Z} \oplus i\mathbb{Z}) \times \mathbb{Z} \cong \mathbb{Z}^3$ is a discrete, cocompact subgroup of S, but of course not any automorphism in $\mathrm{Aut}(\Lambda) \cong \mathrm{GL}(3, \mathbb{Z})$ can be extended to one of S.

3. Preliminaries

3.1. Basic properties of algebraic groups

Recall that the general linear group $\mathrm{GL}(n, \mathbb{C})$ has a so called Zarisky topology; a subset $\mathsf{G} \subset \mathrm{GL}(n, \mathbb{C})$ is called Zarisky closed if and only if it is the zero set of a collection of polynomials in the coefficients a_{ij} and in $\det(a_{ij})^{-1}$. An algebraic subgroup G of $\mathrm{GL}(n, \mathbb{C})$ is a subgroup for which the underlying set is Zarisky closed. It is a well-known and elementary fact that for a subgroup $\mathsf{G} \subset \mathrm{GL}(n, \mathbb{C})$ the Zarisky closure of G is again a group. If $\mathsf{G} \subset \mathrm{GL}(n, \mathbb{C})$ is a group and $\mathsf{H} \subset \mathsf{G}$ is a normal subgroup, then the Zarisky closure $\bar{\mathsf{H}}$ of H is a normal subgroup of the Zarisky closure $\bar{\mathsf{G}}$ of G. If in addition G/H is abelian, then $\bar{\mathsf{G}}/\bar{\mathsf{H}}$ is abelian, too.

We refer to the topology of $\mathrm{GL}(n, \mathbb{R})$ induced by the Zarisky topology as the Zarisky topology of $\mathrm{GL}(n, \mathbb{R})$. In particular, the Zarisky closure in $\mathrm{GL}(n, \mathbb{R})$ is defined. A real algebraic group $\mathsf{G} \subset \mathrm{GL}(n, \mathbb{R})$ is a group for which the underlying set is Zarisky closed in $\mathrm{GL}(n, \mathbb{R})$. It is easy to show that a connected, unipotent subgroup of $\mathrm{GL}(n, \mathbb{R})$ is a real algebraic group, see (Raghunathan, 1972, p. 9).

A (real) algebraic group has only finitely many connected components in the Euclidean topology, see (Mostow, 1957).

If V is a real n-dimensional vector space with a given basis, then the general linear group $\mathrm{GL}(V)$ is canonically isomorphic to $\mathrm{GL}(n, \mathbb{R})$. Evidently, the Zarisky topology induced by this identification on $\mathrm{GL}(V)$ does not depend on the choice of the basis. Thus Zarisky topology of $\mathrm{GL}(V)$ has an intrinsic meaning.

In the sequel, we will make use of these elementary facts without further comments. We also emphasize that except for the notion of Zarisky closure all other topological concepts used for subgroups of $\mathrm{GL}(n, \mathbb{C})$ will be with respect to the Euclidean topology. The main reason why real algebraic groups play a role in the proofs of the above theorems is related to the following well-known observation.

Remark 3.1. Let S be a connected, simply connected Lie group, and let $\mathrm{Aut}(\mathsf{S})$ the group of continuous automorphisms of S. The natural representation $\rho\colon \mathrm{Aut}(\mathsf{S}) \to \mathrm{GL}(\mathfrak{s})$ in the Lie algebra of S is faithful, and its image is a real algebraic linear group. In particular, $\mathrm{Aut}(\mathsf{S})$ is a Lie group with finitely many connected components, it contains at least one maximal compact subgroup, and any compact subgroup is conjugate to a subgroup of a given maximal compact subgroup.

Proof. Since S is connected, ρ is faithful. As S is simply connected, the image of ρ is the automorphism group $\mathrm{Aut}(\mathfrak{s})$ of the Lie algebra $\mathfrak{s}$. Checking that $\mathrm{Aut}(\mathfrak{s})$ is an algebraic group is easy: Choose a basis $v_1, \dots, v_n$ of $\mathfrak{s}$ and define $\sum_{h=1}^{n} c_{ijh} v_h := [v_i, v_j]$. Let $A \in \mathrm{GL}(\mathfrak{s})$ be represented by the matrix (a_{ij}) with respect to the basis $v_1, \dots, v_n$. Then $A \in \mathrm{Aut}(\mathfrak{s})$ if and only if for all h, k, l the following holds

$$\sum_{j,h} c_{klh} a_{hj} v_h = A[v_k, v_l] = [Av_k, Av_l] = \sum_{i,j,h} a_{ik} a_{jl} c_{ijh} v_h.$$

Thus $\mathrm{Aut}(\mathfrak{s})$ is a real algebraic group. The last part of the remark is a general fact for Lie groups with finitely many connected components, see (Hochschild, 1965, Ch. XV,Theorem 3.1). □

3.2. Notational conventions and basic facts for semidirect products

Let S and K be Lie groups and $\beta\colon \mathsf{K} \to \mathrm{Aut}(\mathsf{S})$ a continuous homomorphism. On the Cartesian product $\mathsf{S} \times \mathsf{K}$ we introduce a Lie group structure by setting

$$(v, a) \cdot (w, b) := \big(v \cdot \beta(a)(w), ab\big) \quad \text{for } (v, a), (w, b) \in \mathsf{S} \times \mathsf{K}.$$

This Lie group is denoted by $\mathsf{S} \rtimes_\beta \mathsf{K}$. It acts on S via $(\tau, a) \star h = \tau \cdot \beta(a)(h)$. With this notation the group multiplication can be rewritten as $(v, a) \cdot (w, b) = ((v, a) \star w, ab)$.

For a semidirect product $\mathsf{S} \rtimes_\beta \mathsf{K}$ we will always identify K with the group $\{e\} \times \mathsf{K}$ and S with the normal subgroup $\mathsf{S} \times \{e\}$ of $\mathsf{S} \rtimes_\beta \mathsf{K}$.

Let G be a Lie group, S a closed normal subgroup, and let $\mathsf{K} \subset \mathsf{G}$ be a closed subgroup such that $\mathsf{G} = \mathsf{K} \cdot \mathsf{S}$ and $\mathsf{K} \cap \mathsf{S} = \{e\}$. Then the natural action of K on S by conjugation induces a continuous homomorphism $\beta\colon \mathsf{K} \to \mathrm{Aut}(\mathsf{S})$, and $\mathsf{S} \rtimes_\beta \mathsf{K}$ is isomorphic to G via $(\tau, a) \mapsto \tau \cdot a$. Conversely for a semidirect product $\mathsf{S} \rtimes_\beta \mathsf{K}$ the action of K on S by conjugation coincides with the action of K on S induced by the homomorphism β.

There is a natural homomorphism $\pi\colon \mathsf{S} \rtimes_\beta \mathsf{K} \to \mathsf{S} \rtimes \mathrm{Aut}(\mathsf{S})$, $(\tau, a) \mapsto (\tau, \beta(a))$. Clearly, $\mathrm{Ker}(\pi) = \mathrm{Ker}(\beta)$. In particular, the kernel of β is a normal subgroup of $\mathsf{S} \rtimes_\beta \mathsf{K}$. If β is injective, we will often identify K with $\beta(\mathsf{K})$, and then write $\mathsf{S} \rtimes \mathsf{K}$ for $\mathsf{S} \rtimes_\beta \mathsf{K}$.

Lemma 3.2. *a) Let S be a connected, simply connected solvable Lie group, and let K be a compact subgroup of* $\mathrm{Aut}(\mathsf{S})$. *Then the semidirect product $\mathsf{S} \rtimes \mathsf{K}$ contains no nontrivial compact normal subgroup.*

b) Let G be a Lie group, and let S be a connected, simply connected, closed, co-compact solvable normal subgroup of G. Then G is isomorphic to a semidirect product $\mathsf{S} \rtimes_\beta \mathsf{K}$, where K is a maximal compact subgroup of G.

c) Let G be a Lie group that contains no nontrivial compact normal subgroup, and let $\mathsf{S} \subset \mathsf{G}$ be as in b). Then G is isomorphic to a semidirect product $\mathsf{S} \rtimes \mathsf{K}$, where K is a compact subgroup of $\mathrm{Aut}(\mathsf{S})$.

Proof. a). Suppose that L is a compact normal subgroup of $\mathsf{S} \rtimes \mathsf{K}$. Evidently, K is a maximal compact subgroup of $\mathsf{S} \rtimes \mathsf{K}$. Taking into account that $\mathsf{K} \cdot \mathsf{L}$ is also compact, we see that $\mathsf{L} \subset \mathsf{K}$. Hence L acts effectively by conjugation on S. On the other hand, the normal subgroups L and S have trivial intersection, and thus they commute. In combination these facts show that L is the trivial group.

b). Since G has only finitely many connected components, a maximal compact subgroup exists, see (Hochschild, 1965, Ch. XV, Theorem 3.1). Moreover, for a maximal compact subgroup K of G we get $\mathsf{G} = \mathsf{K} \cdot \mathsf{S}$, see (Hochschild, 1965, Ch. XV, Theorem 3.7). On the other hand, $\mathsf{K} \cap \mathsf{S} = \{e\}$, and therefore G is isomorphic to a semidirect product $\mathsf{S} \rtimes_\beta \mathsf{K}$.

c). By b) the group G is isomorphic to a semidirect product $\mathsf{S} \rtimes_\beta \mathsf{K}'$. Furthermore, G contains no nontrivial compact normal subgroup, and hence the natural homomorphism $\mathsf{S} \rtimes_\beta \mathsf{K}' \to \mathsf{S} \rtimes \mathrm{Aut}(\mathsf{S})$ is injective. □

3.3. Some group theory

Lemma 3.3. *Let Π be a group and Λ a polycyclic subgroup of finite index. Then*

a) Any subgroup of Π is finitely generated.
b) There is a strongly polycyclic normal subgroup of finite index in Π.

For a proof of this lemma see (Segal, 1983, Proposition 2,p 2). The next lemma is known as well, see (Dekimpe, 1996, Lemma 3.2.4). However, since it is less standard we include a proof.

Lemma 3.4. *Let Π be a group and suppose that the center of Π has finite index in Π. Then the torsion elements in Π form a characteristic subgroup T, and the factor group Π/T is an abelian torsion free group.*

Proof of Lemma 3.4. Let C denote the center of Π, $\mathsf{C}_\mathbb{Q} := \mathsf{C} \otimes_\mathbb{Z} \mathbb{Q}$, and let $p\colon \mathsf{C} \to \mathsf{C}_\mathbb{Q}$, $h \mapsto h \otimes 1$ be the natural map. Clearly, the kernel of p precisely consists out of the torsion elements in C. Consider the direct product $\Pi \times \mathsf{C}_\mathbb{Q}$ and the central subgroup

$$\Delta := \{(h, -p(h)) \mid h \in \mathsf{C}\}.$$

The projection $\mathrm{pr}\colon \Pi \times \mathsf{C}_\mathbb{Q} \to \mathsf{G} := \Pi \times \mathsf{C}_\mathbb{Q}/\Delta$ maps $\mathsf{C}_\mathbb{Q}$ injectively onto a central subgroup of G. Moreover, the kernel of $\mathrm{pr}_{|\Pi}$ is the torsion group $\mathrm{Ker}(p)$. Therefore it is sufficient to check that the torsion elements in G form a finite characteristic subgroup $\bar{\mathsf{T}}$ with an abelian factor group $\mathsf{G}/\bar{\mathsf{T}}$. Consider the finite group $\mathsf{F} := \mathsf{G}/\mathrm{pr}(\mathsf{C}_\mathbb{Q}) \cong \Pi/\mathsf{C}$ and the exact sequence

$$\{1\} \to \mathsf{C}_\mathbb{Q} \to \mathsf{G} \to \mathsf{F} \to \{1\}.$$

By cohomology theory such a sequence splits, see (Brown, 1982). Hence G is isomorphic to direct product $\mathsf{F} \times \mathsf{C}_\mathbb{Q}$ and the assertion follows. □

For a group G and an integer k we let $\mathsf{G}^k \rtimes \mathsf{S}_k$ denote the semidirect product of the symmetric group of degree k with the k-fold product of G.

Lemma 3.5. *Let Γ be a group, $\Lambda \subset \Gamma$ a normal subgroup of index $k < \infty$. Then Γ is isomorphic to a subgroup of the semidirect product $\Lambda^k \rtimes \mathsf{S}_k$.*

Proof. Let $b_1, \ldots, b_k \in \Gamma$ be representatives of Γ/Λ. Since Λ is a normal subgroup of Γ, we can find for any $g \in \Gamma$ and $i \in \{1, \ldots, k\}$ a unique $\sigma_g(i) \in \{1, \ldots, k\}$ for which $b_i g b_{\sigma_g(i)}^{-1} \in \Lambda$. In fact, $g \mapsto \sigma_g$ defines an anti-homomorphism from Γ to the symmetric group of degree k. Now we define a monomorphism $i\colon \Gamma \to \Lambda^k \rtimes \mathsf{S}_k$ by

$$i(g) := \left(\left(b_1 g b_{\sigma_g(1)}^{-1}, \ldots, b_k g b_{\sigma_g(k)}^{-1}\right), \sigma_{g^{-1}}\right) \quad \text{for all } g \in \Gamma.$$

□

4. The reduction to supersolvable Lie groups

4.1. Supersolvable Lie groups

A well-known theorem of Engel states that a connected Lie group is nilpotent if and only if the adjoint group consists of unipotents. There is a similar characterization of supersolvable Lie groups.

Lemma 4.1. *A connected Lie group* S *is supersolvable if and only if all elements in the adjoint group* $\mathrm{Ad}(\mathsf{S})$ *have only positive eigenvalues.*

Proof. Let $\mathfrak{s}$ denote the Lie algebra of S. We begin with the case of a supersolvable Lie group S. So there are closed normal subgroups $\{e\} = \mathsf{N}_0 \subset \cdots \subset \mathsf{N}_k = \mathsf{S}$ of S with $\dim(\mathsf{N}_{i+1}/\mathsf{N}_i) = 1$. Choose vectors $b_1, \ldots, b_k \in \mathfrak{s}$ such that $b_1, \ldots, b_i$ is a basis of the Lie algebra of N_i. With respect to this basis $\mathrm{Ad}(\mathsf{S})$ is represented by real upper triangular matrices. Taking into account that $\mathrm{Ad}(\mathsf{S})$ is connected, we see that the Eigenvalues of elements in $\mathrm{Ad}(\mathsf{S})$ are real and positive.

Assume now conversely that the eigenvalues of Ad_g are real and positive for all $g \in \mathsf{S}$. Let S' be the maximal solvable normal subgroup of S. Suppose for a moment that $\mathsf{S}' \neq \mathsf{S}$. Then $\mathsf{G} := \mathsf{S}/\mathsf{S}'$ is a semisimple Lie group with trivial center, and therefore it has a nontrivial compact subgroup K. Clearly, for all $g \in \mathsf{K}$ the eigenvalues of $\mathrm{Ad}_g\colon \mathfrak{g} \to \mathfrak{g}$ have absolute value one. On the other hand, the semisimple endomorphism Ad_g has only positive eigenvalues and consequently $\mathrm{Ad}_g = \mathrm{id}$. Thus K is contained in the center of G, a contradiction.

