HOW TO PRODUCE A RICCI FLOW VIA CHEEGER-GROMOLL EXHAUSTION

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Dedicated to Wolfgang T. Meyer on the occasion of his 75th birthday

Abstract. We prove short time existence for the Ricci flow on open manifolds of nonnegative complex sectional curvature. We do not require upper curvature bounds. By considering the doubling of convex sets contained in a Cheeger-Gromoll convex exhaustion and solving the singular initial value problem for the Ricci flow on these closed manifolds, we obtain a sequence of closed solutions of the Ricci flow with nonnegative complex sectional curvature which subconverge to a solution of the Ricci flow on the open manifold. Furthermore, we find an optimal volume growth condition which guarantees long time existence, and we give an analysis of the long time behaviour of the Ricci flow. Finally, we construct an explicit example of an immortal nonnegatively curved solution of the Ricci flow with unbounded curvature for all time.

1. Introduction and main results

The Ricci flow was introduced by R. Hamilton in [24] as a method to deform or evolve a Riemannian metric g given on a fixed n-dimensional manifold M according to the following partial differential equation:

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)) \tag{1.1}$$

over a time interval $I \subset \mathbb{R}$, with the initial condition $g(0) = g$. The first basic question, without which a theory about the Ricci flow does not even make sense, is to ensure that equation (1.1) admits a solution at least for a short time. This was already completely settled for closed manifolds (i.e. compact and without boundary) by Hamilton in [24]. In dimension 2, short time existence for a non-compact surface (which may be incomplete and with curvature unbounded above and below) was established by Giesen and Topping in [18] using ideas from [45].

The non-compact case for $n \geq 3$, even asking the manifold to be complete, is much harder and in full generality appears to be hopeless: for instance, it is difficult to imagine how to construct a solution to (1.1) starting at a manifold built by attaching in a smooth way spherical cylinders with radius becoming smaller and smaller (say of radius $1/k$ with $k \in \mathbb{N}$). Hence to achieve short time existence one needs to prevent similar situations by adding extra conditions on the curvature. In this spirit, W. X. Shi proved in [41] that the Ricci flow starting on an open (i.e. complete and non-compact) manifold with bounded curvature (i.e. with $\sup_M |\mathrm{R}_g| \le k_0 < \infty$) admits a solution for a time interval $[0, T(n, k_0)]$ also with bounded curvature.

Later on M. Simon (cf. [43]), assuming further that the manifold has nonnegative curvature operator ($\mathrm{R}_g \ge 0$) and is non-collapsing ($\inf_M \mathrm{vol}_g (B_g(\cdot, 1)) \ge v_0 > 0$), was able to extend Shi's solution for a time interval $[0, T(n, v_0)]$, with curvature bounded above by $\frac{c(n,v_0)}{t}$ for positive times. Although $T(n, v_0)$ does not depend on an upper curvature bound, such a bound is still an assumption needed to guarantee short time existence.

The present paper manages to remove any restriction on upper curvature bounds for open manifolds with nonnegative complex sectional curvature (see Definition 3.1) which, by Cheeger, Gromoll and Meyer [10, 21], admit an exhaustion by convex sets C_ℓ. We are able to construct a Ricci flow with nonnegative complex sectional curvature on the closed manifold obtained by gluing two copies of C_ℓ along the common boundary, and whose 'initial metric' is the natural singular metric on the double. By passing to a limit we obtain

Theorem 1. *Let (M^n, g) be an open manifold with nonnegative (and possibly unbounded) complex sectional curvature ($K_g^{\mathbb{C}} \ge 0$). Then there exists a constant $\mathcal{T}$ depending on n and g such that* (1.1) *has a smooth solution on the interval $[0, \mathcal{T}]$, with $g(0) = g$ and with $g(t)$ having nonnegative complex sectional curvature.*

Using that by Brendle [4] the trace Harnack inequality in [26] holds for compact manifolds with $K^{\mathbb{C}} \ge 0$, it follows that the above solution on the open manifold satisfies the trace Harnack estimate as well. This solves an open question posed by Chow, Lu and Ni [14, Problem 10.45].

The proof of Theorem 1 is easier if $K_g^{\mathbb{C}} > 0$ since then by Gromoll and Meyer [21] M is diffeomorphic to $\mathbb{R}^n$. In the general case, we need additional tools; for instance, we prove the following result, which extends a theorem by Noronha [34] for manifolds with $\mathrm{R}_g \ge 0$.

Theorem 2. *Let (M^n, g) be an open, simply connected Riemannian manifold with nonnegative complex sectional curvature. Then M splits isometrically as $\Sigma \times F$, where Σ is the k-dimensional soul of M and F is diffeomorphic to $\mathbb{R}^{n-k}$.*

In the nonsimply connected case M is diffeomorphic to a flat Euclidean vector bundle over the soul. Thus combining with the classification in [6] of compact manifolds with $K^{\mathbb{C}} \ge 0$, we deduce that any open manifold of $K^{\mathbb{C}} \ge 0$ admits a complete nonnegatively curved locally symmetric metric $\hat{g}$, i.e. $K_{\hat{g}} \ge 0$, $\nabla R_{\hat{g}} \equiv 0$.

It is not hard to see that, given any open manifold (M, g) with bounded curvature and $K_g^{\mathbb{C}} > 0$, for any closed discrete countable subset $S \subset M$ one can find a deformation $\bar{g}$ of g in an arbitrary small neighborhood U of S such that $\bar{g}$ and g are C^1-close, $(M, \bar{g})$ has unbounded curvature and $K_{\bar{g}}^{\mathbb{C}} > 0$. The following result, which is very much in spirit of Simon [43], shows that this sort of local deformations will be smoothed out instantaneously by our Ricci flow.

Corollary 3. *Let (M^n, g) be an open manifold with $K_g^{\mathbb{C}} \geq 0$. If*

$$\inf\big\{\mathrm{vol}_g(B_g(p,1)) : p \in M\big\} = v_0 > 0, \tag{1.2}$$

then the curvature of $(M, g(t))$ is bounded above by $\frac{c(n,v_0)}{t}$ for $t \in (0, \mathcal{T}(n, v_0)]$.

In the case of a nonnegatively curved surface, this volume condition is always satisfied (see [15]), so any such surface can be deformed by (1.1) to one with bounded curvature. For $n \geq 3$ the lower volume bound in Corollary 3 is essential:

Theorem 4. *a) There is an immortal 3-dimensional nonnegatively curved complete Ricci flow $(M, g(t))_{t\in[0,\infty)}$ with unbounded curvature for each t.*

b) There is an immortal 4-dimensional complete Ricci flow $(M, g(t))_{t\in[0,\infty)}$ with positive curvature operator such that the curvature of $(M, g(t))$ is bounded if and only if $t \in [0, 1)$.

Higher dimensional examples can be obtained by crossing with a Euclidean factor. Part b) shows that even if the initial metric has bounded curvature one can run into metrics with unbounded curvature. The following result gives a precise lower bound on the existence time for (1.1) in terms of supremum of the volume of balls, instead of infimum as in Corollary 3 and [43]. We emphasize that this is new even in the case of initial metrics of bounded curvature.

Corollary 5. *In each dimension there is a universal constant $\varepsilon(n) > 0$ such that for each complete manifold (M^n, g) with $K_g^{\mathbb{C}} \geq 0$ the following holds: If we put*

$$\mathcal{T} := \varepsilon(n) \cdot \sup\Big\{\tfrac{\mathrm{vol}_g(B_g(p,r))}{r^{n-2}} \mid p \in M, r > 0\Big\} \in (0, \infty],$$

then any complete maximal solution of Ricci flow $(M, g(t))_{t\in[0,T)}$ with $K_{g(t)}^{\mathbb{C}} \geq 0$ and $g(0) = g$ satisfies $\mathcal{T} \leq T$.

If M has a volume growth larger than r^{n-2}, then Corollary 5 ensures the existence of an immortal solution. Previously (cf. [40]) long time existence was only known in the case of Euclidean volume growth under the stronger assumptions $\mathrm{R}_g \geq 0$ and bounded curvature. We highlight that our volume growth condition cannot be further improved: indeed, as the Ricci flow on the metric product $\mathbb{S}^2 \times \mathbb{R}^{n-2}$ exists only for a finite time, the power $n - 2$ is optimal. For $n = 3$ we can even determine exactly the extinction time depending on the structure of the manifold:

Corollary 6. *Let (M^3, g) be an open manifold with $K_g \geq 0$ and soul Σ. If $(M, g(t))_{t\in[0,T)}$ is a maximal complete solution of (1.1) with $g(0) = g$ and $K_{g(t)}^{\mathbb{C}} \geq 0$, then*

$$T = \begin{cases} \frac{\mathrm{area}(\Sigma)}{4\pi\chi(\Sigma)} & \textit{if} \quad \dim\Sigma = 2 \\ \infty & \textit{if} \quad \dim\Sigma = 1 \\ \frac{1}{8\pi}\lim_{r\to\infty}\frac{\mathrm{vol}_g(B_g(p,r))}{r} & \textit{if} \quad \Sigma = \{p_0\} \end{cases}.$$

In the case $\Sigma = \{p_0\}$, if $T < \infty$, then (M, g) is asymptotically cylindrical and $(M, g(t))$ has bounded curvature for $t > 0$.

By Corollary 5 a finite time singularity T on open manifolds with $K^{\mathbb{C}} \geq 0$ can only occur if the manifold collapses uniformly as $t \to T$. For immortal solutions we will also give an analysis of the long time behaviour of the flow: In the case of an initial metric with Euclidean volume growth we remark that a result of Simon and Schulze [40] can be adjusted to see that a suitable rescaled Ricci flow subconverges to an expanding soliton, see Remark 7.3. If the initial manifold does not have Euclidean volume growth, then by Theorem 7.5 any immortal solution can be rescaled suitably so that it subconverges to a steady soliton (different from the Euclidean space).

2. Structure of the paper and strategy of proof

Section 3 contains the background material that we use repeatedly throughout the paper. The definition of nonnegative complex sectional curvature, which implies nonnegative sectional curvature and has the advantage to be invariant under the Ricci flow, can be found in subsection 3.1. Subsection 3.2 is about the basics of open nonnegatively curved manifolds.

Section 4 carries out the proof of Theorem 1 for the particular case of a manifold (M, g) with $K_g^{\mathbb{C}} > 0$, which is an easier scenario since there is a *smooth* strictly convex proper function $\beta : M \to [0, \infty[$. The idea (developed within the proof of Proposition 4.1) is to show that the doubling $D(C_i)$ of the compact sublevel $C_i = \beta^{-1}([0, i])$ admits a metric with $K^{\mathbb{C}} \geq 0$. We actually prove that after replacing C_i by the graph of a convex function defined on C_i (a reparametrization of β) the doubling is a smooth closed manifold (M_i, g_i) with $K_{g_i}^{\mathbb{C}} > 0$. The sequence (M_i, g_i) converges to (M, g). The key is now to establish two important properties for the Ricci flows of (M_i, g_i): (1) there is a lower bound (independent of i) for the maximal times of existence T_i (Proposition 4.3), and (2) we can find arbitrarily large balls around the soul point p_0 where the curvature has an upper bound of the form C/t (here C depends on the distance to p_0, see Proposition 4.6). The crucial tool for (1) is a result by Petrunin (Theorem 4.2) which also allows to conclude that the evolved unit balls around the soul are uniformly non-collapsed (Corollary 4.4). For the proof of (2) we use a fruitful point-picking technique by Perelman [35], and we also need to obtain an improved version of 11.4 in [35] (Lemma 4.5). All these results ensure that we can perform suitable compactness arguments to prove Theorem 1 for the positively curved case (Theorem 4.7).

Several additional difficulties arise when we just assume $K_g^{\mathbb{C}} \geq 0$. For instance, the soul is not necessarily a point. A harder issue is that the sublevels of a Busemann function $C_\ell = b^{-1}((-\infty, \ell])$ have non-smooth boundary. Thus there is no obvious smoothing of the doubling $D(C_\ell)$ with $K^{\mathbb{C}} \geq 0$. Section 5 gathers the technical results we will need to apply in Section 6 to overcome the extra complications of the general case of Theorem 1: we prove Theorem 2, which essentially reduces the problem to the situation where the soul is a point; we establish two estimates for abstract solutions of a Riccati equation which are used later to give a quantitative estimate of the convexity of the sublevels C_ℓ in terms of the curvature (Lemmas 5.2 and 5.3); in Proposition 5.5 we get curvature estimates in terms of volume and lower sectional curvature; finally, we include a technical result (Lemma 5.6) about how to perform a smoothing process for $C^{1,1}$ hypersurfaces with bounds on the principal curvatures in the support sense by C^∞ hypersurfaces where the bounds change with an arbitrarily small error.

All the auxiliary results from Section 5 are employed in Section 6 to give a complete proof of Theorem 1. First, we prove upper and lower estimates for the Hessian of $d^2(\cdot, C_\ell)$ (see Proposition 6.1 and Corollary 6.2), and then we reparametrize such a distance function to give a sequence of functions whose graphs $D_{\ell,k}$, after a smoothing process, give C^∞ closed manifolds converging to the double $D(C_\ell)$. The sets $D_{\ell,k}$ are not anymore convex, but we have a precise control on the complex sectional curvatures of the induced metrics $g_{\ell,k}$ (see Proposition 6.3). In Proposition 6.6 we prove that such curvature control survives for some time for the Ricci flows starting on $(D_{\ell,k}, g_{\ell,k})$. As a consequence we get, for all large ℓ, a solution of the Ricci flow on $D(C_\ell)$ with $K^{\mathbb{C}} \geq 0$, and whose 'initial metric' is the natural singular metric on the double. The rest of the proof is then essentially analogous to Section 4.

Corollary 3, 5 and 6 are proved in Section 7 and Theorem 4 is proved in Section 8

We end with three appendices containing additional background about open nonnegatively curved manifolds (Appendix A), results for convex sets in Riemannian manifolds (Appendix B) and results about smooth convergence and curvature estimates for the Ricci flow (Appendix C).

3. Basic background material

3.1. **About the relevant curvature condition.** We first need to introduce

Definition 3.1. Let (M^n, g) be a Riemannian manifold, and consider its complexified tangent bundle $T^{\mathbb{C}}M := TM \otimes \mathbb{C}$. We extend the curvature tensor R and the metric g at p to $\mathbb{C}$-multilinear maps $\mathrm{R}\colon (T_p^{\mathbb{C}}M)^4 \to \mathbb{C}$, $g\colon (T_p^{\mathbb{C}}M)^2 \to \mathbb{C}$. The complex sectional curvature of a 2-dimensional complex subspace σ of $T_p^{\mathbb{C}}M$ is defined by

$$K^{\mathbb{C}}(\sigma) = \mathrm{R}(u, v, \bar{v}, \bar{u}) = g(\mathrm{R}(u \wedge v), \overline{u \wedge v}),$$

where u and v form any unitary basis for σ, i.e. $g(u, \bar{u}) = g(v, \bar{v}) = 1$ and $g(u, \bar{v}) = 0$. We say M has nonnegative complex sectional curvature if $K^{\mathbb{C}} \geq 0$.

The manifold has nonnegative isotropic curvature if $K^{\mathbb{C}}(\sigma) \geq 0$ for any isotropic plane $\sigma \subset T_p^{\mathbb{C}}M$, i.e. $g(v, v) = 0$ for all $v \in \sigma$.

Remark 3.2. Here we collect some relevant features known about the above curvature condition (see [5] and [33] for the proofs).

(a) If g has strictly (pointwise) 1/4-pinched sectional curvature, then $K_g^{\mathbb{C}} > 0$.

(b) Nonnegative curvature operator ($R_g \geq 0$) implies $K_g^{\mathbb{C}} \geq 0$, which in turn gives nonnegative sectional curvature ($K_g \geq 0$). For $n \leq 3$ the converse holds.
(c) $K^{\mathbb{C}}_{(M,g)} \geq 0$ if and only if $(M,g) \times \mathbb{R}^2$ has nonnegative isotropic curvature.
(d) The positivity and nonnegativity of $K^{\mathbb{C}}$ is preserved under the Ricci flow.
(e) Let (M,g) be closed with $K_g^{\mathbb{C}} > 0$. Then g is deformed by the normalized Ricci flow to a metric of positive constant sectional curvature, as time goes to infinity.

Proposition 3.3. *Let (M^n, g) be closed with $K_g^{\mathbb{C}} \geq 0$. If M is homeomorphic to a sphere, then the Ricci flow $g(t)$ with $g(0) = g$ has $K^{\mathbb{C}}_{g(t)} > 0$ for any $t > 0$.*

Proof. Clearly g cannot be Ricci flat as this would give a flat metric on a sphere. Moreover, since M is a sphere the metric is irreducible and neither Kähler nor Quaternion-Kähler. If (M,g) is a locally symmetric space we could use a result of [3] to see that (M,g) is round. Combining all this with the holonomy classification of Berger [1] we deduce that g as well as $g(t)$ has $\mathrm{SO}(n)$ holonomy. Now the statement follows from the proof of [6, Proposition 10]. □

3.2. **Cheeger-Gromoll convex exhaustion.** Let (M,g) be a nonnegatively curved open manifold. A ray is a unit speed geodesic $\gamma\colon [0,\infty) \to M$ such that $\gamma_{[0,s]}$ is a minimal geodesic for all $s > 0$. Fix $o \in M$, and consider the set of rays

$$\mathcal{R} = \{\gamma : [0,\infty) \to M : \gamma \text{ is a ray with } \gamma(0) = o\}.$$

Recall that

$$b = \sup_{\gamma \in \mathcal{R}} \Big\{ \lim_{s\to\infty} \big(s - d_g(\gamma(s), \cdot)\big) \Big\}$$

is called the Busemann function of M. By the work of Cheeger, Gromoll and Meyer [21, 10] b is a convex function, that is, for any geodesic $c(s) \in M$ the function $s \mapsto b \circ c(s)$ is convex. Equivalently one can say that b satisfies $\nabla^2 b \geq 0$ in the support sense (cf. Definition B.4).

The following properties of the sublevels $C_\ell := b^{-1}((-\infty, \ell])$ will be used throughout the paper:

(1) Each C_ℓ is a totally convex compact set,
(2) $\dim C_\ell = n$ for all $\ell > 0$, $\cup_{\ell > 0} C_\ell = M$,
(3) $s < \ell$ implies $C_s \subset C_\ell$ and $C_s = \{x \in C_\ell : d_g(x, \partial C_\ell) \geq \ell - s\}$,
(4) each C_ℓ, $\ell > 0$, has the structure of an embedded submanifold of M with smooth totally geodesic interior and (possibly non-smooth) boundary.

