

# MANIFOLDS WITH POSITIVE CURVATURE OPERATORS ARE SPACE FORMS

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The Ricci flow has been introduced by Hamilton in 1982 [H1] in order to prove that a compact three-manifold admitting a Riemannian metric of positive Ricci curvature is a spherical space form. In dimension four Hamilton showed that compact four-manifolds with positive curvature operators are spherical space forms as well [H2]. More generally, the same conclusion holds for compact four-manifolds with 2-positive curvature operators [Che]. Recall that a curvature operator is called 2-positive, if the sum of its two smallest eigenvalues is positive. In arbitrary dimensions Huisken [Hu] described an explicit open cone in the space of curvature operators such that the normalized Ricci flow evolves metrics whose curvature operators are contained in that cone into metrics of constant positive sectional curvature.

Hamilton conjectured that in all dimensions compact Riemannian manifolds with positive curvature operators must be space forms. In this paper we confirm this conjecture. More generally, we show the following

**Theorem 1.** *On a compact manifold the normalized Ricci flow evolves a Riemannian metric with 2-positive curvature operator to a limit metric with constant sectional curvature.*

The theorem is known in dimensions below five [H3], [H1], [Che]. Our proof works in dimensions above two: we only use Hamilton's maximum principle and Klingenberg's injectivity radius estimate for quarter pinched manifolds. Since in dimensions above two a quarter pinched orbifold is covered by a manifold (see Proposition 5.2), our proof carries over to orbifolds.

This is no longer true in dimension two. In the manifold case it is known that the normalized Ricci flow converges to a metric of constant curvature for any initial metric [H3], [Cho]. However, there exist two-dimensional orbifolds with positive sectional curvature which are not covered by a manifold. On such orbifolds the Ricci flow converges to a nontrivial Ricci soliton [CW].

Let us mention that a 2-positive curvature operator has positive isotropic curvature. Micallef and Moore [MM] showed that a simply connected compact manifold with positive isotropic curvature is a homotopy sphere. However, their techniques do not allow to get restrictions for the fundamental groups or the differentiable structure of the underlying manifold.

We turn to the proof of Theorem 1. The (unnormalized) Ricci flow is the geometric evolution equation

$$\frac{\partial g}{\partial t} = -2 \operatorname{Ric}(g)$$

for a curve  $g_t$  of Riemannian metrics on a compact manifold  $M^n$ . Using moving frames, this leads to the following evolution equation for the curvature operator  $R_t$

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of  $g_t$  (cf. [H2]):

$$\frac{\partial \mathbf{R}}{\partial t} = \Delta \mathbf{R} + 2(\mathbf{R}^2 + \mathbf{R}^\#).$$

Here  $\mathbf{R}_t: \Lambda^2 T_p M \rightarrow \Lambda^2 T_p M$  and identifying  $\Lambda^2 T_p M$  with  $\mathfrak{so}(T_p M)$  we have

$$\mathbf{R}^\# = \text{ad} \circ (\mathbf{R} \wedge \mathbf{R}) \circ \text{ad}^*,$$

where  $\text{ad}: \Lambda^2(\mathfrak{so}(T_p M)) \rightarrow \mathfrak{so}(T_p M)$  is the adjoint representation. Notice that in our setting the curvature operator of the round sphere of radius one is the identity.

We denote by  $S_B^2(\mathfrak{so}(n))$  the vectorspace of curvature operators, that is the vectorspace of selfadjoint endomorphisms of  $\mathfrak{so}(n)$  satisfying the Bianchi identity. Hamilton's maximum principle asserts that a closed convex  $O(n)$ -invariant subset  $C$  of  $S_B^2(\mathfrak{so}(n))$  which is invariant under the ordinary differential equation

$$(1) \quad \frac{d\mathbf{R}}{dt} = \mathbf{R}^2 + \mathbf{R}^\#$$

defines a Ricci flow invariant curvature condition; that is, the Ricci flow evolves metrics on compact manifolds whose curvature operators at each point are contained in  $C$  into metrics with the same property.

In dimensions above four there are relatively few applications of the maximum principle, since in these dimensions the ordinary differential equation (1) is not well understood. By analyzing how the differential equation changes under linear equivariant transformations, we provide a general method for constructing new invariant curvature conditions from known ones.

Any equivariant linear transformation of the space of curvature operators respects the decomposition

$$S_B^2(\mathfrak{so}(n)) = \langle I \rangle \oplus \langle \text{Ric}_0 \rangle \oplus \langle W \rangle$$

into pairwise inequivalent irreducible  $O(n)$ -invariant subspaces. Here  $\langle I \rangle$  denotes multiples of the identity,  $\langle W \rangle$  the space of Weyl curvature operators and  $\langle \text{Ric}_0 \rangle$  are the curvature operators of traceless Ricci type. Given a curvature operator  $\mathbf{R}$  we let  $\mathbf{R}_I$  and  $\mathbf{R}_{\text{Ric}_0}$  denote the projections onto  $\langle I \rangle$  and  $\langle \text{Ric}_0 \rangle$ , respectively. Furthermore let  $\text{Ric}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  denote the Ricci tensor of  $\mathbf{R}$  and  $\text{Ric}_0$  the traceless part of  $\text{Ric}$ .

**Theorem 2.** *For  $a, b \in \mathbb{R}$  consider the equivariant linear map*

$$l_{a,b}: S_B^2(\mathfrak{so}(n)) \rightarrow S_B^2(\mathfrak{so}(n)) ; \mathbf{R} \mapsto \mathbf{R} + 2(n-1)a\mathbf{R}_I + (n-2)b\mathbf{R}_{\text{Ric}_0}$$

and let

$$D_{a,b} := l_{a,b}^{-1}((l_{a,b}\mathbf{R})^2 + (l_{a,b}\mathbf{R})^\#) - \mathbf{R}^2 - \mathbf{R}^\#.$$

Then

$$\begin{aligned} D_{a,b} &= ((n-2)b^2 - 2(a-b)) \text{Ric}_0 \wedge \text{Ric}_0 + 2a \text{Ric} \wedge \text{Ric} + 2b^2 \text{Ric}_0^2 \wedge \text{id} \\ &+ \frac{\text{tr}(\text{Ric}_0^2)}{n + 2n(n-1)a} (nb^2(1-2b) - 2(a-b)(1-2b+nb^2)) I. \end{aligned}$$

The key fact about the difference  $D_{a,b}$  of the pulled back differential equation and the differential equation itself is that it does not depend on the Weyl curvature.

Let us now explain why Theorem 2 allows us to construct new curvature conditions which are invariant under the ordinary differential equation (1): We consider the image of a known invariant curvature condition  $C$  under the linear map  $l_{a,b}$  for

suitable constants  $a, b$ . This new curvature condition is invariant under the ordinary differential equation, if  $l_{a,b}^{-1}((l_{a,b}R)^2 + (l_{a,b}R)^\#)$  lies in the tangent cone  $T_{\mathbb{R}C}$  of the known invariant set  $C$ . By assumption  $R^2 + R^\#$  lies in that tangent cone, and hence it suffices to show  $D_{a,b} \in T_{\mathbb{R}C}$ . Since this difference does not depend on the Weyl curvature, it can be solely computed from the Ricci tensor.

