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Semifree actions of free groups

Martin Hils

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Abstract We study countable universes similar to a free action of a group G. It turns out that this is equivalent to the study of free semi-actions of G, with two universes being transformable iff one corresponding free semi-action can be obtained from the other by a finite alteration. In the case of a free group G (in finitely many or countably many generators), a classification is given.

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1 Introduction

In [2], the concept of the universe of a first order structure is introduced. This provides the appropriate framework to study the boolean algebra of parameter-definable sets of a (first order) structure without sticking to a particular language. In the category of universes, transformations play the role of isomorphisms, whereas similarity replaces the usual elementary equivalence between first order structures.

Several fundamental questions about universes are raised in [2]. Probably the most intriguing one is Problem 7 in [2], the classification problem for countable universes similar to an uncountably categorical universe.

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M. Hils
Institut Camille Jordan (Lyon 1), Université Claude Bernard,
21, avenue Claude Bernard, 69622 Villeurbanne-Cedex, France

Present Address:
M. Hils (✉)
Institut für Mathematik, Humboldt-Universität zu Berlin, 10099 Berlin, Germany
e-mail: hils@mathematik.hu-berlin.de

There is an example of a two-dimensional universe having exactly two similar non-transformable countable universes. It is the universe of the following first order $\mathcal{L}$-structure, where $\mathcal{L} := \{s, P\}$: the function s is a bijection without cycles, P is an infinite coinfinite unary predicate such that in every s-orbit at most one element is in P. See [2, 8.5] for details. Note that in this structure, both dimensions are given by minimal types with a trivial pregeometry.

The present study of the universe of a free G-action (G some countable group) grew out of the attempt to find a similar example which is uncountably categorical or even strongly minimal. Since a free group action is the prototype of a trivial strongly minimal theory, it seemed to be very natural to look at these universes.

The main result of the paper is Theorem 1, the classification of universes similar to the free action of a free group $F(k)$ (for $1 \leq k \leq \omega$). If $k \neq 1, \omega$, we show that there is no "prime universe": the class of countable non-transformable universes similar to the free $F(k)$-action is given by $(\mathbb{Z} \cup \{\infty\}, <)$. In particular, this shows that the most obvious analogue of the Baldwin-Lachlan theorem is false for uncountably categorical universes.

The paper is organised as follows. In Sect. 2, we gather some general facts about universes. After that, free semi-actions and semi-free actions of a group G are introduced in Sect. 3. We then show, in Sect. 4, that the universes similar to an (infinite) free action of some group G are exactly the ones associated to free semi-actions of G. Moreover, properties of (finitely generated) groups such as amenability and the number of ends are discussed with regard to the classification problem. In Sect. 5, we prove our main result. We equally settle the classification problem for finitely generated abelian groups. Finally, we list some open problems.

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2 Universes

We give a brief summary about the concept of a universe in model theory as introduced by Poizat [2].

Definition 2.1 A *universe* U is given by an infinite set M (the *base set of the universe*) and subfamilies $\mathcal{D}_n(U) \subseteq \mathcal{P}(M^n)$ for every $n \geq 1$ (the *definable subsets* in the sense of U) satisfying the following properties:

1. Boolean Combinations: $\mathcal{D}_n(U)$ is a boolean algebra for all $n \geq 1$.
2. Product: If $X \in \mathcal{D}_n(U)$ and $Y \in \mathcal{D}_m(U)$, then $X \times Y \in \mathcal{D}_{m+n}(U)$. All diagonals $\Delta_{i,j} := \{(x_1, \ldots, x_n) \in M^n \mid x_i = x_j\}$ are definable.
3. Projection: If $X \in \mathcal{D}_{n+1}(U)$, then $\Pi(X) \in \mathcal{D}_n(U)$, where $\Pi : M^{n+1} \to M^n$ is the projection on the first n coordinates.
4. Parameters: The singleton $\{m\}$ is in $\mathcal{D}_1(U)$ for any $m \in M$.

If $\mathcal{M} = (M, R_i)$ is an infinite first order $\mathcal{L}$-structure, the universe attached to $\mathcal{M}$, denoted $U := \mathcal{U}_{\mathcal{L}}(\mathcal{M})$, is simply defined to consist of the base set M

together with all $\mathcal{L}$-definable (*with parameters*) subsets of M^n, $n \geq 1$. Such an $\mathcal{M}$ is called a *generating structure* for U. The minimal cardinality of a set of relations generating U is called the *width* of U. The universe U is *thin* if its width is equal to 1. This means that there is a finite set of definable sets generating U.

We now define some notions in the realm of universes which are analogous to isomorphism, elementary extension and elementary equivalence, respectively, for first order structures.

Definition 2.2 Let U and U' be universes, with base sets M and M', respectively.

1. We call U *transformable* to U' if there is a bijection $\Phi : M \to M'$ of the corresponding base sets such that $\Phi(\mathcal{D}_n(U)) = \mathcal{D}_n(U')$ for all n. Such a Φ is called a *transformation* between U and U'. Notation: $\Phi : U \simeq U'$.
2. An inclusion $M \subseteq M'$ gives rise to an *elementary extension of universes* $U \preccurlyeq U'$ if there are a signature $\mathcal{L}$ and $\mathcal{L}$-structures $\mathcal{M} \preccurlyeq_{\mathcal{L}} \mathcal{M}'$ such that $U = \mathcal{U}_{\mathcal{L}}(\mathcal{M})$ and $U' = \mathcal{U}_{\mathcal{L}}(\mathcal{M}')$.
3. The universe U is *similar* to U' if U and U' have a common elementary extension.