The family C_ℓ is part of the Cheeger-Gromoll convex exhaustion used for the soul construction (see some more details in Appendix A). For us only the structure of C_ℓ for $\ell \to \infty$ is of importance. If (M,g) has positive rather than nonnegative sectional curvature, then $\nabla^2 e^b > 0$ holds in the support sense. By a local smoothing procedure one can then show

Theorem 3.4 (Greene-Wu, [20])**.** *If (M^n, g) is an open manifold with $K_g > 0$, then there exists a smooth proper strictly convex function $\beta\colon M \to [0, \infty[$.*

The main reason why the proof of Theorem 1 is quite a bit easier in the positively curved case is this theorem. In the nonnegatively curved case we will have to work with the sublevels of the Busemann function instead.

4. Manifolds with positive complex sectional curvature

4.1. **Approximating sequence for the initial condition.** Let (M^n, g) be an open manifold with $K_g^{\mathbb{C}} > 0$. On M we can consider a function β as described in Theorem 3.4. Since β is proper, the global minimum is attained and we may assume that its value is 0. Since β is strictly convex, $\beta^{-1}(0)$ consists of a single point p_0, and clearly p_0 is the only critical point of β. Hence the sublevel set

$$C_i = \{x \in M \,:\, \beta(x) \leq i\} \tag{4.1}$$

is a convex set with a smooth boundary for all $i > 0$. Recall that β is obtained essentially from a smoothing of a Busemann function b. Thus we may assume that for each i there is some ℓ_i so that C_i has Hausdorff distance ≤ 1 to $b^{-1}((-\infty, \ell_i])$.

The goal is to construct a pointed sequence of closed manifolds converging to (M, g, p_0). The first attempt would be to consider the double $D(C_i)$ of C_i (which is obtained by gluing together two copies of C_i along the identity map of the boundary). However, $D(C_i)$ is usually not a smooth Riemannian manifold. To overcome this, we adapt to our setting ideas from [30, 22] which roughly consist in modifying the metric in a small inner neighborhood of the boundary ∂C_i to form a cylindrical end so that the gluing is well defined.

Proposition 4.1. *Let (M^n, g) be an open manifold with $K_g^{\mathbb{C}} > 0$ and soul point p_0. Then there exists a collection $\{(M_i, g_i, p_0)\}_{i\geq 1}$ of smooth closed n-dimensional pointed manifolds with $K_{g_i}^{\mathbb{C}} > 0$ satisfying*

$$(M_i, g_i, p_0) \longrightarrow (M, g, p_0) \qquad as \quad i \to \infty$$

in the sense of the smooth Cheeger-Gromov convergence (cf. Definition C.1).

Proof. For each fixed i, consider C_i as in (4.1). The goal is to modify the metric $g|_{C_i}$ within $C_i \setminus C_{i-\varepsilon}$. For that aim, let us choose any real function φ_i such that

(a) φ_i is smooth on $(-\infty, i)$ and continuous at i,
(b) $\varphi_i \equiv 0$ on $(-\infty, i - \varepsilon]$ and $\varphi_i(i) = 1$.
(c) φ_i', φ_i'' are positive on $(i - \varepsilon, i)$,
(d) φ_i^{-1} has all left derivatives vanishing at 1,

By (d) the derivative $\varphi_i'(s)$ tends to ∞ for $s \to i$. Now take $u_i := \varphi_i \circ \beta$ and put

$$\begin{aligned} G_i &= \{(x, u_i(x)) \,:\, x \in C_i\} \\ \tilde{G}_i &= \{(x, 2 - u_i(x)) \,:\, x \in C_i\} \end{aligned}$$

Note that the submanifolds G_i and $\tilde{G}_i$ are isometric and (d) ensures that they paste smoothly together to a C^∞ closed hypersurface $D(C_i) = G_i \cup \tilde{G}_i$ of $M \times \mathbb{R}$.

Clearly the induced metric of G_i can be regarded as a deformation of the metric on C_i. Given the properties of φ_i and β, it is straightforward to check that u_i is a convex function. Using this and that $M \times \mathbb{R}$ has nonnegative complex sectional curvature, we deduce that $(M_i, g_i) := D(G_i)$ has nonnegative complex sectional curvature as well.

Notice that $C_{i-\varepsilon}$ can be seen as a subset of M_i for all $i > 0$, which immediately implies that (M_i, g_i, p_0) converges to (M, g, p_0) in the Cheeger-Gromov sense. We now use the short time existence of the Ricci flow on M_i (cf. [24]), and choose $t_i > 0$ so small that $(M_i, g_i(t_i), p_0)$ still converges to (M, g, p_0).

Since M_i is a topological sphere, we can employ Proposition 3.3 to conclude that $K^{\mathbb{C}}_{g_i(t_i)} > 0$. Thus $g_{i,new} = g_i(t_i)$ is a solution of our problem. □

4.2. **Ricciflowing the approximating sequence.** Consider $\{(M_i, g_i, p_0)\}$ the sequence of closed, positively curved manifolds obtained above. For each fixed i we can construct a Ricci flow $(M_i, g_i(t))$ defined on a maximal time interval $[0, T_i)$, with $T_i < \infty$, and such that $g_i(0) = g_i$.

4.2.1. *A uniform lower bound for the lifespans.* The first difficulty to address is that the curvature of g_i will tend to infinity as $i \to \infty$, so it may happen that the maximal time of existence of the flow T_i goes to zero as i tends to infinity. Then our next concern is to prove that the times T_i admit a uniform lower bound $T_i \geq \mathcal{T} > 0$ for all i. The key to achieve such a goal is to estimate the volume growth of unit balls around p_0. For such an estimate, we make a strong use of

Theorem 4.2 (Petrunin, cf. [37]). *Let (M^n, g) be a complete manifold with $K_g \geq -1$. Then for any p in M*

$$\int_{B_g(p,1)} \mathrm{scal}_g \; d\mu_g \leq C_n,$$

for some constant C_n depending only on the dimension.

Proposition 4.3. *Let (M, g) and (M_i, g_i, p_0) be as in Proposition 4.1. Then there exists a constant $\mathcal{T} > 0$, depending on n, and $V_0 := \mathrm{vol}_g\,(B_g(p_0, 1))$ (but independent of i), such that the Ricci flows $(M_i, g_i(t))$ with $g_i(0) = g_i$ are defined on $[0, \mathcal{T}]$, and satisfy $K^{\mathbb{C}}_{g_i(t)} > 0$ for all $t \in [0, \mathcal{T}]$.*

Proof. For each i, (M_i, g_i) is a closed n-manifold; so the classical short time existence theorem in [24] ensures that there exists some $T_i > 0$ and a unique maximal Ricci flow $(M_i, g_i(t))$ defined on $[0, T_i)$ with $g_i(0) = g_i$. Moreover, $K^{\mathbb{C}}_{g_i(t)} > 0$, since this is true for $t = 0$ by Proposition 4.1, and positive complex sectional curvature is preserved under the Ricci flow (cf. Remark 3.2 (d)).

Observe that $\mathrm{Ric}_{g_i(t)} > 0$ implies $B_{g_i(0)}(p_0, 1) \subset B_{g_i(t)}(p_0, 1)$. Using the evolution equation of the Riemannian volume element $d\mu_{g_i(t)}$ under the Ricci flow and applying Theorem 4.2, we get

$$\frac{\partial}{\partial t}\mathrm{vol}_{g_i(t)}\left(B_{g_i(0)}(p_0,1)\right) = -\int_{B_{g_i(0)}(p_0,1)} \mathrm{scal}_{g_i(t)}\, d\mu_{g_i(t)} \geq -C_n. \tag{4.2}$$

Hence

$$\mathrm{vol}_{g_i(t)}\left(B_{g_i(0)}(p_0,1)\right) - \mathrm{vol}_{g_i(0)}\left(B_{g_i(0)}(p_0,1)\right) \geq -C_n t. \tag{4.3}$$

On the other hand, as $K^{\mathbb{C}}_{g_i} > 0$, we know (cf. Remark 3.2 (e)) that the volume of $(M_i, g_i(t))$ vanishes completely at the maximal time T_i so that

$$T_i \geq \frac{\mathrm{vol}_{g_i(0)}\left(B_{g_i(0)}(p_0,1)\right)}{C_n} \overset{i\to\infty}{\longrightarrow} \frac{\mathrm{vol}_g\left(B_g(p_0,1)\right)}{C_n} =: 2\mathcal{T}.$$

□

As a consequence, we obtain a uniform (independent of t and i) lower bound for the volume of unit balls centered at the soul point:

Corollary 4.4. *For the sequence of pointed Ricci flows $(M_i, g_i(t), p_0)_{t\in[0,\mathcal{T}]}$ from Proposition 4.3, we can find a constant $v_0 = v_0(n, V_0)$ satisfying*

$$\mathrm{vol}_{g_i(t)}\left(B_{g_i(t)}(p_0,1)\right) \geq v_0 > 0 \qquad \textit{for any} \quad t \in [0, \mathcal{T}].$$

Proof. Using again (4.2) and $t \leq \mathcal{T} := V_0/(2C_n)$, we obtain

$$\begin{aligned}\mathrm{vol}_{g_i(t)}\left(B_{g_i(t)}(p_0,1)\right) &\geq \mathrm{vol}_{g_i(t)}\left(B_{g_i(0)}(p_0,1)\right) \geq \mathrm{vol}_{g_i(0)}\left(B_{g_i(0)}(p_0,1)\right) - C_n t \\ &\geq \frac{3}{4}V_0 - C_n\mathcal{T} = \frac{V_0}{4} =: v_0 > 0.\end{aligned}$$

□

4.2.2. *Interior curvature estimates around the soul point.* The first step in order to get a limiting Ricci flow starting on (M, g) from the sequence $(M_i, g_i(t))$ is to obtain uniform (independent of i, but maybe depending on time and distance to p_0) curvature estimates. We first need an improved version of [35, 11.4]:

Lemma 4.5. *Let $(M^n, g(t))$, $t \in (-\infty, 0]$ be an open, non-flat ancient solution of the Ricci flow. Assume further that $g(t)$ has bounded curvature operator, and that $K^{\mathbb{C}}_{g(t)} \geq 0$. Then $\lim_{r\to\infty} \frac{\mathrm{vol}_{g(t)}\left(B_{g(t)}(\,\cdot\,,r)\right)}{r^n}$ vanishes for all t.*

Proof. The k-noncollapsed assumption from 11.4 in [35] was already removed in [32]. So it only remains to ensure that we can relax $\mathrm{R}_{g(t)} \geq 0$ to $K^{\mathbb{C}}_{g(t)} \geq 0$. One can go through the original proof and check that the only instances in which one needs the full $\mathrm{R}_{g(t)} \geq 0$ (instead of just $K_{g(t)} \geq 0$) is when one applies Hamilton's trace Harnack inequality (cf. [26]) or Hamilton's strong maximum principle in [25]. But under our weaker assumption we can replace them by Brendle's trace Harnack in [4] and the strong maximum principle of Brendle and Schoen [6, Proposition 9] (see also the Appendix of [47]). The rest of the proof proceeds verbatim as the original one. □

Proposition 4.6. *Consider the Ricci flows $(M_i, g_i(t))$, with $t \in [0, \mathcal{T}]$, coming from Proposition 4.3. For any $D > 0$ there exists a constant $C_D > 0$ such that*

$$\operatorname{scal}_{g_i(t)}(x) \leq \frac{C_D}{t} \qquad \text{for all} \quad i \geq 1, \quad x \in B_{g_i(t)}(p_0, D) \quad \text{and} \quad t \in (0, \mathcal{T}].$$

Proof. Assume, on the contrary, that we can find a constant $D_0 > 0$ so that there exist indices $i_k \geq 1$ (for brevity, let us denote as $(M_k, g_k(t))$ the corresponding subsequence $(M_{i_k}, g_{i_k}(t))$), and sequences of times $t_k \in (0, \mathcal{T})$ and points $p_k \in B_k(p_0, D_0)$ (hereafter $B_k = B_{g_k(t_k)}$, $\operatorname{scal}_k = \operatorname{scal}_{g_k(t_k)}$ and $d_k = d_{g_k(t_k)}$) satisfying

$$\operatorname{scal}_k(p_k) > 4^k / t_k. \tag{4.4}$$

Claim 1. We can find a sequence of points $\{\bar{p}_k\}_{k \geq k_0}$ which satisfy (4.4) and

$$\operatorname{scal}_{g_k(t)}(p) \leq 8 \operatorname{scal}_k(\bar{p}_k) \quad \text{for all} \quad \begin{cases} p \in B_k\big(\bar{p}_k, \frac{k}{\sqrt{\operatorname{scal}_k(\bar{p}_k)}}\big), \\ t \in \big[t_k - \frac{k}{\operatorname{scal}_k(\bar{p}_k)}, t_k\big] \end{cases}$$

with $d_k(\bar{p}_k, p_0) \leq D_0 + 1$.

Notice that it is enough to prove

$$\operatorname{scal}_k(p) \leq 4 \operatorname{scal}_k(\bar{p}_k) \quad \text{for all} \quad p \in B_k\big(\bar{p}_k, \tfrac{k}{\sqrt{\operatorname{scal}_k(\bar{p}_k)}}\big), \tag{4.5}$$

with $d_k(\bar{p}_k, p_0) \leq D_0 + 1$. In fact, as $K^{\mathbb{C}}_{g_k(t)} \geq 0$, we can apply the trace Harnack inequality in [4] (which, in particular, gives $\frac{\partial}{\partial t}(t \operatorname{scal}_{g(t)}) \geq 0$). This yields for any $t \in \big[t_k - \frac{k}{\operatorname{scal}_k(\bar{p}_k)}, t_k\big]$

$$\operatorname{scal}_{g_k(t)} \leq \frac{t_k}{t} \operatorname{scal}_k \leq \frac{t_k}{t_k - k/\operatorname{scal}_k(\bar{p}_k)} \operatorname{scal}_k < 2 \operatorname{scal}_k,$$

where we have used that (4.4) implies $\frac{k}{\operatorname{scal}_k(\bar{p}_k)} < t_k \frac{k}{4^k} < \frac{t_k}{4}$.

So our goal is to find $\bar{p}_k$ satisfying (4.4) and (4.5). If (4.5) does not hold for $\bar{p}_k = p_k$, it means that there exists a point $x_1 \in B_k\big(p_k, \frac{k}{\sqrt{\operatorname{scal}_k(p_k)}}\big)$ such that $\operatorname{scal}_k(x_1) > 4 \operatorname{scal}_k(p_k)$. Next, check if (4.5) holds for $\bar{p}_k = x_1$, namely, if

$$\operatorname{scal}_k(p) \leq 4 \operatorname{scal}_k(x_1) \qquad \text{for all} \quad p \in B_k\big(x_1, \tfrac{k}{\sqrt{\operatorname{scal}_k(x_1)}}\big).$$

In case this is not satisfied, we iterate the process and, accordingly, we construct a sequence of points $\{x_j\}_{j \geq 2}$ such that

$$x_j \in B_k\Big(x_{j-1}, \frac{k}{\sqrt{\operatorname{scal}_k(x_{j-1})}}\Big) \quad \text{and} \quad \operatorname{scal}_k(x_j) > 4 \operatorname{scal}_k(x_{j-1}).$$

Thus $(x_j)_{j \in \mathbb{N}}$ is a Cauchy sequence and a straightforward computation shows that it stays in the relatively compact ball $B_k(p_0, D_0 + 1)$. Because of

$$\lim_{j \to \infty} \operatorname{scal}_k(x_j) = \infty$$

this gives a contradiction. In conclusion, there exists $\ell \in \mathbb{N}$ such that $\bar{p}_k$ can be taken to be x_ℓ.

Now from Claim 1 it follows that $B_k(p_0, r) \subset B_k(\bar{p}_k, r + D_0 + 1)$. Then for $r \in [D_0 + 3/2, D_0 + 2]$, using Corollary 4.4 with $\mathrm{vol}_k = \mathrm{vol}_{g_k(t_k)}$, we get

$$\frac{\mathrm{vol}_k\left(B_k(\bar{p}_k, r)\right)}{r^n} \geq \frac{\mathrm{vol}_k\left(B_k(p_0, r - D_0 - 1)\right)}{r^n} \geq \mathrm{vol}_k\left(B_k(p_0, 1)\right)\left(\frac{r - D_0 - 1}{r}\right)^n$$
$$\geq \frac{v_0/2^n}{(D_0+2)^n} =: \tilde{v}_0 > 0.$$

Next, Bishop-Gromov's comparison theorem ensures that the above conclusion is true even for smaller radius:

$$\frac{\mathrm{vol}_k\left(B_k(\bar{p}_k, r)\right)}{r^n} \geq \tilde{v}_0 > 0 \quad \text{for} \quad 0 < r \leq D_0 + 2. \tag{4.6}$$

After a parabolic rescaling of the metric

$$\tilde{g}_k(s) = Q_k g(\,\cdot\,, t_k + sQ_k^{-1}) \qquad \text{for} \qquad Q_k = \mathrm{scal}_k(\bar{p}_k) > 4^k/\mathcal{T},$$

using $K_{\tilde{g}_k(s)} > 0$, Claim 1 says that for $k \geq k_0$

$$|\mathrm{R}|_{\tilde{g}_k(s)} \leq \mathrm{scal}_{\tilde{g}_k(s)} \leq 8 \quad \text{on} \quad B_{\tilde{g}_k(0)}(\bar{p}_k, k) \quad \text{for all} \quad s \in [-k, 0]. \tag{4.7}$$

In addition, as the volume ratio in (4.6) is scale-invariant, we have

$$\frac{\mathrm{vol}_{\tilde{g}_k(0)}\left(B_{\tilde{g}_k(0)}(\bar{p}_k, r)\right)}{r^n} \geq \tilde{v}_0 > 0 \quad \text{for} \quad 0 < r \leq (D_0+2)\sqrt{Q_k}. \tag{4.8}$$

Combining this and (4.7) with Theorem C.3 gives

$$\mathrm{inj}_{\tilde{g}_k(0)}(\bar{p}_k) \geq c(n, \tilde{v}_0).$$

Joining the above estimate to (4.7), we are in a position to apply Hamilton's compactness (cf. Theorem C.2) to the pointed sequence

$$(M_k, \tilde{g}_k(s), \bar{p}_k), \qquad s \in [-k, 0]$$

to obtain a subsequence converging, in the smooth Cheeger-Gromov sense to a smooth limit solution of the Ricci flow

$$(M_\infty, g_\infty(t), p_\infty) \qquad t \in (-\infty, 0]$$

which is complete, non compact (since the diameter with respect to $\tilde{g}_k(s)$ tends to infinity with k because $Q_k \to \infty$), non-flat (as $\mathrm{scal}_{g_\infty(0)}(p_\infty) = 1$), of bounded curvature (more precisely, $|\mathrm{R}|_{g_\infty(t)} \leq 8$ on $M_\infty \times (-\infty, 0]$), and with $K^{\mathbb{C}}_{g_\infty(t)} \geq 0$. Moreover, from (4.8) and volume comparison, we have

$$\tilde{v}_0 \leq \frac{\mathrm{vol}_{g_\infty(0)}\left(B_{g_\infty(0)}(p_\infty, r)\right)}{r^n} \leq \omega_n \qquad \text{for all} \quad r > 0.$$

Therefore, the limit of the volume ratio as $r \to \infty$ also lies between two positive constants, which contradicts Lemma 4.5. $\square$

4.3. **Proof of short time existence for the positively curved case.**

Theorem 4.7. *Let (M^n, g) be an open manifold with $K_g^{\mathbb{C}} > 0$ (and possibly unbounded curvature). Then there exists $\mathcal{T} > 0$ and a sequence of closed Ricci flows $(M_i, g_i(t), p_0)_{t\in[0,\mathcal{T}]}$ with $K^{\mathbb{C}}_{g_i(t)} > 0$ which converge in the smooth Cheeger-Gromov sense to a complete limit solution of the Ricci flow*

$$(M, g_\infty(t), p_0) \qquad \textit{for} \quad t \in [0, \mathcal{T}],$$

with $g_\infty(0) = g$.