Using this technique we construct a continuous family of invariant cones joining the invariant cone of 2-positive curvature operators and the invariant cone of positive multiples of the identity operator. Then a standard ode-argument shows that from any such family a generalized pinching set can be constructed – a concept which is slightly more general than Hamilton's concept of pinching sets in [H2]. In Theorem 5.1 we show that Hamilton's convergence result carries over to our situation, completing the proof of Theorem 1.

We expect that Theorem 2 and its Kähler analogue should give rise to further applications. This will be the subject of a forthcoming paper.

## 1. ALGEBRAIC PRELIMINARIES

For a Euclidean vector space  $V$  we let  $\Lambda^2V$  denote the exterior product of  $V$ . We endow  $\Lambda^2V$  with its natural scalar product; if  $e_1, \dots, e_n$  is an orthonormal basis of  $V$  then  $e_1 \wedge e_2, \dots, e_{n-1} \wedge e_n$  is an orthonormal basis of  $\Lambda^2V$ . Notice that two linear endomorphisms  $A, B$  of  $V$  induce a linear map

$$A \wedge B: \Lambda^2V \rightarrow \Lambda^2V; \quad v \wedge w \mapsto \frac{1}{2}(A(v) \wedge B(w) + B(v) \wedge A(w)).$$

We will identify  $\Lambda^2\mathbb{R}^n$  with the Lie algebra  $\mathfrak{so}(n)$  by mapping the unit vector  $e_i \wedge e_j$  onto the linear map  $L(e_i \wedge e_j)$  of rank two which is a rotation with angle  $\pi/2$  in the plane spanned by  $e_i$  and  $e_j$ . Notice that under this identification the scalar product on  $\mathfrak{so}(n)$  corresponds to  $\langle A, B \rangle = -1/2 \operatorname{tr}(AB)$ .

For  $n \geq 4$  there is a natural decomposition of

$$S^2(\mathfrak{so}(n)) = \langle I \rangle \oplus \langle \operatorname{Ric}_0 \rangle \oplus \langle W \rangle \oplus \Lambda^4(\mathbb{R}^n)$$

into  $O(n)$ -invariant, irreducible and pairwise inequivalent subspaces. An endomorphism  $R \in S^2(\mathfrak{so}(n))$  satisfies the first Bianchi identity if and only if  $R$  is an element in  $S_B^2(\mathfrak{so}(n)) = \langle I \rangle \oplus \langle \operatorname{Ric}_0 \rangle \oplus \langle W \rangle$ . Given a curvature operator  $R \in S_B^2(\mathfrak{so}(n))$  we let  $R_I, R_{\operatorname{Ric}_0}$  and  $R_W$ , denote the projections onto  $\langle I \rangle, \langle \operatorname{Ric}_0 \rangle$  and  $\langle W \rangle$ , respectively. Moreover, let

$$\operatorname{Ric}: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

denote the Ricci tensor of  $R$ ,  $\operatorname{Ric}_0$  the traceless Ricci tensor and

$$(2) \quad \bar{\lambda} := \operatorname{tr}(\operatorname{Ric})/n \quad \text{and} \quad \sigma := \|\operatorname{Ric}_0\|^2/n.$$

Then

$$(3) \quad R_I = \frac{\bar{\lambda}}{n-1} \operatorname{id} \wedge \operatorname{id} \quad \text{and} \quad R_{\operatorname{Ric}_0} = \frac{2}{n-2} \operatorname{Ric}_0 \wedge \operatorname{id}.$$

Hamilton observed in [H2] that next to the map  $(R, S) \mapsto \frac{1}{2}(RS + SR)$  there is a second natural  $O(n)$ -equivariant bilinear map

$$\#: S^2(\mathfrak{so}(n)) \times S^2(\mathfrak{so}(n)) \rightarrow S^2(\mathfrak{so}(n)); \quad (R, S) \mapsto R \# S$$

given by

$$(4) \quad \langle (\mathbf{R} \# \mathbf{S})(h), h \rangle = \frac{1}{2} \sum_{\alpha, \beta=1}^N \langle [\mathbf{R}(b_\alpha), \mathbf{S}(b_\beta)], h \rangle \cdot \langle [b_\alpha, b_\beta], h \rangle$$

for  $h \in \mathfrak{so}(n)$  and an orthonormal basis  $b_1, \dots, b_N$  of  $\mathfrak{so}(n)$ . The factor  $1/2$  stems from that fact that we are using the scalar product  $-1/2 \operatorname{tr}(AB)$  instead of  $-\operatorname{tr}(AB)$  as in [H2]. We would like to mention that  $\mathbf{R} \# \mathbf{S} = \mathbf{S} \# \mathbf{R}$  can be described invariantly

$$\mathbf{R} \# \mathbf{S} = \operatorname{ad} \circ (\mathbf{R} \wedge \mathbf{S}) \circ \operatorname{ad}^*,$$

where  $\operatorname{ad}: \Lambda^2 \mathfrak{so}(n) \rightarrow \mathfrak{so}(n), u \wedge v \mapsto [u, v]$  denotes the adjoint representation and  $\operatorname{ad}^*$  is its dual. Following Hamilton we set

$$\mathbf{R}^\# = \mathbf{R} \# \mathbf{R}.$$

We will also consider the trilinear form

$$(5) \quad \operatorname{tri}(\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3) = \operatorname{tr}((\mathbf{R}_1 \mathbf{R}_2 + \mathbf{R}_2 \mathbf{R}_1 + 2 \mathbf{R}_1 \# \mathbf{R}_2) \cdot \mathbf{R}_3).$$

The authors learned from Huisken that  $\operatorname{tri}$  is symmetric in all three components. In fact using (4) it is straightforward to check that

$$\operatorname{tr}(2(\mathbf{R}_1 \# \mathbf{R}_2) \cdot \mathbf{R}_3) = \sum_{\alpha, \beta, \gamma=1}^N \langle [\mathbf{R}_1(b_\alpha), \mathbf{R}_2(b_\beta)], \mathbf{R}_3(b_\gamma) \rangle \cdot \langle [b_\alpha, b_\beta], b_\gamma \rangle.$$

Since the right hand side is clearly symmetric in all three components this gives the desired result. Huisken also observed that the ordinary differential equation (1) is the gradient flow of the function

$$P(R) = \frac{1}{3} \operatorname{tr}(R^3 + R R^\#) = \frac{1}{6} \operatorname{tri}(R, R, R).$$

Finally we recall that if  $e_1, \dots, e_n$  denotes an orthonormal basis of eigenvectors of  $\operatorname{Ric}$ , then

$$(6) \quad \operatorname{Ric}(R^2 + R^\#)_{ij} = \sum_k \operatorname{Ric}_{kk} R_{kijk}$$

where  $R_{kijk} = \langle \mathbf{R}(e_i \wedge e_k), e_j \wedge e_k \rangle$ , see [H1], [H2].