Note that if $U \preccurlyeq U'$, then for any generating $\mathcal{L}$-structure $\mathcal{M}$ of U, there is $\mathcal{M} \preccurlyeq_{\mathcal{L}} \mathcal{M}'$ such that $\mathcal{U}(\mathcal{M}') = U'$.

Fact 2.3 [2, 7.2] *Similarity between universes is an equivalence relation.*

The concept of types does not make sense in universes, and so a fortiori saturation cannot be defined, either. On the other hand, κ*-compactness* survives as a reasonable notion (every family $\mathcal{F}$ of definable sets with the finite intersection property has non-empty intersection, provided $|\mathcal{F}| < \kappa$).

Fact 2.4 [2, 7.4]

1. *Let U be a κ-compact universe of width smaller than κ, and suppose that $U' = \mathcal{U}_{\mathcal{L}}(\mathcal{M}')$ is similar to U, where $\mathcal{M}'$ is an $\mathcal{L}$-structure, for some $\mathcal{L}$ with $|\mathcal{L}| < \kappa$. Then there is $\mathcal{M} \equiv_{\mathcal{L}} \mathcal{M}'$ such that $\mathcal{U}_{\mathcal{L}}(\mathcal{M}) = U$.*
2. *Suppose that $\mathcal{M}$ is a κ-saturated $\mathcal{L}$-structure. Then any generating $\mathcal{L}'$-structure of $\mathcal{U}_{\mathcal{L}}(\mathcal{M})$ is κ-saturated (in the language $\mathcal{L}'$), provided $|\mathcal{L}'| < \kappa$.*

A universe U is called κ-categorical, if any two universes U_1, U_2 which are similar to U and of cardinality κ are transformable. Here, by the cardinality of a universe we mean the cardinality of its base set. Concerning categoricity, we have the following (part (1) is [2, 8.2], and (2) can be easily deduced from Sects. 7 and 8 of the same paper):

Proposition 2.5 1. *Let U be a universe of finite or countable width. If U is κ-categorical for some uncountable κ, it is λ-categorical for every uncountable λ.*
2. *Let $\mathcal{L}$ be countable and M an infinite $\mathcal{L}$-structure whose theory is uncountably categorical. Then, $\mathcal{U}_{\mathcal{L}}(M)$ is an uncountably categorical universe.*

This is Morley's categoricity theorem for universes of finite or countable width. In fact, it can easily be reduced to the version for first order structures. As we already mentioned in the introduction, the classification of countable universes similar to an $\aleph_1$-categorical one is an open problem. It is not clear what could be the general version of a Baldwin–Lachlan type theorem about universes.

3 Semifree actions and free semi-actions

Let M be an infinite set and $Sym(M)$ the group of permutations of M. For $f, f' \in Sym(M)$ set $f \sim f'$ if f and f' coincide almost everywhere. Let $\pi : Sym(M) \to Sym(M)/{\sim}$ be the canonical map. For $f \in Sym(M)$, call $f/{\sim}$ the *germ* of f.

A subgroup $G \leq Sym(M)/{\sim}$ is said to be *trivialisable*, if there is a homomorphism $\lambda : G \to Sym(M)$ such that $\pi \circ \lambda = \text{id}$. This means that there is a system of representatives for G closed under multiplication (also called a *lift*).

Given M and $G \leq Sym(M)/{\sim}$ we build the first order structure $\mathcal{M} := (M, f_i, i \in I)$, where $f_i \in \pi^{-1}(G)$ for all $i \in I$ and $\langle f_i/{\sim}, i \in I\rangle = G$. Since any two generating sets are interdefinable (with parameters in M), the choice of $(f_i)_{i\in I}$ will not be relevant in the sequel.

Note that if $\mathcal{M} \preccurlyeq \mathcal{M}^*$, then for every word w one has $w(f_{i_1}, \ldots, f_{i_m}) \sim \text{id}$ iff $w(f^*_{i_1}, \ldots, f^*_{i_m}) \sim \text{id}^*$. Thus, it is clear what we mean by $G \leq Sym(M^*)/{\sim}$.

Definition 3.1 Let $\mathcal{M} = (M, f_i, i \in I)$ be a structure as described above, with $G := \langle f_i/{\sim}, | \, i \in I\rangle \leq Sym(M)/{\sim}$.

- $\mathcal{M}$ is called a *semi-action* (of G) if there is an elementary extension $\mathcal{M}^* \succcurlyeq \mathcal{M}$ such that $G \leq Sym(M^*)/{\sim}$ can be trivialised.
- The semi-action is *free* if there exists $\mathcal{M}^* \succcurlyeq \mathcal{M}$ such that $G \leq Sym(M^*)/{\sim}$ can be trivialised in a way that the action one obtains is free.
- A free semi-action $\mathcal{M}$ that can be trivialised on M to a (not necessarily free) G-action is called a *semifree action*.

Remark 3.2 If G is finitely presented, then every semi-action $\mathcal{M}$ of G can be trivialised (on M).