Proof. Consider the sequence $(M_i, g_i(t))$, with $t \in [0, \mathcal{T}]$, coming from Proposition 4.3. Take some convex compact set $C_{j+1} = \beta^{-1}((-\infty, j+1]) \subset M$ from the convex exhaustion endowed with the Riemannian metric g. By the construction in Proposition 4.1, we can view C_{j+1} also as a subset of M_i for $i \geq j+2$. Moreover, the metric $g_i(0)$ on C_{j+1} converges to g in the C^∞ topology. By Proposition 4.6 there is some constant L_j with

$$|\mathrm{R}_{g_i(t)}| \leq \tfrac{L_j}{t} \qquad \text{on } B_{g_i(t)}(C_{j+1}, 1) \quad \text{for all } t \in (0, \mathcal{T}] \quad \text{and} \quad i \geq j+2. \tag{4.9}$$

Since the metric $g_i(0)$ converges on C_j in the C^∞ topology to g we can choose $\rho > 0$ so small that

$$|\mathrm{R}_{g_i(0)}| \leq \rho^{-2} \qquad \text{on} \quad C_{j+1} \qquad \text{for} \quad i \geq j+2. \tag{4.10}$$

After possibly decreasing ρ we may assume that the ρ-neighborhood of C_j is contained in C_{j+1} with respect to the metric $g_i(0)$ for $i \geq j+2$. Combining the inequalities (4.9) and (4.10) we are now in a position to apply Theorem C.5 in order to deduce that for some constant $\hat{L}_j > 0$ we have

$$|\mathrm{R}_{g_i(t)}| \leq \hat{L}_j \quad \text{on} \quad C_j \quad \text{for all} \quad t \in [0, \mathcal{T}] \quad \text{and} \quad i \geq j+2. \tag{4.11}$$

Combining this with an extension of Shi's estimate as stated in Theorem C.4, we reach furthermore

$$|\nabla^k \mathrm{R}_{g_i(t)}| \leq \hat{L}_{j,k} \quad \text{on} \quad C_j \quad \text{for all} \quad t \in [0, \mathcal{T}] \quad \text{and} \quad i \geq j+2.$$

From here, by standard arguments as in [28] (see Lemma 2.4 and remarks after it), we have that the metrics $g_i(t)$ on C_j have all space and time derivatives uniformly bounded. Hence one can apply the Arzelà-Ascoli-Theorem to deduce that after passing to a subsequence $g_i(t)$ converges to $g_\infty(t)$ in the C^∞ topology on $C_j \times [0, \mathcal{T}] \subset M \times \mathbb{R}$.

Doing this for all $j \in \mathbb{N}$ and applying the usual diagonal sequence argument we can, after passing to subsequence, assume that $g_i(t)$ converges in the C^∞ topology to a limit metric $g_\infty(t)$ on $C_j \times [0, \mathcal{T}]$ for all j. By construction $g_\infty(t)$ is a solution of the Ricci flow on M with initial metric $g_\infty(0) = g$. The completeness of $g_\infty(t)$ is a consequence of the following Lemma. ∎

Lemma 4.8. *There exists $L > 0$ such that $B_{g_\infty(t)}(p_0, r) \subset B_{g_\infty(0)}(p_0, 2r + L(t+1))$ for all positive r and $t \in [0, \mathcal{T}]$.*

Proof. This will follow by proving a uniform estimate for (M_i, g_i, p_0). Since (M_i, g_i) is the double of a convex set, it has a natural $\mathbb{Z}_2$-symmetry which comes from switching the two copies of the double. As the Ricci flow on closed manifolds is unique, this symmetry is preserved by the Ricci flow. Thus the middle of $(M_i, g_i(t))$, being the fixed point set of an isometry, remains a totally geodesic hypersurface N_i. It is now fairly easy to estimate how the distance of p_0 to N_i changes in time: Let L_1 be a bound on the eigenvalues of the Ricci curvature on $B_{g_i(t)}(p_0, 1)$ for all i and all $t \in [0, \mathcal{T}]$.

If $c(s)$ is a minimal geodesic in $(M_i, g_i(t))$ from p_0 to N_i then it follows for the left derivative of $r_i(t) = d_{g_i(t)}(p_0, N_i)$ that

$$\begin{aligned} \tfrac{d}{dt} r_i(t) &\geq -\int_0^{r_i(t)} \mathrm{Ric}_{g_i(t)}(\dot{c}(s), \dot{c}(s))\, ds \geq -L_1 - \int_1^{r_i(t)} \mathrm{Ric}_{g_i(t)}(\dot{c}(s), \dot{c}(s))\, ds \\ &\geq -L_1 - (n-1) = -L_2, \end{aligned}$$

where we used the second variation formula in the last inequality.

If we put $D_i = d_{g_i(0)}(p_0, N_i)$ then we obtain $d_{g_i(t)}(p_0, N_i) \geq D_i - L_2 t$. Recalling that $d_{g_i(0)}(p, N_i) \geq d_{g_i(t)}(p, N_i)$, for any $r > 0$ we can find i big enough so that

$$B_{g_i(t)}(p_0, r) \quad \subset \quad \{p \in C_i \cap M_i \mid d_{g_i(0)}(p, N_i) \geq D_i - L_2 t - r\}.$$

Next recall that β is essentially a smoothing of a Busemann function and by assumption the level set $\beta^{-1}(i)$ has Hausdorff distance at most 1 to $b^{-1}(\ell_i)$ for a suitable ℓ_i. Combining this with the previous inclusion and that we modified the metric in C_i in a controlled way we deduce

$$\begin{aligned} B_{g_i(t)}(p_0, r) &\subset \left\{p \in b^{-1}((-\infty, \ell_i]) \mid d_{g_i(0)}(p, b^{-1}(\ell_i)) \geq \ell_i - L_2 t - r - L_3\right\} \\ &= b^{-1}\big((-\infty, L_3 + r + L_2 t]\big) \end{aligned}$$

where $L_3 = 3 + \mathrm{diam}_g(b^{-1}((-\infty, 0]))$. Finally, applying Lemma A.3 gives

$$b^{-1}\big((-\infty, L_3 + r + L_2 t]\big) \subset B_{g(0)}(p_0, 2r + L(1+t))$$

for a suitable large L. $\square$

For some of the applications we will need a version of Lemma 4.8 for abstract solutions of the Ricci flow with $K^{\mathbb{C}}_{g(t)} \geq 0$.

Lemma 4.9. *Let $(M, g(t))_{t \in [0, \mathcal{T}]}$ be a solution of the Ricci flow with $K^{\mathbb{C}}_{g(t)} \geq 0$. Suppose that $(M, g(t))$ is complete for $t \in [0, \mathcal{T})$. If $p_0 \in M$, then for some $C > 0$*

$$B_{g(t)}(p_0, R) \subset B_{g(0)}(p, R + C(1+t)) \text{ for all } R \geq 0,\ t \in [0, \mathcal{T}]$$

In particular, $g(\mathcal{T})$ is complete as well.

Proof. There is nothing to prove in the compact case and thus we may assume that $(M, g(0))$ is open. After rescaling we may assume that the closure of $B_{g(\mathcal{T})}(p_0, 1)$ is compact and that $K_{g(t)} \leq 1$ on $B_{g(t)}(p_0, 1)$.

We define $b_t\colon M \to \mathbb{R}$ by $b_t(q) := \limsup_{p\to\infty}\big(d_{g(t)}(p,p_0) - d_{g(t)}(p,q)\big)$. Notice that similarly to the Busemann function in subsection 3.2 b_t is convex, proper and bounded below. In the sequel $\frac{d}{dt}$ refers to a right hand side Dini derivative. Similar to the proof of the previous lemma, it suffices to show

Claim. $\frac{d}{dt}b_t(q) \geq -4(n-1)$ for all $q \in M$.

Choose $q_k \to \infty$ with $b_t(q) = \lim_{k\to\infty} d_{g(t)}(q_k,p_0) - d_{g(t)}(q_k,q)$. We may assume that the Busemann function b_t is differentiable at q_k. Let c_k (resp. γ_k) be a unit speed geodesic from q_k to p_0 (resp. to q). We claim that the angle between $\dot c_k(0)$ and $\dot\gamma_k(0)$ converges to 0.

Notice that $b_t \geq \tilde b_t$, where $\tilde b_t$ is the Busemann function of $(M, g(t))$ defined with respect to all rays emanating from p_0. Therefore we deduce from Lemma A.3 that $b_t(q_k) \geq (1-\delta_k)d(p_0,q_k)$ for some sequence $\delta_k \to 0$. After possibly adjusting δ_k we also may assume that $b_t(q_k) - b_t(q) \geq (1-\delta_k)d(q,q_k)$. Since $s \mapsto b_t(c_k(s))$ and $s \mapsto b_t(\gamma_k(s))$ are convex 1-Lipschitz functions, we get that $\langle \nabla b_t(q_k), \dot c_k(0)\rangle \leq -(1-\delta_k)$ and $\langle \nabla b_t(q_k), \dot\gamma_k(0)\rangle \leq -(1-\delta_k)$. Thus $\langle \dot c_k(0), \dot\gamma_k(0)\rangle \to 1$ as claimed.

By Toponogov triangle comparison theorem $d(c_k(1), \gamma_k(1)) \to 0$. If we put $\tilde q_k := c_k(1)$, then $b_t(q) = \lim_{k\to\infty} d_{g(t)}(\tilde q_k, p_0) - d_{g(t)}(\tilde q_k, q)$.

We can now use the second variation formula combined with the curvature bounds on $B_{g(t)}(p_0,1)$ to see that $\frac{d}{dt}d_{g(t)}(\tilde q_k,p_0) \geq -4(n-1)$. Since $d_{g(t)}(\tilde q_k, q)$ is decreasing in t, we deduce $\frac{d}{dt}\lim_{k\to\infty} d_{g(t)}(\tilde q_k,p_0) - d_{g(t)}(\tilde q_k,q) \geq -4(n-1)$. This in turn implies that the right derivative of $b_t(q)$ is bounded below by $-4(n-1)$ as claimed. ∎

5. Miscellanea of auxiliary results for the general case

5.1. A splitting theorem for open manifolds with $K^{\mathbb{C}} \geq 0$.

Theorem 5.1. *Let (M^n, g) be an open, simply connected Riemannian manifold with $K^{\mathbb{C}}_g \geq 0$. Then M splits isometrically as $\Sigma \times F$, where Σ is the k-dimensional soul of M and F is diffeomorphic to $\mathbb{R}^{n-k}$. In particular, F carries a complete metric of nonnegative complex sectional curvature.*

Proof. By Theorem A.2 due to M. Strake, it is enough to show that the normal holonomy group of the soul Σ is trivial, which in turn is equivalent (by a modification of [36, Section 8.4]) to prove

$$\langle \mathrm{R}(e_1,e_2)v_1,v_2\rangle = 0 \quad \text{for all} \quad e_1,e_2 \in T_p\Sigma \quad \text{and all} \quad v_1,v_2 \in \nu_p\Sigma.$$

With such a goal, we look at the curvature tensor on the 4-dimensional space $N = \mathrm{span}\{e_1,e_2,v_1,v_2\}$, namely, we consider $\tilde{\mathrm{R}} = \mathrm{R}|_{\Lambda^2 N}$. It is well-known that one has the orthogonal decomposition $\Lambda^2 N = \Lambda^2_+ \oplus \Lambda^2_-$ into the eigenspaces of the Hodge star operator $*$ with eigenvalues ± 1. This gives a block decomposition

$$\tilde{\mathrm{R}} = \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}$$

with respect to the bases

$$\{b_1^\pm = e_1 \wedge e_2 \pm v_1 \wedge v_2,\ b_2^\pm = e_1 \wedge v_1 \mp e_2 \wedge v_2,\ b_3^\pm = e_1 \wedge v_2 \pm e_2 \wedge v_1\}$$

for $\Lambda^2_\pm$. By Cheeger and Gromoll [10, Theorem 3.1] the mixed curvatures vanish, i.e. $\tilde{\mathrm{R}}(e_i \wedge v_j, e_i \wedge v_j) = 0$. Thus

$$a_{22} + a_{33} + c_{22} + c_{33} = 0.$$

On the other hand, it is well known (cf. [29]) that nonnegative isotropic curvature of $\tilde{\mathrm{R}}$ implies that the numbers $a_{22} + a_{33}$ and $c_{22} + c_{33}$ are nonnegative. Consequently,

$$a_{22} + a_{33} = 0 = c_{22} + c_{33}. \tag{5.1}$$

Since $\tilde{\mathrm{R}}$ is a 4-dimensional curvature operator with nonnegative sectional curvature, a result by Thorpe (see e.g. [38, Proposition 3.2]) ensures that we can find a $\lambda \in \mathbb{R}$ such that

$$\tilde{\mathrm{R}} + \lambda \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} A + \lambda I & B \\ {}^tB & C - \lambda I \end{pmatrix}$$

is a positive semidefinite matrix. Combining this with (5.1), we obtain $\lambda = 0$, so $\tilde{\mathrm{R}}$ itself is a nonnegative operator. Then $v_i \wedge e_j$ are in the kernel of $\tilde{\mathrm{R}}$. Finally, the first Bianchi identity yields

$$\langle \mathrm{R}(e_1 \wedge e_2), v_1 \wedge v_2 \rangle = -\langle \mathrm{R}(v_1 \wedge e_1), e_2 \wedge v_2 \rangle - \langle \mathrm{R}(e_2 \wedge v_1), e_1 \wedge v_2 \rangle = 0.$$

□

5.2. **Some preliminary estimates for Riccati operators.** In the space of self-adjoint endomorphisms $S(\mathbb{R}^n)$, $A \leq B$ if $\langle Av, v \rangle \leq \langle Bv, v \rangle$ for every $v \in \mathbb{R}^n$.

Lemma 5.2. *Let $A(s) \in S(\mathbb{R}^n)$ be a nonnegative solution of the Riccati equation*

$$A'(s) + A^2(s) + R(s) = 0, \tag{5.2}$$

with $R(s) \geq 0$. Assume also that R and $|R'|$ are bounded for $s \in [0,1]$ by constants C_R and $C_{R'} > 0$, respectively. Then there exists $A_0 \in S(\mathbb{R}^n)$ satisfying

$$A(0) \geq A_0, \qquad A_0 \leq C_R \tag{5.3}$$

and we can find an $\varepsilon_0 = \varepsilon_0(C_R, C_{R'}) > 0$ so that

$$\langle A_0 w, w \rangle \geq \varepsilon_0 \langle R(0)w, w \rangle^2 \qquad \textit{for all} \quad w \in \mathbb{R}^n \quad \textit{with} \quad |w| = 1. \tag{5.4}$$

Proof. From (5.2), we can write

$$A(s) - A(0) = -\int_0^s \left(A^2(\xi) + R(\xi)\right)\, d\xi \leq -\int_0^s R(\xi)\, d\xi.$$

As this is valid for any s, using $A(s) \geq 0$ and $R(s) \geq 0$, we conclude

$$A(0) \geq \int_0^\infty R(\xi)\, d\xi \geq \int_0^1 R(\xi)\, d\xi =: A_0.$$

Clearly, $A_0 \leq C_R$. Next, take $w \in \mathbb{R}^n$ with $|w| = 1$, define $C := \max\{C_R, C_{R'}\}$ and compute

$$\langle R(s)w, w\rangle = \left\langle \big(R(0) + sR'(\xi)\big)\, w, w\right\rangle \geq r(0) - C\, s,$$

where $r(0) = \langle R(0)w, w\rangle$ and $r(0)/C \leq C_R/C \leq 1$. This allows to estimate

$$\begin{aligned}\langle A_0 w, w\rangle &= \int_0^1 \langle R(\xi)w, w\rangle\, d\xi \geq \int_0^1 \max\{0, r(0) - C\xi\}\, d\xi \\ &\geq \int_0^{r(0)/C} (r(0) - Cs)\; ds = \frac{r(0)^2}{2C},\end{aligned}$$

which gives the result taking $\varepsilon_0 = \frac{1}{2C}$. □

Lemma 5.3. *Let $A(s) \in S(\mathbb{R}^n)$ be a solution of (5.2). Suppose that $|R(s)| \leq C_R$ and $|R'(s)| \leq C_{R'}$ for small s. If there exists $\varepsilon_0 > 0$ and $A_0 \in S(\mathbb{R}^n)$ satisfying (5.3) and (5.4), then we can find an $s_0 = s_0(C_R) > 0$ such that*

$$A(s) \geq -Cs^2\,\mathrm{Id} \qquad \text{for all} \qquad s \in (0, s_0], \tag{5.5}$$

for some $C = C(C_R, C_{R'}) > 0$.

Proof. Using Riccati comparison (see e.g. [17]) we can assume without loss of generality that $A(0) = A_0$. Next, we do a Taylor expansion for A and use (5.2) at $s = 0$:

$$\begin{aligned}A(s) &\geq A(0) + sA'(0) - C\, s^2\,\mathrm{Id} \geq A(0) - s(A(0)^2 + R(0)) - C\, s^2\,\mathrm{Id} \\ &\geq (1 - sC_R)A_0 - sR(0) - Cs^2\,\mathrm{Id},\end{aligned}$$

which comes from (5.3). Notice that C depends on C_R and $C_{R'}$. Choose now any $w \in \mathbb{R}^n$ with $|w| = 1$. Then, for $s \leq \frac{1}{2C_R} =: s_0$, our assumption (5.4) yields

$$\begin{aligned}\langle A(s)w, w\rangle &\geq \frac{1}{2}\langle A_0 w, w\rangle - s\,\langle R(0)w, w\rangle - Cs^2 \geq \frac{1}{2}\varepsilon_0 r(0)^2 - sr(0) - Cs^2 \\ &= \left(\sqrt{\frac{\varepsilon_0}{2}}r(0) - \frac{s}{\sqrt{2\varepsilon_0}}\right)^2 - \left(C + \frac{1}{2\varepsilon_0}\right)s^2 \geq -\widetilde{C}(\varepsilon_0, C)\, s^2.\end{aligned}$$

□

5.3. **Curvature estimates in terms of volume.** It will be useful for technical purposes to have in mind the following

Lemma 5.4. *Let (M^n, g) be a Riemannian manifold with $K_g \geq -\Lambda^2 \geq -1$. Then there is a constant C depending on n such that*

$$\frac{\mathrm{vol}_g\,(B_g(p, R))}{\mathrm{vol}_g\,(B_g(p, r))} \leq \left(\frac{R}{r}\right)^n (1 + C(\Lambda R)^2)$$

for all $p \in M$ and all $0 < r \leq R \leq 2$.