## 2. A NEW ALGEBRAIC IDENTITY FOR CURVATURE OPERATORS

The main aim of this section is to prove Theorem 2. A computation using (3) shows that the linear map  $l_{a,b}: S_B^2(\mathfrak{so}(n)) \rightarrow S_B^2(\mathfrak{so}(n))$  given in Theorem 2 satisfies

$$l_{a,b}(R) = R + 2b \operatorname{Ric} \wedge \operatorname{id} + 2(n-1)(a-b) R_I.$$

The bilinear map  $\#$  induces a linear  $O(n)$ -equivariant map given by  $R \mapsto R \# I$ . The normalization of our parameters is related to the eigenvalues of this map.

**Lemma 2.1.** *Let  $R \in S_B^2(\mathfrak{so}(n))$ . Then*

$$R + R \# I = (n-1) R_I + \frac{n-2}{2} R_{\operatorname{Ric}_0} = \operatorname{Ric} \wedge \operatorname{id}.$$

*Proof.* One can write

$$(7) \quad \mathbf{R} + \mathbf{R}\#I = \frac{1}{4}((\mathbf{R}+I)^2 + (\mathbf{R}+I)^\# - (\mathbf{R}-I)^2 - (\mathbf{R}-I)^\#).$$

The result on the eigenvalues of the map corresponding to the subspaces  $\langle \text{Ric}_0 \rangle$  and  $\langle I \rangle$  now follows from equation (6) by a straightforward computation. For  $n = 4$  one verifies directly that  $\langle \mathbf{W} \rangle$  is in the kernel of the map  $\mathbf{R} \mapsto \mathbf{R} + \mathbf{R}\#I$ . Since there is a natural embedding of the Weyl curvature operators in  $S_B^2(\mathfrak{so}(4))$  to the Weyl curvature operators in  $S_B^2(\mathfrak{so}(n))$  this implies the same result for  $n \geq 5$ .  $\square$

We say that a curvature operator  $\mathbf{R}$  is of Ricci type, if  $\mathbf{R} = \mathbf{R}_I + \mathbf{R}_{\text{Ric}_0}$ .

**Lemma 2.2.** *Let  $\mathbf{R} \in S_B^2(\mathfrak{so}(n))$  be a curvature operator of Ricci type, and let  $\bar{\lambda}$  and  $\sigma$  be as in (2). Then*

$$\begin{aligned} \mathbf{R}^2 + \mathbf{R}^\# &= \frac{1}{n-2} \text{Ric}_0 \wedge \text{Ric}_0 + \frac{2\bar{\lambda}}{(n-1)} \text{Ric}_0 \wedge \text{id} - \frac{2}{(n-2)^2} (\text{Ric}_0^2)_0 \wedge \text{id} \\ &\quad + \frac{\bar{\lambda}^2}{n-1} I + \frac{\sigma}{n-2} I. \end{aligned}$$

Moreover

$$\begin{aligned} (\mathbf{R}^2 + \mathbf{R}^\#)_W &= \frac{1}{n-2} (\text{Ric}_0 \wedge \text{Ric}_0)_W \\ \text{Ric}(\mathbf{R}^2 + \mathbf{R}^\#) &= -\frac{2}{n-2} (\text{Ric}_0^2)_0 + \frac{n-2}{n-1} \bar{\lambda} \text{Ric}_0 + \bar{\lambda}^2 \text{id} + \sigma \text{id}. \end{aligned}$$

*Proof.* By equation (3)

$$\mathbf{R} = \mathbf{R}_I + \mathbf{R}_{\text{Ric}_0} = \frac{\bar{\lambda}}{(n-1)} I + \frac{2}{(n-2)} \text{Ric}_0 \wedge \text{id}.$$

Using the abbreviation  $\mathbf{R}_0 = \mathbf{R}_{\text{Ric}_0}$  we have

$$\mathbf{R}^2 + \mathbf{R}^\# = \mathbf{R}_0^2 + \mathbf{R}_0^\# + \frac{2\bar{\lambda}}{(n-1)} (\mathbf{R}_0 + \mathbf{R}_0 \#I) + \frac{\bar{\lambda}^2}{(n-1)^2} (I + I^\#).$$

Since the last two summands are known by Lemma 2.1, we may assume that  $\mathbf{R} = \mathbf{R}_{\text{Ric}_0}$ . Let  $\lambda_1, \dots, \lambda_n$  denote the eigenvalues of  $\text{Ric}_0$  corresponding to an orthonormal basis  $e_1, \dots, e_n$  of  $\mathbb{R}^n$ . The curvature operator  $\mathbf{R}$  is diagonal with respect to  $e_1 \wedge e_2, \dots, e_{n-1} \wedge e_n$  and we denote by  $\mathbf{R}_{ij} = \frac{\lambda_i + \lambda_j}{n-2}$  the corresponding eigenvalues for  $1 \leq i < j \leq n$ . Inspection of (4) shows that also  $\mathbf{R}^2 + \mathbf{R}^\#$  is diagonal with respect to this basis. We have

$$\begin{aligned} (\mathbf{R}^2 + \mathbf{R}^\#)_{ij} &= \mathbf{R}_{ij}^2 + \sum_{k \neq i,j} \mathbf{R}_{ik} \mathbf{R}_{jk} \\ &= \frac{(\lambda_i + \lambda_j)^2}{(n-2)^2} + \frac{1}{(n-2)^2} \sum_{k \neq i,j} (\lambda_i + \lambda_k)(\lambda_j + \lambda_k) \\ &= \frac{\lambda_i \lambda_j}{(n-2)} + \frac{n\sigma - \lambda_i^2 - \lambda_j^2}{(n-2)^2} \end{aligned}$$

as claimed.

The second identity follows immediately from the first. To show the last identity notice that the Ricci tensor of  $\text{Ric}_0 \wedge \text{Ric}_0$  is given by  $-\text{Ric}_0^2$ . A computation shows the claim.  $\square$

*Proof of Theorem 2.* We first verify that  $D = D_{a,b}$  does not depend on the Weyl curvature of  $R$ . We view  $D$  as quadratic form in  $R$ . Then

$$B(R, S) := \frac{1}{4}(D(R+S) - D(R-S))$$

is the corresponding bilinear form.