Proof Let $G = \langle g_1, \ldots, g_n \,|\, R_1, \ldots, R_m\rangle$, and let $\mathcal{M}$ be a semi-action of G. W.l.o.g. $\mathcal{M} = (M, f_1, \ldots, f_n)$, where $f_i/{\sim} = g_i$. Suppose that this semi-action is trivialised on $\mathcal{M}^* \succcurlyeq \mathcal{M}$. Thus, there are $h^*_i \sim f^*_i$ such that $g_i \mapsto h^*_i$ gives rise to a lift of G to $Sym(M^*)$, i.e. $R_j(h^*_1, \ldots, h^*_n) = \text{id}^*$ for $1 \leq j \leq m$ holds.

Since $F := \{x \in M^* \,|\, h^*_i(x) \neq f^*_i(x) \text{ for some } i\}$ is a finite set (of cardinality N, say), and since there are only finitely many relations to be considered, the fact that there is a lift of G obtained by altering the f_i only on the set $\{x_1, \ldots, x_N\}$ can be expressed by a first order formula. So G can be trivialised on M, since $\mathcal{M} \preccurlyeq \mathcal{M}^*$. Clearly, if G were merely finitely generated we would get a partial type on a finite tuple, such that every solution of this type gives rise to a trivialisation of the semi-action. □

Lemma 3.3 *Let $\mathcal{M} = (M, f_i, i \in I)$ be a semi-action.*

1. *$Th(\mathcal{M})$ has quantifier elimination in the language where all f_i and f_i^{-1} are named.*
2. *$\mathcal{M}$ is strongly minimal iff for all $f \not\sim f' \in \langle f_i \mid i \in I\rangle$ the set $\{x \in M \mid f(x) = f'(x)\}$ is finite. In particular, a free semi-action is strongly minimal.*

Proof Part (1) is standard (using a back-and-forth). In fact, this is even true for any structure built from a set of bijections and their inverses. It need not be a semi-action. Now, (2) follows immediately from (1). □

We now consider a strongly minimal semi-action $\mathcal{M}$ with group of germs G. Let $\mathcal{M} \preccurlyeq \mathcal{M}^*$ and $\alpha \in M^* \setminus M$. Since α is generic, for $f, f' \in F := \langle f_i \mid i \in I\rangle$ one has $f(\alpha) = f'(\alpha)$ iff $f \sim f'$. Thus, M^* is the disjoint union of M and a set A^* equipped with a free G-action, the action of F on A^* being induced by π. We further infer from the above lemma that $\mathrm{acl}(\emptyset)$ is given by the union of all F-orbits that are not equal to regular G-orbits. Clearly, the models are classified by the number of regular G-orbits.

4 Universes and free semi-actions

In this section, we study universes of (semi-)free (semi-)actions and transformations between them. The following lemma shows why free semi-actions provide the appropriate framework for the study of universes of free group actions. If G is a group and κ, λ are cardinal numbers, let us denote by $\mathcal{U}_G(\kappa)$ the universe associated to a free G-action with κ orbits, and by $\mathcal{U}_G(\kappa)\dot{+}\lambda$ the universe of a G-action with κ regular and λ trivial orbits.

Lemma 4.1 *The class of universes similar to the universe of an infinite free action of some group G is exactly the class of universes associated to (infinite) free semi-actions of G.*

Proof Clearly, by definition, the universe associated to an infinite free semi-action of G is similar to the universe $\mathcal{U}_G(\kappa)$ of an infinite free G-action. On the other hand, in order to study the similarity class, it is sufficient to look at elementary restrictions of universes of $\mathrm{U}_G(\lambda)$ for some λ. Suppose $V \preccurlyeq \mathcal{U}_G(\lambda) =: U$, and choose a language $\mathcal{L}$ generating V. Thus, in particular $\mathcal{M} := \mathcal{M}_{\mathcal{L}}(V) \preccurlyeq \mathcal{M}_{\mathcal{L}}(U) =: \mathcal{N}$, where our notation means the structure attached to the language $\mathcal{L}$, i.e. to a set of definable sets in the corresponding universe. We will use the following easy fact (the proof of which is left to the reader):

Fact 4.2 *Let T be a trivial strongly minimal theory. Then every germ of definable (with parameters) bijections has a representative over the prime model of T.*

Thus, for every $g \in G$ there is an $\mathcal{L}$-definable (with parameters in M) bijection f_g such that $g \sim f_g$ inside $Sym(N)$. Let $\mathcal{L}'$ be the signature with constants for elements of M and function symbols for the $(f_g)_{g \in G}$. Obviously, the G-action on U can be recovered from $\mathcal{L}'(N)$, showing that $\mathcal{L}'$ is generating for U. A fortiori, every $\mathcal{L}(N)$-definable set can be $\mathcal{L}'(N)$-defined. Since $M \preccurlyeq_{\mathcal{L}} N$, the same is true for $\mathcal{L}(M)$ w.r.t. $\mathcal{L}'(M)$, showing that V is the universe of a free semi-action of G.

In [2], the following notions are introduced:

Definition 4.3 Let T be a complete $\mathcal{L}$-theory and $\mathcal{C}$ the class of all universes similar to $\mathcal{U}(\mathcal{M})$, where $\mathcal{M}$ is some model of T.

1. T is *ubiquitous* (for $\mathcal{C}$) if every universe in $\mathcal{C}$ equals $\mathcal{U}(\mathcal{M}')$ for some $\mathcal{M}' \models T$.
2. T is *classifying* if for all models $\mathcal{M}$ and $\mathcal{M}'$ of T, the following holds: $\mathcal{U}_{\mathcal{L}}(\mathcal{M}) \simeq \mathcal{U}_{\mathcal{L}}(\mathcal{M}')$ iff $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{M}'$.