Proof. Let us use the Taylor expansion for $\sinh\rho$, with $0 \le \rho \le 2$, to deduce

$$\rho^m \le (\sinh\rho)^m = \left(\sum_{j=0}^{\infty} \frac{\rho^{2j+1}}{(2j+1)!}\right)^m \le \rho^m(1+\widetilde{C}\rho^2)^m = \rho^m \sum_{j=0}^{m} \binom{m}{j} (\widetilde{C}\rho^2)^j$$
$$\le \rho^m(1+C\rho^2).$$

Now Bishop-Gromov's comparison theorem says that

$$\frac{\mathrm{vol}_g\,(B_g(p,R))}{\mathrm{vol}_g\,(B_g(p,r))} \le \frac{\displaystyle\int_0^R [\sinh(\Lambda\rho)]^{n-1}\,d\rho}{\displaystyle\int_0^r [\sinh(\Lambda\rho)]^{n-1}\,d\rho} \le \frac{\displaystyle\int_0^R (\Lambda\rho)^{n-1}(1+C(\Lambda\rho)^2)\,d\rho}{\displaystyle\int_0^r (\Lambda\rho)^{n-1}\,d\rho}$$
$$\le \frac{(\Lambda R)^n/(\Lambda n) + C(\Lambda R)^{n+2}/(\Lambda(n+2))}{(\Lambda r)^n/(\Lambda n)} \le \frac{R^n}{r^n}(1+C(\Lambda R)^2).$$

□

Proposition 5.5. *For any $\varepsilon > 0$ we can find positive constants δ, κ, T such that if $(M^n, g(t))$ is a compact Ricci flow with*

$$K_{g(t)} \ge -\kappa \quad \text{on } [0,\bar{t}] \qquad \text{and} \qquad \frac{\mathrm{vol}_{g(0)} B_{g(0)}(\cdot, r)}{r^n} \ge (1-\delta)\omega_n,$$

for some $r \in (0,1]$, then

$$|\mathrm{R}|_{g(t)} \le \frac{\varepsilon}{t} \qquad \text{on} \quad [0,\bar{t}] \cap [0, r^2 T(\varepsilon)]. \tag{5.6}$$

Proof. By rescaling it suffices to prove the statement for $r = 1$. Arguing by contradiction, suppose that there is an $\varepsilon > 0$ and a sequence of Ricci flows $(M_i, g_i(t))$ defined on $[0, \bar{t}_i]$ satisfying

$$K_{g_i(t)} \ge -\frac{1}{(n-1)i} \quad \text{and} \quad \mathrm{vol}_{g_i(0)} B_{g_i(0)}(p,1) \ge \left(1 - \frac{1}{i}\right)\omega_n \tag{5.7}$$

for all $p \in M_i$ and all i. But assume that we can also find a sequence of points and times $\{(p_i, t_i)\}$ such that

$$Q_i := |\mathrm{R}|_{g_i(t_i)}(p_i) = \max_{q \in M_i} |\mathrm{R}|_{g_i(t_i)}(q) > \varepsilon/t_i \qquad \text{with} \quad t_i \to 0. \tag{5.8}$$

Next, we aim to show that the volume estimate in (5.7) survives for some time. From (5.7) and the evolution of $d_{g_i(t)}$ under (1.1), we deduce that

$$B_{g_i(t)}\big(p, e^{t/i}\big) \subset B_{g_i(t+\tau)}\big(p, e^{(t+\tau)/i}\big) \qquad \text{for all} \quad 0 \le t < t+\tau \le \bar{t}_i.$$

Accordingly

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{vol}_{g_i(t)}\big(B_{g_i(t)}(p,e^{t/i})\big) &\geq \frac{\partial}{\partial \tau}\Big|_{\tau=0}\mathrm{vol}_{g_i(t+\tau)}\big(B_{g_i(t)}(p,e^{t/i})\big)\\ &= -\int_{B_{g_i(t)}(p,\,e^{t/i})}\mathrm{scal}_{g_i(t)}\;d\mu_{g_i(t)}\\ &\geq -C_n e^{(n-2)t/i},\end{aligned}$$

which follows from Petrunin's estimate (cf. Theorem 4.2). This leads to

$$\mathrm{vol}_{g_i(t)}\big(B_{g_i(t)}(p,e^{t/i})\big) \geq \mathrm{vol}_{g_i(0)}\big(B_{g_i(0)}(p,1)\big) - C_n e^{(n-2)t/i}\,t \geq (1-\eta_i)\,\omega_n$$

for all $t \in [0,t_i]$ and with $\eta_i \to 0$. Next, we can apply Lemma 5.4 to conclude

$$\begin{aligned}\mathrm{vol}_{g_i(t)}\left(B_{g_i(t)}(p,r)\right) &\geq \tfrac{1}{1+C\frac{e^{2t/i}}{i}}\left(\tfrac{r}{e^{t/i}}\right)^n \mathrm{vol}_{g_i(t)}\left(B_{g_i(t)}(p,e^{t/i})\right)\\ &\geq \tfrac{e^{-nt_i/i}}{1+C\frac{e^{2t_i/i}}{i}}\,(1-\eta_i)r^n\omega_n\\ &= (1-\mu_i)r^n\omega_n \end{aligned}\tag{5.9}$$

for all $0 < r \leq e^{t/i}$, $t \in [0,t_i]$ with $\mu_i \to 0$.

Now consider the rescaled solution to the Ricci flow $\bar{g}_i(t) = Q_i g(t_i + t/Q_i)$. Doing a time-picking argument, we can assume without loss of generality that

$$|\mathrm{R}|_{\bar{g}_i(t)} \leq 4 \quad \text{on } B_{\bar{g}_i(0)}(p_i,2) \quad \text{and} \quad t \in [-t_iQ_i/2, 0] \supset [-\varepsilon/2, 0],$$

where the latter is true by (5.8). In addition,

$$|\mathrm{R}|_{\bar{g}_i(0)}(p_i) = 1. \tag{5.10}$$

A standard application of Shi's derivative estimates gives on $B_{\bar{g}_i(0)}(p_i,1)$

$$|\nabla^\ell \mathrm{R}|_{\bar{g}_i(t)} \leq \frac{C(n,\ell)}{(t+\varepsilon/2)^{\ell/2}}; \quad \text{in particular} \quad |\nabla^\ell \mathrm{R}|_{\bar{g}_i(0)} \leq C(n,\ell,\varepsilon). \tag{5.11}$$

After passing to a subsequence we may assume that $B_{\bar{g}_i(0)}(p_i,1)$ converges to a nonnegatively curved limit ball $B_{\bar{g}_\infty}(p_\infty,1)$ satisfying (5.10) and (5.11). In particular $\mathrm{vol}_{\bar{g}_\infty}(B_{\bar{g}_\infty}(p_\infty,1)) < \omega_n$. On the other hand, it is immediate from (5.9) that $\mathrm{vol}_{\bar{g}_\infty}(B_{\bar{g}_\infty}(p_\infty,1)) \geq \omega_n$ – a contradiction. □

5.4. Smoothing $C^{1,1}$ hypersurfaces.

Lemma 5.6. *Let M^n be a smooth Riemannian manifold, H a $C^{1,1}$ hypersurface, and N a unit normal field. Suppose we have bounds $C_1 \leq A_H \leq C_2$ on the principal curvatures of H in the support sense. Then we can find a sequence of smooth hypersurfaces H_i converging in the C^1-topology to H such that $C_1 - \frac{1}{i} \leq A_{H_i} \leq C_2 + \frac{1}{i}$. If H is invariant under the isometric action of a compact Lie group G on M, then one can assume in addition that H_i is invariant under the action as well.*

Proof. First, we give the proof in the case of a compact hypersurface.

We consider a small tubular neighborhood $U = B_{r_0}(H)$ of H. By assumption $U \setminus H$ has two components U_+ and U_-. We consider the function $f\colon U \to \mathbb{R}$ which is defined on $U_+ \cup H$ as the distance to H and on U_- as minus the distance to H. Clearly f is a $C^{1,1}$-function. Moreover, it is easy to deduce that for each ε we can find r such that

$$C_1 - \varepsilon \leq \nabla^2 f \leq C_2 + \varepsilon \quad \text{on} \quad B_r(H)$$

holds in the support sense. Furthermore, we know of course $|\nabla f| \equiv 1$. Now it is not hard to see that for each $\varepsilon > 0$ we can find a smooth function f_ε on $B_{r/2}(H)$ satisfying

$$\begin{aligned} C_1 - 2\varepsilon \leq \nabla^2 f_\varepsilon &\leq C_2 + 2\varepsilon \quad \text{and} \\ |f_\varepsilon - f| + |\nabla f - \nabla f_\varepsilon| &\leq \min\{\varepsilon, r/10\} \qquad \text{on } B_{r/2}(H). \end{aligned}$$

It is now straightforward to check that for $\varepsilon_i = \frac{1}{10i(|C_1|+|C_2|+1)}$ we can put $H_i := f_{\varepsilon_i}^{-1}(0)$ and check the claimed bounds on the principal curvatures of H_i.

If H is invariant under the isometric action of a compact Lie group G, then $f(gp) = f(p)$ for all $p \in H$. By putting $\tilde{f}_\varepsilon(p) := \frac{1}{\mathrm{vol}(\mathsf{G})} \int_{\mathsf{G}} f_\varepsilon(gp)\, d\mu(g)$ we obtain a G-invariant function with the same bounds on the Hessian as f_ε. We can then define H_i as before.

If the hypersurface is not compact one uses a (G-invariant) compact exhaustion and argues as before. □

6. The general case

6.1. Estimates on the Hessian of the squared distance function.

Proposition 6.1. *Let (M^n, g) be an open manifold with $K_g \geq 0$, let C_ℓ be a sublevel set of the Busemann function (see subsection 3.2), and $p \in \partial C_\ell$. For each unit normal vector $v \in N_p C_\ell$ there is a smooth hypersurface S supporting ∂C_ℓ at p from the outside such that $T_p S$ is given by the orthogonal complement of v, and the second fundamental form A of S satisfies*

$$u \geq \langle A_v w, w\rangle \geq c\,\mathrm{R}(w, v, v, w)^2 \quad \text{for all} \quad w \in T_p S \quad \text{with} \quad |w| = 1, \tag{6.1}$$

for some positive constants c and u depending on C_ℓ.

Proof. We fix $r > 0$ smaller than a quarter of the convexity radius of $C_{\ell+r}$. Proposition B.1 by Yim ensures that any element of $N_p C_\ell$ can be obtained, up to scaling, as (hereafter we use Einstein sum convention)

$$\alpha^i u_i \qquad \text{with} \quad |u_i| = 1, \quad \alpha_0 + \ldots + \alpha_k = 1 \quad \text{and} \quad \alpha_i \geq 0,$$

where each $u_i \in \mathrm{span}(T_p C_\ell)$ is such that $\gamma_i(s) = \exp_p(s u_i)$ is the minimal geodesic from p to a point of $\partial C_{\ell+2r}$. As $\dim(C_\ell) = n$, we can choose $k \leq n$ by Carathéodory's Theorem (cf. [39]).

Consider $q_i := \gamma_i(r) \in C_{\ell+r}$ and the hyperplane $V_i \subset T_{q_i}M$ perpendicular to $\gamma_i'(r)$. Since $\gamma_i'(r) \in N_{q_i}C_{\ell+r}$ it follows that $H_i := \exp(B_r(0) \cap V_i)$ is a smooth hypersurface supporting $C_{\ell+r}$ from the outside.

Then $\varphi_i := r+\ell-d(H_i,\cdot)$ is a lower support function of the Busemann function b at p (which can be seen using e.g. Lemma A.4 by Wu). Note that $\nabla^2\varphi_i|_{\gamma_i(s)}$ is a positive semidefinite solution of a Riccati equation for $s \in [0,r]$. So we clearly have upper bounds (just depending on C_ℓ) for $\nabla^2\varphi_i|_p$. Lemma 5.2 now yields

$$\nabla^2\varphi_i|_p(w,w) \geq \varepsilon_0 \mathrm{R}(w,u_i,u_i,w)^2 \tag{6.2}$$

for all unit vectors $w \in T_pM$.

As mentioned above there is some $\lambda > 0$ such that $\lambda v = \alpha^i u_i$ with $\sum_i \alpha_i = 1$. Define $\phi = \alpha^i\varphi_i$, which is a function whose gradient is λv. Since it is a convex combination of lower support functions for b at p, ϕ is also a lower support function for b at p; therefore, $b^{-1}((-\infty,\ell]) \cap B_r(p) \subset \phi^{-1}((-\infty,\ell]) \cap B_r(p)$. Consequently, if we define S as the level set $\phi^{-1}(\ell) \cap B_r(p)$, then T_pS is orthogonal to v and S supports C_ℓ at p from the outside. Moreover, the second fundamental form of S at p is proportional to $\nabla^2\phi|_p$, and from (6.2) we have

$$\nabla^2\phi|_p(w,w) = \alpha^i\nabla^2\varphi_i|_p(w,w) \geq \varepsilon_0\, \alpha^i R_w(u_i,u_i)^2. \tag{6.3}$$

Next, using $K_g(\alpha_i u_i - \alpha_j u_j, w) \geq 0$ we can estimate the curvature:

$$R_w(\alpha^i u_i, \alpha^j u_j) \leq \frac{1}{2}\sum_{i,j}\left(\alpha_i^2 R_w(u_i,u_i) + \alpha_j^2 R_w(u_j,u_j)\right) \leq (n+1)(\alpha^i)^2 R_w(u_i,u_i).$$

Now, combining a discrete version of Hölder's inequality applied to (6.3), that $\alpha_i \leq 1$ and the above computation, we reach

$$\begin{aligned}\nabla^2\phi|_p(w,w) &\geq \frac{\varepsilon_0}{n+1}\Big(\sum_i \sqrt{\alpha_i}\, R_w(u_i,u_i)\Big)^2 \geq \frac{\varepsilon_0}{n+1}\Big(\sum_i \alpha_i^2\, R_w(u_i,u_i)\Big)^2 \\ &\geq \frac{\varepsilon_0}{(n+1)^2} R_w(\alpha^i u_i, \alpha^j u_j)^2 = c(n,\varepsilon_0) R_{\alpha^i u_i}(w,w)^2.\end{aligned}$$

Finally, the statement follows since the second fundamental form of S satisfies $\langle A_v w, w\rangle \geq c(n,\varepsilon_0)\lambda^3 R_v(w,w)^2$, and it is easy to see that λ is bounded below by a constant just depending on C_ℓ. □

Corollary 6.2. *Consider (M^n, g) and $C = C_\ell$ as in Proposition 6.1. Then there exists a neighborhood U of C_ℓ such that $f = d^2(\cdot, C)$ is a $C^{1,1}$ function on U and satisfies the following estimates*

$$-\lambda f^{3/2} \leq \nabla^2 f \leq 2 \qquad \text{on} \qquad U \tag{6.4}$$

in the support sense, for some positive constant $\lambda = \lambda(C)$.

Proof. By a result of Walter (see Theorem B.3) we can find a tubular neighborhood U of C, such that f is $C^{1,1}$ on U.

Let $q \in U$ and let $p \in \partial C$ denote a point with $d(q,p) = d(q,C)$. Clearly $d(\cdot,p)^2$ is an upper support function of f at q and thus $\nabla^2 f|_q \leq 2$.

In order to get the lower bound, we consider a minimal unit speed geodesic $\gamma(s)$ from p to q. The initial direction $v = \gamma'(0)$ is a normal vector, and by Proposition 6.1 we can find a hypersurface S touching C from the outside at p such that T_pS is normal to v, and the second fundamental form of S is bounded by $u \geq \langle A_v w, w\rangle \geq c\,\mathrm{R}(v,w,w,v)^2$ for any unit vector w.

Notice that $a^2 := d(S,\cdot)^2$ is a lower support function of f at q. Since $A(s) = \nabla^2 a|_{\gamma(s)}$ satisfies a Riccati equation with $A(0) = A_v$, we can employ Lemma 5.3 (for which we can take $A_0 = A_v$, since the latter is bounded above) to conclude $A(s) \geq -Cs^2$. Consequently, $\nabla^2 f|_q \geq -2C\,a(q)\,d(p,q)^2 = -2Cf^{3/2}(q)$. □

6.2. **A sequence of graphical sets with controlled curvatures.** For any $r > 0$ and any set $S \subset M$ consider the tubular neighborhood $B_r(S) = \cup_{p\in S}\overline{B}_r(p)$.

Proposition 6.3. *Let $C \subset (M^n, g)$ be a sublevel set of the Busemann function (see subsection 3.2). Then we can construct a sequence $\{D_k\}_{k=1}^{\infty}$ of C^∞ closed hypersurfaces of $B_1(C) \times [0,1]$ which converges in the Gromov-Hausdorff sense to the double of C, and whose principal curvatures λ_i satisfy*

$$-\frac{b}{k^2} \leq \lambda_i \leq B\,k \tag{6.5}$$

for all $1 \leq i \leq n$ and some positive constants b, B depending on C. Hence, if we endow D_k with the induced Riemannian metric g_k, we get the curvature estimates

$$-\frac{\tilde{b}}{k} \leq K^{\mathbb{C}}_{g_k} \qquad \text{and} \qquad |\mathrm{R}_{g_k}| \leq \tilde{B}k^2 \qquad \text{on} \qquad D_k. \tag{6.6}$$

Proof. In a first important step we will construct a closed $C^{1,1}$ hypersurface D_k so that (6.5) holds for its principal curvatures in the support sense. Define $\phi_k = \frac{1}{k}\,\phi(k^2 f)$, where as before $f = d^2(\,\cdot\,,C)$ and $\phi : [0,1] \to [0,1]$ is a smooth function satisfying

(a) $\phi \equiv 0$ on $[0,1/4]$ and $\phi(1) = 1$;
(b) on $(1/4,1)$: ϕ', ϕ'' are positive and $\phi'' \leq \alpha(\phi')^3$ for some finite $\alpha > 0$;
(c) ϕ^{-1} has all left derivatives vanishing at 1.