Let  $S = W \in \langle W \rangle$ . We have to show  $B(R, W) = 0$  for all  $R \in S_B^2(\mathfrak{so}(n))$ . We start by considering  $R \in \langle W \rangle$ . Then  $l_{a,b}(R \pm W) = R \pm W$ . It follows from formula (6) for the Ricci curvature of  $R^2 + R^\#$  that  $(R \pm W)^2 + (R \pm W)^\#$  has vanishing Ricci tensor. Hence  $(R \pm W)^2 + (R \pm W)^\#$  is a Weyl curvature operator and accordingly fixed by  $l_{a,b}^{-1}$ .

Next we consider the case that  $R = I$  is the identity. Using the polarization formula (7) for  $W$  we see that  $B(I, W)$  is a multiple of  $W + W \# I$ , which is zero by Lemma 2.1.

It remains to consider the case of  $R \in \langle \text{Ric}_0 \rangle$ . Using the symmetry of the trilinear form  $\text{tri}$  defined in (5) we see for each  $W_2 \in \langle W \rangle$  that

$$\text{tri}(W, R, W_2) = \text{tri}(W, W_2, R) = 0$$

as  $W W_2 + W_2 W + 2W \# W_2$  lies in  $\langle W \rangle$  and  $R \in \langle \text{Ric}_0 \rangle$ . Combining this with  $\text{tri}(W, R, I) = 0$  gives that  $W R + R W + 2W \# R \in \langle \text{Ric}_0 \rangle$ . Using once more that  $l := l_{a,b}$  is the identity on  $\langle W \rangle$  we see that

$$l(W)l(R) + l(R)l(W) + 2l(W)\#l(R) = l(W R + R W + 2W \# R).$$

This clearly proves  $B(R, W) = 0$ .

Thus, for computing  $D$  we may assume that  $R_W = 0$ . So let  $R = R_I + R_{\text{Ric}_0}$ . We next verify that both sides of the equation have the same projection to the space  $\langle W \rangle$  of Weyl curvature operators. Recall that  $l_{a,b}^{-1}$  induces the identity on  $\langle W \rangle$  and that  $\text{Ric}_0(l_{a,b}(R)) = (1 + (n-2)b)\text{Ric}_0$ . Then using the second identity in Lemma 2.2 we see that

$$\begin{aligned} D_W &= \frac{1}{n-2}((1 + (n-2)b)^2 - 1)(\text{Ric}_0 \wedge \text{Ric}_0)_W \\ &= ((n-2)b^2 + 2b)(\text{Ric}_0 \wedge \text{Ric}_0)_W. \end{aligned}$$

It is straightforward to check that the right hand side in the asserted identity for  $D$  has the same projection to  $\langle W \rangle$ .

It remains to check that both sides of the equation have the same Ricci tensor. Because of  $\text{Ric}(l_{a,b}(R)) = (1 + (n-2)b)\text{Ric}_0 + (1 + 2(n-1)a)\bar{\lambda}\text{id}$ , the third identity in Lemma 2.2 implies

$$\begin{aligned} \text{Ric}(D) &= -2b(\text{Ric}_0^2)_0 + 2(n-2)a\bar{\lambda}\text{Ric}_0 + 2(n-1)a\bar{\lambda}^2\text{id} \\ &\quad + \frac{2(n-2)b + (n-2)^2b^2 - 2(n-1)a}{1 + 2(n-1)a}\sigma\text{id} \\ (8) \quad &= -2b\text{Ric}_0^2 + 2(n-2)a\bar{\lambda}\text{Ric}_0 + 2(n-1)a\bar{\lambda}^2\text{id} \\ &\quad + \frac{2(n-1)b + (n-2)^2b^2 - 2(n-1)a(1-2b)}{1 + 2(n-1)a}\sigma\text{id}. \end{aligned}$$

A straightforward computation shows that the same holds for the Ricci tensor of the right hand side in the asserted identity for  $D$ . This completes the proof.  $\square$

**Corollary 2.3.** *We keep the notation of Theorem 2, and let  $\sigma, \bar{\lambda}$  be as in (2). Suppose that  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $\text{Ric}_0$  corresponding to an orthonormal basis  $e_1, \dots, e_n$ . Then  $e_i \wedge e_j$  ( $i < j$ ) is an eigenvector of  $D_{a,b}$  corresponding to the eigenvalue*

$$d_{ij} = ((n-2)b^2 - 2(a-b))\lambda_i\lambda_j + 2a(\bar{\lambda} + \lambda_i)(\bar{\lambda} + \lambda_j) + b^2(\lambda_i^2 + \lambda_j^2) + \frac{\sigma}{1+2(n-1)a}(nb^2(1-2b) - 2(a-b)(1-2b+nb^2)).$$

Furthermore,  $e_i$  is an eigenvector of the Ricci tensor of  $D_{a,b}$  with respect to the eigenvalue

$$r_i = -2b\lambda_i^2 + 2a\bar{\lambda}(n-2)\lambda_i + 2a(n-1)\bar{\lambda}^2 + \frac{\sigma}{1+2(n-1)a}(n^2b^2 - 2(n-1)(a-b)(1-2b)).$$

Notice that  $\lambda_i + \bar{\lambda}$  are the eigenvalues of the Ricci tensor  $\text{Ric}$ . The first formula follows immediately from Theorem 2, the second from (8).

### 3. NEW INVARIANT SETS

We call a continuous family  $C(s)_{s \in [0,1]} \subset S_B^2(\mathfrak{so}(n))$  of closed convex  $O(n)$ -invariant cones of full dimension a pinching family, if

- (1) each  $R \in C(s) \setminus \{0\}$  has positive scalar curvature,
- (2)  $R^2 + R^\#$  is contained in the interior of the tangent cone of  $C(s)$  at  $R$  for all  $R \in C(s) \setminus \{0\}$  and all  $s \in (0, 1)$ ,
- (3)  $C(s)$  converges in the pointed Hausdorff topology to the one-dimensional cone  $\mathbb{R}^+I$  as  $s \rightarrow 1$ .

The main aim of this section is to prove

**Theorem 3.1.** *There is a pinching family  $C(s)_{s \in [0,1]}$  of closed convex cones such that  $C(0)$  is the cone of 2-nonnegative curvature operators.*

As before a curvature operator is called 2-nonnegative if the sum of its smallest two eigenvalues is nonnegative. It is known that the cone of 2-nonnegative curvature operators is invariant under the ordinary differential equation (1) (see [H4]). The pinching family that we construct for this cone is defined piecewise by three subfamilies. Each cone in the first subfamily is the image of the cone of 2-nonnegative curvature operators under a linear map. In fact we have the following general result.

**Proposition 3.2.** *Let  $C \subset S_B^2(\mathfrak{so}(n))$  be a closed convex  $O(n)$ -invariant subset which is invariant under the ordinary differential equation (1). Suppose that  $C \setminus \{0\}$  is contained in the half space of curvature operators with positive scalar curvature, that each  $R \in C$  has nonnegative Ricci curvature and that  $C$  contains all nonnegative curvature operators of rank 1. Then for  $n \geq 3$  and*

$$b \in (0, \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}] \quad \text{and} \quad 2a = 2b + (n-2)b^2$$

the set  $l_{a,b}(C)$  is invariant under the vector field corresponding to (1) as well. In fact, it is transverse to the boundary of the set at all boundary points  $R \neq 0$ .