Hence $\mathsf{S} = \mathsf{S}'$ is a solvable Lie group. Let T be a maximal compact subgroup of S. As above we deduce that T is central, and therefore it is sufficient to check that the factor group S/T is supersolvable. According to (Hochschild, 1965, Ch. XV, Theorem 3.1) S/T is diffeomorphic to an Euclidean space, and thus we may assume that S itself is simply connected. By Lie's theorem we can find a vector v in the complexification $\mathfrak{s}_{\mathbb{C}}$ of $\mathfrak{s}$ which is an eigenvector of $\mathrm{Ad}(\mathsf{S})$. But then the conjugate $\bar{v}$ of v is also an eigenvector of $\mathrm{Ad}(\mathsf{S})$. Taking into account that all eigenvalues are real, we see that any vector in $\mathrm{span}_{\mathbb{C}}(\bar{v}, v)$ is an eigenvector of $\mathrm{Ad}(\mathsf{S})$. The intersection of $\mathrm{span}_{\mathbb{C}}(\bar{v}, v)$ and $\mathfrak{s} \subset \mathfrak{s}_{\mathbb{C}}$ is nontrivial, so there is a vector $w \in \mathfrak{s}$ which is an eigenvector of $\mathrm{Ad}(\mathsf{S})$. But then $\mathbb{R} \cong \exp(\mathbb{R}w) = \mathsf{A}$ is a closed normal subgroup of S, and in order to show that S is supersolvable, we just have to check that S/A is supersolvable. Now the statement follows by induction on the dimension of S. □

Lemma 4.2. *a) A maximal compact subgroup* T *of a connected supersolvable Lie group* S *is unique and central, and the factor group* S/T *is simply connected.*

b) Let S_1 *and* S_2 *be connected supersolvable normal Lie subgroups of a Lie group* G*. Then* $\mathsf{S}_1 \cdot \mathsf{S}_2$ *is a connected supersolvable normal Lie subgroup, too.*

c) Let G *be a Lie group. The maximal connected supersolvable normal Lie subgroup* S *of* G *is closed. If* G *contains no nontrivial compact normal subgroup, then* S *is simply connected.*

Proof. a). This is an immediate consequence of the proof of Lemma 4.1.

b). Let $\mathsf{S}_3 = \mathsf{S}_1 \cdot \mathsf{S}_2$, $\mathsf{H} = \mathsf{S}_1 \cap \mathsf{S}_2$, and let $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3$ and $\mathfrak{h}$ be the corresponding Lie algebras. The adjoint representation of S_3 induces representations in $\mathfrak{s}_1/\mathfrak{h}$ and $\mathfrak{s}_2/\mathfrak{h}$. Because of the inclusion $[\mathfrak{s}_1, \mathfrak{s}_2] \subset \mathfrak{h}$ the natural representation of S_1 in $\mathfrak{s}_2/\mathfrak{h}$ is trivial. Moreover, $\mathrm{Ad}_{g|\mathfrak{s}_1}$ has only positive eigenvalues for $g \in \mathsf{S}_1$, and hence $\mathrm{Ad}_{g|\mathfrak{s}_3}$ has only positive eigenvalues. Similarly, $\mathrm{Ad}_{g|\mathfrak{s}_3}$ has only positive eigenvalues for $g \in \mathsf{S}_2$. By Lie's theorem the eigenvalues of $\mathrm{Ad}_{g|\mathfrak{s}_3}$ are positive for all $g \in \mathsf{S}_2 \cdot \mathsf{S}_1 = \mathsf{S}_3$. Therefore S_3 is supersolvable, see Lemma 4.1.

c). Let $\tilde{\mathsf{S}}$ be the maximal connected solvable normal subgroup of G. Clearly, $\tilde{\mathsf{S}}$ is a closed subgroup of G. Furthermore, S is the maximal connected supersolvable normal subgroup of $\tilde{\mathsf{S}}$. So without loss of generality G is a solvable Lie group. Set

$$\mathsf{S}' := \{g \in \mathsf{G} \mid \mathrm{Ad}_g \text{ has only positive eigenvalues}\}.$$

By Lie's theorem S' is a closed normal subgroup of G, and Lemma 4.1 exhibits the identity component S'_0 of S' as a supersolvable Lie group. Let $\mathfrak{s}$ and $\mathfrak{g}$ be the Lie algebras corresponding to S and G. Since S contains the maximal connected nilpotent normal subgroup of G, it follows that G/S is abelian. Hence the natural representation of G in $\mathfrak{g}/\mathfrak{s}$ is trivial. Now Lemma 4.1 gives $\mathsf{S} \subset \mathsf{S}'$ and thereby $\mathsf{S} = \mathsf{S}'_0$. Thus S is closed. According to a) there is a unique maximal compact central subgroup T of S, and S/T is simply connected. Notice that the characteristic subgroup T of S is normal in G. Consequently, S itself is simply connected provided that G contains no nontrivial compact normal subgroups. □

4.2. Existence of cocompact supersolvable subgroups

Lemma 4.3. *Let* $\mathsf{S} \subset \mathrm{GL}(n, \mathbb{C})$ *be a group containing a solvable subgroup of finite index. Then the matrices in* S *that have only positive eigenvalues form a normal subgroup* R *of* S*.*

Proof. Evidently, the set R is invariant under conjugation, and therefore we only have to show that R is a subgroup. Notice that the Zarisky closure $\bar{\mathsf{S}}$ of S is solvable up to finite index, too. By replacing S by $\bar{\mathsf{S}}$ if necessary, we may assume that $\mathsf{S} = \bar{\mathsf{S}}$ is an algebraic group. In particular, S is a Lie group with finitely many connected components, and the identity component S_0 is solvable. Because of Lie's theorem it is sufficient to verify the assertion under the additional assumption that S_0 is contained in the group of upper triangular matrices.

Let S be a matrix in S that has only positive eigenvalues. The group S has only finitely many connected components, and hence S^k is an upper triangular matrix for some positive integer k. Then $e_1 \in \mathbb{C}^n$ is an eigenvector of S^k. Since the eigenvalues of S are positive, it follows that e_1 is also an eigenvector of S. Similarly, we can deduce from the fact that $U_i := \mathrm{span}_{\mathbb{C}}\{e_1, \dots, e_i\}$ is an invariant under S^k, that U_i is an invariant under S as well, $i = 1, \cdots, n$. But this proves that S is an upper triangular matrix. Thus the set R consists of upper triangular matrices, and now it is trivial to check that R is a subgroup of S. □

Lemma 4.4. *Let* $\mathsf{S} \subset \mathrm{GL}(n, \mathbb{R})$ *be a real algebraic group with a solvable identity component* S_0*, and let* R *be the set of matrices in* S *that have only positive eigenvalues. Then* R *is a connected, simply connected, cocompact normal subgroup of* S*, and* S *is isomorphic to a semidirect product of a compact group and* R.

Proof. By Lemma 4.3 the set R is a normal subgroup of S. Because of Lie's theorem there is an one-dimensional subspace $\mathbb{C}v$ of $\mathbb{C}^n$ that is invariant under S_0. By restriction we get a homomorphism $r\colon \mathsf{S}_0 \to \mathrm{GL}(\mathbb{C}v) = \mathbb{C}^*$. We denote by $\mathbb{R}^+$ the group of positive real numbers. Evidently, $r(\mathsf{S}_0) \cap \mathbb{R}^+$ is cocompact in $r(\mathsf{S}_0)$.

Let $\mathsf{S}' := r^{-1}(\mathbb{R}^+)$. By construction v is an eigenvector for all $g \in \mathsf{S}'$, and the corresponding eigenvalues are real and positive. Analogously to the proof of Lemma 4.3 we can find an eigenvector $w \in \mathbb{R}^n \setminus \{0\}$ of S'.

Denote by $\mathsf{H}_2 \subset \mathsf{S}$ the subset consisting precisely of the matrices in S that have w as an an eigenvector. Clearly, H_2 is a real algebraic subgroup of S, and $\mathsf{H}_2 \supset \mathsf{S}'$ is a cocompact subgroup of S.

Consider the natural representation $h\colon \mathsf{H}_2 \to \mathrm{GL}(\mathbb{R}^n/\mathbb{R}w)$. Similarly to above there is a vector $w_2 \in \mathbb{R}^n/\mathbb{R}w$ and a cocompact real algebraic subgroup H_3 such that w_2 is an eigenvector of $h(\mathsf{H}_3)$. Combining this with a simple induction argument we see that S contains a real algebraic cocompact subgroup H_n which is in $\mathrm{GL}(n, \mathbb{R})$ conjugate to a group of upper triangular matrices.

The identity component H_{n0} of H_n is cocompact in S, too. Since the matrices in H_{n0} have only positive eigenvalues, they are contained in R, and thus the identity component R_0 of R is cocompact in S. Taking into account that R is closed, we see that R/R_0 is finite. By (Hochschild, 1965, Ch. XV) R contains a maximal compact subgroup K, and the quotient R/K is a connected, simply connected manifold. On the other hand, we infer from the definition of R that R contains no nontrivial compact subgroup at all, and hence R itself is connected and simply connected. The remaining part of the lemma is a direct consequence of Lemma 3.2. □

Lemma 4.5. *Let* S *be a connected, simply connected solvable Lie group,* N *the maximal connected nilpotent normal subgroup of* S*, and let* $\mathrm{Int}(\mathsf{N}) \subset \mathrm{Aut}(\mathsf{S})$ *be the group of inner automorphisms of* S *induced by elements of* N*. There is a toral subgroup* $\mathsf{T} \subset \mathrm{Aut}(\mathsf{S})$ *such that*

(i) $\mathrm{Int}(\mathsf{N}) \cdot \mathsf{T}$ *is normal in* $\mathrm{Aut}(\mathsf{S})$,
(ii) the maximal connected, simply connected supersolvable normal subgroup R *of the semidirect product* $\mathsf{S} \rtimes \mathsf{T}$ *is cocompact,*
(iii) T *also can be viewed as a subgroup of* $\mathrm{Aut}(\mathsf{R})$*, and under this identification* $\mathsf{S} \rtimes \mathsf{T}$ *is isomorphic to the semidirect product* $\mathsf{R} \rtimes \mathsf{T}$.

Proof. We identify the groups $\mathrm{Aut}(\mathsf{S})$ and $\mathrm{Aut}(\mathfrak{s})$ in natural fashion, compare proof of Remark 3.1. Observe that under this identification the group $\mathrm{Int}(\mathsf{N})$ equals $\mathrm{Ad}(\mathsf{N}) \subset \mathrm{Ad}(\mathsf{S}) \subset \mathrm{Aut}(\mathfrak{s})$. By Remark 3.1 $\mathrm{Aut}(\mathfrak{s})$ is a real algebraic group, and hence it contains the Zarisky closure Z of $\mathrm{Ad}(\mathsf{S})$ in $\mathrm{GL}(\mathfrak{s})$ as a subgroup. Recall that $\mathrm{Ad}(\mathsf{S})$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{s})$, and accordingly the same is valid for Z. The connected, unipotent group $\mathrm{Ad}(\mathsf{N})$ is Zarisky closed in $\mathrm{GL}(\mathfrak{s})$, and since S/N is abelian, $\mathsf{Z}/\mathrm{Ad}(\mathsf{N})$ is abelian, too.

Let Z_0 be the identity component of Z. The abelian group $\mathsf{Z}_0/\mathrm{Ad}(\mathsf{N})$ contains a unique maximal toral subgroup $\tilde{\mathsf{T}}$. Consider the preimage M of $\tilde{\mathsf{T}}$ under the projection $\mathsf{Z}_0 \to \mathsf{Z}_0/\mathrm{Ad}(\mathsf{N})$. Since $\mathrm{Ad}(\mathsf{N})$ and Z_0 are normal subgroups of $\mathrm{Aut}(\mathfrak{s})$ and $\tilde{\mathsf{T}}$ is a characteristic subgroup of $\mathsf{Z}_0/\mathrm{Ad}(\mathsf{N})$, it follows that M is a normal subgroup of $\mathrm{Aut}(\mathfrak{s})$, too. Choose a maximal compact subgroup T of M. Lemma 3.2 yields the equation $\mathsf{M} = \mathsf{T} \cdot \mathrm{Ad}(\mathsf{N})$, and thus T satisfies condition (i).

Using that Z_0/M is a vector group we see that T is maximal compact in Z_0 as well. By Lemma 4.4 the solvable real algebraic group Z contains a connected, cocompact normal subgroup Y such that A has only positive eigenvalues for $A \in \mathsf{Y}$. Moreover, $\mathsf{Z}_0 = \mathsf{T} \cdot \mathsf{Y}$. In other words, for all $g \in \mathsf{Z}_0$ there is an element $h \in \mathsf{T}$ such that $h \cdot g$ has only positive eigenvalues.

Via the natural identification $\mathrm{Aut}(\mathfrak{s}) = \mathrm{Aut}(\mathsf{S})$ the group T becomes a subgroup of $\mathrm{Aut}(\mathsf{S})$. Let $\mathfrak{g}$ be the Lie algebra of the semidirect product $\mathsf{S} \rtimes \mathsf{T}$, $\mathfrak{t}$ and $\mathfrak{s}$ the subalgebras corresponding to the subgroups T and S.

$$\tilde{\mathsf{R}} := \{g \in \mathsf{S} \rtimes \mathsf{T} \mid \mathrm{Ad}_g \text{ has only positive eigenvalues}\}.$$

Because of Lie's theorem $\tilde{\mathsf{R}}$ is a normal subgroup of the solvable group $\mathsf{S} \rtimes \mathsf{T}$. The adjoint map Ad_g induces the identity on the abelian Lie algebra $\mathfrak{g}/\mathfrak{s}$ for all $g \in \mathsf{S} \rtimes \mathsf{T}$. This consideration shows that the eigenvalues of Ad_g are positive if and only if all eigenvalues of $\mathrm{Ad}_{g|\mathfrak{s}}\colon \mathfrak{s} \to \mathfrak{s}$ are positive. Taking into account that $\mathrm{Ad}_{g|\mathfrak{s}} \in \mathsf{Z}_0$, we see that there is an element $h \in \mathsf{T}$ such that all eigenvalues of Ad_{hg} are positive. Thus $\mathsf{S} \rtimes \mathsf{T} = \mathsf{T} \cdot \tilde{\mathsf{R}}$.

For $h \in \mathsf{T} \setminus \{e\}$ the map $\mathrm{Ad}_{h|\mathfrak{s}}$ is a nontrivial semisimple endomorphism with eigenvalues of absolute value 1. Hence $\mathsf{T} \cap \tilde{\mathsf{R}} = \{e\}$. Consequently, $\mathsf{S} \rtimes \mathsf{T}/\mathsf{T}$ is homeomorphic to $\tilde{\mathsf{R}}$. In particular, $\tilde{\mathsf{R}}$ is a connected, simply connected, co-compact supersolvable normal subgroup. Therefore $\tilde{\mathsf{R}} \subset \mathsf{R}$, and equality holds, since the factor group $\mathsf{R}/\tilde{\mathsf{R}}$ is a connected, simply connected, compact solvable Lie group. Lemma 3.2 allows us to regard T as a subgroup of $\mathrm{Aut}(\mathsf{R})$, and $\mathsf{R} \rtimes \mathsf{T}$ is then isomorphic to $\mathsf{S} \rtimes \mathsf{T}$. □

4.3. Proof of Theorem 1

Choose a subgroup $\mathsf{T} \subset \mathrm{Aut}(\mathsf{S})$ as stated in Lemma 4.5. Since $\mathrm{Int}(\mathsf{N}) \cdot \mathsf{T}$ is a normal subgroup of $\mathrm{Aut}(\mathsf{S})$, it follows that $\mathsf{T}' = (\mathrm{Int}(\mathsf{N}) \cdot \mathsf{T}) \cap \mathsf{K}$ is maximal compact in $\mathrm{Int}(\mathsf{N}) \cdot \mathsf{T}$. Thus T is conjugate to T' and $\mathsf{S} \rtimes \mathsf{T}$ is isomorphic to $\mathsf{S} \rtimes \mathsf{T}'$. Hence T' also satisfies the conclusion of Lemma 4.5, and we can assume $\mathsf{T} = \mathsf{T}'$. But then T is a normal subgroup of K and $\mathsf{S} \rtimes \mathsf{T}$ is a normal subgroup of $\mathsf{S} \rtimes \mathsf{K}$. Consequently, the maximal connected supersolvable normal subgroup R of $\mathsf{S} \rtimes \mathsf{T}$ is normal in $\mathsf{S} \rtimes \mathsf{K}$. Using that R is cocompact, we can deduce from Lemma 4.2 that R is the maximal connected, simply connected supersolvable normal subgroup of $\mathsf{S} \rtimes \mathsf{K}$ as well. By Lemma 3.2 K may be viewed as subgroup of $\mathrm{Aut}(\mathsf{R})$ and $\mathsf{R} \rtimes \mathsf{K}$ and $\mathsf{S} \rtimes \mathsf{K}$ are isomorphic.

Clearly, we can find an isomorphism $\iota\colon \mathsf{R} \rtimes \mathsf{K} \to \mathsf{S} \rtimes \mathsf{K}$. with $\iota_{|\mathsf{K}} = \mathrm{id}$. Set $(f(g), a(g)) := \iota(g)$ for $g \in \mathsf{R}$. It is straightforward to check that $f\colon \mathsf{R} \to \mathsf{S}$ is an equivariant diffeomorphism. Let g be a left invariant metric on S for which $g_{|e}$ is invariant under the natural representation of K. The natural action of $\mathsf{S} \rtimes \mathsf{K}$ on (S, g) is isometric. Consequently, the natural action of $\mathsf{R} \rtimes \mathsf{K}$ on (S, f^*g) is isometric, where f^*g denotes the pull back metric. In particular, f^*g is left invariant on R.

5. Discrete, cocompact subgroups

5.1. Characterizations of subgroups

We have seen in Remark 2.4 that Corollary 8.25 in Raghunathan (1972) is not correct. However, in view of Theorem 1 part b) of the following proposition can be regarded as a weak version of its statement:

Proposition 5.1. *Let* S *be a connected, simply connected supersolvable Lie group,* K *a compact subgroup of* $\mathrm{Aut}(\mathsf{S})$, $\Gamma \subset \mathsf{S} \rtimes \mathsf{K}$ *a discrete, cocompact subgroup, and let* N *be the maximal connected nilpotent normal subgroup of* S.

a) *Let ρ be the natural representation of* $\mathsf{S} \rtimes \mathsf{K}$ *in the Lie algebra* $\mathfrak{n}$ *of* N. *Then for* $g \in \mathsf{S} \rtimes \mathsf{K}$ *the endomorphism* $\rho(g)$ *is unipotent, if and only if* $g \in \mathsf{N}$. *In particular, the kernel of* ρ *coincides with the center of* N.
b) $\mathsf{N} \cap \Gamma$ *is a lattice in* N.
c) *The group* Γ *is polycyclic up to finite index and* $\mathsf{N} \cap \Gamma$ *is the nilradical of* Γ.
d) *The subgroup* $\Gamma^+ := \mathsf{S} \cap \Gamma$ *of* Γ *has the following algebraic characterization: The nilradical* $\mathrm{nil}(\Gamma)$ *is torsion free, and the natural action of* Γ *on* $\mathrm{nil}(\Gamma)$ *induces an action on its Malcev completion* $\tilde{\mathsf{N}}$. *Let*

$$\tilde{\rho}\colon \Gamma \to \mathrm{GL}(\tilde{\mathfrak{n}})$$

be the corresponding representation in the Lie algebra of $\tilde{\mathsf{N}}$. *Then* $g \in \Gamma^+$ *if and only if all eigenvalues of* $\tilde{\rho}(g)$ *are real and positive.*

Part d) of the proposition is a generalization of the main result in (Dekimpe, 1997), where the special case of a finite group K was considered.

