

ORIGINAL PAPER

Some definable types that cannot be amalgamated

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Deutsche Forschungsgemeinschaft, Grant/Award Numbers: EXC 2044-390685587, HI 2004/1-1; Ministero dell'Istruzione, dell'Università e della Ricerca, Grant/Award Number: PRIN 2017: Prot. 2017NWTM8RPRIN

We exhibit a theory where definable types lack the amalgamation property.

If $q_0(x, y)$ and $q_1(x, z)$ are types over a given set A , both extending the same type $p(x) \in S(A)$, it is an easy exercise to show that there is $r(x, y, z) \in S(A)$ extending $q_0(x, y)$ and $q_1(x, z)$ simultaneously. In other words, in every theory, types have the *amalgamation property*. Suppose now that p, q_0, q_1 all belong to some special class \mathcal{K} of types, and consider the following question: among the amalgams r as above, is there always one which belongs to \mathcal{K} ? In this short note, we prove that, for \mathcal{K} the class of definable types, the answer is in general negative.

By a fundamental result of Shelah, a complete theory is stable if and only if all types over models are definable. Definable types, and the tightly related notion of *stable embeddedness*, recently attracted considerable attention in unstable contexts as well. For instance, Hrushovski isolated a criterion for elimination of imaginaries in terms of density of definable types, which yielded a simplified proof of the classification of imaginaries in algebraically closed valued fields [7], and similar classification results in other (enriched) henselian valued fields [4, 5, 11]. In o-minimal theories, stable embeddedness of elementary substructures corresponds to relative Dedekind completeness [9], and in benign theories of henselian valued fields stable embeddedness obeys an Ax–Kochen–Ershov principle [2, 12]. Definable types are also central in Hrushovski and Loeser's celebrated work on Berkovich analytifications [6]: their *stable completions* of algebraic varieties are certain spaces of definable types which, crucially, form *strict pro-definable sets*.

This brings us to the main motivation for the present paper. If T is stable, then definable types over M may be seen as a pro-definable set in M^{eq} (this is a special case of [6, Lemma 2.5.1]), albeit this pro-definability need not be strict: it follows from work of Poizat on *belles paires* [10] that, if T is stable, then definable types over models form strict pro-definable sets if and only if T is nfcp, if and only if all belles paires of models of T are ω -saturated (cf. [3, § 3.1]). In order to establish strict pro-definability for other spaces of definable types, Cubides, Ye and the first author [3] recently introduced *beautiful pairs* in an arbitrary L -theory T . Poizat's belles paires are beautiful, and his theory generalises smoothly to unstable T ,

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provided the latter satisfies certain assumptions: an extension property called (EP), and the amalgamation property (AP) for definable types.

In stable theories, (AP) and (EP) always hold. Similarly in o-minimal theories, where (AP) follows from a result of Baisalov–Poizat [1] (cf. [3, § 3.2]). In benign valued fields, there is an Ax–Kochen–Ershov type reduction for (AP) [3, § 8]. In [3, Corollary 2.4.16] an example of a (dp-minimal) theory satisfying (AP) but not (EP) was found, namely the levelled binary tree with level set ω . Whether there is a theory where (AP) fails is left open in [3].

Building on the aforementioned tree, we construct such an example.

1 | THE THEORY

Models M of the theory in which our counterexample lives are four-sorted, and are roughly obtained as follows. We start with a binary tree $T(M)$, with discrete level set $L(M)$. We then introduce two levelled sets $A(M)$ and $B(M)$, both with the same level set as the tree, namely $L(M)$, and cover each level x of $T(M)$ with a generic surjection from the cartesian product of the x th levels of $A(M)$ and $B(M)$. In this section, we spell out this construction in detail.

Definition 1.1. Let L be the following language.