Notice that (c) implies that ϕ' and ϕ'' tend to infinity at 1. Hereafter, ϕ, ϕ' and ϕ'' will always be evaluated at $k^2 f$ (without saying it explicitly).

Consider the tubular neighborhood U from Corollary 6.2, and take

$$G_k = \{(p, \phi_k(p)) \,:\, p \in B_{1/k}(C) \cap U\}$$

which is a hypersurface of the cylinder $B_1(C) \times [0,1/k]$. Observe that by (a) G_k can be written as the union of $\big(B_{1/(2k)}(C) \cap U\big) \times \{0\}$ (which is totally geodesic in the cylinder) and the graphical annulus

$$A_k = \big\{(p, \phi_k(p)) \,:\, p \in U \ \text{and} \ \tfrac{1}{2k} \leq d(p,C) \leq \tfrac{1}{k}\big\} \tag{6.7}$$

whose second fundamental form h is given by

$$h = \frac{\nabla^2\phi_k}{\sqrt{1+|\nabla\phi_k|^2}}, \quad \text{where} \quad \begin{array}{l} \nabla\phi_k = k\,\phi'\nabla f = 2k\phi' d\nabla d \\ \nabla^2\phi_k = k^3\phi''\nabla f\otimes\nabla f + k\,\phi'\nabla^2 f \end{array}$$

We need to estimate the principal curvatures of A_k to prove (6.5). With such a goal, take $e_1 = \frac{\nabla d}{\sqrt{1+\langle\nabla d,\nabla\phi_k\rangle^2}}$ and complete to form a basis $\{e_i\}$ orthonormal with respect to the metric induced on the graph $\tilde g = g + \nabla\phi_k\otimes\nabla\phi_k$, and which diagonalizes h. Notice that

$$\frac{k^3\phi''\,\langle\nabla f, e_1\rangle^2}{\sqrt{1+(2\,k\,\phi'\,d)^2}} \le \frac{k^2}{2d}\frac{\phi''}{\phi'}\frac{\langle\nabla f,\nabla d\rangle^2}{1+\langle\nabla d,\nabla\phi_k\rangle^2} \le \frac{k^2}{2d}\frac{\phi''}{(\phi')^3}\frac{1}{k^2} \le \frac{\alpha}{2d}$$

and

$$\Lambda := \frac{k\,\phi'\nabla^2 f(e_i,e_i)}{\sqrt{1+(2\,k\,\phi'\,d)^2}} \le \frac{1}{2d}\nabla^2 f(e_i,e_i) \le \frac{1}{d},$$

which comes from (6.4). Therefore, from (6.7) we obtain $\lambda_i \le \alpha\,k + 2k =: Bk$.

On the other hand, using $\phi'' \ge 0$, (6.4) and (6.7), we have

$$\lambda_i = h(e_i,e_i) \ge \Lambda \ge -\lambda\,f^{3/2}\frac{1}{2d} = -\frac{\lambda}{2}d^2 \ge -\frac{\lambda}{2k^2}.$$

All the previous computations are true at almost every point of A_k, since Corollary 6.2 ensures that f is $C^{1,1}$ on U and thus twice differentiable almost everywhere. At the remaining points all the above estimates are still valid in the support sense (just redo the proof substituting f by its support functions).

Clearly the hypersurfaces $D_k = D(G_k)$ converge in the Gromov-Hausdorff sense to the double $D(C)$ of the convex set C. Employing Lemma 5.6, after increasing b and B slightly, we can find a smooth hypersurface $\widetilde{D}_k$ which is C^1 close to D_k such that the estimate (6.5) remains valid. Clearly we can assume that $\widetilde{D}_k$ still converges to $D(C)$. Finally, rename $D_k = \widetilde{D}_k$, and notice that (6.6) now follows from (6.5) and the Gauß equations. ☐

Observe that each (D_k, g_k) constructed above is not anymore nonnegatively curved, but we have a precise control of its curvature given by (6.6). Using the short time existence theory from [24], we have the following immediate

Corollary 6.4. *There exists $T_k > 0$ such that $(D_k, g_k(t))$ is a sequence of solutions to the Ricci flow for $t \in [0, T_k)$ starting at the smooth closed manifolds (D_k, g_k) from Proposition 6.3.*

6.3. **Curvature estimates for the Ricci flow of our initial sequence of smoothings.** We consider a fixed convex exhaustion $C_\ell = b^{-1}((-\infty,\ell])$ as in subsection 3.2. For each C_ℓ we apply Proposition 6.3 with $C = C_\ell$, let $(D_{\ell,k}, g_{\ell,k}(t))$ denote the Ricci flow from Corollary 6.4 and put $g_{\ell,k} = g_{\ell,k}(0)$. Moreover, when a constant B depends on C_ℓ we will write B_ℓ to denote $B(C_\ell)$.

Our next concern is to extend the curvature estimates in (6.6) at least for a short interval of time, where the important point is that the length of such an interval is independent of k. It is somewhat surprising that we can only prove this if ℓ is large enough. Ultimately this in turn is due to the following

Lemma 6.5. *Let $(D_{\ell,k}, g_{\ell,k})$ be the closed smooth manifolds constructed in Proposition 6.3, and take $p \in D_{\ell,k}$. Then we can find $r = r(\ell) \in (0,1]$ (possibly converging to 0 with $\ell \to \infty$) and $\eta_\ell \to 0$ independent of k such that*

$$\frac{\mathrm{vol}_{g_{\ell,k}}\big(B_{g_{\ell,k}}(p,r)\big)}{r^n} \geq (1-\eta_\ell)\,\omega_n \qquad \textit{for all } k. \tag{6.8}$$

Proof. As the manifolds $(D_{\ell,k}, g_{\ell,k})$ converge to the double $D(C_\ell)$ of C_ℓ in the Gromov-Hausdorff sense, the continuity of volumes (see e.g. [9, Theorem 5.9] by Cheeger and Colding) gives $\lim_{k\to\infty} \mathrm{vol}_{g_{\ell,k}}\big(B_{g_{\ell,k}}(p_k,r)\big) = \mathcal{H}^n_{D(C_\ell)}(B(p_\infty,r))$. Thus it suffices to prove that small balls in $D(C_\ell)$ have nearly Euclidean volume provided that ℓ is large.

This essentially follows from Lemma B.2 by Guijarro and Kapovitch which ensures that, for $p \in \partial C_\ell$ with large ℓ, T_pC_ℓ is close to a half-space, and so $\mathrm{vol}\{v \in T_pC_\ell : |v| < r\} = \frac{1}{2}(\omega_n - \varepsilon_\ell)r^n$ with $\varepsilon_\ell \to 0$ as $\ell \to \infty$. As C_ℓ is a convex set in a Riemannian manifold, we can find for each $p \in D(C_\ell)$ a number $r(p)$ small enough so that the volume of a geodesic ball $B(p,r)$ in $D(C_\ell)$ is $\geq (\omega_n - 2\varepsilon_\ell) r(p)^n$.

To remove the dependence on p, we choose a finite subcover $\bigcup_{i=1}^k B(p_i, \varepsilon_\ell\, r_i)$ of $\bigcup_{p\in D(C_\ell)} B(p, \varepsilon_\ell\, r(p))$, where $r_i = r(p_i)$ and we take $r_0 = \min_i r_i$. Then any $q \in D(C_\ell)$ is contained in $B(p_i, \varepsilon_\ell\, r_i)$ for some i. Notice that $B(p_i, r_i) \subset B(q, (1+\varepsilon_\ell)r_i)$ and thus $\mathrm{vol}\,(B(q,(1+\varepsilon_\ell)r_i)) \geq (\omega_n - 2\varepsilon_\ell)r_i^n$. Finally, apply volume comparison to get $\frac{\mathrm{vol}(B(q,r_0))}{r_0^n} \geq \frac{\omega_n - 2\varepsilon_\ell}{(1+\varepsilon_\ell)^n}$. □

Proposition 6.6. *There exists some $\ell_0 > 0$ and for each $\ell \geq \ell_0$ exists a time $T_\ell > 0$ (independent of k) such that for the Ricci flow $(D_{\ell,k}, g_{\ell,k}(t))$ constructed in Corollary 6.4 we have*

$$K^{\mathbb{C}}_{g_{\ell,k}(t)} \geq -\frac{1}{\sqrt{k}} \qquad \textit{and} \qquad |\mathrm{R}|_{g_{\ell,k}(t)} \leq \frac{1}{t} \tag{6.9}$$

for all $t \in (0, T_\ell]$ and all sufficiently large k.

Proof. Unless otherwise stated, all the curvature quantities hereafter correspond to $g_{\ell,k}(t)$. We consider a maximal solution $(D_{\ell,k}, g_{\ell,k}(t))$ of the Ricci flow with $t \in [0, T_{\ell,k})$. By (6.6) there is some constant B_ℓ such that

$$K^{\mathbb{C}}(0) \geq -\tfrac{B_\ell}{k}, \qquad \text{and} \qquad |\mathrm{R}(0)| \leq B_\ell\, k^2. \tag{6.10}$$

Henceforth we will restrict our attention to $k \geq 4B_\ell^2$.

We define $t_{\ell,k}$ as the minimal time for which we can find some complex plane σ in $T^{\mathbb{C}}D_{\ell,k}$ with $K^{\mathbb{C}}(t_{\ell,k})(\sigma) = -\frac{1}{\sqrt{k}}$. If such a time does not exist, we put

$t_{\ell,k} = T_{\ell,k}$. In particular, we have for the usual sectional curvature

$$K(t) \geq -\tfrac{1}{\sqrt{k}} \qquad \text{for all} \qquad t \in [0, t_{\ell,k}]. \tag{6.11}$$

We put

$$u(t) := 4n(|\mathrm{R}(t)| + 1).$$

Using the initial estimate (6.10) it is not hard to obtain a doubling estimate for $u(t)$. In fact, the application of [14, Lemma 6.1] gives

$$u(t) \leq L_\ell\, k^2 \qquad \text{for all} \qquad t \in \big[0, \tfrac{1}{L_\ell k^2}\big] \tag{6.12}$$

for some positive constant L_ℓ.

By Lemma 6.5, for any $\delta > 0$ we can find an ℓ_0 such that for each $\ell \geq \ell_0$ there is $r = r(\ell)$ with $\mathrm{vol}_{g(0)}(B_{g(0)}(p, r)) \geq (1 - \delta)\, r^n \omega_n$. Combining this with (6.11) and Proposition 5.5 we deduce that, for $\ell \geq \ell_0$, there is some $\bar{t}_\ell$ and $k_0 = k_0(\ell)$ such that

$$u(t) \leq \frac{1}{10t} \quad \text{for all} \quad t \in [0, t_{\ell,k}] \cap [0, \bar{t}_\ell] \quad \text{and all} \quad k \geq k_0. \tag{6.13}$$

Thus the inequalities (6.9) hold for $t \in [0, t_{\ell,k}] \cap [0, \bar{t}_\ell]$ and it suffices to check that $t_{\ell,k}$ is bounded away from 0 for $k \to \infty$. In particular, it is enough to consider hereafter $k \geq k_0$ with $t_{\ell,k} < \min\{T_{\ell,k}, \bar{t}_\ell\}$.

In order to get a lower bound on $t_{\ell,k}$ we have to estimate $K^{\mathbb{C}}(t)$ from below. Consider the algebraic curvature operator $\widetilde{\mathrm{R}} := \mathrm{R} + \lambda(t) I$, where $I_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ represents the curvature operator of the standard unit sphere, and $\lambda(t) \geq 0$. Under the Ricci flow, $\widetilde{\mathrm{R}}$ evolves according to

$$\left(\frac{\partial}{\partial t} - \Delta\right) \widetilde{\mathrm{R}} = \lambda'(t) I + 2(\mathrm{R}^2 + \mathrm{R}^\sharp).$$

Next, recall the formula (cf. [2, Lemma 2.1])

$$(\mathrm{R} + \lambda I)^2 + (\mathrm{R} + \lambda I)^\sharp = \mathrm{R}^2 + \mathrm{R}^\sharp + 2\lambda \mathrm{Ric} \wedge \mathrm{id} + \lambda^2 (n-1)\, I.$$

It is easy to see that

$$2\mathrm{Ric} \wedge \mathrm{id} + \lambda(n-1)\, I \leq \tfrac{u(t)}{2} I$$

holds provided that $\lambda \leq 1$. We reach

$$\left(\frac{\partial}{\partial t} - \Delta\right) \widetilde{\mathrm{R}} \geq 2\big(\widetilde{\mathrm{R}}^2 + \widetilde{\mathrm{R}}^\sharp\big) + \big[\lambda'(t) - \lambda(t) u(t)\big] I.$$

As the ODE $\mathrm{R}' = \mathrm{R}^2 + \mathrm{R}^\#$ preserves $K^{\mathbb{C}} \geq 0$, we can use the maximum principle to ensure that: If we define $\lambda(t)$ as the solution of the initial value problem

$$\begin{aligned} \lambda'(t) &= \lambda(t) u(t) \\ \lambda(0) &= \tfrac{B_\ell}{k} \quad \text{(from (6.10))} \end{aligned},$$

it follows that $\widetilde{\mathrm{R}}(t)$ has nonnegative complex sectional curvature for $t \in [0, t_{\ell,k}]$. Hence

$$K^{\mathbb{C}}(t) \geq -\tfrac{B_\ell}{k} e^{\int_0^t u(\tau) d\tau} \qquad \text{for all } t \in [0, t_{\ell,k}].$$

Combining this with (6.12) and (6.13) we deduce

$$\begin{aligned} K^{\mathbb{C}}(t) &\geq -e\,\tfrac{B_\ell}{k} && \text{for } t \in \big[0, \tfrac{1}{k^2 L_\ell}\big] \cap [0, t_{\ell,k}], \quad \text{and} \\ K^{\mathbb{C}}(t) &\geq -e\,\tfrac{B_\ell}{k}\left(L_\ell k^2 t\right)^{\frac{1}{10}} && \text{for } t \in \big[\tfrac{1}{k^2 L_\ell}, t_{\ell,k}\big]. \end{aligned}$$

Using that by construction the minimum of $K^{\mathbb{C}}(t_{\ell,k})$ is given by $-\frac{1}{\sqrt{k}}$, we obtain the desired uniform lower bound on $t_{\ell,k}$.

□

6.4. **Reduction to the positively curved case.** We have the following improvement of Proposition 4.1.

Proposition 6.7. *Let (M^n, g) be an open manifold with $K_g^{\mathbb{C}} \geq 0$ whose soul is a point p_0. Then there is a sequence of closed manifolds (M_i, g_i, p_0) with $K_{g_i}^{\mathbb{C}} > 0$ converging in the Cheeger-Gromov sense to (M, g, p_0).*

Proof. Consider the sets $C_\ell = \{b \leq \ell\}$ from subsection 3.2. Summing up, from Proposition 6.3 and Corollary 6.4 we have a sequence $(D_{\ell,k}, g_{\ell,k}(t))$ of Ricci flows satisfying (6.9) on $(0, T_\ell]$ for all $\ell \geq \ell_0$. Using Petrunin's result (Theorem 4.2) similar to the proof of Proposition 5.5, we see that the volume estimate (6.8) remains valid for $(D_{\ell,k}, g_{\ell,k}(t))$ provided we double the constant η_ℓ and we assume $t \in [0, t_\ell]$. This, combined with (6.9), allows to apply Theorem C.3 by Cheeger, Gromov and Taylor to reach a uniform lower bound for the injectivity radius $\mathrm{inj}_{g_{\ell,k}(\bar{t})} \geq c(\ell, \bar{t})$, for any $\bar{t} \in (0, t_\ell]$.

Then, we can apply Hamilton's compactness (Theorem C.2) to get a compact limiting Ricci flow $(D_{\ell,\infty}, g_{\ell,\infty}(t))$ on $(0, T_\ell]$ with $K^{\mathbb{C}}_{g_{\ell,\infty}(t)} \geq 0$. Arguing e.g. as in [43, Theorem 9.2], we deduce that $(D_{\ell,\infty}, d_{g_{\ell,\infty}(t)})$ converges (in the Gromov-Hausdorff sense as $t \to 0$) to $(D(C_\ell), d_{g_\ell})$; in particular, $D_{\ell,\infty}$ is homeomorphic to the sphere $D(C_\ell)$. By Proposition 3.3 $K^{\mathbb{C}}_{g_{\ell,\infty}(t)} > 0$ for all $t \in (0, T_\ell]$.

On the other hand, for any $\varepsilon > 0$ we can view $C_{\ell-\varepsilon}$ as a subset of $D_{\ell,k}$ for all $k \in \mathbb{N} \cup \infty$. Combining Theorem C.5 and a generalization of Shi's estimate (see Theorem C.4) with (6.9) we see that on $C_{\ell-\varepsilon}$ the metric $g_{\ell,\infty}(t)$ converges for $t \to 0$ in the C^∞ topology to g. We choose t_ℓ so close to zero that $(C_{\ell-\varepsilon}, g_{\ell,\infty}(t_\ell))$ converges in the C^∞ topology to (M, g, p_0) as $\ell \to \infty$. □

If the soul of the manifold of Theorem 1 is a point, one can now deduce the conclusion of Theorem 1 completely analogously to Section 4 using Proposition 6.7 in place of Proposition 4.1.

Proof of Theorem 1. If M is not simply connected, we consider its universal cover $\widetilde{M}$. The goal is to construct a Ricci flow $(\widetilde{M}, g(t))$ on $\widetilde{M}$ for which each $g(t)$ is invariant under $\mathrm{Iso}(\widetilde{M}, g(0))$. By Theorem 2, $\widetilde{M}$ splits isometrically as $\Sigma^k \times F$, where Σ is closed and F is diffeomorphic to $\mathbb{R}^{n-k}$ with $K^{\mathbb{C}}_{g_F} \geq 0$. By [10, Corollary 6.2] of Cheeger and Gromoll F splits isometrically as $F = \mathbb{R}^q \times F'$

where $\mathbb{R}^q$ is flat and F' has a compact isometry group. Clearly there is a Ricci flow $(\mathbb{R}^q \times \Sigma, g(t))$ which is invariant under $\mathrm{Iso}(\mathbb{R}^q \times \Sigma)$ and thus it suffices to find a Ricci flow $(F', g(t))$ which is invariant under $\mathrm{Iso}(F')$. Using [10, Corollary 6.3] by Cheeger and Gromoll, we can find $o \in F'$ which is a fixed point of $\mathrm{Iso}(F')$. We now define the Busemann function on F' with respect to this base point. Then all sublevel sets C_ℓ, the doubles $D(C_\ell)$ and the smoothings of the double $D_{\ell,k}$ come with a natural isometric action of $\mathrm{Iso}(F')$.