We first study some easy properties of a group G leading to classifying and/or ubiquitous free G-actions. For brevity, a group G is called ubiquitous (classifying) if the infinite free G-action is. From now on, all groups will be countable and infinite, so the properties "ubiquitous" and "classifying" have to be checked only on countable universes, since the corresponding universes are uncountably categorical by Proposition 2.5(2). Note that any transformation $\Phi : U \simeq U'$ induces an isomorphism $\Phi/\!\sim$ between the groups of germs of definable bijections in U and U', respectively.

Lemma 4.4 *Suppose that $U = \mathcal{U}_G(\kappa)\dot{+}\lambda \simeq \mathcal{U}_G(\kappa')\dot{+}\lambda' = U'$. Then, there is a transformation $\Phi : U \simeq U'$ such that $\Phi/\!\sim = \mathrm{id} : G \to G$, where G is identified with the group of germs of definable bijections in U and U', respectively.*

Proof Let $\Phi_0 : U \simeq U'$ be any transformation. Then, $\Phi_0/\!\sim \in \mathrm{Aut}(G)$ (with the obvious identifications). Note that for every $\alpha \in \mathrm{Aut}(G)$ there is a self-transformation Ψ_α of U' such that $\Psi_\alpha/\!\sim = \alpha$. Just use α on every regular orbit and the identity on trivial orbits. Thus, composing Φ_0 with a suitable self-transformation of U', we get a transformation $\Phi : U \simeq U'$ as desired. □

Remark 4.5 Let G be a group containing a subgroup H of finite index which is classifying. Then, G is classifying, too.

Proof Suppose that $\Phi : U = \mathcal{U}_G(m) \simeq \mathcal{U}_G(m') = U'$ is a transformation, for some $m, m' \in \mathbb{N}^*$. Using Lemma 4.4, we may assume that Φ induces the identity on G. Put $i := [G : H]$, and let $\mathcal{L}_G \supseteq \mathcal{L}_H$ be the signatures for G-actions and H-actions, respectively. Then, $U = \mathcal{U}_{\mathcal{L}_G}(M)$ and $\mathrm{U}' = \mathcal{U}_{\mathcal{L}_G}(M')$, where M and M' are free G-actions with m and m' regular orbits, respectively.

Since one regular G-orbit gives rise to i regular H-orbits, one gets $\mathcal{U}_{\mathcal{L}_H}(M) = \mathcal{U}_H(i \cdot m)$ and $\mathcal{U}_{\mathcal{L}_H}(M') = \mathcal{U}_H(i \cdot m')$, and so using the same map (on points) we get $\Phi : \mathcal{U}_H(i \cdot m) \simeq \mathcal{U}_H(i \cdot m')$. Since H is classifying, this means $i \cdot m = i \cdot m'$, so $m = m'$ and G is shown to be classifying. □

Let us mention that it is not clear in general if the roles of G and H can be interchanged in Remark 4.5.

Let Γ be an infinite graph with finite valency, and let K be a finite subgraph. Put $n(K)$ equal to the number of infinite connected components of $\Gamma \setminus K$. The *number of ends* of Γ is defined as *end*$(\Gamma) := \sup\{n(K) \mid K \subseteq_\omega \Gamma\}$. If G is a finitely generated (infinite) group, we define $end(G) := end(\Gamma)$, where Γ is the Cayley graph of G for some finite system of generators. It is easy to see that this definition does not depend on the particular choice of the generators. In fact, the definition of $end(\Gamma)$ is invariant under quasi-isometries, and Cayley graphs with respect to different (finite) systems of generators are quasi-isometric. Groups of the form $G_1 \times G_2$, where both G_1 and G_2 are infinite, have one end. For $G := \mathbb{Z} \times G_0$ with G_0 some finite group, one has $end(G) = 2$. For $k \geq 2$, the free group on k generators has infinitely many ends. More generally, if G_1 and G_2 are non-trivial groups, not both isomorphic to $\mathbb{Z}/2$, then $end(G_1 * G_2) = \infty$.

Proposition 4.6 *Let G be finitely generated with* $end(G) = 1$. *Then, the following holds:*

1. *The group G is classifying.*
2. *Every semi-free G-action is free.*
3. *If G is finitely presented, then G is ubiquitous.*

Proof Note that (3) follows from (2) together with Remark 3.2.

In order to show (2), we consider $V \preccurlyeq \mathcal{U}_G(m) =: U$, where we suppose that V be given by a semi-free G-action, for $G = \langle g_1, \ldots, g_n \rangle$. By Fact 2.4 we may assume that $m \in \mathbb{N} \cup \{\aleph_0\}$, but it is easy to see that it suffices to check elementary restrictions of $\mathcal{U}_G(m)$ for all finite m. This is due to finite width, but we will not use this. Let $g_i' \sim g_i$ be those bijections defining the semi-free G-action on V, and define the *exceptional locus* for this alteration as follows:

$$E := \{x \in U \mid g_i(x) \neq g_i'(x) \text{ or } g_i^{-1}(x) \neq (g_i')^{-1}(x) \text{ for some } i\}.$$

Let $\tilde{E}$ be the union of E and all finite connected components of $U \setminus E$, so $\tilde{E}$ is finite.