Proof. Let $\mathfrak{g}$ be the Lie algebra of $\mathsf{S} \rtimes \mathsf{K}$, $\mathfrak{s}$, $\mathfrak{n}$ and $\mathfrak{k}$ the subalgebras corresponding to the subgroups S, N and K. Since N is a normal subgroup of $\mathsf{S} \rtimes \mathsf{K}$, the adjoint representation induces a homomorphism

$$\rho\colon \mathsf{S} \rtimes \mathsf{K} \to \mathrm{GL}(\mathfrak{n}).$$

We claim that an element $g \in \mathsf{S} \rtimes \mathsf{K}$ is contained in S if and only if all eigenvalues of $\rho(g)$ are positive. By Lemma 4.1 the elements in $\rho(\mathsf{S})$ have only positive eigenvalues. Let $\tilde{\mathsf{S}} \subset \mathsf{S} \rtimes \mathsf{K}$ be the set of elements that are mapped by ρ onto endomorphisms that have only positive eigenvalues. For an element $g = (\tau, a) \in \tilde{\mathsf{S}} \subset \mathsf{S} \rtimes \mathsf{K}$ we consider the group S' generated by g and S. Evidently, S' is solvable, and from Lemma 4.3 we obtain the inclusion $\mathsf{S}' \subset \tilde{\mathsf{S}}$. Thus all eigenvalues of $\rho(a) = \rho(\tau^{-1})\rho(g)$ are positive. On the other hand, $\rho(a)$ is contained in the compact group $\rho(\mathsf{K})$ and accordingly has to be the identity. This proves that $\tilde{\mathsf{S}} = \mathsf{S} \rtimes \mathrm{Ker}(\rho_{|\mathsf{K}}) \subset \mathsf{S} \rtimes \mathsf{K}$. In particular, it just remains to check that $\mathrm{Ker}(\rho_{|\mathsf{K}}) = \{e\}$.

Now, let $g = (\tau, a) \in \mathrm{Ker}(\rho)$. Clearly, $\mathrm{Ker}(\rho) \subset \tilde{\mathsf{S}} = \mathsf{S} \rtimes \mathrm{Ker}(\rho_{|\mathsf{K}})$. Therefore $a \in \mathrm{Ker}(\rho)$ and as a consequence $\tau \in \mathrm{Ker}(\rho)$. The kernel of $\rho_{|\mathsf{S}}$ is easily recognized as the center C of N. So the kernel $\mathrm{Ker}(\rho) = \mathsf{C} \times \mathrm{Ker}(\rho_{|\mathsf{K}})$ is a direct product, and $\mathrm{Ker}(\rho_{|\mathsf{K}})$ is a characteristic subgroup of $\mathrm{Ker}(\rho)$. Because of Lemma 3.2 the compact normal subgroup $\mathrm{Ker}(\rho_{|\mathsf{K}})$ of $\mathsf{S} \rtimes \mathsf{K}$ is trivial.

a). Let $g \in \mathsf{S} \rtimes \mathsf{K}$. From the above consideration we deduce that if $\rho(g)$ is unipotent, then $g \in \mathsf{S}$. Since the eigenvalues of $\rho(\mathsf{S})$ are real, the group $\rho(\mathsf{S})$ can be represented by real upper triangular matrices. It follows that the unipotent elements in $\rho(\mathsf{S})$ form a connected normal subgroup U. Taking into account that the kernel of ρ is the center of N, we see that $\rho^{-1}(\mathsf{U})$ is a connected nilpotent normal subgroup of S, and hence $\rho^{-1}(\mathsf{U}) = \mathsf{N}$.

b). By a theorem of Auslander the identity component D_0 of the closure D of $\Gamma \cdot \mathsf{S}$ is a solvable group, see (Raghunathan, 1972, Theorem 8.24). Evidently, $\Gamma \cap \mathsf{D}_0$ is discrete and cocompact in D_0. It is an immediate consequence of a) that the maximal connected nilpotent normal subgroup of D is N. By a theorem of Mostow (Raghunathan, 1972, Theorem 3.3) the group $\mathsf{N} \cap \Gamma$ is a lattice in N.

c). The group D has only finitely many connected components. In particular, $\Gamma \cap \mathsf{D}_0$ is of finite index in Γ. Moreover, the group $\Gamma \cap \mathsf{D}_0$, being discrete in the connected solvable Lie group D_0, is polycyclic. Clearly, $\Gamma \cap \mathsf{N}$ belongs to the nilradical $\mathrm{nil}(\Gamma)$. Using that $\Gamma \cap \mathsf{N}$ is a lattice in N, we deduce that $\mathrm{nil}(\Gamma) \cdot \mathsf{N}$ is nilpotent. Consequently, $\rho(\mathrm{nil}(\Gamma))$ consists of unipotents, and by a) $\mathrm{nil}(\Gamma) \subset \mathsf{N}$.

d). Since $\mathrm{nil}(\Gamma)$ is a lattice in N, we can identify $\tilde{\mathsf{N}}$ with N. The natural representation $\tilde{\rho}$ coincides under this identification with $\rho_{|\Gamma}$. Now d) follows from the analogous statement on ρ which we have proved above. □

5.2. The Proof of Theorem 3

$a) \Rightarrow b)$. In view of Theorem 1 we may assume that S is supersolvable. Let N be the maximal connected nilpotent normal subgroup of S, and let $\mathsf{H} \subset \mathsf{K}$ be the kernel of the induced action of K on S/N. Proposition 5.1 exhibits $\Gamma_N := (\mathsf{N} \rtimes \mathsf{H}) \cap \Gamma$ as a cocompact subgroup of $\mathsf{N} \rtimes \mathsf{H}$. Moreover, H acts effectively on N, and hence Γ_N is an almost crystallographic group.

The kernel of the projection $\pi : \mathsf{S} \rtimes \mathsf{K} \to (\mathsf{S}/\mathsf{N}) \rtimes (\mathsf{K}/\mathsf{H})$ contains $\Gamma_N \subset \Gamma$ as a cocompact subgroup, and accordingly the image $\Gamma_A := \pi(\Gamma)$ is a discrete, cocompact subgroup of $(\mathsf{S}/\mathsf{N}) \rtimes (\mathsf{K}/\mathsf{H})$. Evidently, S/N is isomorphic to $\mathbb{R}^d$ for some d. Since the action of (K/H) on S/N is effective, it follows that $(\mathsf{S}/\mathsf{N}) \rtimes (\mathsf{K}/\mathsf{H})$ is isomorphic to a closed, cocompact subgroup of $\mathbb{R}^d \rtimes \mathrm{O}(d)$. Thus Γ_A is a crystallographic group.

$b) \Rightarrow c)$. By assumption it is sufficient to prove that there is a subnormal series $\{e\} = \Gamma_0 \subset \cdots \subset \Gamma_{n-1} = \Gamma_N$ with crystallographic factors Γ_i/Γ_{i-1}. We argue by induction on $\mathrm{rank}(\Gamma_N)$. By the definition of an almost crystallographic group there is a connected, simply connected nilpotent Lie group N and a compact group $\mathsf{K} \subset \mathrm{Aut}(\mathsf{N})$ such that Γ_N is a discrete cocompact subgroup of $\mathsf{N} \rtimes \mathsf{K}$. Proposition 5.1 exhibits $\Gamma_N \cap \mathsf{N}$ as the nilradical $\mathrm{nil}(\Gamma_N)$ of Γ_N, and also as a lattice in N.

Let $\mathsf{H} \subset \mathsf{K}$ be the kernel of the natural action of K on the factor group $\mathsf{N}/[\mathsf{N}, \mathsf{N}]$. Notice that the action of the compact group H on $[\mathsf{N}, \mathsf{N}]$ is effective. Moreover, the commutator group of $\mathrm{nil}(\Gamma)$ is a lattice in $[\mathsf{N}, \mathsf{N}]$, and hence $\Gamma_N' := [\mathsf{N}, \mathsf{N}] \rtimes \mathsf{H} \cap \Gamma_N$ is an almost crystallographic group. As above we see that the factor group Γ_N/Γ_N' is crystallographic. Because of $\mathrm{rank}(\Gamma_N') < \mathrm{rank}(\Gamma_N)$ this completes the proof.

$c) \Rightarrow d)$. Since crystallographic groups are abelian up to finite index, Γ is polycyclic up to finite index by a standard argument, see (Segal, 1978, Proposition 2, p.2). By induction on n we can assume that Γ_{n-1} does not contain any nontrivial finite normal subgroup. Suppose now that E is a finite normal subgroup of Γ. By the induction hypothesis the intersection $\mathsf{E} \cap \Gamma_{n-1}$ is trivial. Thus the projection $\Gamma \to \Gamma/\Gamma_{n-1}$ maps E isomorphically onto a finite normal subgroup of Γ/Γ_{n-1}. But a crystallographic group does not contain any nontrivial finite normal subgroup, and consequently E is the trivial group.

d) $\Rightarrow$ *e*). Notice that

$$\mathsf{A} := \{g \in \Gamma \mid \text{the centralizer of } g \text{ has finite index in } \Gamma\}.$$

is a characteristic subgroup of Γ. Since A is by Lemma 3.3 finitely generated, it is evident from the definition that the center of A is of finite index in A. By Lemma 3.4 the torsion elements in A form a group E. Using that the center of E is a finitely generated subgroup of finite index we see that E is finite. But E is normal in Γ and hence trivial. Therefore $\mathsf{A} \cong \mathsf{A}/\mathsf{E}$, an abelian torsion free group, is free abelian.

The factor group Γ/A is polycyclic up to finite index, and by Lemma 3.3 it contains a strongly polycyclic normal subgroup H of finite index. Let Λ be the preimage of H under the projection $\pi : \Gamma \to \Gamma/\mathsf{A}$. Evidently, Λ is a strongly polycyclic normal subgroup of finite index in Γ. Moreover, the centralizer of Λ is contained in $\mathsf{A} \subset \Lambda$.

e) $\Rightarrow$ *d*). Suppose that E is a finite normal subgroup of Γ. The torsion free group Λ has trivial intersection with E. Using that both E and Λ are normal subgroups, we deduce that they commute. So E is contained in the centralizer of Λ which is by assumption a subgroup of Λ. Hence E is the trivial group.

Trivially *f*) implies *a*); so it remains to prove the implication *d*) $\Rightarrow$ *f*), and this will be done together with Theorem 5 in Subsect. 7.4 below.

6. Further preparations

Lemma 6.1. *Let $A \in M(n, \mathbb{C})$, and let $\lambda_1, \dots, \lambda_k$ be the pairwise different eigenvalues of A. Suppose that the numbers* $\exp(\lambda_1), \dots, \exp(\lambda_k)$ *are pairwise different, too. Then there is a polynomial $p \in \mathbb{C}[X]$ satisfying $p(\exp(A)) = A$. If in addition $A \in M(n, \mathbb{R})$, we can choose $p \in \mathbb{R}[X]$.*

Proof. Choose a decomposition $A = S + N$, where S is semisimple, N is nilpotent and $SN = NS$. Then $\exp(A) = \exp(S) \cdot \exp(N)$, $\exp(S)$ is semisimple, $\exp(N)$ is unipotent, and the matrices $\exp(S)$ and $\exp(N)$ commute. It is a well-known fact that such a decomposition is unique and that there exist polynomials p_1, p_2 satisfying $p_1(\exp(A)) = \exp(S)$ and $p_2(\exp(A)) = \exp(N)$. Thus it suffices to find polynomials $q_1, q_2 \in \mathbb{C}[X]$ satisfying $q_1(\exp(S)) = S$ and $q_2(\exp(N)) = N$.

We choose $q_1 \in \mathbb{C}[X]$ with $q_1(\exp(\lambda_i)) = \lambda_i$, $i = 1, \dots, k$. Using that S is conjugate to a diagonal matrix, we deduce that $q_1(\exp(S)) = S$. Next we define

$$q_2(X) \;:=\; \sum_{i=1}^{n} (-1)^{i+1}\, \tfrac{(X-1)^i}{i}$$

and obtain

$$\begin{aligned} q_2(\exp(N)) &= \sum_{i=1}^{\infty} (-1)^{i+1}\, \tfrac{(\exp(N)-I)^i}{i} \\ &= \log(\exp(N)) \;=\; N. \end{aligned}$$

Suppose in addition that $A \in M(n, \mathbb{R})$. By the first part there is a polynomial $q \in \mathbb{C}[X]$ with $q(\exp(A)) = A$. Let $\bar{q}$ denote the conjugate polynomial of q. Then $p(X) \;:=\; \frac{1}{2}(q(X) + \bar{q}(X))$ is a polynomial in $\mathbb{R}[X]$ satisfying $p(\exp(A)) = A$. □

6.1. The exponential map of $\mathsf{S} \rtimes_\beta \mathsf{K}$

Proposition 6.2. *Let* S *be a connected, simply connected supersolvable Lie group,* K *a compact Lie group,* $\beta\colon \mathsf{K} \to \mathrm{Aut}(\mathsf{S})$ *a continuous homomorphism,* $\mathsf{G} := \mathsf{S} \rtimes_\beta \mathsf{K}$*, and let* $\mathfrak{k}$*,* $\mathfrak{s}$ *and* $\mathfrak{g}$ *be the Lie algebras corresponding to* K*,* S *and* G.

a) *There is a neighborhood* U *of* 0 *in* $\mathfrak{k}$ *such that the set* $U + \mathfrak{s} := \{u + v \mid u \in U, v \in \mathfrak{s}\}$ *is invariant under the adjoint representation, and* $\exp\colon U + \mathfrak{s} \to \mathsf{S} \rtimes \mathsf{T}$ *is a diffeomorphism onto its image* $\exp(U + \mathfrak{s}) = \mathsf{S} \times \exp(U) \subset \mathsf{G}$*. Moreover, for* $v \in U + \mathfrak{s}$ *there is a polynomial* $q \in \mathbb{R}[X]$ *satisfying* $q(\mathrm{Ad}_{\exp(v)}) = \mathrm{ad}_v$.
b) *Let* $U + \mathfrak{s}$ *be as in a). Suppose that for a Lie subgroup* R *of* G *the elements* $\exp^{-1}_{|U+\mathfrak{s}}(\mathsf{R})$ *are contained in the Lie algebra* $\mathfrak{r}$ *of* R*. Then* R *is a closed subgroup with finitely many connected components and* $\mathsf{S}' := \mathsf{R} \cap \mathsf{S}$ *is a connected, cocompact subgroup of* R*. Furthermore,* R *is isomorphic to a semidirect product* $\mathsf{S}' \rtimes_{\beta'} \mathsf{L}$*, where* L *is maximal compact in* R.
c) *The centralizer* R *of a subgroup* $\mathsf{H} \subset \mathsf{G}$ *satisfies the hypothesis of b).*
d) *The normalizer* R *of an analytic subgroup* $\mathsf{H} \subset \mathsf{G}$ *satisfies the hypothesis of b).*

Proof. a). It is easy to see that there is an open, connected neighborhood $U \subset \mathfrak{k} \subset \mathfrak{g}$ of 0 in $\mathfrak{k}$ for which the following three conditions are satisfied.

(i) $\exp\colon U \to \mathsf{K}$ is a diffeomorphism onto its image.
(ii) U is invariant under the adjoint representation of K.

(iii) For $u \in U$ the imaginary parts of the eigenvalues of the adjoint map $\mathrm{ad}_u : \mathfrak{g} \to \mathfrak{g}$ lie strictly between $-\pi$ and π.

We claim that the conclusion of a) is true as soon as $U \subset \mathfrak{k}$ is an open set satisfying the above conditions. Note that for $u \in U$ and $s \in \mathfrak{s}$ the Lie subalgebra generated by u and s is solvable. Since all eigenvalues of $\mathrm{ad}_s : \mathfrak{g} \to \mathfrak{g}$ are real, it follows from Lie's theorem that the imaginary parts of the eigenvalues of $\mathrm{ad}_{u+s} : \mathfrak{g} \to \mathfrak{g}$ lie strictly between $-\pi$ and π. By Lemma 6.1 this implies the existence of a polynomial $p \in \mathbb{R}[X]$ with $p(\mathrm{Ad}_{\exp(u+s)}) = p(\exp(\mathrm{ad}_{u+s})) = \mathrm{ad}_{u+s}$.