Using the Bianchi identity it is straightforward to check that a nonnegative curvature operator of rank 1 corresponds up to a positive factor and a change of basis in  $\mathbb{R}^n$  to the curvature operator of  $\mathbb{S}^2 \times \mathbb{R}^{n-2}$ . The condition that  $C$  contains all these operators is equivalent to saying that  $C$  contains the cone of geometrically nonnegative curvature operators. A curvature operator is geometrically nonnegative if it can be written as the sum of nonnegative curvature operators of rank 1. In dimensions above 4 this cone is strictly smaller than the cone of nonnegative curvature operators. Although we will not need it, we remark that the cone of geometrically nonnegative curvature operators is invariant under (1) as well.

*Proof.* We have to prove that for each  $R \in C$  the curvature operator

$$(9) \quad X_{a,b} = l_{a,b}^{-1}(l_{a,b}(R)^2 + l_{a,b}(R)^\#)$$

lies in the tangent cone  $T_R C$  of  $C$  at the point  $R$ . Notice that by assumption we have  $R^2 + R^\# \in T_R C$ . Thus it suffices to show that  $D_{a,b} = X_{a,b} - R^2 - R^\#$  lies in  $T_R C$ . Since  $C$  contains all nonnegative curvature operators of rank 1, we can establish this by showing that  $D_{a,b}$  is positive for  $b > 0$ . Looking at the formula for the eigenvalues of  $D_{a,b}$  in Corollary 2.3 this amounts to showing that

$$0 \leq b^2(n(1-2b) - (n-2)(1-2b+nb^2))$$

holds in the given range. This is a straightforward computation.  $\square$

Let us remark that the intersection of two closed convex  $O(n)$ -invariant cones, which are invariant under the ordinary differential equation (1), have the same properties as the given cones.

**Corollary 3.3.** *In order to prove Theorem 3.1, it suffices to establish the existence of a pinching family  $C(s)_{s \in [0,1]}$  with  $C(0)$  being the cone of nonnegative curvature operators.*

*Proof.* Suppose  $n \geq 4$ . Notice that the cone  $C$  of 2-nonnegative curvature operators satisfies the assumptions of Proposition 3.2. We plan to show that the family of closed invariant cones from Proposition 3.2 can be extended to a pinching family. By the above remark it suffices to show that  $l_{a(b),b}(C \setminus \{0\})$  is contained in the open cone of positive curvature operators where  $b$  is the maximal allowed value from Proposition 3.2. In fact then we can extend the family from Proposition 3.2 to a pinching family by defining it on the second part of the interval as a reparameterization of the pinching family  $(C(s) \cap C')_{s \in [0,1]}$  where  $C' := l_{a(b),b}(C)$ .

Let  $R \in C \setminus \{0\}$ . Recall that by (7) we have  $l_{a,b}(R) = R + 2b \operatorname{Ric} \wedge \operatorname{id} + h R_I$  for  $h := 2(n-1)(a-b)$ . The smallest eigenvalue of  $R$  is by a standard estimate larger than or equal to  $-\frac{2 \operatorname{tr}(R)}{n(n-1)-2}$ . Moreover, since the sum of the two smallest eigenvalues of  $R$  is nonnegative the smallest eigenvalue of  $\operatorname{Ric}$  is bounded from below by  $(n-3)$  times the absolute value of the smallest eigenvalue of  $R$ . Thus in order to show that  $l_{a,b}(R) > 0$  it is sufficient to prove  $h > (1-2b)\frac{n(n-1)}{n(n-1)-2}$ . This is equivalent to

$$(n-2)b^2 > (1-2b)\frac{n}{(n+1)(n-2)}.$$

By the definition of  $b$  we have  $(n-2)b^2 = \frac{2}{n}(1-2b)$ . This shows the claim for  $n \geq 4$ . For  $n = 3$  Theorem 3.1 is well known.  $\square$



It remains to construct a pinching family for the cone of nonnegative curvature operators. This pinching family will be defined up to parameterization piecewise by two subfamilies in the next two lemmas.

**Lemma 3.4.** *For  $b \in [0, 1/2]$  put*

$$a = \frac{(n-2)b^2 + 2b}{2 + 2(n-2)b^2} \quad \text{and} \quad p = \frac{(n-2)b^2}{1 + (n-2)b^2}.$$

*Then the set*

$$I_{a,b} \left( \left\{ \mathbf{R} \in S_B^2(\mathfrak{so}(n)) \mid \mathbf{R} \geq 0, \text{Ric} \geq p(b) \frac{\text{tr}(\text{Ric})}{n} \right\} \right)$$

*is invariant under the vector field corresponding to (1). In fact, for  $b \in (0, 1/2]$  it is transverse to the boundary of the set at all boundary points  $\mathbf{R} \neq 0$ .*

*Proof.* Put

$$C(p) := \left\{ \mathbf{R} \in S_B^2(\mathfrak{so}(n)) \mid \mathbf{R} \geq 0, \text{Ric} \geq p(b) \frac{\text{tr}(\text{Ric})}{n} \right\}.$$

It suffices to check that for  $\mathbf{R} \in C(p) \setminus \{0\}$  the pulled back vector field  $X_{a,b}$  defined in (9) is in the interior of the tangent cone of  $C(p)$  at  $\mathbf{R}$ .

In the first step we verify that  $X_{a,b}$  is positive definite for  $b \in (0, 1/2]$ . Since  $\mathbf{R}^2 + \mathbf{R}^\#$  is positive semi-definite, we can establish  $X_{a,b} > 0$  by showing  $D_{a,b} > 0$ . Since by assumption  $\mathbf{R} \in C(p)$  we have the following estimate for the eigenvalues of  $\text{Ric}_0$ :

$$\lambda_i \geq -(1-p)\bar{\lambda}.$$

Next, observe that

$$2(a-b) = \frac{1-2b}{1+(n-2)b^2}(n-2)b^2.$$

We use the notation of Theorem 2 and Corollary 2.3. Rewriting  $d_{ij}$  gives

$$\begin{aligned} (10) \quad d_{ij} &= \frac{2a}{1-p}((1-p)\bar{\lambda} + \lambda_i)((1-p)\bar{\lambda} + \lambda_j) + 2ap\bar{\lambda}^2 + b^2(\lambda_i^2 + \lambda_j^2) \\ &\quad + \frac{n(1+(n-2)b^2) - (n-2)(1-2b+nb^2)}{(1+2(n-1)a)(1+(n-2)b^2)} \sigma b^2(1-2b) \\ &> \frac{2+2(n-2)b}{(1+2(n-1)a)(1+(n-2)b^2)} \sigma b^2(1-2b) \\ &\geq 0. \end{aligned}$$