Claim The G-action on U via $g_1', \ldots, g_n'$ is free.

To show the claim, for $g \in G$ denote g' the bijection of U which we obtain in the following way: if $g = w(g_1, \ldots, g_n)$, then $g' := w(g_1', \ldots, g_n')$. Since the alteration defines a G-action, this assignment is independent of the particular choice of w.

For every $e \in \tilde{E}$ we choose y in $G \cdot e \setminus \tilde{E}$ from the unique infinite component. If $g \cdot y = e$, we let $\sigma(e) := g' \cdot y$ (note that σ does not depend on the choice of y). It is now easy to see that σ is a permutation of $\tilde{E}$, so (extending σ identically to all of U) can be considered as an element of $Sym_{fin}(U)$. We pretend that for all $g \in G$, the equality $g = \sigma^{-1} g' \sigma$ holds. For $x \in U$ arbitrary, choose $y \in U \setminus \tilde{E}$ and $g_0 \in G$ such that $x = g_0 \cdot y$. We then compute (use the definition of σ twice)

$$\begin{aligned} g \cdot x &= (gg_0) \cdot y = \sigma^{-1}\sigma(gg_0) \cdot y = \sigma^{-1}(g'g_0') \cdot y \\ &= \sigma^{-1}g'(g_0' \cdot y) = (\sigma^{-1}g')(\sigma \cdot g_0 \cdot y) = (\sigma^{-1}g'\sigma) \cdot x. \end{aligned}$$

Thus, the alteration is a free G-action, and (2) is shown. On the other hand, the number of orbits does not change in the above proof (and σ is an isomorphism of the original G-action with the altered version), so (1) follows, too.

The preceding proof can be readily modified to show a bit more.

Remark 4.7 Let G be finitely generated with $end(G) = 1$. Then $\mathcal{U}_G(\kappa)\dot{+}\lambda \simeq \mathcal{U}_G(\kappa')\dot{+}\lambda'$ iff $\kappa = \kappa'$ and $\lambda = \lambda'$.

We observe that the content of this remark does not hold for the group of integers $\mathbb{Z}$, since $\mathcal{U}_{\mathbb{Z}}(1) \simeq \mathcal{U}_{\mathbb{Z}}(1)\dot{+}n$ for all $n \in \mathbb{N}$. If e.g. $n = 1$, just alter the action of a generator s of $\mathbb{Z}$ on $\mathbb{Z} = \mathcal{U}_{\mathbb{Z}}(1)$ to s', putting $s'(0) := 2$ and $s'(1) := 1$ and s' equal to s outside $\{0, 1\}$. Nevertheless, there is a class of groups (containing $\mathbb{Z}$), where things are not too strange.

Recall that a (discrete) group G is called *amenable* if there is a finitely additive (right-)invariant probability measure on G. All finite and all abelian groups are amenable. The class of amenable groups is closed under subgroups, quotients, extensions and direct limits. In particular, every solvable group is amenable, and a group G is amenable if and only if all finitely generated subgroups of G are amenable. For background on amenable groups we refer to [1].

Proposition 4.8 *Let G be a finitely generated infinite amenable group. Suppose that $\mathcal{U}_G(m)\dot{+}n \simeq \mathcal{U}_G(m')\dot{+}n'$. Then $m = m'$. In particular, G is classifying.*

Proof Suppose there is a transformation $\Phi : U := \mathcal{U}_G(m)\dot{+}n \simeq U' := \mathcal{U}_G(m')\dot{+}n'$, with underlying sets M and M'. Let G be generated by $(g_1, \ldots, g_k)$ and let μ be a right-invariant probability measure on G. Using Lemma 4.4, we may assume that $g' := \Phi^{-1}(g)$ and g have the same germ for all $g \in G$. Now choose a finite set $K \subseteq M$ containing the trivial orbits of U, the preimage under Φ of the trivial orbits of U' and the exceptional locus of the alteration (computed in the generators $(g_1, \ldots, g_k)$). Let $A_1, \ldots, A_N$ be the connected components of $M \setminus K$. Every time we think of the altered G-action on M induced by Φ, we will write G' and g', respectively.

For $x \in M$ with $G \cdot x \supseteq A_i$ we put $G_i(x) := \{g \in G \mid g \cdot x \in A_i\}$. Note that if y is another element from the same orbit, say $y = g \cdot x$, then $G_i(y) = G_i(x)g^{-1}$. The same is true for the altered action. Let $x', y' \in M$ with $G' \cdot x' \supseteq A_i$ and suppose that $y' = g' \cdot x'$. Putting $G'_i(x') := \{g \in G \mid g' \cdot x' \in A_i\}$, we compute $G'_i(y') = G'_i(x')g^{-1}$ (as subsets of G). As A_i is connected and disjoint from the exceptional locus, one has $G_i(a_i) = G'_i(a_i)$ for all $a_i \in A_i$.

Let $O_1, \ldots, O_m \subseteq M$ be the regular orbits for the unaltered G-action, and $O'_1, \ldots, O'_{m'} \subseteq M$ the regular orbits for the (altered) G'-action. For $k = 1, \ldots, m$ choose $x_k \in O_k$, similarly $x'_k \in O'_k$.

The family $F := \{A_1, \ldots, A_N\}$ can be partitioned into m subfamilies $F_1, \ldots, F_m$ in such a way that for $k = 1, \ldots, m$ the set $\cup_{A_i \in F_k} A_i$ is a cofinite subset of $O_k = G \cdot x_k$. This means that $\{G_i(x_k) \mid A_i \in F_k\}$ is a partition of a cofinite set of G $(1 \leq k \leq m)$.