Let N be the maximal connected nilpotent normal subgroup of S, and let C be the center of N. Clearly, C is a characteristic subgroup of S. Thus the homomorphism $\beta : \mathsf{K} \to \mathrm{Aut}(\mathsf{S})$ induces a homomorphism $\bar{\beta} : \mathsf{K} \to \mathrm{Aut}(\mathsf{S}/\mathsf{C})$. Consider the natural projection $\pi : \mathsf{S} \rtimes_\beta \mathsf{K} \to (\mathsf{S}/\mathsf{C}) \rtimes_{\bar{\beta}} \mathsf{K}$. We let $\mathfrak{s}/\mathfrak{c}$ denote the Lie algebra of the supersolvable Lie group S/C. By induction on the dimension of S we can assume that $\exp : U + \mathfrak{s}/\mathfrak{c} \to (\mathsf{S}/\mathsf{C}) \rtimes_{\bar{\beta}} \mathsf{K}$ is a diffeomorphism onto its image $\exp(U + \mathfrak{s}/\mathfrak{c}) = (\mathsf{S}/\mathsf{C}) \times \exp(U)$. For $v \in \mathfrak{g}$ we have $\pi(\exp(v)) = \exp(\pi_*(v))$, and hence it is sufficient to prove that for any $v \in U + \mathfrak{s}$ the map

$$f : \mathfrak{c} \to \mathsf{C}, \quad c \mapsto \exp(-v)\exp(v + c)$$

is a diffeomorphism. But this is an elementary computation: We identify the simply connected abelian group C canonically with its Lie algebra $\mathfrak{c}$. Then

$$f(c) \;=\; \sum_{k=0}^{\infty} \tfrac{(-ad_v)^k}{(k+1)!}\, c \quad \text{for } c \in \mathfrak{c} = \mathsf{C}.$$

In fact, this identity follows immediately from the formula for the differential of the exponential mapping, see (Helgason, 1978, Theorem 1.7). In order to show that f is a diffeomorphism, it remains to check that the linear map $h(\mathrm{ad}_v) = \sum_{k=0}^{\infty} \frac{(-\mathrm{ad}_{v|\mathfrak{c}})^k}{(k+1)!}$ is invertible. Let $\lambda_1, \ldots, \lambda_k$ be the different eigenvalues of ad_v. Then the eigenvalues of $h(\mathrm{ad}_v)$ are given by $\sum_{k=0}^{\infty} \frac{(-\lambda_i)^k}{(k+1)!}$, $i = 1, \ldots, k$. For $\lambda_i = 0$ this number is 1, and for $\lambda_i \neq 0$ this number equals $\frac{1-\exp(-\lambda_i)}{\lambda_i}$. Since the imaginary parts of the numbers $\lambda_1, \ldots, \lambda_k$ lie strictly between $-\pi$ and π, the eigenvalues of $h(\mathrm{ad}_v)$ are different from 0, and thus $h(\mathrm{ad}_v)$ is invertible.

b). Let $\pi : \mathsf{G} \to \mathsf{K}$ be the natural projection. By hypothesis any element in $\exp^{-1}_{|U+\mathfrak{s}}(\mathsf{R})$ is contained in the Lie algebra of R. Therefore any element of $\exp^{-1}_{|U}(\pi(\mathsf{R}))$ is contained in the Lie algebra of $\pi(\mathsf{R})$. Hence $\pi(\mathsf{R})$ is embedded and accordingly closed in K. Clearly, $\mathsf{S}' := \mathsf{R} \cap \mathsf{S}$ is a connected, closed subgroup of S. The compactness of $\pi(\mathsf{R}) \cong \mathsf{R}/\mathsf{S}'$ implies that R has only finitely many connected components. The last part of b) is a direct consequence of Lemma 3.2.

c). Let $g \in (\mathsf{S} \times \exp(U)) \cap \mathsf{R}$ and $v = \exp^{-1}_{|U+\mathfrak{s}}(g)$. Observe that for $h \in \mathsf{H}$ the vector $\mathrm{Ad}_h v$ lies in $U + \mathfrak{s}$ and that $\exp(\mathrm{Ad}_h v) = h\exp(v)h^{-1} = \exp(v)$. By a) $\mathrm{Ad}_h v = v$ for all $h \in \mathsf{H}$. Consequently, v is contained in the Lie algebra of R.

d). Let $\mathfrak{h}$, $\mathfrak{r}$ be the Lie algebras corresponding to H and R. Suppose that g is in $\mathsf{R} \cap (\mathsf{S} \times \exp(U))$ and put $v := \exp^{-1}_{|U+\mathfrak{s}}(g)$. Evidently, Ad_g leaves $\mathfrak{h}$ invariant. Since there is a polynomial $p \in \mathbb{R}[X]$ with $p(\mathrm{Ad}_g) = \mathrm{ad}_v$, it follows that $\mathfrak{h}$ is an invariant subspace of ad_v as well. Using that H is connected, we find $v \in \mathfrak{r}$. □

6.2. A global correspondence between subgroups and subalgebras

Proposition 6.3. *We keep the notations of Proposition 6.2. There is an open neighborhood U of 0 in $\mathfrak{k}$ satisfying the conclusion of Proposition 6.2 a) and for which moreover the following is true: For any subgroup H of $\mathsf{S} \rtimes_\beta \mathsf{K}$ the subspace*

$$\mathfrak{b} := \mathrm{span}_{\mathbb{R}}\big(\exp^{-1}_{|U+\mathfrak{s}}(\mathsf{H})\big) \subset \mathfrak{g}$$

is a subalgebra, the Lie group B corresponding to $\mathfrak{b}$ is closed in G, $\mathsf{B} \cap \mathsf{H}$ is cocompact in B, and $\mathsf{B}\cap\mathsf{S}$ is a connected, simply connected, cocompact subgroup of B.

Proof. At first we want to define the set U occurring in the proposition. There is a biinvariant metric $\langle\cdot,\cdot\rangle$ on K such that for any $x \in \mathfrak{k}$ with $\exp(x) = e$ the quantity $\langle x, x\rangle$ is an integer. In fact, if K is semisimple, one can set $\langle x, y\rangle := -\frac{1}{4\pi^2}B(x, y)$, where B is the Killing form of $\mathfrak{k}$. In the general case one can define $\langle\cdot,\cdot\rangle$ as the pullback metric of a locally faithful representation $\rho\colon \mathsf{K} \to (\mathrm{SO}(d), g)$ where $g := -\frac{1}{4\pi^2}B_{\mathfrak{so}(d)}$.

It is easy to see that with respect to the above metric the following holds: Let L be a normal subgroup of K and $\mathfrak{l} \subset \mathfrak{k}$ the corresponding ideal. Then the orthogonal complement $\mathfrak{l}^\perp$ of $\mathfrak{l}$ is an ideal in $\mathfrak{k}$, and the analytic Lie subgroup corresponding to $\mathfrak{l}^\perp$ is a *compact* normal subgroup of K, too.

Consider the ball $U := B_r(0)$ of radius r around 0 in $\mathfrak{k}$. By shrinking r if necessary, it is possible to assume that $U + \mathfrak{s}$ satisfies the conclusion of Proposition 6.2 a). Furthermore, a theorem of Jordan (Raghunathan, 1972, proof of Theorem 8.29) allows us to require that for any finite subgroup F of K the group generated by $\exp(U) \cap \mathsf{F}$ is abelian.

We claim that then the proposition is correct with $U = B_r(0)$. Henceforth we can assume that K is connected. Since we can replace H by its closure if necessary, we only have to verify the proposition for any closed subgroup $\mathsf{H} \subset \mathsf{G}$. The subgroup of H generated by $(\mathsf{S} \times \exp(U)) \cap \mathsf{H}$ is a normal subgroup of finite index in H, and without loss of generality H itself is generated by elements in $\mathsf{S} \times \exp(U)$. We argue by induction on $\dim(\mathsf{G})$. Evidently, the proposition is correct if $\dim(\mathsf{G}) = 1$. The induction conclusion is divided in five steps.

Step 1. It is sufficient to prove the proposition under the following two additional assumptions:

(1) The centralizer of H is contained in the center of G.
(2) A connected Lie subgroup $\mathsf{Z} \subset \mathsf{G}$ is normalized by H if and only if it is normal in G.

If (1) is not true, then there is a subgroup $\mathsf{Z}_1 \subset \mathsf{G}$ such that the centralizer G' of Z_1 contains H but not G. In fact, one can define Z_1 as the centralizer of H.

If (2) is not true, then there is a connected Lie subgroup $\mathsf{Z} \subset \mathsf{G}$ such that the normalizer G' of Z contains H but not G.

In either case G' has strictly smaller dimension than G. By Proposition 6.2 the group G' is a semidirect product of a compact subgroup K' and of the connected, simply connected supersolvable group $\mathsf{S}' := \mathsf{G}' \cap \mathsf{S}$.

The compact group K' is conjugate to a subgroup of K. Since the set $U + \mathfrak{s}$ is invariant under the adjoint representation of G, it is allowed to replace H by a conjugate subgroup. So we may assume that $\mathsf{K}' \subset \mathsf{K}$.

Let $\mathfrak{k}' \subset \mathfrak{k}$ and $\mathfrak{s}' \subset \mathfrak{s}$ be the Lie algebras of K' and S'. From the definition of U and from the induction hypothesis we infer that we can apply the proposition for $\mathsf{G}' = \mathsf{S}' \rtimes_\beta \mathsf{K}'$ with $U' := U \cap \mathfrak{k}'$. Finally, we have by Proposition 6.2 c), d)

$$\exp^{-1}_{|U+\mathfrak{s}}(\mathsf{H}) \;\subset\; U' \oplus \mathfrak{s}',$$

and hence the assertion follows.

Step 2. The proposition is correct provided that H is abelian.

Without loss of generality the centralizer of H is contained in the center of G, see Step 1. Taking into account that H is abelian we see that the centralizer of H is G. Thus G itself is abelian. But for an abelian group G the proposition is clearly correct.

Step 3. The proposition is correct if H contains a noncompact, closed, abelian normal subgroup A.

By Step 2

$$\mathfrak{b}' \;:=\; \operatorname{span}_{\mathbb{R}}\big(\exp^{-1}_{|U+\mathfrak{s}}(\mathsf{A})\big)$$

is a Lie algebra, the group B' corresponding to $\mathfrak{b}'$ is closed, $\mathsf{A} \cap \mathsf{B}'$ is cocompact in B', and $\mathsf{C} := \mathsf{B}' \cap \mathsf{S}$ is a connected, cocompact subgroup of B'. Since A is noncompact, the group C is nontrivial. From the definition of $\mathfrak{b}'$ we deduce that H normalizes B'. The additional assumption (2) in Step 1 says that B' is normal in G. This implies that C is normal in G, too. Let $\hat{\mathsf{H}}$ be the closure of $\mathsf{H} \cdot \mathsf{C}$. Evidently,

$$\mathfrak{b} \;=\; \operatorname{span}_{\mathbb{R}}\big(\exp^{-1}_{|U+\mathfrak{s}}(\mathsf{H})\big) \;=\; \operatorname{span}_{\mathbb{R}}\big(\exp^{-1}_{|U+\mathfrak{s}}(\hat{\mathsf{H}})\big).$$

Furthermore, H is cocompact in $\hat{\mathsf{H}}$. Thus it is sufficient to prove the proposition for $\hat{\mathsf{H}}$ instead of H. In other words, without loss of generality $\mathsf{C} \subset \mathsf{H}$.

Consider the homomorphism $\bar{\beta}\colon \mathsf{K} \to \mathrm{Aut}(\mathsf{S}/\mathsf{C})$ induced by β and the projection $\pi\colon \mathsf{S} \rtimes_\beta \mathsf{K} \to (\mathsf{S}/\mathsf{C}) \rtimes_{\bar{\beta}} \mathsf{K}$. By the induction hypothesis the proposition is known for the pair $\pi(\mathsf{H}) \subset (\mathsf{S}/\mathsf{C}) \rtimes_{\bar{\beta}} \mathsf{K}$. For $\mathsf{H} \subset \mathsf{S} \rtimes_\beta \mathsf{K}$ the assertion follows trivially.

Step 4. The proposition is correct provided that the identity component H_0 of H is nontrivial.

We begin with the case of a noncompact identity component H_0. The maximal connected solvable normal subgroup R of H_0 is cocompact in H_0, and hence it is itself a noncompact, closed normal subgroup of H. The commutator group $[\mathsf{R}, \mathsf{R}]$ is a connected subgroup of S. Thus if R is not abelian, the center of the nilpotent commutator group $[\mathsf{R}, \mathsf{R}]$ is a noncompact, closed, abelian normal subgroup of H. In either case the assertion follows from Step 3.

Suppose now that H_0 is compact. By Step 1 it is sufficient to prove the proposition under the additional assumption that H_0 is normal in G. In particular, $\mathsf{H}_0 \subset \mathsf{K}$. As explained above the orthogonal complement L of H_0 in K is a compact subgroup, too. Moreover, L and H_0 commute, and as a consequence H_0 and $\mathsf{S} \rtimes_\beta \mathsf{L}$ commute.

Let $\mathfrak{l}$ be the Lie algebra of L, $U' := U \cap \mathfrak{l}$, $\mathsf{H}' := \mathsf{H} \cap (\mathsf{S} \rtimes_\beta \mathsf{L})$, and let

$$\mathfrak{b}' := \mathrm{span}_{\mathbb{R}}\big(\exp^{-1}_{|U' \oplus \mathfrak{s}}(\mathsf{H}')\big).$$

By the induction hypothesis $\mathfrak{b}'$ is a Lie algebra, the Lie group B' corresponding to $\mathfrak{b}'$ is a closed subgroup of $\mathsf{S} \rtimes_\beta \mathsf{L}$ and $\mathsf{B}' \cap \mathsf{S}$ is connected and cocompact in B'. It remains to check that $\mathfrak{b} = \mathfrak{b}' \oplus \mathfrak{h}$. Clearly, $\mathfrak{b} \supset \mathfrak{b}' \oplus \mathfrak{h}$, and in order to get the converse relation, we consider a vector $v \in \exp^{-1}_{|U \oplus \mathfrak{s}}(\mathsf{H})$. Then v decomposes as a sum $v = u_1 + u_2 + u_3$ where $u_1 \in \mathfrak{h}$, $u_2 \in \mathfrak{l}$ and $u_3 \in \mathfrak{s}$. By construction the norm of the vector $u_1 + u_2 \in U = B_r(0)$ is strictly less than r. Since u_1 and u_2 are orthogonal, this implies $\|u_2\| < r$, and accordingly $u_2 \in U'$. Using that H_0 and $\mathsf{S} \rtimes_\beta \mathsf{L}$ commute, we see $\exp(u_1 + u_2 + u_3) = \exp(u_1) \cdot \exp(u_2 + u_3)$. Hence $\exp(u_2 + u_3) \in \mathsf{H}$ and $u_2 + u_3 \in \mathfrak{b}'$. But this proves $v \in \mathfrak{b}' \oplus \mathfrak{h}$.

A closed subgroup of G with trivial identity component is discrete, and consequently we can complete the proof of the proposition by showing:

Step 5. The proposition is correct for a discrete group H.

If H is finite, then it is conjugate to a subgroup of K, and without loss of generality $\mathsf{H} \subset \mathsf{K}$. Evidently, $\mathsf{H} \cap \mathsf{S} \times \exp(U) = \mathsf{H} \cap \exp(U)$. By definition of U the group generated by $\exp(U) \cap \mathsf{H}$ is abelian, and now Step 2 yields the assertion.

Assume now that H is not finite. By a theorem of Auslander (Raghunathan, 1972, Theorem 8.24) there is a subgroup of finite index in H which is a discrete subgroup of a connected solvable Lie group. In particular, H is polycyclic up to finite index. We employ Lemma 3.3 in order to find a strongly polycyclic normal subgroup H' of finite index in H. Let A be the center of the nilradical of H'. Evidently, A is an infinite, abelian normal subgroup of H, so the claim follows from Step 3. □

6.3. The Lie hull of a subgroup

Definition 6.4. Let $\mathsf{S} \rtimes_\beta \mathsf{K}$ be as in Proposition 6.2. For a subgroup $\mathsf{H} \subset \mathsf{S} \rtimes_\beta \mathsf{K}$ the Lie hull of H in $\mathsf{S} \rtimes_\beta \mathsf{K}$ is defined as the group R that corresponds in the sense of part a) of the following theorem to H. A subgroup H is said to be a Lie hull (in $\mathsf{S} \rtimes_\beta \mathsf{K}$) if and only if H coincides with its Lie hull.

Theorem 6.5. *Let $\mathsf{G} = \mathsf{S} \rtimes_\beta \mathsf{K}$, $\mathfrak{s}$, $\mathfrak{k}$ and $\mathfrak{g}$ be as in Proposition 6.2.*

a) For a subgroup $\mathsf{H} \subset \mathsf{G}$ there is a unique closed subgroup $\mathsf{R} \subset \mathsf{G}$ satisfying: H is a cocompact subgroup of R, $\mathsf{S}' := \mathsf{R} \cap \mathsf{S}$ is a connected, cocompact normal subgroup of R and R coincides with the closure of $\mathsf{H} \cdot \mathsf{S}'$. Furthermore, the group R is then isomorphic to a semidirect product $\mathsf{S}' \rtimes_{\beta'} \mathsf{L}$ where L is compact.

b) Let $\tilde{\mathsf{H}} \subset \mathsf{H} \subset \mathsf{G}$ be subgroups, and let $\tilde{\mathsf{R}}$ and R be the corresponding Lie hulls in G. If $\tilde{\mathsf{H}}$ is a normal subgroup of H, then $\tilde{\mathsf{R}}$ is a normal subgroup of R. Moreover, we have in that case that the commutator group

$$[\mathsf{R}, \tilde{\mathsf{R}}] := \left\langle \{ghg^{-1}h^{-1} \mid g \in \mathsf{R}, h \in \tilde{\mathsf{R}}\} \right\rangle$$

is the Lie hull of $[\mathsf{H}, \tilde{\mathsf{H}}]$ in G. In particular, if H is abelian (nilpotent, solvable), then R is abelian (nilpotent, solvable), too.

c) Assume that $\mathsf{H} \subset \mathsf{G}$ is a closed subgroup such that G is the Lie hull of H in G. Let S_1 be another connected, simply connected supersolvable Lie group, $\mathsf{K}_1 \subset \mathrm{Aut}(\mathsf{S}_1)$ a compact subgroup, and let $\varphi\colon \mathsf{H} \to \mathsf{S}_1 \rtimes \mathsf{K}_1$ be a continuous homomorphism mapping H onto a cocompact subgroup of $\mathsf{S}_1 \rtimes \mathsf{K}_1$. Then there is a unique continuous extension of φ to a continuous homomorphism $\bar{\varphi}\colon \mathsf{G} \to \mathsf{S}_1 \rtimes \mathsf{K}_1$. Moreover, $\bar{\varphi}(\mathsf{S}) = \mathsf{S}_1$.