1. L has four sorts A , B , T , and L .
2. T has a binary relation \leq_T , a binary function \sqcap , a unary function pred_T , and constants g_T, r .
3. L has a binary relation \leq_L , a unary function pred_L , and constants $g_L, 0$.
4. There are functions $\ell_T : T \rightarrow L$, $\ell_A : A \rightarrow L$, and $\ell_B : B \rightarrow L$.
5. There is a function $f : A \times B \rightarrow T$.

Definition 1.2. Let T be the L -theory expressing the following properties.

- (i) $0 \neq g_L$, and $(L \setminus \{g_L\}, \leq_L)$ is a discrete linear order with smallest element 0 and no largest element, with predecessor function pred_L , with the convention that $\text{pred}_L(0) = 0$. The “garbage” point g_L is not \leq_L -related to anything, and $\text{pred}_L(g_L) = g_L$.
- (ii) $(T \setminus \{g_T\}, \leq_T, \sqcap)$ is a meet-tree, viewed as a semilinear order \leq_T with associated meet function \sqcap , root r , binary ramification,¹ and, for every fixed element, its set of predecessors is a discrete linear order, with predecessor function pred_T , with the convention that $\text{pred}_T(r) = r$. The “garbage” point g_T behaves similarly to the garbage point g_L .
- (iii) ℓ_T is a surjective level function $T \setminus \{g_T\} \rightarrow L \setminus \{g_L\}$, extended by $\ell_T(g_T) = g_L$. For every fixed element $t \in T \setminus \{g_T\}$, the restriction of ℓ_T to the set of predecessors of t defines an order isomorphism onto an initial segment of $L \setminus \{g_L\}$ (in particular, $\ell_T \circ \text{pred}_T = \text{pred}_L \circ \ell_T$). Moreover, for any t from $T \setminus \{g_T\}$ and any y in L with $\ell_T(t) \leq_L y$ there is t' in T with $t \leq_T t'$ such that $\ell_T(t') = y$.
- (iv) g_L is not in the image of $\ell_A : A \rightarrow L$, nor in that of $\ell_B : B \rightarrow L$.
- (v) For every $c \in L \setminus \{g_L\}$ the fibers $\ell_A^{-1}(c)$ and $\ell_B^{-1}(c)$ are infinite.
- (vi) $f(a, b) = g_T$ if and only if $\ell_A(a) \neq \ell_B(b)$.
- (vii) If $\ell_A(a) = \ell_B(b)$, then $\ell_T(f(a, b)) = \ell_A(a)$.
- (viii) At any level, f defines a *generic surjection*: for any $c \in L \setminus \{g_L\}$, any t_1, \dots, t_n from $T \setminus \{g_T\}$ and any pairwise distinct a_1, \dots, a_n from A such that $\ell_T(t_i) = c = \ell_A(a_i)$ for all i , there are infinitely many b from B such that, for all i , we have $f(a_i, b) = t_i$; similarly with the roles of A and B interchanged.

Recall that a meet-tree together with a linear order and a map satisfying (iii) above is called a *levelled tree*.

Proposition 1.3. The following properties hold.

1. T is complete and admits quantifier elimination.
2. The union of the definable sets $T \setminus \{g_T\}$ and $L \setminus \{g_L\}$ is stably embedded, with induced structure a pure levelled (binary) meet-tree. In particular, $L \setminus \{g_L\}$ is stably embedded with induced structure a pure ordered set.

¹ We fix binary ramification for simplicity, but this is not important: any fixed finite ramification will work.

Proof. It is easy to see that T is consistent. It is enough to prove quantifier elimination: (2) is a direct consequence of the latter; as for completeness, it follows (cf., e.g., [8, Proposition 18.4]) from quantifier elimination and the fact that, given an arbitrary model of T , the L -substructure with underlying set the interpretations of the closed L -terms over \emptyset embeds in every other model.

Let N_0 and N_1 be models of T with N_0 countable and N_1 \aleph_1 -saturated, and let M be a common L -substructure of N_0 and N_1 . It is an easy exercise to M -embed N_0 into N_1 , yielding quantifier elimination. \square

The theory induced on $T \setminus \{g_T\}$ and $L \setminus \{g_L\}$ is thus precisely the one used in [3, Fact 2.4.15].