Since the Ricci flow on compact Riemannian manifolds is unique, the Ricci flow $(D_{\ell,k}, g_{\ell,k}(t))$ is invariant under $\mathrm{Iso}(F')$; hence the same holds for the limit $(D_{\ell,\infty}, g_{\ell,\infty}(t))$, and finally for the limiting Ricci flow on F'.

In summary, there is a Ricci flow $(\widetilde{M}, g(t))$ with $K^{\mathbb{C}}_{g(t)} \geq 0$ which is invariant under $\mathrm{Iso}(\widetilde{M}, g(0))$ and so descends to a solution on M. □

7. Applications

7.1. **Proof of Corollary 3.** Arguing as before, it is enough to consider the case where the soul is a point. Redoing the arguments from the proof of Proposition 4.3 and using (1.2), we deduce that our Ricci flow exists until time $\mathcal{T} = \frac{v_0}{2C_n}$. Plugging this and (1.2) into a reasoning like in Corollary 4.4, we reach

$$\frac{\mathrm{vol}_{g(t)}\left(B_{g(t)}(p,r)\right)}{r^n} \geq \frac{v_0}{2} > 0 \qquad \text{for } r \in (0,1], \quad p \in M, \quad t \in [0, \mathcal{T}]. \tag{7.1}$$

Now assume that the claim about bounded curvature does not hold, i.e., there exists a sequence of Ricci flows $(M_i, g_i(t))$ constructed as in Theorem 1 (in particular, $K^{\mathbb{C}}_{g_i(t)} \geq 0$ and $(M_i, g_i(t))$ satisfies a trace Harnack inequality) and points $(p_i, t_i) \in M_i \times (0, \mathcal{T})$ with $\mathrm{scal}_{g_i(t_i)} > 4^i/t_i$. By means of a point picking argument as in the proof of Proposition 4.6 on the relatively compact set $B_{g_i(t_i)}(p_i, 1)$, we get a sequence of points $\{\bar{p}_i\}_{i \geq i_0}$ such that, after parabolic rescaling of the metric with factor $Q_i = \mathrm{scal}_{g_i(t_i)}(\bar{p}_i)$, we get for the rescaled metric $\tilde{g}_i(s)$

$$|\mathrm{R}|_{\tilde{g}_i(s)} \leq 8 \qquad \text{on} \qquad B_{\tilde{g}_i(0)}(\bar{p}_i, i) \qquad \text{for } s \in [-i, 0].$$

By the scaling invariance of (7.1), the corresponding estimate holds with $B_{\tilde{g}_i(0)}(\bar{p}_i, r)$ for any $0 < r \leq \sqrt{Q_i}$. The rest of the proof goes exactly as the remaining steps in the proof of Proposition 4.6.

7.2. **Estimates for the extinction time.** We first need a scale invariant version of Petrunin's estimate (Theorem 4.2).

Lemma 7.1. *Let (M^n, g) be an open manifold with $K_g \geq 0$. Then for any $p \in M$ and $r > 0$, there exists a constant $C_n > 0$ such that*

$$\int_{B_g(p,r)} \mathrm{scal}_g \, d\mu_g \leq C_n \, r^{n-2}.$$

Proof. For any $r > 0$, consider the rescaled metric $\tilde{g} = \frac{1}{r^2} g$. Since $K_{\tilde{g}} = \frac{1}{r^2} K_g \geq 0$, we are in position to apply Theorem 4.2 to $(M, \tilde{g})$ which gives

$$C_n \geq \int_{B_{\tilde{g}}(p,1)} \operatorname{scal}_{\tilde{g}} \, d\mu_{\tilde{g}} = \int_{B_g(p,r)} r^{2-n} \operatorname{scal}_g \, d\mu_g,$$

where for the last equality we have used the identities $d\mu_{\tilde{g}} = r^{-n} \, d\mu_g$, $\operatorname{scal}_{\tilde{g}} = r^2 \operatorname{scal}_g$ and $B_{\tilde{g}}(p,1) = B_g(p,r)$. □

Lemma 7.2. *Suppose* $(M^n, g(t))_{t \in [0,T)}$ *is a maximal solution of the Ricci flow with* $K^{\mathbb{C}}_{g(t)} \geq 0$*. If* $T < \infty$*, then*

$$\limsup_{t \to T} \sup \Big\{ \tfrac{\operatorname{vol}_{g(t)}(B_{g(t)}(p,r))}{r^{n-2}} \mid p \in M, r > 0 \Big\} = 0.$$

Proof. We assume on the contrary that we can find $v_0 > 0$, $x_j \in M$, $t_j \to T$ and $r_j > 0$ satisfying $\operatorname{vol}_{g(t_j)}(B_{g(t_j)}(x_j, r_j)) \geq v_0 r_j^{n-2}$. We fix some $(\bar{x}, \bar{t}, \bar{r}) = (x_{j_0}, t_{j_0}, r_{j_0})$ with $(T - \bar{t}) \leq \frac{v_0}{2C_n}$, where C_n is the constant in Lemma 7.1.

Now we can use Petrunin's result as in Lemma 7.1 in order to estimate

$$\begin{aligned} \operatorname{vol}_{g(t)}(B_{g(t)}(\bar{p}, \bar{r})) &\geq \operatorname{vol}_{g(t)}(B_{g(\bar{t})}(\bar{p}, \bar{r})) \\ &\geq (v_0 - C_n(t - \bar{t}))\bar{r}^{n-2} \geq \tfrac{v_0}{2} \bar{r}^{n-2} \qquad \text{for } t \in [\bar{t}, \mathcal{T}). \end{aligned}$$

This in turn allows us to prove, similarly to Proposition 4.6, that for each D there is a C_D with

$$|\mathrm{R}_{g(t)}| \leq C_D \qquad \text{on } B_{g(t)}(\bar{p}, D), \qquad t \in \big[\bar{t}, \mathcal{T}\big).$$

As in the proof of Theorem 4.7 we get also bounds on the derivatives of $\mathrm{R}_{g(t)}$. This in turn shows that $g(t)$ converges smoothly to a smooth limit metric $g(\mathcal{T})$. By Lemma 4.9 $g(\mathcal{T})$ is complete and thus we can extend the Ricci flow by applying Theorem 1 to $(M, g(\mathcal{T}))$ – a contradiction. □

Proof of Corollary 5. Consider a maximal Ricci flow $(M, g(t))_{t \in [0,T)}$ with $K^{\mathbb{C}} \geq 0$ and suppose on the contrary that

$$T < \tfrac{1}{C_n} \sup \Big\{ \tfrac{\operatorname{vol}_{g(0)}\left(B_{g(0)}(p,r)\right)}{r^{n-2}} \mid p \in M, r > 0 \Big\},$$

where C_n is the constant from Lemma 7.1.

By assumption we can choose $r > 0$ and $p \in M$ with

$$\operatorname{vol}_{g(0)}\big(B_{g(0)}(p,r)\big) > C_n T r^{n-2}.$$

Using Petrunin's estimate (as restated in Lemma 7.1) we deduce

$$\operatorname{vol}_{g(t)}\big(B_{g(t)}(p,r)\big) \geq \operatorname{vol}_{g(t)}\big(B_{g(0)}(p,r)\big) > C_n(T - t)\, r^{n-2}.$$

Combining with Lemma 7.2 this gives a contradiction.

□

Proof of Corollary 6. Any open nonnegatively curved manifold with a two dimensional soul Σ is locally isometric to $\Sigma \times \mathbb{R}$ and the Ricci flow exists exactly until $\frac{\mathrm{area}(\Sigma)}{4\pi\chi(\Sigma)} \in (0,\infty]$. If $\dim(\Sigma)=1$, then the universal cover splits off a line and Corollary 5 ensures the existence of an immortal solution. So it only remains to consider the case that the soul is a point.

If $T<\infty$, using Corollary 5 we know that

$$\limsup_{r\to\infty} \tfrac{\mathrm{vol}_g(B_g(p,r))}{r} = L < \infty.$$

By Lemma A.3 there is a sequence $\eta_i \searrow 1$ such that for a base point $o\in M$ the following holds

$$B_g(o,i)\subset C_i \subset B_g(o,i\eta_i) \qquad \text{for all} \quad i\geq 1,$$

where C_i is the sublevel set $b^{-1}((-\infty,i])$ of the Busemann function at o (see subsection 3.2). In particular, $\frac{\mathrm{vol}_g(C_i)}{\mathrm{vol}_g(B_g(o,i))}\to 1$.

Clearly $\mathrm{vol}(C_i)=\mathrm{vol}(C_0)+\int_0^i \mathrm{area}(\partial C_t)\,dt$. Moreover, by work of Sharafutdinov (see e.g. [49, Theorem 2.3]) there is an 1-Lipschitz map $\partial C_b\to\partial C_a$ for $a\leq b$. Accordingly, the area of ∂C_i is monotonously increasing and thus

$$0<\lim_{r\to\infty}\mathrm{area}(\partial C_r)=\lim_{r\to\infty}\tfrac{\mathrm{vol}_g(B_g(p,r))}{r}=L.$$

This yields that $D:=\lim_{r\to\infty}\mathrm{diam}(\partial C_r)<\infty$. In fact, suppose for a moment $D=\infty$. Choose $a>0$ so large that $\mathrm{area}(\partial C_a)\geq \frac{3}{4}L$. Since $\mathrm{diam}(\partial C_r)$ tends to infinity while the area converges to L, we can find for each $\varepsilon>0$ an r and a circle of length $\leq\varepsilon$ in ∂C_r which subdivides ∂C_r into two regions of equal area. If we consider the image of this circle under the 1-Lipschitz map $\partial C_r\to\partial C_a$ (for $r\geq a$), we get a closed curve of length $\leq\varepsilon$ which subdivides ∂C_a in two regions such that each of them has area at least $L/4$. Since ε was arbitrary this gives a contradiction.

As it is the boundary of a totally convex set, ∂C_r is a nonnegatively curved Alexandrov space (cf. Buyalo [7]). Combining compactness and Sharafutdinov retraction, ∂C_r converges for $r\to\infty$ to a nonnegatively curved Alexandrov space S. Moreover, for any sequence $p_i\in M$ converging to infinity we have $\lim_{GH,i\to\infty}(M,g,p_i)\to S\times\mathbb{R}$. Thus M is asymptotically cylindrical.

In particular, M is volume non-collapsed, and from Corollary 3 we deduce that $(M,g(t))$ has bounded curvature $\leq\frac{C}{t}$ for positive times. It is now easy to extract from the sequence $(M,g(t),p_i)$ a subsequence converging to $(N,g_\infty(t))$. Topologically the nonnegatively curved manifold N is homeomorphic to $S\times\mathbb{R}$ – a manifold with two ends. Thus $(N,g_\infty(t))$ splits isometrically as $(\mathbb{S}^2,\bar g(t))\times\mathbb{R}$.

From Lemma 7.2 one can deduce that $\lim_{t\to T}\mathrm{vol}_{\bar g(t)}(\mathbb{S}^2)=0$. By Gauß Bonnet $\lim_{t\to 0}\mathrm{vol}_{\bar g(t)}(\mathbb{S}^2)=8\pi T=L$.

□

Remark 7.3. Let (M, g) be an open manifold with $K_g^{\mathbb{C}} \geq 0$ and Euclidean volume growth. By Corollary 3 the curvature of our Ricci flow $g(t)$ starting on (M, g) is bounded for positive times. Following the work of Schulze and Simon [40], with Hamilton's Harnack inequality replaced by [4], one can show that there is a sequence of positive numbers $c_i \to 0$ such that $\lim_{i\to\infty}(M, c_i g(t/c_i)) = (M, g_\infty(t))$ is a Ricci flow ($t \in (0,\infty)$) whose 'initial metric' (Gromov Hausdorff limit of $(M, d_{g_\infty(t)})$ for $t \to 0$) is the cone at infinity of (M, g). Moreover, $(M, g_\infty(t))$ is an expanding gradient Ricci soliton.

7.3. **Long time behaviour of the Ricci flow.** We will only consider solutions which satisfy the trace Harnack inequality. Notice that this is automatic if we consider a solution coming out of the proof of Theorem 1.

Lemma 7.4. *Let $(M^n, g(t))$ be a non flat immortal solution of the Ricci flow with $K^{\mathbb{C}} \geq 0$ satisfying the trace Harnack inequality. If $(M, g(0))$ does not have Euclidean volume growth, then for $p_0 \in M$*

$$\limsup_{t\to\infty} \frac{\mathrm{vol}_{g(t)}(B_{g(t)}(p_0, \sqrt{t}))}{\sqrt{t}^n} = 0.$$

Proof. Suppose on the contrary that we can find $t_k \to \infty$ and $\varepsilon > 0$ with $\mathrm{vol}_{g(t_k)}(B_{g(t_k)}(p_0, \sqrt{t_k})) \geq \varepsilon\sqrt{t_k}^n$. Analogous to Proposition 4.6 one can show that there is some universal $\mathcal{T} > 0$ such that for the rescaled flow $\tilde{g}_k(t) = \frac{1}{t_k} g(t_k + t\cdot t_k)_{t\in[-1,\infty)}$ we have that $\mathrm{scal}_{\tilde{g}_k(t)} \leq \frac{C}{t}$ on $B_{\tilde{g}_k(t)}(p_0, 1)$ for $t \in (0, \mathcal{T}]$ where C is independent of k. Using this for $t = \mathcal{T}$ and combining with the Harnack inequality, we find a universal constant C_2 with $\mathrm{scal}_{\tilde{g}_k(0)} \leq C_2$ on $B_{\tilde{g}_k(0)}(p_0, 1)$.

Thus we obtain that $\mathrm{scal}_{g(t_k)} \leq \frac{C_2}{t_k}$ on $B_{g(t_k)}(p_0, \sqrt{t_k})$. Combining with the Harnack inequality and using $B_{g(0)} \subset B_{g(t_k)}$ we deduce that $\mathrm{scal}_{g(t)} \leq \frac{C_2}{t}$ on M. By Hamilton [27, Editor's note 24] this implies that $B_{g(t)}(p_0, \sqrt{t}) \subset B_{g(0)}(p_0, C_3\sqrt{t})$ for a constant $C_3 = C_3(C_2, n)$. Hence

$$\varepsilon\sqrt{t_k}^n \leq \mathrm{vol}_{g(t_k)}\Big(B_{g(0)}(p_0, C_3\sqrt{t_k})\Big) \leq \mathrm{vol}_{g(0)}\Big(B_{g(0)}(p_0, C_3\sqrt{t_k})\Big),$$

which means that $g(0)$ has Euclidean volume growth – a contradiction. $\square$

Theorem 7.5. *Let $(M^n, g(t))$ be a non flat immortal Ricci flow with $K^{\mathbb{C}} \geq 0$ satisfying the trace Harnack inequality. If $(M, g(0))$ does not have Euclidean volume growth, then for $p_0 \in M$ there is a sequence of times $t_k \to \infty$ and a rescaling sequence Q_k such that for $\tilde{g}_k(t) = Q_k g(t_k + \frac{t}{Q_k})$ the following holds. The rescaled flow $(M, \tilde{g}_k(t), p_0)$ converges in the Cheeger-Gromov sense to a steady soliton $(M_\infty, \tilde{g}_\infty(t))$ which is not isometric to $\mathbb{R}^n$.*

Proof. For $t \in [0,\infty)$ we define $Q(t) > 0$ as the minimal number for which

$$\mathrm{vol}_{g(t)}\Big(B_{g(t)}\big(p_0, \tfrac{1}{\sqrt{Q(t)}}\big)\Big) = \tfrac{1}{2}\omega_n \frac{1}{\sqrt{Q(t)}^n}$$

We can choose $\varepsilon_k \to 0$ and $t_k \to \infty$ with $\frac{\partial}{\partial t}_{|t=t_k} \operatorname{scal}_{g(t)}(p_0) \le \varepsilon_k \operatorname{scal}_{g(t_k)}(p_0)^2$. In fact otherwise it would be easy to deduce that a finite time singularity occurs.

By Lemma 7.4 the rescaled Ricci flow $\tilde g_k(t) = Q_k\, g(t_k + \frac{t}{Q_k})$ with $Q_k = Q(t_k)$ is defined on an interval $[-T_k, \infty)$ with $T_k \to \infty$. Moreover, $\operatorname{vol}_{\tilde g_k(0)}\big(B_{\tilde g_k(0)}(p_0,1)\big) = \frac{\omega_n}{2}$ and $\frac{\partial}{\partial t}_{|t=0} \operatorname{scal}_{\tilde g_k(t)}(p_0) \le \varepsilon_k \operatorname{scal}_{\tilde g_k(0)}(p_0)^2$.

Arguing as in the proof of Lemma 7.4 one can show that there is some $\mathcal{T} > 0$ such that for each r there is a constant C_r for which $\operatorname{scal}_{\tilde g_k(\mathcal{T})} \le C_r$ on $B_{\tilde g_k(\mathcal{T})}(p_0, r)$. Using the Harnack inequality after possibly increasing C_r we may assume that $\operatorname{scal}_{\tilde g_k(t)} \le C_r$ on $B_{\tilde g_k(\mathcal{T})}(p_0, r)$ for all $t \in [-T_k/2, \mathcal{T}]$. Shi's estimate also give bounds on the derivative of the curvature tensor on $B_{\tilde g_k(\mathcal{T})}(p_0, r) \times [-T_k/4, \mathcal{T}/2]$.