Proof. a). Choose a neighborhood $U \subset \mathfrak{k}$ of 0 as in Proposition 6.3. Then

$$\mathfrak{b} := \operatorname{span}_{\mathbb{R}}\left(\exp^{-1}_{|U+\mathfrak{s}}(\mathsf{H})\right)$$

is a Lie algebra, and the corresponding Lie group B is closed in G. Furthermore, $\mathsf{H} \cap \mathsf{B}$ is cocompact in B, and $\mathsf{S}' := \mathsf{B} \cap \mathsf{S}$ is a connected, cocompact subgroup of B. Finally, it is evident from the definition of $\mathfrak{b}$ that H normalizes B and that $\mathsf{H} \cap \mathsf{B}$ has finite index in H. Clearly, H also normalizes S', and thus the closure R of $\mathsf{H} \cdot \mathsf{S}'$ has the claimed properties.

Assume now that $\tilde{\mathsf{R}}$ is another group satisfying the conclusion of a). Then

$$\tilde{\mathfrak{b}} := \operatorname{span}_{\mathbb{R}}\left(\exp_{|U+\mathfrak{s}}^{-1}(\tilde{\mathsf{R}})\right)$$

is a Lie algebra, $\mathsf{S}' \subset \mathsf{B}$ is cocompact in the Lie group $\tilde{\mathsf{B}}$ corresponding to $\tilde{\mathfrak{b}}$. More precisely, S' is a connected cocompact subgroup of $\tilde{\mathsf{B}} \cap \mathsf{S}$. The orbit space $(\tilde{\mathsf{B}} \cap \mathsf{S})/\mathsf{S}'$ being both compact and contractible must be trivial, and hence $\tilde{\mathsf{B}} \cap \mathsf{S} = \mathsf{S}'$. Similarly we obtain the equation $\tilde{\mathsf{R}} \cap \mathsf{S} = \tilde{\mathsf{B}} \cap \mathsf{S}$, so $\mathsf{S}' = \tilde{\mathsf{R}} \cap \mathsf{S}$. But this implies $\tilde{\mathsf{R}} = \mathsf{R}$.

Let L be a maximal compact subgroup of R. From Subsect. 3.2 we infer that R is canonically isomorphic to a semidirect product $\mathsf{S}' \rtimes_{\beta'} \mathsf{L}$, where $\beta' \colon \mathsf{L} \to \operatorname{Aut}(\mathsf{S}')$ is induced by conjugation.

b). The following obvious fact will be used subsequently without further comments: Let $\mathsf{B} \subset \mathsf{G}$ be a Lie hull, $\mathsf{H} \subset \mathsf{B}$ a subgroup, and let $\mathsf{L} \subset \mathsf{B}$ be a maximal compact subgroup. Then B is canonically isomorphic to a semidirect product $(\mathsf{B} \cap \mathsf{S}) \rtimes_{\beta'} \mathsf{L}$, and under this identification the Lie hull of H in B coincides with the Lie hull of H in G.

So we may assume that $\mathsf{R} = \mathsf{G}$. We first want to prove that $\tilde{\mathsf{R}}$ is a normal subgroup of G. Let M be the normalizer of $\tilde{\mathsf{R}}$, and let N be the normalizer of the identity component $\tilde{\mathsf{R}}_0$ of $\tilde{\mathsf{R}}$. Evidently, $\mathsf{H} \subset \mathsf{M} \subset \mathsf{N}$. From Proposition 6.2 we deduce that N is a Lie hull and thus $\mathsf{N} = \mathsf{G}$.

Since $\tilde{\mathsf{R}}_0$ is normal in G, the group $\tilde{\mathsf{K}}_0 := \tilde{\mathsf{R}}_0 \cap \mathsf{K}$ is maximal compact in $\tilde{\mathsf{R}}_0$. This implies that $\tilde{\mathsf{R}}_0 = \tilde{\mathsf{S}} \rtimes_\beta \tilde{\mathsf{K}}_0 \subset \mathsf{S} \rtimes_\beta \mathsf{K}$ where $\tilde{\mathsf{S}} := \tilde{\mathsf{R}}_0 \cap \mathsf{S} = \tilde{\mathsf{R}} \cap \mathsf{S}$. Evidently, β induces a homomorphism $\bar{\beta} \colon \mathsf{K}/\tilde{\mathsf{K}}_0 \to \operatorname{Aut}(\mathsf{S}/\tilde{\mathsf{S}})$. Consider the natural projection $\pi : \mathsf{S} \rtimes_\beta \mathsf{K} \to (\mathsf{S}/\tilde{\mathsf{S}}) \rtimes_{\bar{\beta}} (\mathsf{K}/\tilde{\mathsf{K}}_0)$ and observe that $\pi(\mathsf{M})$ is the normalizer of $\pi(\tilde{\mathsf{R}})$ in $\pi(\mathsf{G})$. In particular, the centralizer $\mathsf{C}(\pi(\tilde{\mathsf{R}}))$ of the finite group $\pi(\tilde{\mathsf{R}})$ is of finite index in $\pi(\mathsf{M})$. Thus $\mathsf{C}(\pi(\tilde{\mathsf{R}}))$ is cocompact in $\pi(\mathsf{G})$, too. By Proposition 6.2 c) the group $\mathsf{C}(\pi(\tilde{\mathsf{R}})) \cap \mathsf{S}/\tilde{\mathsf{S}}$ is a connected, cocompact subgroup of $\mathsf{S}/\tilde{\mathsf{S}}$, and consequently these two groups coincide. Hence $\mathsf{S} \subset \mathsf{M}$. Since G is the closure of $\mathsf{H} \cdot \mathsf{S}$, we find $\mathsf{M} = \mathsf{G}$.

In other words, $\tilde{\mathsf{R}}$ is normal in $\mathsf{G} = \mathsf{R}$, as claimed. It follows that $\tilde{\mathsf{K}} := \tilde{\mathsf{R}} \cap \mathsf{K}$ is maximal compact in $\tilde{\mathsf{R}}$ and that $\tilde{\mathsf{R}} = \tilde{\mathsf{S}} \rtimes_\beta \tilde{\mathsf{K}} \subset \mathsf{S} \rtimes_\beta \mathsf{K}$ where $\tilde{\mathsf{S}} = \tilde{\mathsf{R}} \cap \mathsf{S}$. Now it is easy to see that $[\mathsf{G}, \tilde{\mathsf{R}}] \cap \mathsf{S}$ is a connected group and that the factor group $[\mathsf{G}, \tilde{\mathsf{R}}] \big/ ([\mathsf{G}, \tilde{\mathsf{R}}] \cap \mathsf{S})$ is isomorphic to the compact group $[\mathsf{K}, \tilde{\mathsf{K}}]$. This proves that $[\mathsf{G}, \tilde{\mathsf{R}}]$ is a Lie hull, and accordingly the Lie hull R' of $[\mathsf{H}, \tilde{\mathsf{H}}]$ is contained in $[\mathsf{G}, \tilde{\mathsf{R}}]$.

In order to get the converse relation we observe that $[\mathsf{H}, \tilde{\mathsf{H}}]$ is normal in H. Its Lie hull R' is normal in G, the Lie hull of H. So $\mathsf{R}' = \mathsf{S}' \rtimes_\beta \mathsf{K}'$ where $\mathsf{K}' = \mathsf{R}' \cap \mathsf{K}$ and $\mathsf{S}' = \mathsf{R}' \cap \mathsf{S}$. Consider the natural projection

$$\mathrm{pr} \colon \mathsf{G} = \mathsf{S} \rtimes_\beta \mathsf{K} \to (\mathsf{S}/\mathsf{S}') \rtimes_{\hat{\beta}} (\mathsf{K}/\mathsf{K}') =: \hat{\mathsf{G}}.$$

Evidently, the groups $\hat{\mathsf{G}} = \mathrm{pr}(\mathsf{G})$ and $\mathrm{pr}(\tilde{\mathsf{R}})$ are the Lie hulls of $\mathrm{pr}(\mathsf{H})$ and $\mathrm{pr}(\tilde{\mathsf{H}})$ in $\hat{\mathsf{G}}$. Moreover, $\mathrm{pr}(\tilde{\mathsf{H}})$ is contained in the center of $\mathrm{pr}(\mathsf{H})$. From Proposition 6.2 we deduce that the centralizer of $\mathrm{pr}(\tilde{\mathsf{H}})$ is a Lie hull, and thus $\mathrm{pr}(\tilde{\mathsf{H}})$ is contained in the center of $\hat{\mathsf{G}}$. Since the center of $\hat{\mathsf{G}}$ is also a Lie hull, we see that the Lie hull $\mathrm{pr}(\tilde{\mathsf{R}})$ of $\mathrm{pr}(\tilde{\mathsf{H}})$ is contained in the center of $\hat{\mathsf{G}}$, too. Consequently, $[\mathsf{G}, \tilde{\mathsf{R}}] \subset \mathrm{Ker}(\mathrm{pr}) = \mathsf{R}'$.

c). Set $\mathsf{G}_1 := \mathsf{S}_1 \rtimes \mathsf{K}_1$. At first, we want to check that there is no nontrivial compact subgroup of G_1 which is normalized by $\varphi(\mathsf{H})$. Assume that L is a compact subgroup of $\mathsf{S}_1 \rtimes \mathsf{K}_1$ normalized by $\varphi(\mathsf{H})$. Let R_1 be the Lie hull of $\varphi(\mathsf{H}) \cdot \mathsf{L} \subset \mathsf{S}_1 \rtimes \mathsf{K}_1$. Using that $\mathsf{R}_1 \cap \mathsf{S}_1$ is a connected cocompact subgroup of $\mathsf{S}_1 \subset \mathsf{G}_1$, we obtain the inclusion $\mathsf{S}_1 \subset \mathsf{R}_1$ and for that reason $\mathsf{R}_1 = \mathsf{S}_1 \rtimes \mathsf{K}_1'$, where $\mathsf{K}_1' = \mathsf{K}_1 \cap \mathsf{R}_1$. Clearly, the compact group $\mathsf{L} \subset \mathsf{S}_1 \rtimes \mathsf{K}_1$ is a Lie hull. Since L is a normal subgroup of $\varphi(\mathsf{H}) \cdot \mathsf{L}$, it follows from b) that L is normal in R_1. By Lemma 3.2 the group $\mathsf{R}_1 \cong \mathsf{S}_1 \rtimes \mathsf{K}_1'$ does not contain any nontrivial compact normal subgroup and hence $\mathsf{L} = \{e\}$.

Consider the product group

$$\mathsf{P} := \mathsf{G} \times \mathsf{G}_1 = (\mathsf{S} \times \mathsf{S}_1) \rtimes_{\beta \times \mathrm{id}} (\mathsf{K} \times \mathsf{K}_1)$$

and the graph $\tilde{\mathsf{H}} := \{(h, \varphi(h)) \mid h \in \mathsf{H}\}$ of φ. We view $\mathsf{S} \times \mathsf{S}_1$, G_1 and G in the natural fashion as subgroups of P.

Let $\tilde{\mathsf{R}}$ be the Lie hull of $\tilde{\mathsf{H}}$ in P. Since $\tilde{\mathsf{H}}$ is cocompact in $\tilde{\mathsf{R}}$, it follows that $\tilde{\mathsf{R}} \cap \mathsf{G}_1$ is a compact normal subgroup of $\tilde{\mathsf{R}}$. Furthermore, $\tilde{\mathsf{R}} \cap \mathsf{G}_1$ is also normalized by the image of the projection $\mathrm{pr}_1 \colon \tilde{\mathsf{R}} \to \mathsf{G}_1$ onto the second factor of P. In particular, $\varphi(\mathsf{H}) \subset \mathrm{pr}_1(\tilde{\mathsf{R}})$ normalizes the compact group $\tilde{\mathsf{R}} \cap \mathsf{G}_1$. As explained above this implies that $\tilde{\mathsf{R}} \cap \mathsf{G}_1 = \{e\}$.

Consequently, the projection $\mathrm{pr} \colon \tilde{\mathsf{R}} \to \mathsf{G}$ onto the first factor of P is injective. We claim that pr is surjective as well. Clearly, $\mathrm{pr}(\tilde{\mathsf{R}}) \supset \mathsf{H}$ is cocompact in G. The group $\tilde{\mathsf{S}} := \tilde{\mathsf{R}} \cap (\mathsf{S} \times \mathsf{S}_1)$ is a connected, cocompact subgroup of $\tilde{\mathsf{R}}$, and hence $\mathrm{pr}(\tilde{\mathsf{S}})$ is a connected, cocompact subgroup of S. Therefore $\mathrm{pr}(\tilde{\mathsf{S}}) = \mathsf{S}$, so $\mathrm{pr}(\tilde{\mathsf{R}})$ contains the closure of $\mathsf{H} \cdot \mathsf{S}$ which is by hypothesis equal to G.

Thus pr is an isomorphism, and $\bar{\varphi} := \mathrm{pr}_1 \circ \mathrm{pr}^{-1}$ is a continuous extension of φ. Moreover, the connected, cocompact subgroup $\bar{\varphi}(\mathsf{S})$ of S_1 coincides with S_1.

Suppose now that $\hat{\varphi}$ is another continuous extension of φ. Consider the graph $\hat{\mathsf{G}} := \{(g, \hat{\varphi}(g)) \mid g \in \mathsf{G}\}$ of $\hat{\varphi}$, and let $\hat{\mathsf{R}}$ be the Lie hull of $\hat{\mathsf{G}}$ in P. Evidently, $\tilde{\mathsf{R}} \subset \hat{\mathsf{R}}$. As above we can show that $\hat{\mathsf{R}}$ is again a graph. Since the three groups $\tilde{\mathsf{R}} \subset \hat{\mathsf{R}} \supset \hat{\mathsf{G}}$ are graphs, we find $\tilde{\mathsf{R}} = \hat{\mathsf{R}} = \hat{\mathsf{G}}$ and in particular $\hat{\varphi} = \bar{\varphi}$. □

7. Proofs of the main results

Recall that until now we have not verified Theorem 3 in full generality. However, we have seen that each of the conditions in Theorem 3 implies condition d). In this section we say that Γ is a polycrystallographic group if and only if Γ matches the condition d) of Theorem 3. Notice that with this definition the statements of the Corollary 4 and Theorem 5 make sense, so we can prove them. Also remark that Theorem 5 then implies the missing implication "$d) \Rightarrow f)$" of Theorem 3.

7.1. The proof of Theorem 2

By replacing K_i by a compact subgroup if necessary, we may assume that $\mathsf{S}_i \rtimes \mathsf{K}_i$ is the Lie hull of $\Gamma_i \subset \mathsf{S} \rtimes \mathsf{K}_i$. According to Theorem 6.5 there is an isomorphism $\varphi\colon \mathsf{S}_1 \rtimes \mathsf{K}_1 \to \mathsf{S}_2 \rtimes \mathsf{K}_2$ that extends ι.

The group $\varphi(\mathsf{K}_1)$ is not necessarily equal to K_1. However, $\varphi(\mathsf{K}_1)$ is maximal compact in $\mathsf{S}_2 \rtimes \mathsf{K}_2$, and for a suitable $\tau \in \mathsf{S}_2$ we have $\tau^{-1}\varphi(\mathsf{K}_1)\tau = \mathsf{K}_2$. The affine diffeomorphism

$$f\colon \mathsf{S}_1 \to \mathsf{S}_2, \quad v \mapsto \varphi(v)\cdot\tau$$

is easily seen to be equivariant, and hence we are done.

7.2. The proof of Corollary 4

The uniqueness part of Corollary 4 is a direct consequence of Theorem 6.5 c). The polycrystallographic group Γ is polycyclic up to finite index, and by (Raghunathan, 1972, Theorem 4.28) there is a subgroup Λ of finite index in Γ which is isomorphic to lattice in a connected, simply connected solvable Lie group S'. By passing from Λ to a subgroup of finite index if necessary, we may assume that Λ is a normal subgroup of Γ.

There is a toral subgroup T of $\mathrm{Aut}(\mathsf{S}')$ such that $\mathsf{T} \rtimes \mathsf{S}'$, is isomorphic to a semidirect product $\mathsf{G} := \tilde{\mathsf{S}} \rtimes \mathsf{T}$, where $\tilde{\mathsf{S}}$ is a connected, simply connected supersolvable Lie group, see Lemma 4.5. Let k be the index of Λ in Γ, G^k the k-fold product of G, $\tilde{\mathsf{S}}^k$ the k-fold product of $\tilde{\mathsf{S}}$, and let S_k be the symmetric group of degree k. By Lemma 3.5 there is an injective homomorphism

$$\psi\colon \Gamma \to \Lambda^k \rtimes \mathsf{S}_k \subset \mathsf{G}^k \rtimes \mathsf{S}_k.$$

Clearly, the semidirect product $\mathsf{G}^k \rtimes \mathsf{S}_k$ is isomorphic to $(\tilde{\mathsf{S}})^k \rtimes \tilde{\mathsf{K}}$, where $\tilde{\mathsf{K}} \cong \mathsf{T}^k \rtimes \mathsf{S}_k$ is a compact subgroup of $\mathrm{Aut}(\tilde{\mathsf{S}}^k)$.