Lemma 1.4. For all $M \models T$, all linearly ordered definable subsets of $T(M)$ have a maximum.

Proof. This follows from quantifier elimination. Alternatively, one may use that no infinite branch is definable in the standard binary meet-tree $(2^{<\omega}, \omega)$, e.g., since for any $n \in \omega$ and branches $s, s' \in 2^\omega$ with $s \upharpoonright_n = s' \upharpoonright_n$ there is an automorphism σ over $2^{<n}$ with $\sigma(s) = s'$. \square

2 | THE TYPES

Let T be the theory defined in the previous section, and $\mathfrak{U} \models T$ a monster model. Failure of amalgamation of definable types boils down to the following phenomenon. All elements y of A with level larger than $L(\mathfrak{U})$ have the same, definable, type, and similarly for z in B ; nevertheless, if such y and z have the *same* infinite level, then $f(y, z)$ can be used to produce an externally definable subset of $T(\mathfrak{U})$ which is not definable. More formally, we proceed as follows.

Definition 2.1. Define the following sets of $L(\mathfrak{U})$ -formulas.

1. $p(x)$ is the global type of an element x of sort L such that $x > L(\mathfrak{U})$.
2. $q_A(x, y)$ restricts to p on x , and says that y is an element of sort A with $\ell_A(y) = x$.
3. $q_B(x, z)$ restricts to p on x , and says that z is an element of sort B with $\ell_B(y) = x$.

Lemma 2.2. All of p, q_A, q_B are complete types over \mathfrak{U} which are \emptyset -definable.

Proof. Consistency and \emptyset -definability are clear. As for completeness, we argue as follows.

1. Completeness of $p(x)$ follows from Proposition 1.3(2).
2. As for $q_A(x, y)$, note that, since y has a new level, it cannot be in $A(\mathfrak{U})$. Again because $\ell_A(y)$ is new, for all $b \in B(\mathfrak{U})$ we must have $f(y, b) = g_T$. By quantifier elimination, this is enough to determine a complete type.
3. The argument for $q_B(x, z)$ is symmetrical. \square

Proposition 2.3. The types $q_A(x, y)$ and $q_B(x, z)$ cannot be amalgamated over $p(x)$ into a definable type. In other words, no completion of $q_A(x, y) \cup q_B(x, z)$ is definable.

Proof. Suppose $r(x, y, z)$ is a completion of $q_A(x, y) \cup q_B(x, z)$. Then $r(x, y, z) \vdash \ell_A(y) = x = \ell_B(z)$, thus $r(x, y, z) \vdash \ell_T(f(y, z)) = x$. Since $p(x)$ is not realised, $f(y, z)$, having a new level, cannot be in $T(\mathfrak{U})$. Consider the set $\{d \in T(\mathfrak{U}) \mid r(x, y, z) \vdash f(y, z) > d\}$. If $r(x, y, z)$ is definable, then this set is definable. As a set of predecessors, it must be linearly ordered, hence have a maximum by Lemma 1.4. But then $f(y, z) \notin \mathfrak{U}$ contradicts binary ramification. \square

Corollary 2.4. In T , global definable types do not have the amalgamation property.

This partially answers [3, Question 9.3.1], asking whether there is such a theory which, additionally, has uniform definability of types; note that T does not. Moreover, T is easily shown to have IP.

Question 2.5. Is there a NIP theory where global definable types do not have the amalgamation property?

ACKNOWLEDGMENTS

Martin Hils was partially supported by the German Research Foundation (DFG) via HI 2004/1-1 (part of the French-German ANR-DFG project GeoMod) and under Germany's Excellence Strategy EXC 2044-390685587, 'Mathematics Münster: Dynamics-Geometry-Structure'. Rosario Mennuni is supported by the project PRIN 2017: "Mathematical Logic: models, sets, computability" Prot. 2017NWTM8RPRIN.

CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

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