Analogously, there is a partition of F into m' subfamilies $F'_1, \ldots, F'_{m'}$, ultimately leading to partitions $\{G'_i(x'_k) \mid A_i \in F'_k\}$ of cofinite subsets of G (for $1 \leq k \leq m'$).

By what we have said (using an element $a_i \in A_i$ to do the transition), it follows that for $i = 1, \ldots, N$, the set $G'_i(x'_k)$ is a right-translate of $G_i(x_k)$, so

$\mu(G_i'(x_i')) = \mu(G_i(x_i))$, as μ is right-invariant. Thus,

$$m = m\mu(G) = \sum_{i=1}^{N} \mu(G_i(x_i)) = \sum_{i=1}^{N} \mu(G_i'(x_i')) = m'\mu(G) = m',$$

since finite subsets of G have measure 0 and μ is finitely additive. □

5 Classification of semifree actions of free groups

In this section, we will study semifree actions of the free group on k generators $F(k)$, for $k \in \mathbb{N}^* \cup \{\omega\}$.

We will tacitly use the following easy fact, the proof of which is left to the reader (essentially, one has to use Lemma 4.4).

Fact 5.1 *For $i = 1, 2$, let $U_i := \mathcal{U}_G(\kappa_i)\dot{+}\lambda_i$ and $U_i' := \mathcal{U}_G(\kappa_i')\dot{+}\lambda_i'$. Suppose that $U_1 \simeq U_1'$ and $U_2 \simeq U_2'$. Then, $\mathcal{U}_G(\kappa_1 + \kappa_2)\dot{+}(\lambda_1 + \lambda_2) \simeq \mathcal{U}_G(\kappa_1' + \kappa_2')\dot{+}(\lambda_1' + \lambda_2')$.*

The following easy observation is crucial:

Remark 5.2 Let $k \in \mathbb{N}^* \cup \{\omega\}$, and set $G := F(k)$. Then $\mathcal{U}_G(1) \simeq \mathcal{U}_G(k)\dot{+}1$. In particular, $\mathcal{U}_G(n)\dot{+}m$ is semifree for all $n \in \mathbb{N}^* \cup \{\omega\}$ and $m \in \mathbb{N}$.

Proof Suppose that $F(k)$ is freely generated by elements $(g_i)_{1\le i\le k}$. Choose $e \in \mathcal{U}_G(1) =: U$ arbitrary and define $g_i' \sim g_i$ in the following way:

Put $g_i'(e) := e$, $g_i'(g_i^{-1} \cdot e) := g_i \cdot e$ and leave g_i unaltered outside $\{e, g_i^{-1} \cdot e\}$. It is straightforward to check that the $(g_i')_{1\le i\le k}$ define a G-action on U which consists of k regular orbits and one trivial orbit (the set $\{e\}$). □

Lemma 5.3 *Let $U := \mathcal{U}_{F(k)}(n)$, for some finite k. Then every elementary restriction of U is of the form $\mathcal{U}_{F(k)}(n')\dot{+}m'$.*

Proof Let $G := \langle g_1, \ldots, g_k\rangle = F(k)$. Consider $V \preccurlyeq U$, where the embedding is elementary with respect to a G-action via $(g_i')_{i\le k}$, such that (on U) one has $g_i' \sim g_i$.

Let $E \subseteq U$ be a finite set containing the exceptional locus of this alteration and such that it contains an element from every regular orbit. Define $\tilde{E}$ as the set of all $x \in U$ lying on the shortest path (for the unaltered action) from e_1 to e_2 for some $e_1, e_2 \in E$. By construction, $\tilde{E}$ is a finite superset of E, and it contains the shortest path between any two of its elements. Now, $U \setminus \tilde{E}$ is the finite union of its connected components $A_1, \ldots, A_N$ which are all infinite. For every such component A_j there is a unique element $x_j \in A_j$ such that $h_j \cdot x_j \in \tilde{E}$ for some (unique) $h_j \in \{g_1, \ldots, g_k, g_1^{-1}, \ldots, g_k^{-1}\}$. Thus, the set of connected components is in 1:1 correspondence with $S := S^1(\tilde{E}) := \{x \in U \mid dist(x, \tilde{E}) = 1\}$, where U is equipped with the word metric *dist* with respect to the generators $(g_1, \ldots, g_k)$. For $x_j \in S$ there is some $n \in \mathbb{N}^*$ such that $(h_j')^n(x_j) \notin \tilde{E}$, since h_j' is bijective.

Obviously, if n is minimal such, $(h_j')^n(x_j)$ is in S. It is equal to some $x_{\sigma(j)}$, where $\sigma \in S_N$ with $\sigma^2 = \mathrm{id}$ and $\sigma(j) \neq j$ for $j = 1, \ldots, N$, so N is even.