Let R be the Lie hull of $\psi(\Gamma)$ in $(\tilde{\mathsf{S}})^k \rtimes \tilde{\mathsf{K}}$, see Definition 6.4. Then $\psi(\Gamma)$ is a discrete, cocompact subgroup of R. Furthermore, R is isomorphic to a semidirect product $\mathsf{S} \rtimes_\beta \tilde{\mathsf{K}}'$, where $\mathsf{S} \subset \tilde{\mathsf{S}}^k$ is a connected, simply connected supersolvable Lie group and $\tilde{\mathsf{K}}'$ is a compact group acting on S by continuous automorphism via β. We identify $\Gamma \cong \psi(\Gamma)$ with a discrete, cocompact subgroup of $\mathsf{S} \rtimes_\beta \tilde{\mathsf{K}}'$.

Consider the natural projection $\pi : \mathsf{S} \rtimes_\beta \tilde{\mathsf{K}}' \to \mathsf{S} \rtimes \beta(\tilde{\mathsf{K}}) \subset \mathsf{S} \rtimes \mathrm{Aut}(\mathsf{S})$. The kernel of $\pi_{|\Gamma}$, being both discrete and compact, is finite and hence trivial because Γ satisfies condition d) of Theorem 3. Thus $\iota := \pi_{|\Gamma}$ maps Γ isomorphically onto a discrete, cocompact subgroup of $\mathsf{S} \rtimes \beta(\tilde{\mathsf{K}})$ such that $\iota(\Gamma) \cdot \mathsf{S}$ is dense in $\mathsf{S} \rtimes \beta(\tilde{\mathsf{K}})$.

7.3. A criterion for connectivity

Let V be a real or complex vector space. We recall that an endomorphism $A \in \mathrm{GL}(V)$ is called net if the multiplicative subgroup of $\mathbb{C}^*$ generated by the eigenvalues of A does not contain any nontrivial root of unity. We need a slightly different definition: Let $A \in \mathrm{GL}(V)$, and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be the eigenvalues of A. We call A absolutely net if and only if the group generated by $\frac{\lambda_1}{|\lambda_1|}, \ldots, \frac{\lambda_k}{|\lambda_k|}$ does not contain any nontrivial root of unity.

For a polycrystallographic group Γ the nilradical $\mathrm{nil}(\Gamma)$ is torsion free; in fact this follows immediately if one applies Proposition 5.1 to the image of the embedding $\iota\colon \Gamma \to \mathsf{S} \rtimes \mathsf{K}$ of Corollary 4. Hence the conjugate action of Γ on $\mathrm{nil}(\Gamma)$ induces a representation

$$\rho\colon \Gamma \to \mathrm{GL}(\mathfrak{n}) \tag{7.1}$$

in the Lie algebra $\mathfrak{n}$ of the Malcev completion of $\mathrm{nil}(\Gamma)$. We call an element $g \in \Gamma$ absolutely net if and only if $\rho(g)$ is absolutely net.

Proposition 7.1. *Let Γ be a polycrystallographic group, and let $\iota\colon \Gamma \to \mathsf{S} \rtimes \mathsf{K}$ be an embedding satisfying the assumptions of Corollary 4. If $g \in \Gamma$ is absolutely net, then $\iota(g)$ is contained in the identity component of $\mathsf{S} \rtimes \mathsf{K}$.*

Corollary 7.2. *Let $\iota\colon \Gamma \to \mathsf{S} \rtimes \mathsf{K}$ be as above.*

a) If Γ is generated by elements which are absolutely net, then $\mathsf{S} \rtimes \mathsf{K}$ is connected. In particular, $\Gamma/\mathrm{nil}(\Gamma)$ is then abelian.
b) The number of connected components of $\mathsf{S} \rtimes \mathsf{K}$ is bounded by a constant only depending on the rank of Γ.

Corollary 7.3. *Let Γ be a polycrystallographic group that is generated by elements which are absolutely net. Suppose moreover that $\Gamma/\mathrm{nil}(\Gamma)$ is free abelian. Then Γ is isomorphic to a discrete, cocompact subgroup of a connected, simply connected solvable Lie group.*

Of course, Corollary 7.3 and Corollary 7.2 a) can be regarded as corrections of the two theorems of Auslander from which we have seen in Example 2.1 that the original versions are not correct.

Lemma 7.4. *Let $d = (d_1, \dots, d_k) \in (\mathsf{S}^1)^k$. Suppose that the subgroup of $\mathsf{S}^1 \subset \mathbb{C}^*$ generated by $d_1, \dots, d_k \in \mathsf{S}^1$ is torsion free. Then the closure of the cyclic group generated by d is a connected subgroup of $(\mathsf{S}^1)^k$.*

Proof of Lemma 7.4. We argue by induction on k. If $k = 1$ and $d \neq e$, then the group generated by d_1 is dense in S^1. Suppose now that the lemma is known for $k - 1 \geq 1$. Choose real numbers $\varphi_1, \dots, \varphi_k$ such that $\exp(2\pi i \varphi_i) = d_i \in \mathsf{S}^1 \subset \mathbb{C}$. If $1, \varphi_1, \dots, \varphi_k \in \mathbb{R}$ are linear independent over $\mathbb{Q}$, then the group generated by d is dense in $\big(\mathsf{S}^1\big)^k$, and we are done. Otherwise there are integers $z_1, \dots, z_k, w \in \mathbb{Z}$ such that $(z_1, \dots, z_k) \neq 0$ and

$$\sum_{i=1}^{n} z_i \varphi_i = w.$$

Clearly, we can assume that the greatest common divisor of $z_1, \dots, z_k, w$ is 1. Suppose for a moment that an integer $m > 1$ divides the numbers $z_1, \dots, z_k$. By construction $\frac{w}{m} \in \mathbb{Q} \setminus \mathbb{Z}$, and hence $\prod_{i=1}^{k} d_i^{z_i/m}$ is a nontrivial root of unity which is impossible. Thus the greatest common divisor of $z_1, \dots, z_k$ is 1, and there are integers $a_1, \dots, a_k$ with $\sum_{i=1}^{k} a_i z_i = w$. Let $\tilde{\varphi}_i = \varphi - a_i$. Then $\exp(2\pi i \tilde{\varphi}_i) = d_i$ and $\sum_{i=1}^{k} z_i \tilde{\varphi}_i = 0$.

In other words, without loss of generality $w = 0$. To prove the induction step we argue by induction on $\sum_{i=1}^{k} |z_i|$. If $\sum_{i=1}^{k} |z_i| = 1$, then $\varphi_{i_0} = 0$ for some i_0, and the assertion follows from the k-induction hypothesis.

Assume now that $\sum_{i=1}^{k} |z_i| > 1$. There are at least two numbers z_i, z_j different from 0 because the greatest common divisor of $(z_1, \dots, z_k)$ is 1. After reordering we have $0 < |z_k| \leq |z_{k-1}|$. Choose a number $n \in \mathbb{Z}$ such that $|z_{k-1} - n z_k| < |z_k|$, consider the Lie group automorphism

$$\begin{aligned} \sigma\colon\ (\mathsf{S}^1)^k &\to (\mathsf{S}^1)^k, \\ \big(b_1, \dots, b_k\big) &\mapsto \big(b_1, \dots, b_{k-1}, b_k b_{k-1}^n\big), \qquad \text{and let} \\ \hat{d} := \big(\hat{d}_1, \dots, \hat{d}_k\big) &:= \sigma(d), \\ \big(\hat{\varphi}_1, \dots, \hat{\varphi}_k\big) &:= \big(\varphi_1, \dots, \varphi_{k-1}, \varphi_k + n\varphi_{k-1}\big). \end{aligned}$$

Evidently, the group generated by $\hat{d}_1, \dots, \hat{d}_k \in \mathsf{S}^1$ coincides with the group generated by $d_1, \dots, d_k$. Since σ is an isomorphism, the closure of the group generated by d is connected if and only if the closure of the group generated by $\hat{d}$ is connected. Moreover, for the numbers $(\hat{z}_1, \dots, \hat{z}_k) := (z_1, \dots, z_{k-1} - n z_k, z_k)$ we have $\sum_{i=1}^{k} \hat{z}_i \hat{\varphi}_i = 0$. By construction $\sum_{i=1}^{k} |\hat{z}_i| < \sum_{i=1}^{k} |z_i|$, and thus the assertion follows from the induction hypothesis. □

Proof of Proposition 7.1. Let N be the maximal connected nilpotent normal subgroup of S, $\mathfrak{n}$ the Lie algebra of N, and let

$$\rho\colon \mathsf{S} \rtimes \mathsf{K} \to \mathrm{GL}(\mathfrak{n})$$

be the natural representation. Proposition 5.1 allows us to identify N with the Malcev completion of $\mathrm{nil}(\Gamma)$ and hence $\rho(\iota(g))$ is absolutely net.

Let $\mathsf{H} \subset \mathsf{S} \rtimes \mathsf{K}$ be the cyclic group generated by $\iota(g)$, and let R be the Lie hull of H in $\mathsf{S} \rtimes \mathsf{K}$, see Definition 6.4. Theorem 6.5 exhibits R as an abelian group and $\mathsf{S}' := \mathsf{R} \cap \mathsf{S}$ as a connected, cocompact subgroup of R. Consider the maximal compact subgroup L of R. Clearly, we can write $\iota(g) \in \mathsf{R}$ uniquely as a product $\iota(g) = a \cdot \tau = \tau \cdot a$ where $a \in \mathsf{L}$ and $\tau \in \mathsf{S}'$.

Let $\lambda_1, \dots, \lambda_k$ be the eigenvalues of $\rho(\iota(g))$. The eigenvalues of $\rho(\tau)$ are positive and the eigenvalues of $\rho(a)$ have absolute value 1. Taking into account that

$$\rho\big(\iota(g)\big) = \rho(a)\rho(\tau) = \rho(\tau)\rho(a),$$

we see that the eigenvalues of $\rho(a)$ are given by $\frac{\lambda_1}{|\lambda_1|}, \dots, \frac{\lambda_k}{|\lambda_k|}$. The closure of the cyclic group generated by $\rho(a)$ is contained in the compact group $\rho(\mathsf{L})$. Combining this with Lemma 7.4 we find that $\rho(a)$ is contained in the identity component of $\rho(\mathsf{S} \rtimes \mathsf{K})$. By Proposition 5.1 the kernel of ρ is given by the center of N. In particular, $\mathrm{Ker}(\rho)$ is connected, and accordingly a is contained in the identity component of $\mathsf{S} \rtimes \mathsf{K}$. Of course, the same is valid for $a \cdot \tau = \iota(g)$. □

For the proof of Corollary 7.2 and for a later application we need the following

Lemma 7.5. *Let Γ be a polycrystallographic group of rank n, and let ρ be as in equation (7.1). Then Γ contains a characteristic subgroup Γ^* of finite index satisfying the following three conditions.*

(i) $\mathrm{nil}(\Gamma) = \mathrm{nil}(\Gamma^*)$ *and* $\Gamma^*/\,\mathrm{nil}(\Gamma)$ *is free abelian.*
(ii) The subgroup of S^1 generated by all elements of the form $\frac{\lambda}{|\lambda|}$, where λ is an eigenvalue of an element in $\rho(\Gamma^*)$, is torsion free.
(iii) The index of Γ^* in Γ is bounded by constant only depending on the rank n.

Proof of Lemma 7.5. Let $\pi : \Gamma \to \Gamma/\,\mathrm{nil}(\Gamma)$ be the projection, and let E be the maximal finite normal subgroup of $\Gamma/\,\mathrm{nil}(\Gamma)$. The preimage $\Gamma_N := \pi^{-1}(\mathsf{E})$ contains no nontrivial finite normal subgroup, because the maximal finite normal subgroup of Γ_N is normal in Γ. Thus Γ_N is a polycrystallographic group. Since $\mathrm{nil}(\Gamma)$ is of finite index in Γ we can employ Proposition 5.1 in order to see that Γ is an almost crystallographic group. It is known that the index of the nilradical in an almost crystallographic group is bounded by a constant only depending on the rank of the group. Hence the order of E is bounded by a constant only depending on $\mathrm{rank}(\mathrm{nil}(\Gamma))$. For that reason there is a free abelian characteristic subgroup $\mathsf{A} \subset \Gamma/\,\mathrm{nil}(\Gamma)$ of controlled finite index. The group $\Gamma_1 := \pi^{-1}(\mathsf{A})$ is a characteristic subgroup of Γ, and $\mathrm{nil}(\Gamma_1) = \mathrm{nil}(\Gamma)$.

As before we let N denote the Malcev completion of $\mathrm{nil}(\Gamma) = \mathrm{nil}(\Gamma_1)$. Since $\mathrm{nil}(\Gamma) \subset \mathsf{N}$ is a lattice in N, the group $\mathsf{D} := \mathrm{span}_{\mathbb{Z}}\big(\exp^{-1}(\mathrm{nil}(\Gamma))\big) \subset \mathfrak{n}$ is a lattice in the Lie algebra $\mathfrak{n}$, see (Raghunathan, 1972, p.34). Clearly, D is invariant under $\rho(\Gamma)$, and thus the characteristic polynomial of $\rho(g)$ is in $\mathbb{Z}[X]$ for all $g \in \Gamma$.

Set $h = \operatorname{rank}(\Gamma_1/\operatorname{nil}(\Gamma))$, and choose elements $a_1, \dots, a_h \in \Gamma$ which project onto a generator system of the free abelian group $\Gamma_1/\operatorname{nil}(\Gamma)$. Consider the eigenvalues $\lambda_{i1}, \dots, \lambda_{ik}$ of $\rho(a_i)$. Since the characteristic polynomial of $\rho(g)$ is in $\mathbb{Z}[X]$, the degree of the field extension $\mathbb{Q} \subset \mathbb{Q}(\lambda_{i1}, \dots, \lambda_{ik})$ is bounded by $k!$. Moreover, $|\lambda_{ij}|^2 \in \mathbb{Q}(\lambda_{i1}, \dots, \lambda_{ik})$, so the degree of the field extension

$$\mathbb{Q} \subset \mathbb{K} := \mathbb{Q}\Big(\big\{\tfrac{\lambda_{ij}}{|\lambda_{ij}|} \;\big|\; i = 1, \dots, h, j = 1, \dots, k\big\}\Big)$$

is at most $(2^k k!)^h$. The roots of unity in $\mathbb{K}$ form a finite cyclic group $\mathsf{C} \subset \mathbb{K}^*$. The inequality $\dim_{\mathbb{Q}}(\mathbb{K}) \leq (2^k k!)^h$ yields that the number $c := \operatorname{ord}(\mathsf{C})$ is bounded from above by a constant only depending on the rank of Γ.

Set $\mathsf{A}' := \{g^c \mid g \in \Gamma_1/\operatorname{nil}(\Gamma)\}$ and $\Gamma^* := \pi^{-1}(\mathsf{A}')$. Evidently, Γ^* is a characteristic subgroup of Γ_1 and Γ. Moreover, the index $(\Gamma : \Gamma^*)$ is bounded by a constant only depending on $\operatorname{rank}(\Gamma)$.

The commutator group of $\rho(\Gamma_1)$ is contained in the unipotent group $\rho(\operatorname{nil}(\Gamma))$. As can be extracted from (Raghunathan, 1972, p.69), this implies that for some basis of the complexification $\mathfrak{n}_{\mathbb{C}}$ the group $\rho(\Gamma_1)$ can be represented by a group of upper triangular matrices. In particular, for any eigenvalue λ of an element in $\rho(\Gamma_1)$ the number $\lambda/|\lambda|$ is contained in $\mathbb{K}^*$.

Let $h \in \Gamma^*$. Choose $a \in \Gamma_1$ such that $a^c \operatorname{nil}(\Gamma) = h \operatorname{nil}(\Gamma)$. If $\lambda_1, \dots, \lambda_k$ are the eigenvalues of $\rho(a)$, then $\lambda_1^c, \dots, \lambda_k^c$ are the eigenvalues of $\rho(h)$. Since the numbers $\frac{\lambda_1^c}{|\lambda_1|^c}, \dots, \frac{\lambda_k^c}{|\lambda_k|^c}$ are contained in the torsion free group $\{z^c \mid z \in \mathbb{K}^*\} \subset \mathbb{K}^*$, the assertion follows. □

Proof of Corollary 7.2. a). Suppose that Γ is generated by elements which are absolutely net. Then $\iota(\Gamma)$ is by Proposition 7.1 contained in the identity component of $\mathsf{S} \rtimes \mathsf{K}$. Taking into account that $\mathsf{S} \rtimes \mathsf{K}$ is the closure of $\iota(\Gamma) \cdot \mathsf{S}$, we see that $\mathsf{S} \rtimes \mathsf{K}$ itself is connected. In other words, K is a torus. By Proposition 5.1 the maximal connected nilpotent normal subgroup N of $\mathsf{S} \rtimes \mathsf{K}$ is contained in S and $\iota(\Gamma) \cap \mathsf{N} = \iota(\operatorname{nil}(\Gamma))$. Since $\mathsf{S} \rtimes \mathsf{K}$ is a connected solvable group, the factor group $\mathsf{S} \rtimes \mathsf{K}/\mathsf{N}$ is abelian, and hence $\Gamma/\operatorname{nil}(\Gamma)$ is abelian, too.

b). Choose a subgroup $\Gamma^* \subset \Gamma$ as stated in Lemma 7.5. The elements in Γ^* are absolutely net, and via Proposition 7.1 this implies that $\iota(\Gamma^*)$ is contained in the identity component of $\mathsf{S} \rtimes \mathsf{K}$. Using that $\mathsf{S} \rtimes \mathsf{K}$ is the closure of $\iota(\Gamma) \cdot \mathsf{S}$, we deduce that the number of connected components of $\mathsf{S} \rtimes \mathsf{K}$ is bounded by the index of Γ^* in Γ. □

Proof of Corollary 7.3. Let N be the maximal connected nilpotent normal subgroup of $\mathsf{S} \rtimes \mathsf{K}$. By Proposition 5.1 $\mathsf{N} \subset \mathsf{S}$ and $\iota(\operatorname{nil}(\Gamma)) = \mathsf{N} \cap \iota(\Gamma)$ is a lattice in N. Consider the projection $\pi \colon \mathsf{S} \rtimes \mathsf{K} \to \mathsf{Q} := \big(\mathsf{S} \rtimes \mathsf{K}\big)/\mathsf{N}$. Since $\iota(\operatorname{nil}(\Gamma))$ is cocompact in the kernel of π, the image $\Lambda := \pi(\iota(\Gamma))$ is discrete.