In the second step we must show that the above Ricci pinching is preserved by the ordinary differential equation (1). Let  $\text{Ric}(X_{a,b})$  denote the Ricci tensor of  $X_{a,b}$ . Assume that  $\lambda_i = -(1-p)\bar{\lambda}$ . We have to show that

$$\text{Ric}(X_{a,b})_{ii} > p \frac{\text{scal}(X_{a,b})}{n} = p \left( (1+2(n-1)a)\bar{\lambda}^2 + \frac{(1+(n-2)b)^2}{1+2(n-1)a} \sigma \right)$$

holds for  $b \in (0, 1/2]$ . We first observe that by (6)

$$\text{Ric}(\mathbf{R}^2 + \mathbf{R}^\#)_{ii} = \sum_{k \neq i} \text{Ric}_{kk} \text{R}_{kiii} \geq \sum_{k \neq i} p\bar{\lambda} \text{R}_{kiii} = p^2 \bar{\lambda}^2.$$

Using formula (10) for  $d_{ij}$  and  $\lambda_i = -(1-p)\bar{\lambda}$  we see that

$$\begin{aligned} \text{Ric}(X_{a,b})_{ii} &\geq p^2\bar{\lambda}^2 + \sum_{j \neq i} d_{ij} \\ &= p^2\bar{\lambda}^2 + 2(n-1)ap\bar{\lambda}^2 + (n-2)b^2(1-p)^2\bar{\lambda}^2 + nb^2\sigma \\ &\quad + \frac{(n-1)\sigma b^2(1-2b)}{(1+2(n-1)a)(1+(n-2)b^2)}(2+2(n-2)b). \end{aligned}$$

By our choice for  $b$  and  $p$  it is straightforward to check that

$$p^2 + (n-2)b^2(1-p)^2 = p.$$

This shows that in the asserted inequality the  $\bar{\lambda}^2$ -terms cancel each other. Since  $\sigma > 0$  it remains to verify

$$nb^2 + \frac{(n-1)b^2(1-2b)}{(1+2(n-1)a)(1+(n-2)b^2)}(2+2(n-2)b) > p \frac{(1+(n-2)b)^2}{1+2(n-1)a}.$$

The identity

$$(1+2(n-1)a) = \frac{1+2(n-1)b+n(n-2)b^2}{1+(n-2)b^2}$$

shows that this is equivalent to

$$\begin{aligned} 0 &< n(1+2(n-1)b+n(n-2)b^2) - (n-2)(1+(n-2)b)^2 \\ &\quad + (n-1)(1-2b)(2+2(n-2)b) \\ &= 2n + 2n(n-2)b. \end{aligned}$$

This shows the claim.  $\square$

We remark that the above sets remain in fact invariant for all  $b > 0$ . For  $b \rightarrow +\infty$  they converge to an invariant set of Einstein curvature operators.

We will now finish the proof of Theorem 3.1 by showing that the cone from Lemma 3.4 for  $b = 1/2$  can be joined by a continuous family of invariant cones with arbitrarily small cones around the identity.

**Lemma 3.5.** *Assume  $b = 1/2$  and put for  $s \geq 0$*

$$a = \frac{1+s}{2} \quad \text{and} \quad p = 1 - \frac{4}{n+2+4s}.$$

*Then the set*

$$l_{a,b} \left( \left\{ R \in S_B^2(\mathfrak{so}(n)) \mid R \geq 0, \text{Ric} \geq p(s) \frac{\text{tr}(\text{Ric})}{n} \right\} \right)$$

*is invariant under the vector field corresponding to (1). In fact, it is transverse to the boundary of the set at all boundary points  $R \neq 0$ .*

Notice that  $\lim_{s \rightarrow \infty} \frac{1}{a} l_{a,b}(R) = 2(n-1)R_I$ . Consequently the cones of the lemma converge to  $\mathbb{R}^+ I$  for  $s \rightarrow \infty$ .

*Proof.* Notice that the formulas in Corollary 2.3 simplify:

$$\begin{aligned} d_{ij} &= \left( \frac{1}{4}(n-2) - s \right) \lambda_i \lambda_j + (s+1)(\bar{\lambda} + \lambda_i)(\bar{\lambda} + \lambda_j) + \frac{1}{4}(\lambda_i^2 + \lambda_j^2) \\ &\quad - \frac{\sigma n s}{4n + 4(n-1)s} \end{aligned}$$

and

$$r_i = -\lambda_i^2 + (s+1)\bar{\lambda}(n-2)\lambda_i + (s+1)(n-1)\bar{\lambda}^2 + \frac{\sigma n^2}{4n+4(n-1)s}.$$

We first verify that  $X_{a,b}$  does preserve the Ricci pinching. We may suppose that  $\lambda_i = -(1-p)\bar{\lambda}$ . We have to show

$$0 \leq p^2\bar{\lambda}^2 - (1-p)^2\bar{\lambda}^2 - (s+1)\bar{\lambda}^2(n-2)(1-p) + (s+1)(n-1)\bar{\lambda}^2 + \frac{\sigma n^2}{4n+4(n-1)s} - p\left((n+(n-1)s)\bar{\lambda}^2 + \frac{n^2}{4n+4(n-1)s}\sigma\right).$$

Because of  $\sigma \geq 0$  we can neglect the terms with  $\sigma$ . Dividing by  $\bar{\lambda}^2$  gives

$$p^2 - (1-p)^2 + (s+1) + (s+1)p(n-2) - p(n+(n-1)s) = s(1-p),$$

which is clearly positive. Notice that this calculation is independent of  $p$ . As before we can complete the proof by showing that  $D_{a,b}$  is positive definite. Using

$$\sigma \leq (n-1)(1-p)^2\bar{\lambda}^2 = \frac{16(n-1)\bar{\lambda}^2}{(n+2+4s)^2}$$

we see that

$$\begin{aligned} d_{ij} &= \frac{n+2}{4}\left(\lambda_i + \frac{4\bar{\lambda}}{n+2}\right)\left(\lambda_j + \frac{4\bar{\lambda}}{n+2}\right) + s\bar{\lambda}\left(\lambda_i + \lambda_j + \frac{8\bar{\lambda}}{n+2+4s}\right) \\ &\quad + \frac{1}{4}(\lambda_i^2 + \lambda_j^2) + \frac{n-2}{n+2}\bar{\lambda}^2 + s\frac{n-6+4s}{n+2+4s}\bar{\lambda}^2 - \frac{\sigma ns}{4n+4(n-1)s} \\ &\geq \left(\frac{n-2}{n+2} + s\frac{n-6+4s}{n+2+4s} - \frac{16(n-1)ns}{(4n+4(n-1)s)(n+2+4s)^2}\right)\bar{\lambda}^2 \\ &> (5+s(n-6)+4s^2-4s)\frac{\bar{\lambda}^2}{n+2+4s} > 0 \end{aligned}$$

where we used  $n \geq 3$  in the last two inequalities.  $\square$

#### 4. CONSTRUCTING A GENERALIZED PINCHING SET FROM A FAMILY OF INVARIANT CONES

We show how to construct from a family of invariant cones a generalized pinching set, similar to Hamilton's concept in [H2]. Let us recall that we denoted by  $S_B^2(\mathfrak{so}(n))$  the space of curvature operators.