After passing to a subsequence we may assume $(M, \tilde g_k(\mathcal{T}/2), p_0)$ converges in the Cheeger-Gromov sense to $(M_\infty, \tilde g_\infty(\mathcal{T}/2), p_\infty)$. By the Arcelà-Ascoli theorem we also may assume that under the same set of local diffeomorphisms the pull backs of $\tilde g_k(t)$ converge to $\tilde g_\infty(t)$, $t \in (-\infty, \mathcal{T}/2]$. Clearly $\tilde g_\infty(t)$ is a solution of the Ricci flow with $K^{\mathbb{C}}_{\tilde g_\infty(t)} \ge 0$. The completeness of $\tilde g_\infty(t)$ follows from the completeness of $\tilde g_\infty(\mathcal{T}/2)$, for $t < \mathcal{T}/2$. Moreover, we have that $\frac{\partial \operatorname{scal}_{\tilde g_\infty(0)}(p_0)}{\partial t} = 0$. If $\tilde g_\infty(0)$ is not flat, we can pass to the universal cover of M_∞ and after spitting off an Euclidean factor we may assume that the Ricci curvature is positive. Recall that for ancient solutions with $K^{\mathbb{C}}_{\tilde g_\infty(t)} \ge 0$ and positive Ricci curvature the Harnack inequality implies that

$$0 \le \frac{\partial \operatorname{scal}_{\tilde g_\infty(t)}}{\partial t} - \frac{1}{2} \operatorname{Ric}^{-1}_{\tilde g_\infty(t)}(\nabla \operatorname{scal}_{\tilde g_\infty(t)}, \nabla \operatorname{scal}_{\tilde g_\infty(t)}),$$

where $\operatorname{Ric}^{-1}_{\tilde g_\infty(t)}$ is the (positive definite) symmetric $(2,0)$-tensor defined by the equation $\operatorname{Ric}^{-1}_{\tilde g_\infty(t)}\big(v, \operatorname{Ric}^{\tilde g_\infty(t)} w\big) = \tilde g_\infty(t)(v,w)$. Since equality holds for one point in space-time one can deduce from a strong maximum principle, that $\tilde g_\infty(t)$ is a steady Ricci soliton. In fact, this only requires minor modification in the proof of a result by Brendle [4, Proposition 14]. We leave the details to the reader. Finally $\operatorname{vol}_{\tilde g_\infty(0)}\big(B_{\tilde g_\infty(0)}(p_0,1)\big) = \frac{1}{2}\omega_n$ and thus the limit is not the Euclidean space. ∎

7.4. **Further Consequences.**

Corollary 7.6. *Let (M,g) be an open manifold with $K^{\mathbb{C}}_g \ge 0$ and $M \cong \mathbb{R}^n$, then there is a sequence g^i of complete metrics on M with $K^{\mathbb{C}}_{g^i} > 0$ converging to g in the C^∞ topology.*

Proof. Consider the de Rham decomposition $M = \mathbb{R}^k \times (M_1, g_1) \times \ldots \times (M_l, g_l)$ of M. Let $g_j(t)$ be a Ricci flow from Theorem 1 on M_j with $g_j(0) = g_j$, and let $g(t)$ be the corresponding product metric on M. We know that $K^{\mathbb{C}}_{g_j(t)} \ge 0$ and clearly $(M_j, g_j(t))$ is irreducible for small $t \ge 0$. Since M_j is diffeomorphic to a Euclidean space, $(M_j, g_j(t))$ cannot be Einstein and we can deduce from Berger's holonomy classification theorem [1] that the holonomy group is either $\mathrm{SO}(n_j)$ or $\mathsf{U}(n_j/2)$, where $n_j = \dim M_j$.

The strong maximum principle implies that either $K^{\mathbb{C}}_{g_j(t)} > 0$ or $(M_j, g_j(t))$ is Kähler. Even in the Kähler case it follows that the (real) sectional curvature is positive, $K_{g_j(t)} > 0$: If there were a real plane with $K_{g_j(t)}(\sigma) = 0$, then by the strong maximum principle we could deduce that either $K_{g_j(t)}(v, Jv) = 0$ for all $v \in TM$ or $K_{g_j(t)}(v, w) = 0$ for all $v, w \in T_pM$ with $\mathrm{span}_{\mathbb{R}}\{v, Jv\} \perp \mathrm{span}_{\mathbb{R}}\{w, Jw\}$. But both conditions imply in $K \geq 0$ that the manifold is flat.

Since $K_{g_j(t)} > 0$, Theorem 3.4 of Greene and Wu gives a strictly convex smooth proper nonnegative function $b_j(t)$ on $(M_j, g_j(t))$. Clearly we can also find such a function on $\mathbb{R}^k$. By just adding these functions we deduce that there is a proper function $b(t)\colon M \to [0, \infty)$ which is strictly convex with respect to the product metric $g(t)$. We now choose a sequence $t_i \to 0$ and $\varepsilon_i \to 0$ and define g^i as the metric on M which is obtained by pulling back the metric on the graph of $\varepsilon_i b(t_i)$ viewed as hypersurface in $(M, g(t_i)) \times \mathbb{R}$. Clearly $K^{\mathbb{C}}_{g^i} > 0$ and if ε_i tends to zero sufficiently fast, then g^i converges to g in the C^∞ topology. □

Remark 7.7. Although a priori we could prove this only for large ℓ, it is true that for each convex set $C_\ell = b^{-1}((-\infty, \ell])$ one can find a Ricci flow on a compact manifold with $K^{\mathbb{C}} \geq 0$ such that $(M, g(t))$ converges to the double $D(C_\ell)$ of C_ℓ for $t \to 0$. In fact, by using Corollary 7.6 one can find a sequence of strictly convex sets $C_{\ell,k}$ in manifolds with $K^{\mathbb{C}} > 0$ which converge in the Gromov-Hausdorff topology to C_ℓ. For strictly convex sets it is not hard to see that one can smooth the double $D(C_{\ell,k})$ without losing $K^{\mathbb{C}} \geq 0$ and thus the result follows.

8. An immortal nonnegatively curved solution of the Ricci flow with unbounded curvature

8.1. **Double cigars.** Recall that Hamilton's cigar is the complete Riemannian surface $(C, g_0) := \left(\mathbb{R}^2, \frac{dx^2+dy^2}{1+x^2+y^2}\right)$, which is rotationally symmetric, positively curved and asymptotic at infinity to a cylinder of radius 1. The Ricci flow starting at (C, g_0) is a gradient steady Ricci soliton (i.e. an eternal self-similar solution).

Definition 8.1. Let $(M, \bar{g})$ and (N, g) be two complete n-dimensional Riemannian manifolds, $p \in M$ and $q \in N$. We say $(M, \bar{g}, p)$ is ε-close to (N, g, q) if

- there is a subset $U \subset N$ with $B_{\frac{1}{\varepsilon}-\varepsilon}(q) \subset U \subset B_{\frac{1}{\varepsilon}+\varepsilon}(q)$ and
- a diffeomorphism $f\colon B_{\frac{1}{\varepsilon}}(p) \to U$ such that $\|\bar{g} - f^*g\|_{C^k} \leq \varepsilon$ for all $k \leq 1/\varepsilon$.

We denote $x_0 \in C$ the tip of Hamilton's cigar, i.e. the unique fixed point of the isometry group $\mathrm{Iso}(C)$, where the maximal curvature of C is attained. We will also consider the rescaled manifolds $(C, \lambda^2 g_0)$.

Definition 8.2. A nonnegatively curved metric g on $\mathbb{S}^2$ is called an (ε, λ)-double cigar if the following holds

- g is invariant under $O(2) \times \mathbb{Z}_2 \subset O(3)$, and
- if $\bar{p}$ is one of the two fixed points of the identity component of $O(2) \times \mathbb{Z}_2$, then $(\mathbb{S}^2, g, \bar{p})$ is ε-close to $(C, \lambda^2 g_0, x_0)$.

An important feature of the definition is that except for nonnegative curvature, we do not make any assumptions on the middle region of the double cigar. In the applications we will have $\operatorname{diam}(\mathbb{S}^2, g) \gg \frac{1}{\varepsilon}$.

We have two easy consequences of compactness results.

Lemma 8.3. *For any λ and $\varepsilon > 0$ there exists some $\delta > 0$ such that: If $(\mathbb{S}^2, g)$ is any (δ, λ)-double cigar and $(\mathbb{S}^2, g(t))$ is a Ricci flow with $g(0) = g$, then $(\mathbb{S}^2, g(t))$ is an (ε, λ)-double cigar for all $t \in [0, 1/\varepsilon]$.*

Lemma 8.4. *Let $\bar{g}$ be a nonnegatively curved metric on $\mathbb{S}^2$, $(\mathbb{S}^2, \bar{g}(t))_{t\in[0,T]}$ the Ricci flow with $\bar{g}(0) = \bar{g}$, and $\bar{p} \in \mathbb{S}^2$. For a given $\varepsilon > 0$ there exists a positive integer $\delta = \delta(\varepsilon, \bar{g})$ such that the following holds.*

Let (M^3, g) be any open nonnegatively curved 3-manifold and $p \in M$ so that (M, g, p) is δ-close to $\big((\mathbb{S}^2, \bar{g}) \times \mathbb{R}, (\bar{p}, 0)\big)$. If $(M, g(t))$ is an immortal nonnegatively curved Ricci flow with $g(0) = g$, then $(M, g(t), p)$ is ε-close to $\big((\mathbb{S}^2, \bar{g}(t)) \times \mathbb{R}, (\bar{p}, 0)\big)$ for all $t \in [0, 1/\varepsilon] \cap [0, T/2]$.

Proof of Lemma 8.3. Suppose on the contrary that for some positive ε and λ we can find a sequence of $(\frac{1}{i}, \lambda)$-double cigars $(\mathbb{S}^2, g_i)$ and times $t_i \in [0, \frac{1}{\varepsilon}]$ such that $(\mathbb{S}^2, g_i(t_i))$ is not an (ε, λ)-double cigar. Here $(\mathbb{S}^2, g_i(t))_{t\in[0,T_i)}$ is the maximal solution of the Ricci flow with $g_i(0) = g_i$. Let $\bar{p}$ denote a fixed point of the identity component of the $O(2) \times \mathbb{Z}_2$-action. By assumption we know that $(\mathbb{S}^2, g_i(t_i), \bar{p})$ is not ε-close to $(C, \lambda^2 g_0, x_0)$.

It is easy to see that the volume of any unit ball in $(\mathbb{S}^2, g_i)$ is bounded below by a universal constant independent of i. Thus we have universal curvature and injectivity radius bounds for all positive times. Moreover, by Gauß Bonnet $T_i \to \infty$. Using furthermore that we have control of the curvature and its derivatives on larger and larger balls around $\bar{p}$, one can deduce that the Ricci flow subconverges to a (rotationally symmetric) limit immortal solution on the cigar $(C, g_\infty(t), x_0)$ with bounded curvature and whose initial metric is $\lambda^2 g_0$. Because of the uniqueness of the Ricci flow (see [13]) it follows that $(C, g_\infty(t), x_0)$ is isometric to $(C, \lambda^2 g_0, x_0)$ for all t. On the other hand, if $t_\infty \in [0, 1/\varepsilon]$ is a limit of a convergent subsequence of t_i, then $(C, g_\infty(t_\infty), x_0)$ is not $\varepsilon/2$-close to $(C, \lambda^2 g_0, x_0)$ – a contradiction. □

Proof of Lemma 8.4. Suppose on the contrary that we can find a sequence (M_i, g_i) of open 3-manifolds with $K_{g_i} \geq 0$ and $p_i \in M_i$ such that (M_i, g_i, p_i) is $\frac{1}{i}$-close to $\big((\mathbb{S}^2, \bar{g}) \times \mathbb{R}, (\bar{p}, 0)\big)$ and a complete immortal Ricci flow $g_i(t)$ with $g_i(0) = g_i$ and $K_{g_i(t)} \geq 0$, such that $(M_i, g_i(t_i), p_i)$ is not ε-close to $\big((\mathbb{S}^2, \bar{g}(t)) \times \mathbb{R}, (\bar{p}, 0)\big)$ for some $t_i \in [0, T/2]$.

Arguing similar to the proof of the previous lemma, we can use Hamilton's compactness theorem to deduce that $(M_i, g_i(t), p_i)$ converges to a limit nonnegatively curved solution on the manifold $\mathbb{S}^2 \times \mathbb{R}$. Clearly the solution is just given by the product solution on $\mathbb{S}^2 \times \mathbb{R}$ and because of uniqueness of the Ricci flow on $\mathbb{S}^2$ we deduce that it is exactly given by $\big((\mathbb{S}^2, \bar{g}(t)) \times \mathbb{R}, (\bar{p}, 0)\big)$ – again this yields a contradiction. $\square$

8.2. **Convex hulls of convex sets.** Let C_0 and C_1 be two closed convex sets of $\mathbb{R}^n$. Then

$$C_\lambda = \{(1-\lambda)x + \lambda y \mid x \in C_0, y \in C_1\}$$

is convex as well. If ∂C_0 and ∂C_1 are smooth compact hypersurfaces of positive sectional curvature, then ∂C_λ is smooth as well: In fact, let N_0 and N_1 denote the unit outer normal fields of C_0 and C_1. By assumption $N_i \colon \partial C_i \to \mathbb{S}^{n-1}$ is a diffeomorphism, $i = 0, 1$. For $z = (1-\lambda)x + \lambda y \in C_\lambda$ and $\lambda \in (0,1)$ the tangent cone $T_z C_\lambda$ contains $T_x C_0$ as well as $T_y C_1$. This in turn implies

$$\partial C_\lambda = \big\{(1-\lambda)x + \lambda N_1^{-1}(N_0(x)) \mid x \in \partial C_0\big\}$$

and thus ∂C_λ is smooth. Furthermore, it is easy to see that ∂C_λ is positively curved as well.

Consider now the convex sets $C_0 \times \{h_0\}$ and $C_1 \times \{h_1\}$ in $\mathbb{R}^{n+1}$. The convex hull C of these two sets is given by

$$C = \big\{(y, (1-\lambda)h_0 + \lambda h_1) \mid y \in C_\lambda, \lambda \in [0,1]\big\}.$$

In particular, we see that the boundary $\partial C \cap (\mathbb{R}^n \times (h_0, h_1))$ is a smooth manifold.

8.3. **Proof of Theorem 4 a).** By Lemma 8.3 we can find a sequence $\varepsilon_i \to 0$ such that any $(\varepsilon_i, \frac{1}{i})$-double cigar $(\mathbb{S}^2, g)$ satisfies the following: The solution of the Ricci flow $g(t)$ with $g(0) = g$ exists on $[0, i]$ and $(\mathbb{S}^2, g(t))$ is a $(2^{-i}, \frac{1}{i})$-double cigar for all $t \in [0, i]$. The sequence ε_i is hereafter fixed.

We now define inductively a sequence of $(\varepsilon_i, \frac{1}{i})$-double cigars S_i so that

(1) S_i has positive curvature and embeds as a convex hypersurface $S_i \subset \mathbb{R}^3$ in such a way that it is invariant under the linear action of $\mathbb{Z}_2 \times \mathrm{O}(2) \subset \mathrm{O}(3)$.
(2) The convex domain bounded by S_{i-1} is contained in the interior of the convex domain bounded by S_i.

It is fairly obvious that one can find $(\varepsilon_i, \frac{1}{i})$-double cigars satisfying (1). In order to accomplish also (2), we choose r_0 such that $S_{i-1} \subset B_{r_0}(0)$. We can find an $(\varepsilon_i, \frac{1}{i})$-double cigar $(\mathbb{S}^2, g)$ and a fixed point $\bar{p} \in \mathbb{S}^2$ of the identity component of $\mathbb{Z}_2 \times \mathrm{O}(2)$ such that

- $B_{1/\varepsilon_i}(\bar{p})$ is isometric to $B_{1/\varepsilon_i}(x_0) \subset (C, \frac{1}{i^2} g_0)$,
- for some $R \gg \frac{1}{\varepsilon_i}$ the set $B_R(\bar{p}) \setminus B_{2/\varepsilon_i}(\bar{p})$ is isometric to a subset of a cone
- and $B_{4R}(\bar{p}) \setminus B_{2R}(\bar{p})$ is isometric to $\mathbb{S}^1 \times [0, 2R)$ for a circle of length $4\pi r_0$.

We can now construct an embedding of this cigar into $\mathbb{R}^3$ such that the surface S_i bounds a convex domain which contains $B_{2r_0}(0)$. By slightly changing the embedding one can ensure that S_i has positive sectional curvature. The sequence S_i of embedded double cigars is now fixed.

We put $S_0 = \{0\}$. We now define inductively a sequence of positive numbers $r_i \to \infty$, and heights $h_i := \sum_{j=0}^{i} r_j$. Denote $C_j \subset \mathbb{R}^4$ the convex hull of $S_{j-1} \times \{h_{j-1}\} \subset \mathbb{R}^4$ and $S_j \times \{h_j\} \subset \mathbb{R}^4$. By choosing r_i large enough we can arrange for the following:

- The union $C_{i-1} \cup C_i$ is convex as well. In fact, C_i converges to the cylinder bounded by $S_{i-1} \times [h_{i-1}, \infty)$ for $r_i \to \infty$. Hence for all large r_i and all $p \in S_{i-1} \times \{h_{i-1}\}$ the union of the tangent cones T_pC_{i-1} and T_pC_i is properly contained in a half space.
- The hypersurface $H_i := \big(\mathbb{R}^3 \times [h_i - 1 - \sqrt{r_i}, h_i - 1]\big) \cap \partial C_i$ is arbitrarily close to a product $S_i \times [h_i - 1 - \sqrt{r_i}, h_i - 1]$ in the C^∞ topology.
- For any open 3-manifold $(M^3, \tilde{g})$ with $K_{\tilde{g}} \geq 0$ containing an open subset U isometric to H_i, for big r_i Lemma 8.4 ensures that, if $(M, \tilde{g}(t))$ is an immortal Ricci flow with $\tilde{g}(0) = \tilde{g}$ and $K_{\tilde{g}(t)} \geq 0$, then for some $p \in U$ we have that $(M, \tilde{g}(t), p)$ is $\frac{1}{(1+\mathrm{diam}(S_i))^i}$-close to $((\mathbb{S}^2, g(t)) \times \mathbb{R}, (\bar{p}, 0))$, where $(\mathbb{S}^2, g(t))$ is a $(2^{-i}, \frac{1}{i})$-double cigar for all $t \in [0, i]$.

By construction, $C = \bigcup_{i \geq 1} C_i$ is a convex set whose boundary ∂C is not smooth but the singularities only occur for points in $\mathbb{R}^3 \times \{h_i\} \cap \partial C$, see subsection 8.2. We can now smooth C as follows: Notice that $\partial C \subset \mathbb{R}^4$ can be defined as the graph of a convex function f on $\mathbb{R}^3$. By construction $T_p \partial C$ is not a half space for all $p \in S_i \times \{h_i\}$ and thus the gradient of f jumps at the level set S_i.

We choose a smooth convex function $g\colon [0, \infty) \to [0, \infty)$ with $g \equiv 0$ on $[1, \infty)$ and $g'' > 0$ on $[0, 1)$. We also define $\varphi\colon [0, \infty) \to \mathbb{R}$ by $\varphi(t) = t + \delta \cdot g(|t - h_i|)$ for some $\delta > 0$ which is to be determined next. Notice that φ is is C^∞ on $[0, h_i]$ and on $[h_i, \infty)$. Moreover $\varphi(t) = t$ for $|t - h_i| > 1$. On the intervals $[h_i - 1, h_i]$ and $[h_i, h_i + 1]$ the function φ is convex. However, at h_i left and right derivative of φ differ by $2\delta g'(0)$. Similarly there is small neighborhood U of S_i such that f is C^∞ on $U \setminus S_i$. Along the level set S_i there is an outer and an inner gradient of f and the norm of the outer gradient is strictly larger than the norm of the inner gradient. Thus we can choose δ so small that $\varphi \circ f$ is still a convex function. In addition, we know that the Hessian of $\varphi \circ f$ is bounded below by a small positive constant in a neighborhood of S_i. Thus we can mollify $\varphi \circ f$ in a neighborhood of S_i and patch things together using a cut off function.