Furthermore, $\Lambda \cong \Gamma/\operatorname{nil}(\Gamma)$ is free abelian. Choose a basis $b_1, \dots, b_h$ of Λ and elements $v_1, \dots, v_h \in \mathfrak{q}$ in the Lie algebra of the abelian group Q satisfying $\exp(v_i) = b_i$. Let A be the Lie group corresponding to the abelian Lie algebra $\operatorname{span}_{\mathbb{R}}(v_1, \dots, v_h)$. Clearly, A contains Λ as a discrete, cocompact subgroup. From $\dim(\mathsf{A}) = \operatorname{rank}(\Lambda)$ we infer that A is simply connected. But then $\pi^{-1}(\mathsf{A})$ is a connected, simply connected solvable Lie group containing $\iota(\Gamma) \cong \Gamma$ as a discrete cocompact subgroup. □

7.4. On the proof of Theorem 5

We need the following simple observation:

Lemma 7.6. *Let* G *be a connected solvable Lie group and* N *the maximal connected nilpotent normal subgroup of* G*. Then the natural homomorphism*

$$\pi : \operatorname{Aut}(\mathsf{G}) \to \operatorname{Aut}(\mathsf{G}/\mathsf{N})$$

has a finite image. Moreover, the order of the image is bounded by a constant only depending on the dimension of G*.*

Proof. Let $\mathfrak{n}$ and $\mathfrak{g}$ be the Lie algebras of N and G, and let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. By Lie's theorem there is a basis of $\mathfrak{g}_{\mathbb{C}}$ with respect to which $\operatorname{ad}(\mathfrak{g})$ is represented by upper triangular matrices. For $v \in \mathfrak{g}$ we let $d(v) = (d_1(v), \dots, d_k(v)) \in \mathbb{C}^k$ denote the diagonal elements of the matrix representing ad_v, where k is the dimension of $\mathfrak{g}$. Clearly, $d\colon \mathfrak{g} \to \mathbb{C}^k$ is a homomorphism with kernel $\mathfrak{n}$.

For $v \in \mathfrak{g}$, $\alpha \in \operatorname{Aut}(\mathsf{G})$ the endomorphisms ad_v and $\operatorname{ad}_{\alpha_*(v)}$ have the same eigenvalues. Thus there is a permutation $\sigma \in \mathsf{S}_k$ with $d(\alpha_*(v)) = (d_{\sigma(1)}(v), \dots, d_{\sigma(k)}(v))$. Since the kernel of d is $\mathfrak{n}$, it follows that the set $\{\alpha_*(v + \mathfrak{n}) \mid \alpha \in \operatorname{Aut}(\mathsf{G})\}$ consists of at most $k!$ elements. Using that α_* (and accordingly α) is determined by the image of a basis of $\mathfrak{g}$, we deduce that the image of π has at most $(k!)^k$ elements. □

Proof of Theorem 5. Choose an embedding $\iota\colon \Gamma \to \mathsf{S} \rtimes \mathsf{K}$ satisfying the assumptions of Corollary 4. We will view ι in the following as inclusion map. The identity component K_0 of K is a torus. Notice that for any subgroup Γ' of finite index the inclusion $\Gamma' \to \Gamma' \cdot (\mathsf{S} \rtimes \mathsf{K}_0)$ matches the assumptions of Corollary 4. In particular, any automorphism: $\sigma : \Gamma' \to \Gamma'$ can be extended uniquely to an automorphism $\bar{\sigma}$ of $\Gamma' \cdot (\mathsf{S} \rtimes \mathsf{K}_0)$.

Let N be the maximal nilpotent normal subgroup of S. By Proposition 5.1 N is also the maximal nilpotent normal subgroup of $\mathsf{S} \rtimes \mathsf{K}_0$, and by Lemma 7.6 the image of the natural projection $\pi : \operatorname{Aut}(\mathsf{S} \rtimes \mathsf{K}_0) \to \operatorname{Aut}(\mathsf{S} \rtimes \mathsf{K}_0/\mathsf{N})$ is finite. Moreover, the order of the image of π is bounded by a constant only depending on $\dim(\mathsf{S} \rtimes \mathsf{K}_0) \leq 2 \cdot \mathrm{rank}(\Gamma)$. Let $\mathfrak{q}$ be the Lie algebra of the abelian Lie group $\mathsf{Q} := \mathsf{S} \rtimes \mathsf{K}_0/\mathsf{N}$.

Choose a subgroup $\Gamma^* \subset \Gamma$ as stated in Lemma 7.5. Observe that Γ^* is necessarily contained in $\Gamma_0 := \Gamma \cap (\mathsf{S} \rtimes \mathsf{K}_0)$ by Proposition 7.1.

Clearly, $\exp\colon \mathfrak{q} \to \mathsf{Q}$ is a homomorphism, and hence $W := \exp^{-1}(\mathrm{pr}(\Gamma^*))$ is a lattice in $\mathfrak{q}$. Set $W_{\mathbb{Q}} := \mathrm{span}_{\mathbb{Q}}(W)$. If a map $\bar{\sigma} \in \mathrm{Aut}(\mathsf{S} \rtimes \mathsf{K}_0)$ leaves a subgroup Γ' of finite index in Γ_0 invariant, then $\pi(\bar{\sigma})_*\colon \mathfrak{q} \to \mathfrak{q}$ leaves a subgroup of finite index in W invariant and thus $\pi(\bar{\sigma})_*(W_{\mathbb{Q}}) = W_{\mathbb{Q}}$.

Claim. Let $\mathsf{E} := \big\{\alpha \in \pi(\mathrm{Aut}(\mathsf{S} \rtimes \mathsf{K}_0)) \mid \alpha_*(W_{\mathbb{Q}}) = W_{\mathbb{Q}}\big\}$. Then

$$\mathsf{H} := \big\langle \{\alpha(\mathrm{pr}(g)) \mid g \in \Gamma^*,\ \alpha \in \mathsf{E}\}\big\rangle$$

is a discrete free abelian subgroup of Q.

In order to show that H is discrete, we just have to verify that the group

$$\bar{\mathsf{W}} := \langle \{\alpha_*(w) \mid w \in W, \alpha \in \mathsf{E}\}\rangle$$

is discrete. But this is trivial because $\bar{\mathsf{W}} \subset W_{\mathbb{Q}}$ is obviously finitely generated. So it remains to check that H is torsion free. Set

$$\bar{\Gamma}^* := \big\langle \{\sigma(g) \mid g \in \Gamma^*, \sigma \in \mathrm{Aut}(\mathsf{S} \rtimes \mathsf{K})\}\big\rangle.$$

Clearly, H is contained in $\mathrm{pr}(\bar{\Gamma}^*)$. Consider the natural representation $\rho\colon \mathsf{S} \rtimes \mathsf{K}_0 \to \mathrm{GL}(\mathfrak{n})$. By definition of Γ^* the multiplicative subgroup $\Theta \subset \mathbb{C}^*$ generated by all eigenvalues of elements in $\rho(\Gamma^*)$ is torsion free. From Lie's theorem we infer that the multiplicative group generated by all eigenvalues of elements in $\rho(\bar{\Gamma}^*)$ coincides with Θ. Now let $g \in \bar{\Gamma}^* \setminus (\bar{\Gamma}^* \cap \mathsf{N})$. By Proposition 5.1 the element $\rho(g)$ is not unipotent. Since the eigenvalues of $\rho(g)$ are not roots of unity, it follows that $\rho(g^k)$ is not unipotent for all positive integers k. In particular, $g^k \notin \mathsf{N}$ for all $k > 0$. Taking into account that N is the kernel of pr, we see that $\mathrm{pr}(\bar{\Gamma}^*) \supset \mathsf{H}$ is torsion free, and hence the above claim is proved.

Notice that H is by definition invariant under the natural action of E on Q. For the free abelian group H we choose a homomorphism

$$\psi\colon \mathsf{H} \to \mathfrak{q} \quad \text{satisfying} \quad \exp\circ\psi = \mathrm{id}\,.$$

Recall that $l := \mathrm{ord}(\mathsf{E}) \leq \mathrm{ord}(\mathrm{image}(\pi))$ is bounded by a constant only depending on $\mathrm{rank}(\Gamma)$. Put $\mathsf{l}\cdot\mathsf{H} := \{g^l \mid g \in \mathsf{H}\}$. The map $\mathsf{H} \to \mathsf{l}\cdot\mathsf{H}$, $g \mapsto g^l$ is an isomorphism, and accordingly we can define a homomorphism $\tilde{\psi}\colon \mathsf{l}\cdot\mathsf{H} \to \mathfrak{q}$ by means of

$$\tilde{\psi}(g^l) := \sum_{\alpha\in\mathsf{E}} \alpha_*\big(\psi(\alpha^{-1}(g))\big) \quad \text{for } g \in \mathsf{H}.$$

Evidently, $\exp \circ \tilde{\psi} = \mathrm{id}$. Furthermore,

$$\tilde{\psi}(\alpha(g)) = \alpha_*(\tilde{\psi}(g)) \quad \text{for } \alpha \in \mathsf{E} \text{ and } g \in \mathsf{I}\cdot\mathsf{H}.$$

Set $\mathfrak{a} := \mathrm{span}_{\mathbb{R}}(\tilde{\psi}(\mathsf{I}\cdot\mathsf{H}))$ and $\mathsf{A} := \exp(\mathfrak{a})$. Clearly, A contains $\mathsf{I}\cdot\mathsf{H}$ as a discrete, cocompact subgroup. From $\dim(\mathsf{A}) = \mathrm{rank}(\mathsf{I}\cdot\mathsf{H})$ we infer that A is a simply connected, closed subgroup of Q. Moreover, A is invariant under the natural action of E on Q. Since $\mathrm{pr}(\Gamma^*) \subset \mathsf{H}$ and $\mathsf{I}\cdot\mathsf{H} \subset \mathsf{A}$, the index of $\mathsf{A} \cap \mathrm{pr}(\Gamma_0)$ in $\mathrm{pr}(\Gamma_0)$ is bounded by a constant only depending on $\mathrm{rank}(\Gamma)$.

Put $\mathsf{R} := \mathrm{pr}^{-1}(\mathsf{A})$. Then $\mathsf{N} \subset \mathsf{R}$ is the maximal nilpotent normal subgroup of R and $\mathrm{nil}(\Gamma) \subset \mathsf{N}$. By Corollary 7.2 we can control the quantity $(\Gamma : \Gamma_0)$, and hence

$$(\Gamma : \mathsf{R} \cap \Gamma) = (\Gamma : \Gamma_0) \cdot \big(\mathrm{pr}(\Gamma_0) : \mathsf{A} \cap \mathrm{pr}(\Gamma_0)\big)$$

is bounded by a constant only depending on the rank of Γ.

Furthermore, for a subgroup Γ' of finite index in Γ and an automorphism $\sigma : \Gamma' \to \Gamma'$ the unique extension $\bar{\sigma} : \Gamma' \cdot \mathsf{S} \rtimes \mathsf{K}_0 \to \Gamma' \cdot \mathsf{S} \rtimes \mathsf{K}_0$ of σ leaves the subgroup R invariant. In particular, Γ normalizes R. The product $\Gamma \cdot \mathsf{R}$ splits as a semidirect product $\mathsf{R} \rtimes \mathsf{F}$ where the order of $\mathsf{F} \subset \mathrm{Aut}(\mathsf{R})$ equals $(\Gamma : \mathsf{R} \cap \Gamma)$, see Lemma 3.2. Moreover, for a subgroup Γ' of finite index in Γ any automorphism $\sigma : \Gamma' \to \Gamma'$ can be extended to an automorphism $\bar{\sigma}_{|\Gamma'\cdot\mathsf{R}}$ of $\Gamma' \cdot \mathsf{R}$.

It remains to check that there is at most one extension of each automorphism. Any automorphism of $\Gamma' \cdot \mathsf{R}$ can be extended to $\Gamma' \cdot (\mathsf{S} \rtimes \mathsf{K}_0)$ by Theorem 6.5. Furthermore, by the same theorem any automorphism $\bar{\sigma}$ of $\Gamma' \cdot (\mathsf{S} \rtimes \mathsf{K}_0)$ is determined by the restriction $\bar{\sigma}_{|\Gamma'}$. In summary, we can say that any automorphism $\bar{\sigma}$ of $\Gamma' \cdot \mathsf{R}$ is determined by the restriction $\bar{\sigma}_{|\Gamma'}$. □

7.5. The proof of Theorem 6

In view of Theorem 1 we may assume that S is supersolvable. Notice that the torsion free group $\Upsilon \subset \mathsf{S} \rtimes \mathsf{K}$ contains no nontrivial compact subgroup, so its identity component is a simply connected solvable Lie group. Let R' be the Lie hull of Υ in $\mathsf{S} \rtimes \mathsf{K}$ and $\mathsf{S}' = \mathsf{S} \cap \mathsf{R}'$. By replacing Υ by a conjugate subgroup if necessary, we may assume that $\mathsf{K}' := \mathsf{R}' \cap \mathsf{K}$ is maximal compact in R'. Then R' is a semidirect product subgroup $\mathsf{R}' = \mathsf{S}' \rtimes_{\beta'} \mathsf{K}' \subset \mathsf{S} \rtimes \mathsf{K}$, where $\beta' : \mathsf{K}' \to \mathrm{Aut}(\mathsf{S}')$ is the restriction.

Clearly, the Lie algebra $\mathfrak{s}'$ is invariant under the natural representation of K' in $\mathfrak{s}$. Since K' is compact, we can find a K'-invariant subspace $\mathfrak{p} \subset \mathfrak{s}$ such that $\mathfrak{s} = \mathfrak{p} \oplus \mathfrak{s}'$. For any such complement we let $\varrho : \mathsf{R}' \to \mathrm{GL}(\mathfrak{p})$ denote the representation that is given by $\varrho((A, \tau))(u) := A_* u$ for $(A, \tau) \in \mathsf{R}' \supset \Upsilon$, $u \in \mathfrak{p}$. We define an action of $\mathsf{R}' \supset \Upsilon$ on $\mathsf{S}' \times \mathfrak{p}$ by using on the first factor the natural action and on the second factor the action induced by the representation

ϱ. Notice that for any decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k$ of K'-invariant subspaces the following map is R'-equivariant.

$$f\colon \mathsf{S}' \times \mathfrak{p} \to \mathsf{S}, \qquad (h, u_1 + \cdots + u_k) \mapsto h \cdot \exp(u_1) \cdots \exp(u_k) \tag{7.2}$$

for $u_i \in \mathfrak{p}_i$, $i = 1, \ldots, k$. We claim that for a suitable choice f is a diffeomorphism as well. From this it is clear that the manifold S/Υ is diffeomorphic to $\big(\mathsf{S}' \times \mathfrak{p}\big)/\Upsilon$. Since $\big(\mathsf{S}' \times \mathfrak{p}\big)/\Upsilon$ is a vector bundle over the compact manifold $B := \mathsf{S}'/\Upsilon$, this completes the proof of a).

In order to prove the existence of a suitable decomposition $\mathfrak{s} = \mathfrak{s}' \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k$ we argue by induction on $\dim(\mathsf{S})$. Let $\mathfrak{c}$ be the center of the nilradical of $\mathfrak{s}$, and let C be the corresponding Lie group. Observe that $\mathsf{C}' := \mathsf{C} \cap \mathsf{S}'$ is a connected normal subgroup of R'. Denote by $\mathfrak{c}'$ the Lie algebra of C' and choose a K'-invariant subspace $\mathfrak{p}_1$ such that $\mathfrak{c} = \mathfrak{c}' \oplus \mathfrak{p}_1$. Next we consider the factor groups S'/C' and S/C and the corresponding Lie algebras $\mathfrak{s}'/\mathfrak{c}'$ and $\mathfrak{s}/\mathfrak{c}$. By our induction hypothesis there is a decomposition $\mathfrak{s}/\mathfrak{c} = \mathfrak{s}'/\mathfrak{c}' \oplus \bar{\mathfrak{p}} = \mathfrak{s}'/\mathfrak{c}' \oplus \bar{\mathfrak{p}}_2 \oplus \cdots \oplus \bar{\mathfrak{p}}_k$ of K'-invariant subspaces such that the map

$$\bar{f}\colon (\mathsf{S}'/\mathsf{C}') \times \bar{\mathfrak{p}} \to \mathsf{S}/\mathsf{C}, \qquad (\bar{h}, \bar{u}_2 + \cdots + \bar{u}_k) \mapsto \bar{h} \cdot \exp(\bar{u}_2) \cdots \exp(\bar{u}_k)$$

is a diffeomorphism. Since K' is compact, there is a K'-invariant subspace $\mathfrak{p}_i$ such that the natural projection $\mathfrak{s} \to \mathfrak{s}/\mathfrak{c}$ maps $\mathfrak{p}_i$ isomorphically onto $\bar{\mathfrak{p}}_i$, $i = 2, \ldots, k$. Evidently, $\mathfrak{s} = \mathfrak{s}' \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k$, and it is straightforward to check that for this decomposition the map f in (7.2) is a diffeomorphism.