Now consider an orbit $X \subseteq U$ for the altered action. By definition, if $A_j \cap X \neq \emptyset$, then $A_j \cup A_{\sigma(j)} \subseteq X$. Since V equals U minus a finite number of (regular) orbits for the altered action, one has $V = \dot{\bigcup}_{i=1}^{l}(A_{j_i} \cup A_{\sigma(j_i)}) \dot{\cup} E_V$, where $E_V := V \cap \tilde{E}$ is finite and $l \geq 1$. We now alter $(g'_\alpha)_{1\leq\alpha\leq k}$ on V, defining $g''_\alpha \restriction E_V := \mathrm{id} \restriction E_V$, $g''_\alpha(x_{j_i}) := x_{\sigma(j_i)}$ if $h_{j_i} = g_\alpha$, else $g''_\alpha := g'_\alpha$. This shows that $V \simeq \mathcal{U}_G(l)\dot{+}|E_V|$. □

Lemma 5.4 *Let k be finite and $G = F(k)$. The following are equivalent:*

1. $\mathcal{U}_G(n)\dot{+}m \simeq \mathcal{U}_G(n')\dot{+}m'$
2. $n - (k-1)m = n' - (k-1)m'$.

In particular, G is classifying.

Proof (2)⇒(1) follows from Remark 5.2. For the other direction, suppose that $\mathcal{U}_G(n)\dot{+}m \simeq \mathcal{U}_G(n')\dot{+}m' =: U$, given by G-actions via $(g_1, \ldots, g_k)$ and $(g'_1, \ldots, g'_k)$, respectively (with $g'_i \sim g_i$ as usual). Choose $\tilde{E} \subseteq U$ finite such that $\tilde{E}$ contains the exceptional locus for the alteration, all trivial orbits for both actions and at least one element from every regular orbit (for both actions), and such that $\tilde{E}$ is closed by shortest paths for both actions. Note that any set $E' \supseteq E$ which is closed by shortest paths for one action is automatically closed by shortest paths for *both* actions. This can be shown by induction on the length of the shortest path, using that outside E' we have $g_i = g'_i$ and $g_i^{-1} = g_i'^{-1}$.

Let $O_1, \ldots, O_n$ be the regular G-orbits in U, and $O'_1, \ldots, O'_{n'}$ the regular orbits for the altered action. Put $\tilde{E}_i := \tilde{E} \cap O_i$. Then,

$$S^1(\tilde{E}_i) = S^1(\tilde{E}) \cap O_i \quad \text{and} \quad S^1(\tilde{E}'_i) = S^1(\tilde{E}) \cap O'_i. \tag{5.1}$$

Now suppose that K is a finite subset of some regular G-orbit which is closed under shortest paths. By induction on the cardinality of K, one shows:

$$|S^1(K)| = (2k-2) \cdot |K| + 2. \tag{5.2}$$

Summing up and using (5.1) as well as (5.2), this yields

$$|S^1(\tilde{E})| = (2k-2)(|\tilde{E}| - m) + 2n = (2k-2)(|\tilde{E}| - m') + 2n',$$

as $S^1(\tilde{E})$ calculated for both actions amounts to the same. □

Theorem 1 (Classification of semifree actions of free groups) *The classes $Cl(G)$ of countable semifree G-actions (up to transformation) with elementary embeddings are as follows:*

1. *If $G := \mathbb{Z}$, then $Cl(G) = (\mathbb{N}^* \cup \{\infty\}, \leq)$.*
2. *If $G := F(k)$ for some finite $k \geq 2$, then $Cl(G) = (\mathbb{Z} \cup \{\infty\}, \leq)$.*
3. *If $G = F(\omega)$, then $Cl(G)$ consists of one element, i.e. the universe of a free G-action is totally categorical.*

Proof Parts (1) and (2) follow from Lemma 5.3 together with Lemma 5.4. That the universe associated to the countable saturated free G-action is not transformable into a universe of the form $\mathcal{U}_G(m)\dot{+}n$ follows from Fact 2.4(2), since the universe is thin.

We now show (3). By Remark 5.2 we know that for $G := F(\omega)$ one has

$$\mathcal{U}_G(1) = \mathcal{U}_G(\omega)\dot{+}1 = \mathcal{U}_G(\omega)\dot{+}(\mathcal{U}_G(\omega)\dot{+}1) = \mathcal{U}_G(\omega)\dot{+}\mathcal{U}_G(1) = \mathcal{U}_G(\omega) = \mathcal{U}_G(1)\dot{+}1,$$

so all countable free G-actions have the same universe. Suppose G is freely generated by $(g_i)_{i<\omega}$. It remains to show the ubiquity of G. So let $V \preccurlyeq \mathcal{U}_G(1)$ be a semifree G-action, say given by $g_i' \sim g_i$. Restricting the language to $(g_1', \ldots, g_k')$, we get a semifree $F(k)$-action on V. The same is true for $F(k+1)$, hence one can finitely alter the $(g_i')_{i\leq k}$ on V to obtain a free $F(k)$-action with infinitely many orbits, since $F(k)$ is of infinite index in $F(k+1)$.