**Theorem 4.1.** *Let  $C(s)_{s \in [0,1]} \subset S_B^2(\mathfrak{so}(n))$  be a continuous family of closed convex  $\text{SO}(n)$ -invariant cones of full dimension, such that  $C(s) \setminus \{0\}$  is contained in the half space of curvature operators with positive scalar curvature. Suppose that for  $R \in C(s) \setminus \{0\}$  the vector field  $X(R) = R^2 + R^\#$  is contained in the interior of the tangent cone of  $C(s)$  at  $R$  for all  $s \in (0,1)$ . Then for  $\varepsilon, h_0 > 0$  there exists a closed convex  $\text{SO}(n)$ -invariant subset  $F \subset S_B^2(\mathfrak{so}(n))$  with the following properties:*

- (1)  $F$  is invariant under the vector field  $X$ .
- (2)  $C(\varepsilon) \cap \{R \mid \text{tr}(R) \leq h_0\} \subset F$ .
- (3)  $F \setminus C(s)$  is relatively compact for all  $s \in [\varepsilon, 1)$ .

We remark that  $F$  is  $\text{O}(n)$ -invariant if the cones are. We also note that the analogue of the theorem holds in the vector space of Kähler curvature operators.

*Proof.* Let  $F$  denote the minimal closed convex  $\mathrm{SO}(n)$ -invariant subset which is invariant under the flow of  $X$  and which contains the set

$$C(\varepsilon) \cap \{\mathbf{R} \mid \mathrm{tr}(\mathbf{R}) \leq h_0\}.$$

Notice that  $F$  is the intersection of all subsets which satisfy the above properties. In particular  $F$  is well defined and  $F \subset C(\varepsilon)$ . We have to prove that for all  $s$  the set  $F \setminus C(s)$  is bounded.

Suppose on the contrary that  $F \setminus C(s)$  is not bounded for some  $s$ . Let  $s_0 \geq \varepsilon$  denote the infimum among all  $s$  with this property. Since the vector field  $X$  is transverse to the boundary of  $C(s_0)$  it is clear that for all small  $\delta$  the cone  $C_\delta(s_0)$  over the convex set

$$\{\mathbf{R} \in C(s_0) \mid \mathrm{tr}(\mathbf{R}) = 1, d(\mathbf{R}, \partial C(s_0)) \geq \delta\}$$

is invariant under  $X$ . For small  $\delta$  the cone  $C_\delta(s_0)$  has maximal dimension and  $C_0(s_0) = C(s_0)$ . We now choose  $\delta_0$  so small that the vector field  $X$  is transverse to the boundary of  $C_\delta(s_0)$  for all  $\mathbf{R} \neq 0$  and for all  $\delta \in [0, \delta_0]$ .

A simple compactness argument shows that there is some constant  $\eta$  such that for each  $R \in C_\delta(s_0)$  the vector field  $X$  has distance at least  $\eta\|R\|^2$  to the boundary of the tangent cone of  $C_\delta(s_0)$  at  $R$ . We note that  $X$  is locally Lipschitz continuous with a Lipschitz constant that grows linearly in  $\|R\|$ . Combining both facts we see that there is some constant  $c > 0$  such that the truncated shifted cone

$$TC_\delta(s_0) := \{\mathbf{R} \mid \mathbf{R} + I \in C_\delta(s_0), \mathrm{tr}(\mathbf{R}) \geq \bar{h}\}$$

is invariant under the flow of  $X$  for all  $\delta \in [0, \delta_0]$ .

Consequently for small  $\delta > 0$  we have that  $C(s_0) \cap \{\mathbf{R} \mid \mathrm{tr}(\mathbf{R}) = \bar{h}\}$  is contained in the interior of  $TC_\delta(s_0)$ . Since the family  $C(s)$  is continuous, we conclude

$$C(\bar{s}) \cap \{\mathbf{R} \mid \mathrm{tr}(\mathbf{R}) = \bar{h}\} \subset TC_\delta(s_0)$$

for some  $\varepsilon \leq \bar{s} < s_0$ . In the case of  $s_0 = \varepsilon$  put  $\bar{s} = \varepsilon$ . By the definition of  $s_0$  we can choose  $k \in \mathbb{N}$  so large that

$$F \cap \{\mathbf{R} \mid \mathrm{tr}(\mathbf{R}) = k\bar{h}\} \subset C(\bar{s}) \cap \{\mathbf{R} \mid \mathrm{tr}(\mathbf{R}) = k\bar{h}\} \subset k \cdot TC_\delta(s_0).$$

The scaled set  $k \cdot TC_\delta(s_0)$  is invariant under the flow of  $X$  too, since  $X(k\mathbf{R}) = k^2X(\mathbf{R})$ . Thus the set

$$F' := \left( F \cap \{\mathbf{R} \mid \mathrm{tr}(\mathbf{R}) \leq k\bar{h}\} \right) \cup \left( F \cap k \cdot TC_\delta(s_0) \right)$$

is convex and invariant under the flow of  $X$ . By assumption  $F' \subset F$ . On the other hand  $TC_\delta(s_0) \setminus C_{\delta/2}(s_0)$  is bounded. By the continuity of the family it follows that  $F \setminus C(s)$  is bounded for all  $s$  which are sufficiently close to  $s_0$ . A contradiction to the choice of  $s_0$ .  $\square$

## 5. PROOF OF THE MAIN RESULT

Using Theorem 3.1, Theorem 1 is an immediate consequence of the following

**Theorem 5.1.** *Let  $C(s)_{s \in [0,1]} \subset S_B^2(\mathfrak{so}(n))$  be a pinching family of closed convex cones,  $n \geq 3$ . Suppose that  $(M, g)$  is a compact Riemannian manifold such that the curvature operator of  $M$  at each point is contained in the interior of  $C(0)$ . Then the normalized Ricci flow evolves  $g$  to a constant curvature limit metric.*

*Proof.* Let  $R_p$  denote the curvature operator of  $(M, g)$  at a point  $p \in M$ . For all  $p \in M$  we have

$$R_p \in \{R \mid \text{scal} \leq h_0\} \cap C(\varepsilon)$$

for a sufficiently small  $\varepsilon > 0$  and a sufficiently large  $h_0$ , since the family of cones is continuous and  $M$  is compact. For this pair  $\varepsilon, h_0$  we consider an invariant set  $F$  as in Theorem 4.1.