Statement b) follows from Lemma 7.7 below. It remains to verify the addition. By our additional assumption the identity component Υ_0 of Υ is contained in S'. Set $\tilde{\Upsilon} := \Upsilon \cap \mathsf{S}'$, and let $\tilde{\mathsf{R}} \subset \mathsf{S}'$ be the Lie hull of $\tilde{\Upsilon}$. Notice that $\tilde{\Upsilon}$ is a normal subgroup of Υ and that the factor group $\Upsilon/\tilde{\Upsilon}$ is abelian up to finite index. From Theorem 6.5 we infer that $\tilde{\mathsf{R}}$ is normal in $\mathsf{S}' \rtimes_{\beta'} \mathsf{K}'$ and that the factor group $\mathsf{Q} := \mathsf{S}' \rtimes_{\beta'} \mathsf{K}'/\tilde{\mathsf{R}}$ has an abelian identity component Q_0. Since $\tilde{\Upsilon}$ is a cocompact subgroup of $\tilde{\mathsf{R}}$ and since $\Upsilon_0 \subset \tilde{\Upsilon}$, it follows that the projection $\mathrm{pr}\colon \mathsf{S}' \rtimes_{\beta'} \mathsf{K}' \to \mathsf{Q}$ maps Υ onto a discrete subgroup of Q.

Analogously to Subsect. 7.4 we can find a simply connected, closed subgroup $\mathsf{A} \subset \mathsf{Q}$ with finitely many connected components that contains $\mathrm{pr}(\Upsilon)$ as a discrete cocompact subgroup. The preimage $\hat{\Upsilon} := \mathrm{pr}^{-1}(\mathsf{A}) \supset \Upsilon$ is a simply connected Lie group with finitely many connected components and with a solvable identity component $\hat{\Upsilon}_0$. By Lemma 3.2 we may identify $\hat{\Upsilon}$ with a semidirect product $\hat{\Upsilon}_0 \rtimes_\beta \mathsf{F}$, where F is a finite group. The representation $\varrho\colon \mathsf{S}' \rtimes_{\beta'} \mathsf{K}' \to \mathrm{GL}(\mathfrak{p})$ restricts to a representation $\varrho\colon \hat{\Upsilon}_0 \rtimes_\beta \mathsf{F} \to \mathrm{GL}(\mathfrak{p})$. Recall that the image of ϱ is relatively compact. In particular, $\varrho(\hat{\Upsilon}_0)$ is abelian. Thus ϱ induces a representation

$$\bar{\varrho}\colon \big(\hat{\Upsilon}_0/[\hat{\Upsilon}_0, \hat{\Upsilon}_0]\big) \rtimes_{\bar{\beta}} \mathsf{F} \to \mathrm{GL}(\mathfrak{p}).$$

The group $\hat{\Upsilon}_0/[\hat{\Upsilon}_0, \hat{\Upsilon}_0]$ is a simply connected abelian group and therefore it is isomorphic to $\mathbb{R}^p$ for some integer p. Set $\bar{\varrho}_\lambda(v, f) := \bar{\varrho}(\lambda \cdot v, f)$ for $\lambda \in [0, 1]$. Clearly, $(\bar{\varrho}_\lambda)_{\lambda\in[0,1]}$ is a smooth family of homomorphisms and the image of $\bar{\varrho}_0$ is the finite group $\varrho(\mathsf{F})$. Because of $\Upsilon \subset \hat{\Upsilon}$ this consideration shows that

$$\varrho_1 := \varrho_{|\Upsilon}\colon \Upsilon \to \mathrm{GL}(\mathfrak{p})$$

can be deformed via a continuous family of representations $\varrho_\lambda\colon \Upsilon \to \mathrm{GL}(\mathfrak{p})$ into a representation ϱ_0 with a finite image, too. Define a new action of Υ on $\mathsf{S}' \times \mathfrak{p}$ by using on the first factor the natural action and on the second factor the action induced by the representation ϱ_0. It is known that the quotient of this action is diffeomorphic to the quotient of the original action. Since the image of ϱ_0 is finite, a finite sheeted covering space of S/Υ is diffeomorphic to the product $(\mathsf{S}'/\Upsilon') \times \mathfrak{p}$, where $\Upsilon' = \mathrm{Ker}(\varrho_0)$ is a subgroup of finite index in Υ.

Lemma 7.7. *Let S be a connected, simply connected supersolvable Lie group, K a compact Lie group, $\mathsf{G} := \mathsf{S} \rtimes_\beta \mathsf{K}$, and let $\Upsilon \subset \mathsf{G}$ be a torsion free cocompact, closed subgroup. Then $\Gamma := \pi_0(\Upsilon) := \Upsilon/\Upsilon_0$ is a torsion free polycrystallographic group and for an embedding $\Gamma \to \hat{\mathsf{S}} \rtimes \hat{\mathsf{K}}$ satisfying the assumptions of Corollary 4 the quotient $\hat{\mathsf{S}}/\Gamma$ is diffeomorphic to the manifold S/Υ.*

Proof. Without loss of generality G is the Lie hull of Υ in G. Since the normalizer of the identity component Υ_0 of Υ is by Proposition 6.2 a Lie hull, the group Υ_0 is normal in G. The Lie hull R of Υ_0 is a normal subgroup of G, too. Furthermore, Υ_0 is cocompact in R, and accordingly $\hat{\Upsilon} := \Upsilon \cdot \mathsf{R}$ is a closed subgroup. We claim that for the natural action of $\hat{\Upsilon}$ on S the $\hat{\Upsilon}$–orbits coincide with the Υ–orbits. In order to prove this it is sufficient to show that R and Υ_0 have the same orbits. The group $\mathsf{L} = \mathsf{R} \cap \mathsf{K}$ is maximal compact in R because R is normal in G. So $\mathsf{R} = \mathsf{S}' \rtimes_\beta \mathsf{L}$ where $\mathsf{S}' = \mathsf{S} \cap \mathsf{R}$. By construction Υ_0 is a connected, cocompact subgroup of R, and hence $\Upsilon_0 \star e = \mathsf{S}' = \mathsf{R}_0 \star e$. Taking into account that Υ_0 and R are normal subgroups of G, we see that $\Upsilon_0 \star v = \mathsf{R} \star v$ for all $v \in \mathsf{S}$.

The projection $\mathrm{pr}\colon \mathsf{S} \rtimes_\beta \mathsf{K} \to (\mathsf{S}/\mathsf{S}') \rtimes_{\bar{\beta}} (\mathsf{K}/\mathsf{L})$ maps Υ onto a discrete, cocompact subgroup $\tilde{\Gamma} := \mathrm{pr}(\Upsilon) = \mathrm{pr}(\hat{\Upsilon})$. Evidently, the quotient $(\mathsf{S}/\mathsf{S}')/\tilde{\Gamma}$ is diffeomorphic to S/Υ. Since the action of Υ on S is free, the action of $\tilde{\Gamma}$ on (S/S') is free as well. In particular, $\tilde{\Gamma}$ is torsion free. Consider the projection

$$\pi\colon (\mathsf{S}/\mathsf{S}') \rtimes_{\bar{\beta}} (\mathsf{K}/\mathsf{L}) \to (\mathsf{S}/\mathsf{S}') \rtimes \bar{\beta}(\mathsf{K}/\mathsf{L}) =: \hat{\mathsf{S}} \rtimes \hat{\mathsf{K}}$$

and the group $\Gamma := \pi(\tilde{\Gamma}) \cong \tilde{\Gamma} \cong \pi_0(\Upsilon)$. Clearly, the inclusion $\Gamma \subset \hat{\mathsf{S}} \rtimes \hat{\mathsf{K}}$ matches the hypothesis of Corollary 4. Finally, the manifold S/Υ is diffeomorphic to $\hat{\mathsf{S}}/\Gamma$. □

8. Further consequences

8.1. Representations of polycrystallographic groups

The aim of this subsection is to prove the following

Corollary 8.1. *Let Γ be a polycrystallographic group, $\iota\colon \Gamma \to \mathsf{S} \rtimes \mathsf{K}$ an embedding satisfying the assumptions of Corollary 4, $\rho\colon \Gamma \to \mathrm{GL}(m, \mathbb{R})$ a representation of Γ, and let Z be the Zarisky closure of $\rho(\Gamma)$ in $\mathrm{GL}(m, \mathbb{R})$. Assume further that L is the maximal compact normal subgroup of Z, and let $\pi : \mathsf{Z} \to \mathsf{Z}/\mathsf{L}$ denote the projection. Then there is a homomorphism $\varphi\colon \mathsf{S} \rtimes \mathsf{K} \to \mathsf{Z}/\mathsf{L}$ with $\varphi \circ \iota = \pi \circ \rho$.*

Proof. Evidently, the identity component of Z is solvable. Let $\mathsf{R} \subset \mathsf{Z}$ be the set of matrices that have only positive eigenvalues. Lemma 4.4 exhibits R as a connected, simply connected normal subgroup of Z. Furthermore, Z is isomorphic to a semidirect product $\mathsf{R} \cdot \mathsf{K}$ where $\mathsf{K} \subset \mathsf{Z}$ is maximal compact.

The group R is supersolvable, and thus the Lie hull of $\rho(\Gamma)$ in $\mathsf{R} \cdot \mathsf{K}$ is defined, see Definition 6.4. In particular, there is a compact group $\mathsf{K}' \subset \mathsf{Z}$ and a connected subgroup $\mathsf{S}' \subset \mathsf{R}$ normalized by K' such that $\rho(\Gamma)$ is cocompact in $\mathsf{K}' \cdot \mathsf{S}'$. Let L' be the maximal compact normal subgroup of $\mathsf{S}' \cdot \mathsf{K}'$. Clearly, $\rho(\Gamma)$ normalizes L'. The Zarisky closure of L' in $\mathrm{GL}(n, \mathbb{R})$ is a compact normal subgroup of Z and accordingly $\mathsf{L}' \subset \mathsf{L}$. Therefore $\pi(\mathsf{S}' \cdot \mathsf{K}')$ contains no compact normal subgroup. Hence $\pi(\mathsf{S}' \cdot \mathsf{K}')$ is by Lemma 3.2 isomorphic to a semidirect product $\mathsf{S}' \rtimes \mathsf{K}''$ with $\mathsf{K}'' \subset \mathrm{Aut}(\mathsf{S}')$. Now Theorem 6.5 c) applies. □

8.2. Some consequences for lattices in supersolvable Lie groups

For a polycrystallographic group Γ we call the group Γ^+ that is characterized in Proposition 5.1 the positive part of Γ. A polycrystallographic group Λ is called a positive polycyclic group if and only if it coincides with its positive part. We have

Remark 8.2. An abstract group Λ is isomorphic to a lattice in a connected, simply connected supersolvable Lie group, if and only if Λ is a positive polycyclic group.

Proof. That the condition is necessary is an immediate consequence of Proposition 5.1. Furthermore, for a positive polycyclic group Λ we can choose an embedding $\iota\colon \Lambda \to \mathsf{S} \rtimes \mathsf{K}$ satisfying the assumptions of Corollary 4. We infer from Proposition 5.1 that $\Lambda \cong \iota(\Lambda)$ is contained in the supersolvable group S. □

For a positive polycyclic group Λ, and a connected, simply connected supersolvable Lie group S containing Λ as a lattice we call S a supersolvable completion of Λ. This definition generalizes the concept of the Malcev completion of a torsion free nilpotent group. From Theorem 6.5 a), c) we obtain:

Corollary 8.3. *Let Λ_i be a positive polycyclic group, and let $\mathsf{S}_i \supset \Lambda_i$ be a supersolvable completion of Λ_i, $i = 1, 2$. Then each homomorphism $\varphi\colon \Lambda_1 \to \Lambda_2$ extends uniquely to a homomorphism $\mathsf{S}_1 \to \mathsf{S}_2$.*

Thus the supersolvable completion of a positive polycyclic group is a functor. Of course, the class of positive polycyclic groups is much smaller than the class of polycrystallographic groups. However, at least for polycrystallographic groups of rank three the difference is not that big:

Remark 8.4. For a polycrystallographic group Γ of rank three the positive part Γ^+ of Γ is of finite index in Γ.

In fact, it is elementary to show that Γ contains a subgroup of finite index which is isomorphic to a semidirect product $(\mathbb{Z}^2) \rtimes_\beta \mathbb{Z}$, where $\beta\colon \mathbb{Z} \to \mathrm{GL}(2, \mathbb{Z})$, $z \mapsto A^z$ for some $A \in \mathrm{GL}(2, \mathbb{Z})$. Thus it just remains to check that there is a positive integer n such that A^n has only positive eigenvalues. But this is a trivial computation.

8.3. Extendable homomorphisms

We have seen in the preceding subsection that the embedding of Corollary 4 induces on the subclass of positive polycyclic groups a functor that maps a positive polycyclic group Λ onto its supersolvable completion. In order to extend this functor to a category containing all polycrystallographic groups as objects, we have to restrict ourselves to a special type of homomorphism.

Definition 8.5. Let Γ_i be a polycrystallographic group, and let $\iota\colon \Gamma \to \mathsf{S}_i \rtimes \mathsf{K}_i$ be an embedding satisfying the assumption of Corollary 4, $i = 1, 2$. A homomorphism $\varphi\colon \Gamma_1 \to \Gamma_2$ is called extendable if there is a homomorphism $\hat\varphi\colon \mathsf{S}_1 \rtimes \mathsf{K}_1 \to \mathsf{S}_2 \rtimes \mathsf{K}_2$ of Lie groups satisfying $\hat\varphi(\mathsf{S}_1) \subset \mathsf{S}_2$ and $\hat\varphi \circ \iota_1 = \iota_2 \circ \varphi$.

If such a homomorphism $\hat\varphi$ exists, then it is unique. In fact one can show that the graph of $\hat\varphi$ is necessarily given by the Lie hull of $\{(\iota_1(g), \iota_2(\varphi(g))) \mid g \in \Gamma_1\}$ in $(\mathsf{S}_1 \rtimes \mathsf{K}_1) \times (\mathsf{S}_2 \rtimes \mathsf{K}_2)$. Clearly, the polycrystallographic groups together with the extendable homomorphisms form a category. Furthermore, with respect to this category the embedding of Corollary 4 is a functor.

Proposition 8.6. *Let $\varphi\colon \Gamma_1 \to \Gamma_2$ be a homomorphism between polycrystallographic groups. Then φ is an extendable homomorphism provided that the image $\Gamma' := \varphi(\Gamma_1)$ satisfies one of the following conditions.*

a) *Γ' is a subgroup of the positive part Γ_2^+ of Γ_2.*
b) *There are subgroups*

$$\Gamma' := \mathsf{N}_0 \subset \cdots \subset \mathsf{N}_k = \Gamma_2$$

such that N_{i-1} is either normal in N_i or of finite index in N_i, $i = 1, \ldots, k$.

Proof. a). By the very definition of the positive part the Lie hull S' of $\iota_2(\Gamma')$ in $\mathsf{S}_2 \rtimes \mathsf{K}_2$ is a connected subgroup of S_2. Now Theorem 6.5 c) applies.

b). We claim that the Lie hull R_i of $\iota_2(\mathsf{N}_i)$ in $\mathsf{S}_2 \rtimes \mathsf{K}_2$ contains no nontrivial compact normal subgroups. We argue by reversed induction on i. By Lemma 3.2 the statement is correct for $\mathsf{R}_k = \mathsf{S}_2 \rtimes \mathsf{K}_2$. Suppose that the Lie hull R_{i+1} of $\iota_2(\mathsf{N}_{i+1})$ contains no nontrivial compact normal subgroup. In the case that N_i is normal in N_{i+1} we can employ Theorem 6.5 to see that R_i is normal in R_{i+1}. Thus the maximal compact normal subgroup of R_i is normal in R_{i+1} and hence trivial. If N_i is of finite index in N_{i+1}, then R_i is of finite index in R_{i+1}, and the assertion follows from Lemma 3.2.

Since the Lie hull R_0 of $\iota_2(\Gamma')$ contains no compact normal subgroups, it follows that R_0 it is isomorphic to a semidirect $\bar{\mathsf{S}} \rtimes \bar{\mathsf{K}}$, where $\bar{\mathsf{S}} \subset \mathsf{S}_2$ is a connected supersolvable Lie group and $\bar{\mathsf{K}} \subset \mathrm{Aut}(\bar{\mathsf{S}})$ is compact. By Theorem 6.5 c) there is a homomorphism $\hat{\varphi}\colon \mathsf{S}_1 \rtimes \mathsf{K}_1 \to \mathsf{R}_0$ with $\hat{\varphi}(\mathsf{S}_1) \subset \bar{\mathsf{S}} \subset \mathsf{S}_2$ and $\hat{\varphi} \circ \iota_1 = \iota_2 \circ \varphi$. □

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