By doing this procedure iteratively for all i we obtain a smooth convex hypersurface H. By construction the volume growth of H is larger than linear and by Corollary 5 we have an immortal solution $g(t)$ of the Ricci flow on H starting with the initial metric.

By construction for any i we can find a point $p \in H$ such that $(H, g(t), p)$ is $\frac{1}{i}$-close to $((\mathbb{S}^2, g(t)) \times \mathbb{R}, (\bar{p}, 0))$ where $(\mathbb{S}^2, g(t))$ is a $(2^{-i}, \frac{1}{i})$-double cigar for all $t \in [0, i]$. In particular, we know that

$$\begin{aligned} \sup\big\{K_{g(t)}(\sigma) \mid \sigma \subset TH\big\} &= \infty \quad \text{and} \\ \inf\big\{\mathrm{vol}_{g(t)}(B_{g(t)}(p,1)) \mid p \in H\big\} &= 0 \quad \text{for all } t. \end{aligned}$$

Remark 8.5. (a) A volume collapsed nonnegatively curved 3-manifold was constructed by Croke and Karcher [15]. Although the details are somewhat different, their example is realized as a convex hypersurface of $\mathbb{R}^4$ as well.

(b) At an informal discussion at UCSD the second named author was asked by Richard Hamilton, whether a nonnegatively curved three dimensional ancient solution with unbounded curvature could exist. During this discussion Hamilton described possible features of a counterexample. The construction in this section is in part inspired by what Hamilton had in mind. Since the construction only gives an immortal solution, Hamilton's question remains open nevertheless.

(c) As said in the introduction, a nonnegatively curved surface evolves immediately to bounded curvature under the Ricci flow. Giesen and Topping [19] gave immortal 2-dimensional Ricci flows with unbounded curvature throughout time.

8.4. **Proof of Theorem 4 b).**

Lemma 8.6. *Let (M, g) be an open manifold with $K^{\mathbb{C}} \geq 0$ and bounded curvature. Then there is an $\varepsilon > 0$ and C such that, for any complete Ricci flow $g(t)$ with $g(0) = g$ and $K^{\mathbb{C}}_{g(t)} \geq 0$, we have $\mathrm{scal}_{g(t)} \leq C$ on the interval $[0, \varepsilon]$.*

Proof. Recall that the injectivity radius of an open nonnegatively curved manifold with bounded curvature is positive. By Corollary 3 for any solution $g(t)$ we know $\mathrm{scal}_{g(t)} \leq \frac{C_1}{t}$ on some interval $(0, \varepsilon]$. We can now use Theorem C.5 to see that $\mathrm{scal}_{g(t)} \leq C$ for all $t \in [0, \varepsilon]$ with some universal $C = C((M, g))$. □

Lemma 8.7. *There is an open 4-manifold (M, g) with nonnegative curvature operator and a constant $v_0 > 0$ such that the following holds*

- $\mathrm{vol}(B_g(p, 1)) \geq v_0$ *for all* $p \in M$.
- *There is a sequence of points $p_k \in M$ such that (M, g, p_k) converges in the Cheeger-Gromov sense to the Riemannian product $\mathbb{S}^2 \times \mathbb{R}^2$ (where $\mathbb{S}^2$ has constant curvature 1 and $\mathbb{R}^2$ is flat).*
- *There is a sequence of points $q_k \in M$ such that (M, g, q_k) converges in the Cheeger-Gromov sense to $\mathbb{R}^4$ endowed with the flat metric.*

Proof. The construction of (M^4, g) is very similar to the one in subsection 8.3. There is a sequence of embedded $(\frac{1}{i}, 1)$-double cigars $S_i \subset \mathbb{R}^3$ so that

- S_i is invariant under an $\mathbb{Z}_2 \times \mathrm{O}(2) \subset \mathrm{O}(3)$-action fixing the origin.
- The interior of the convex domain bounded by S_i contains S_{i-1}.

- S_i contains a subset which is isometric to $\mathbb{S}^1 \times [-R_{i-1}, R_{i-1}]$ where $\mathbb{S}^1$ is a circle of radius $2R_{i-1} = 2\,\mathrm{diam}(S_{i-1}) \to \infty$.

Analogous to subsection 8.3 one can then construct a smooth convex hypersurface $(M^3, g) \subset \mathbb{R}^4$ with $\mathrm{O}(2) \times \mathbb{Z}_2$ symmetry satisfying: There is $p_i \in M$ such that (M^3, p_i) is e^{-R_i}-close to $S_i \times \mathbb{R}$. Moreover, it is clear from the construction that (M^3, g) is uniformly volume non collapsed. We now define M^4 as the unique convex hypersurface in $\mathbb{R}^5$ whose intersection with $\mathbb{R}^4$ is given by (M^3, g) and which has a $\mathrm{O}(3) \times \mathbb{Z}_2$ symmetry. It is straightforward to check that M^4 with the induced metric has the claimed properties. □

Proof of Theorem 4 b). Let (M^4, g) be as in Lemma 8.7. We consider a solution $g(t)$ of the Ricci flow coming out of the proof of Theorem 1. Using (M^4, g, q_i) converges to $\mathbb{R}^4$ in the Cheeger-Gromov sense, it follows that the Ricci flow on the compact approximations $(M_i, g_i(t))$ converging to $(M, g(t))$ exists until $T_i \to \infty$. By the proof of Theorem 1 we can assume that $g(t)$ is immortal. Moreover, it is clear from the proof that $g_i(t)$ and hence $g(t)$ have nonnegative curvature operator. In particular we have an immortal complete solution $g(t)$ with $g(0) = g$ and $g(t)$ satisfies the trace Harnack inequality.

By Corollary 3 it follows that $\mathrm{scal}_{g(t)} \leq \frac{C}{t}$ for $t \in (0, \varepsilon]$. We claim that $g(1)$ has unbounded curvature. Suppose on the contrary that $\mathrm{scal}_{g(1)} \leq C$. The trace Harnack inequality implies that $\mathrm{scal}_{g(t)} \leq \frac{C}{t}$ for $t \in (0, 1]$.

We now consider the sequence (M, g, p_i) converging to $(\mathbb{S}^2 \times \mathbb{R}^2, p_\infty)$ in the Cheeger-Gromov sense. Applying Theorem C.5 it is easy that there is a universal constant C_2 such that for any r we can find i_0 such that $\mathrm{scal}_{g(t)} \leq C_2$ on $B_{g(0)}(p_i, r)$ for all $t \in [0, 1]$ and $i \geq i_0$. By Hamilton's compactness theorem $(M, g(t), p_i)$ subconverges to a solution $g_\infty(t)$ of the Ricci flow on $(\mathbb{S}^2 \times \mathbb{R}^2, p_\infty)$ with bounded curvature such that $g_\infty(0)$ is given by the product metric ($\mathbb{S}^2$ with constant curvature 1), $t \in [0, 1]$. On the other hand, for the the unique solution (with bounded curvature) the curvature blows up at time $1/2$ – a contradiction.

In summary, we can say $\mathrm{scal}_{g(t)}$ is bounded for $t \in (0, \varepsilon]$ and that $\mathrm{scal}_{g(1)}$ is unbounded. By Lemma 8.6 there must be a minimal time $t_0 \in (\varepsilon, 1]$ such that $g(t_0)$ has unbounded curvature. From the trace Harnack inequality it follows that $g(t)$ has unbounded curvature for all $t \geq t_0$.

Thus M^4 endowed with the rescaled Ricci flow $\tilde{g}(t) := \frac{1}{t_0 - \varepsilon/2} g(\varepsilon/2 + t(t_0 - \varepsilon/2))$ satisfies the conclusion of Theorem 4 b). □

Appendix A. Open manifolds of nonnegative curvature

Recall that a set C is called totally convex if for every geodesic segment Γ joining two points in C, we have $\Gamma \subset C$, and that the normal bundle of a submanifold $S \subset M$ is $\nu(S) = \bigcup_p \{v \in T_pM \mid v \perp T_pS\}$. We start with the Soul Theorem.

Theorem A.1 (Cheeger-Gromoll-Meyer, cf. [10, 21]). *Let (M^n, g) be an open manifold with $K_g \geq 0$. Then there is a closed, totally geodesic submanifold $\Sigma \subset M$ which is totally convex and with $0 \leq \dim\Sigma < n$. Σ is called a soul of M, and M is diffeomorphic to $\nu(\Sigma)$. If $K_g > 0$, then a soul of M is a point, and so M is diffeomorphic to $\mathbb{R}^n$.*

Here is a further property about the soul.

Theorem A.2 (Strake, cf. [44]). *Let (M^n, g) be an open manifold with $K_g \geq 0$ and Σ^k be the soul of M. If the holonomy group of $\nu(\Sigma)$ is trivial then M is isometric to $\Sigma \times \mathbb{R}^{n-k}$, where $\mathbb{R}^{n-k}$ carries a complete metric of $K \geq 0$.*

Fix $p \in \Sigma$ and let $d_p = d_g(\,\cdot\,, p)$, where d_g is the Riemannian distance. It is known (see e.g. [16]) that b (see subsection 3.2) and d_p are asymptotically equal:

Lemma A.3. *There exists a function $\theta(s)$ with $\lim_{s\to\infty} \theta(s) = 0$ such that*

$$(1 - \theta \circ d_p)d_p \leq b \leq d_p,$$

and for all $x, y \in M$ it holds $|b(x) - b(y)| \leq d_g(x, y)$.

It is useful to recall that b is indeed the distance from an appropriate set:

Lemma A.4 (Wu, cf. [48]). *Let $a \in \mathbb{R}$ and let $C_a = \{x \in M : b(x) \leq a\}$. Then $b|_{int(C_a)} = a - d(\cdot, \partial C_a)$.*

Appendix B. Convex sets in Riemannian manifolds

Let C be a compact totally convex set (TCS) in a manifold M. We define the tangent cone at $p \in \partial C$ as

$$T_pC = \mathrm{Clos}\{v \in T_pM \,:\, \exp_p(tv/|v|) \in C \text{ for some } t > 0\}.$$

By convexity of C, this is a convex cone in T_pM. The normal space is defined as

$$N_pC = \{v \in \mathrm{span}(T_pC) \,:\, \langle v, w\rangle \leq 0 \text{ for all } w \in T_pC \setminus \{0\}\}.$$

Here is a useful characterization of the normal space.

Proposition B.1 (Yim, cf. [49]). *Let $\{C_a\}$ be a family of TCS. Consider $a > b$ with $a - b < \delta$, where $\delta > 0$ is chosen so that the projection $C_a \to C_b$ is well-defined (i.e. for all $q \in C_a$ there is a unique $q^* \in C_b$ with $d(q, q^*) = d(q, C_b)$). For each $p \in \partial C_b$, N_pC_b is the convex hull of the set of vectors $v \in \mathrm{span}(T_pC)$ such that the geodesic $\gamma(s) = \exp_p(sv/|v|)$ is the shortest path from p to some point in ∂C_a.*

Further details about the structure of the sublevel sets of the Busemann function b are given by

Lemma B.2 (Guijarro-Kapovitch, cf.[23])**.** *Consider $C_\ell = b^{-1}((-\infty, \ell]) \subset M$ and $p \in \partial C_\ell$. Take γ any minimal geodesic from $p = \gamma(0)$ to any point of the soul. Then there exists $\varepsilon(\ell)$, with $\varepsilon(\ell) \to 0$ as $\ell \to \infty$ such that if $v \in T_pM$ is a unit vector with $\angle(v, \dot\gamma(0)) < \frac{\pi}{2} - \varepsilon(\ell)$, then $v \in T_pC_\ell$.*

The following theorem gives the existence of a **tubular neighborhood** U:

Theorem B.3 (Walter, cf. [46])**.** *For each closed locally convex set $A \subset (M, g)$, there is an open set $U \subset A$ such that*

1) For each $q \in U$, there is a unique $q^* \in A$ with $d(q, q^*) = d(q, A)$, and a unique minimal geodesic from q to q^* which lies entirely in U.

2) d_A is C^1 in $U \setminus A$ and twice differentiable almost everywhere in $U \setminus A$.

Let us recall the **Hessian bounds in the support sense**:

Definition B.4 (Calabi, cf. [8])**.** Let $f : (M, g) \to \mathbb{R}$ be continuous. We say that $\nabla^2 f|_p \geq h(p)$ in the support sense, for some function $h : M \to \mathbb{R}$, if for every $\varepsilon > 0$ there exists a smooth function φ_ε defined on a neighborhood of p such that

1) $\varphi_\varepsilon(p) = f(p)$ and $\varphi_\varepsilon \leq f$ in some neighborhood of p.

2) $\nabla^2\varphi_\varepsilon|_p \geq (h - \varepsilon)g_p$.

Such functions φ_ε are called lower support functions of f at p. One can analogously define $\nabla^2 f \leq h$ at p in the support sense.

Appendix C. Miscellanea of Ricci flow results

C.1. **Smooth convergence of manifolds and flows.**

Definition C.1 (Cheeger-Gromov convergence)**.** (a) Consider a sequence of complete manifolds (M_i^n, g_i) and choose $p_i \in M_i$. We say that (M_i, g_i, p_i) converges to the pointed Riemannian n-manifold $(M_\infty, g_\infty, p_\infty)$ if there exists

(1) a collection $\{U_i\}_{i\geq 1}$ of compact sets with $U_i \subset U_{i+1}$, $\cup_{i\geq 1} U_i = M$ and $p_\infty \in \operatorname{int}(U_i)$ for all i, and
(2) $\phi_i : U_i \to M_i$ diffeomorphisms onto their image, with $\phi_i(p_\infty) = p_i$

such that $\phi_i^* g_i \to g_\infty$ smoothly on compact subsets of M_∞, meaning that

$$|\nabla^m(\phi_i^* g_i - g_\infty)| \longrightarrow 0 \qquad \text{as} \quad i \to \infty \quad \text{on} \quad K \quad \text{for all} \quad m \geq 0$$

for every compact set $K \subset M$. Here $|\cdot|$ and ∇ are computed with respect to any fixed background metric.

(b) A sequence of complete evolving manifolds $(M_i, g_i(t), p_i)_{t\in I}$ converges to a pointed evolving manifold $(M_\infty, g_\infty(t), p_\infty)_{t\in I}$ if we have (1) and (2) as before such that $\phi_i^* g_i(t) \to g_\infty(t)$ smoothly on compact subsets of $M_\infty \times I$.

Theorem C.2 (Hamilton, cf. [27])**.** *Let $(M_k, g_k(t), x_k)_{t\in(a,b]}$ be complete n-dimensional Ricci flows, and fix $t_0 \in (a, b]$. Assume the following two conditions:*

(1) For each compact interval $I \subset (a,b]$, there is a constant $C = C(I) < \infty$ so that for all $t \in I$

$$|\mathrm{R}|_{g_k(t)} \leq C \qquad \text{on} \qquad B_{g_k(0)}(x_k, r) \quad \text{for all} \quad k \geq k_0(r).$$

(2) There exists $\delta > 0$ such that $inj_{g_k(t_0)}(x_k) \geq \delta$.

Then, after passing to a subsequence, the solutions converge smoothly to a complete Ricci flow solution $(M_\infty, g_\infty(t), x_\infty)$ of the same dimension, defined on $(a,b]$.

Some authors quote stronger versions of this theorem, where the curvature bound C is allowed to increase arbitrarily with r. However, in the proof one then runs into trouble if one wants to verify completeness of the limit metrics for different times. Lemma 4.9 can be regarded as a way to circumvent this problem.

Under bounded curvature, condition (2) above can be guaranteed by ensuring a lower bound on the volume (see [11, Theorem 4.3]):

Theorem C.3 (Cheeger-Gromov-Taylor)**.** *Let $B_g(p,r)$ be a metric ball in a complete Riemannian manifold (M^n, g) with $\lambda \leq K_g|_{B_g(p,r)} \leq \Lambda$ for some constants λ, Λ. Then, for any constant r_0 such that $4r_0 < \min\{\pi/\sqrt{\Lambda}, r\}$ if $\Lambda > 0$, we have*

$$\mathrm{inj}_g(p) \geq r_0 \left(1 + \frac{V^n_\lambda(2r_0)}{\mathrm{vol}_g\left(B_g(p,r_0)\right)}\right)^{-1},$$

where $V^n_\lambda(\rho)$ denotes the volume of the ball of radius ρ in the n-dimensional space with constant sectional curvature λ.

C.2. **Curvature estimates.** Shi's local derivative estimates ensure that if the curvature is bounded on $B_{g(0)}(p,r) \times [0,T]$, then we also have bounds on all covariant derivatives of the curvature on the smaller set $B_{g(0)}(p,r/2) \times (0,T]$, where such bounds blow up to infinity as $t \to 0$. Such a degeneracy can be avoided by making the stronger assumption of having bounded derivatives of the curvature in the initial metric.

Theorem C.4 (Lu-Tian, [31])**.** *For any positive numbers $\alpha, K, K_\ell, r, n \geq 2, m \in \mathbb{N}$, let M^n be a manifold with $p \in M$, and $g(t)$, $t \in [0,\tau]$ where $\tau \in (0, \alpha/K)$, be a Ricci flow on an open neighborhood $\mathcal{U}$ of p containing $\overline{B}_{g(0)}(p,r)$ as a compact subset. If*

$$|\mathrm{R}_{g(t)}|(x) \leq K \qquad \text{for all } x \in B_{g(0)}(p,r) \quad \text{and} \quad t \in [0,\tau]$$

$$|\nabla^\ell \mathrm{R}_{g(0)}|(x) \leq K(\ell) \qquad \text{for all } x \in B_{g(0)}(p,r) \quad \text{and all} \quad \ell \geq 0,$$

then there exists $C = C(\alpha, K, K(\ell), r, m, n)$ such that

$$|\nabla^m \mathrm{R}_{g(t)}| \leq C \qquad \text{on} \quad \overline{B}_{g(0)}(p, r/2) \times [0,\tau].$$

Next we state a result of Simon [42, Theorem 1.3]. We actually use a simplified and coordinate free version, see also Chen [12, Corollary 3.2]:

Theorem C.5 (M. Simon, B. L. Chen). *Let $(M^n, g(t))$, with $t \in [0, T]$, be a complete Ricci flow. Assume we have the curvature bounds*

$$|\mathrm{R}|_{g(0)} \leq \rho^{-2} \qquad \text{on} \qquad B_{g(0)}(p, \rho) \tag{C.1}$$

and

$$|\mathrm{R}|_{g(t)}(x) \leq K/t \qquad \text{for} \qquad x \in B_{g(0)}(p, \rho) \qquad \text{and} \qquad t \in (0, T]. \tag{C.2}$$

Then there exists a constant C depending only on n such that

$$|\mathrm{R}|_{g(t)}(x) \leq 4\, e^{CK} \rho^{-2} \qquad \text{for all} \qquad x \in B_{g(0)}(p, \rho/2), \quad t \in [0, T].$$

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