By the maximum principle the Ricci flow evolves  $g$  to metrics  $g_t$  whose curvature operators at each point are contained in  $F$ . We do also know that the solution of the Ricci flow exists as long as the curvature does not tend to infinity. Furthermore it follows from the maximum principle that the Ricci flow exists only on a finite time interval  $t \in [0, t_0)$ . By Shi [Sh] it follows from the maximum principle applied to the evolution equation for the  $i$ -th derivatives of the curvature operator that

$$\max \|\nabla^i R_t\|^2 \leq C_i \max \|R_t\|^{i+2}$$

for all  $t \in [t_0/2, t_0)$ .

We now rescale each metric  $g_t$  to a metric  $\tilde{g}_t$  such that the maximal sectional curvature is equal to 1. From the above estimates we get a priori bounds for all derivatives of the curvature tensor of the metric  $\tilde{g}_t$  for  $t \in [t_0/2, t_0)$ .

Next, we pick a point  $p_t \in (M, g_t)$  such that the sectional curvature attains its maximum in the ball  $B_\pi(p_t)$  of radius  $\pi$  around  $p_t$ . We pull the metric via the exponential map back to the ball of radius  $\pi$  in  $T_{p_t}M$ . By choosing a linear isometry  $\mathbb{R}^n \rightarrow T_{p_t}M$  we identify this ball with the ball  $B_\pi(0) \subset \mathbb{R}^n$  and denote by  $\bar{g}_t$  the induced metric on  $B_\pi(0)$ . From the above estimates on the derivatives of the curvature tensor it is clear that for any sequence  $(t_k)$  in  $[0, t_0)$  converging to  $t_0$  there is a subsequence of  $(\bar{g}_{t_k})$  converging in the  $C^\infty$  topology to a limit metric.

Let now  $\lambda_j$  denote the scaling factors of these metrics  $\bar{g}_{t_j}$  which by assumption tend to infinity. At each point of  $M$  the curvature operator of the limit metric is contained in the set

$$\bigcap \frac{1}{\lambda_j^2} F = \mathbb{R}^+ I.$$

Thus the limit metric on  $B_\pi(0)$  has pointwise constant sectional curvature. Since  $n \geq 3$ , it has constant curvature one by Schur's theorem.

Since the sequence was arbitrary, the minimal sectional curvature converges on a ball of radius  $\pi$  around  $p_t$  in  $(M, g_t)$  to 1 as well as  $t$  tends to  $t_0$ . Notice that this argument works for all  $p_t \in B_\pi(q_t)$ , where  $q_t$  denotes a point where the sectional curvature attains its maximum 1. Therefore the minimal sectional curvature converges on the ball of radius  $2\pi$  around  $q_t$  to 1 as well. By the theorem of Bonnet Myers  $\text{diam}(M, \tilde{g}_t) \leq 3\pi/2$  for large  $t < t_0$  and consequently, also the minimum of the sectional curvature of  $(M, \tilde{g}_t)$  tends to 1 for  $t \rightarrow t_0$ .

In the case of manifolds one is done since by Klingenberg's injectivity radius estimate [CE] collapse can not occur. Alternatively, one can use the fact that  $(M, g_t)$  satisfies the assumption of Huisken's theorem [Hu] for suitable large  $t$ . In the case of orbifolds one has to use additionally Proposition 5.2 from below.  $\square$

Let us remark that collapse in the above situation can also be ruled out by applying Perelman's local injectivity radius estimate for the Ricci flow [Pe].

**Proposition 5.2.** *Let  $(X, g)$  be a compact orbifold with sectional curvature  $K$ . If  $n \geq 3$  and  $g$  is strictly quarter pinched, that is  $1/4 < K \leq 1$ , then  $X$  is the quotient of a Riemannian manifold by a finite isometric group action.*

*Proof.* By replacing  $X$  by a cover if necessary we may assume that  $X$  is not a nontrivial quotient of an orbifold by a finite group action. We then have to show that  $X$  is a manifold. Recall that the frame bundle  $FX$  of the orbifold  $X$ , endowed with the connection metric of  $g$ , is a Riemannian manifold. We consider an  $\mathrm{SO}(n)$  orbit  $\mathrm{SO}(n)v$  in  $FX$ . Clearly the normal exponential map of the orbit  $\mathrm{SO}(n)v$  has a focal radius  $\geq \pi$ . Similarly to Klingenberg's injectivity radius estimate we show below that the normal exponential map of the orbit  $\mathrm{SO}(n)v$  has injectivity radius  $\geq \pi$ . Since the orbit was arbitrary, this rules out exceptional orbits and hence  $X$  is then a manifold.

From the assumption that  $X$  is not a nontrivial quotient it follows that the natural map  $\pi_1(\mathrm{SO}(n)) \rightarrow \pi_1(FX)$  is surjective. This implies that the space  $\Omega_{\mathrm{SO}(n)v}FX$  of all curves starting and ending in  $\mathrm{SO}(n)v$  is connected. The critical levels of the energy functional in  $\Omega_{\mathrm{SO}(n)v}FX$  are in one to one correspondence to the geodesic loops in the orbifold.

Suppose on the contrary that the injectivity radius of the normal exponential map of  $\mathrm{SO}(n)v$  is equal to  $r < \pi$ . It is then easy to see that there is a horizontal geodesic  $c$  of length  $2r$  in  $\Omega_{\mathrm{SO}(n)v}FX$ . Analogously to Klingenberg's long homotopy lemma one can show that every path  $c_s$  in  $\Omega_{\mathrm{SO}(n)v}FX$  that connects  $c = c_0$  with a constant curve  $c_1$  satisfies  $L(c_s) \geq 2\pi$  for some  $s$ . In other words the space of paths of energy  $< 2\pi^2$  is not connected.

On the other hand it is straightforward to check that the critical points of the energy function with energy  $\geq 2\pi^2$  have indices at least  $n - 1 \geq 2$ . But then by a standard degenerate Morse theory argument the loop space  $\Omega_{\mathrm{SO}(n)v}FX$  itself is not connected – a contradiction.  $\square$

- Remark 5.3.*
1. The main difference between the two-dimensional and the higher dimensional case is that in dimension two, Schur's theorem fails.
  2. Proposition 5.2 does not remain valid in dimension two either. In fact given any positive  $\delta < 1$ , there is a  $\delta$  pinched two-dimensional orbifold  $X$  which is not the quotient of a manifold: Consider two discs of constant curvature 1 and with totally geodesic boundary. Divide out the cyclic group of order  $(p + 1)$  from the first disc and the cyclic group of order  $p$  from the second. After scaling the first disc by the factor  $\frac{p+1}{p}$  the two orbifolds can be glued along their common boundary. By smoothing this example for some large  $p$  one obtains the claimed result.
  3. The space of 3-positive curvature operators is not invariant under the ordinary differential equation (1) for  $n \geq 4$ .

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