The idea is to adapt the proof of Lemma 5.3 and do the alterations step by step. That this can be done is the content of the following lemma. □

Lemma 5.5 *Let $\mathcal{M}$ be a semifree $F(k+1)$-action with infinitely many regluar $F(k+1)$-orbits, where $F(k+1) = \langle g_1, \ldots, g_{k+1}\rangle$. Suppose that $(g_1, \ldots, g_k)$ gives rise to a free $F(k)$-action on M. Then there is some $g_{k+1}' \sim g_{k+1}$ such that the $F(k+1)$-action via $(g_1, \ldots, g_k, g_{k+1}')$ is free.*

Proof From the hypotheses we infer that one can finitely alter $g_1, \ldots, g_{k+1}$ on M in order to obtain a free $F(k+1)$-action. Let $E \subseteq M$ be the exceptional locus of such an alteration, and choose a finite set $\tilde{E} \supseteq E$ which is closed by shortest paths (for both actions in sight). We now define $g_{k+1}' \sim g_{k+1}$ as follows: on $\tilde{E}$, g_{k+1}' is the identity, and for $x \in M \setminus \tilde{E}$ one puts $g_{k+1}'(x) := g_{k+1}^n(x)$, where $n \in \mathbb{N}^*$ is minimal such that $g_{k+1}^n(x) \notin \tilde{E}$. Let $\tilde{E} = \{e_1, \ldots, e_N\}$. Choose N new regular $F(k+1)$-orbits in $M \setminus \tilde{E}$, say $X_1 = F(k+1)\cdot x_1, \ldots, X_N = F(k+1)\cdot x_N$. Finally, alter g_{k+1}' again to the following bijection g_{k+1}'':

- $g_{k+1}''(x_i) := e_i$ for $i = 1, \ldots, N$,
- $g_{k+1}''(e_i) := g_{k+1}'(x_i)[= g_{k+1}(x_i)]$ for $i = 1, \ldots, N$ and
- $g_{k+1}''(x) := g_{k+1}'(x)$ else.

It is fairly easy to see that $(g_1, \ldots, g_k, g_{k+1}'')$ defines a free $F(k+1)$-action on M. □

In particular, we see that there is a totally categorical universe interpreting non-omega-categorical universes. Already in [2], an example of an omega-categorical universe that interprets a non-omega-categorical one appears.

Before we finish the paper raising a series of questions, we mention the following:

Proposition 5.6 *Finitely generated abelian groups are ubiquitous and classifying.*

Proof Let G be finitely generated and abelian. If $rk(G) \geq 2$, then $end(G) = 1$, so we conclude by Proposition 4.6. Since the result is trivial for finite groups, the only remaining case is $G \simeq \mathbb{Z} \times A_0$, where A_0 is finite abelian. In fact, we even show that for an arbitrary finite group G_0, the group $G := \mathbb{Z} \times G_0$ is classifying and ubiquitous. Let s be a generator of $\mathbb{Z}$ and $G_0 = \{g_1, \ldots, g_N\}$.

By Lemma 4.8, G is classifying. Now, consider $U := \mathcal{U}_G(n)$ and an elementary restriction $V \preccurlyeq U$. Combining Lemma 4.1 with Remark 3.2, we see that V is given by a semifree G-action $\mathcal{M}$. Moreover, due to the finiteness of G_0, a straight forward refinement of the argument given in the proof of 3.2 shows that we may suppose that the underlying G_0-action on M is free. Suppose that the G-action $\mathcal{M}$ is given by $s' \sim s$ and $g'_1 \sim g_1, \ldots, g'_N \sim g_N$.

The semi-free $\mathbb{Z}$-action on U given by s' consists of Nn regular $\mathbb{Z}$-orbits $O'_1, \ldots, O'_{Nn}$ and a finite number of finite orbits $K_1, \ldots, K_\ell$.

We show:

(I) For $1 \neq g_k \in G_0$ and any $x \in O'_i$, the element $g'_k \cdot x$ is in O'_j for some $j \neq i$. In other words, via $g'_1, \ldots, g'_N$, the group G_0 freely permutes the regular $\langle s' \rangle$-orbits.

(II) Let $K := \cup_{i=1}^{\ell} K_i$. Then, $|K|$ is divisible by N.

It is clear that K is invariant under $g'_1, \ldots, g'_N$. Since the corresponding G_0-action is free, it follows that $|K|$ is divisible by $|G_0| = N$. This shows (II). Now consider $1 \neq g_k \in G_0$ and $x \in O'_i$. By what we have said, $g'_k \cdot x \in O'_j$ for some j. If $j = i$, then $g'_k \cdot x = (s')^{z_0} \cdot x$ for some $z_0 \in \mathbb{Z}$, and so $g'_k \cdot ((s')^n \cdot x) = ((s')^n (s')^{z_0}) \cdot x = (s')^{z_0} \cdot ((s')^n \cdot x)$ for all n. This means that g'_k and $(s')^{z_0}$ coincide on infinitely many elements, and thus, using semi-freeness, $g'_k \sim s'$, a contradiction. This proves (I).

Since $U \setminus V$ is a union of regular G-orbits, the $\mathbb{Z}$-action on M consists of Nm regular orbits (for some $1 \leq m \leq n$) together with K. Let s'' be the identity map on K and equal to s' outside K. The G-semi-action on M given by s'' and the g'_k is then still a semifree action, and (I) and (II) are true for this action, too. Now define a final alteration on M as follows: every G_0-orbit from K is put "transversally" into a block of N regular $\mathbb{Z}$-orbits that are permuted by G_0 (for this we have to use (I)) in a way to obtain a regular G-orbit. □

6 Open problems

In this final section we gather some open problems concerning free semi-actions of groups.

Pb.1: Find a (finitely generated) group G and a free semi-action $\mathcal{M}$ of G that can not be trivialised to an action on M.

Pb.2: Is there a finitely generated group G which is not classifying?

Pb.3: What can be said in general for larger classes of groups, e.g. hyperbolic groups or more generally automatic groups?

Pb.4: Find a reasonable cohomological description for the set of all free semi-actions.

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