MODEL THEORY OF VALUED FIELDS

MARTIN HILS

1. Introduction

These lecture notes contain the material of my course 'Model Theory of Valued Fields' as given during the second week of the Münster Month in Model Theory.

Here is a short overview of the notes. We start with a section on the model theory of ordered abelian groups (Section 2). After a short preliminary section on the model theory of valued fields (Section 3), we then present some fundamental results about the theory ACVF of algebraically closed non-trivially valued fields in Section 4. At the end of the section, we point to some more recent and advanced topics in the study of ACVF.

Section 5 is devoted to Kaplansky theory. The results of this section play a key role in our treatment of the Ax-Kochen-Ershov principle on henselian valued fields of equal-characteristic 0 in Section 6. Our approach to the AKE principle is via Pas' Theorem on the elimination of field quantifiers for henselian valued fields of equal-characteristic 0 in the three-sorted Denef-Pas language with angular components. We finish the notes with a glimpse on p-adic model theory. The text is organised so that similarities between the various contexts are highlighted.

The algebraic results on valued fields presented in Franziska Jahnke's notes will be used throughout the text. Some of the material we present here (e.g., Pas' Theorem) will play a crucial role in Immanuel Halupczok's introductory notes on motivic integration.

2. Model Theory of Ordered Abelian Groups

We consider ordered abelian groups in the language $\mathcal{L}_{\text{oag}} = \{0, \leq, +\}$. The theory of all ordered abelian groups in $\mathcal{L}_{\text{oag}}$ is denoted by OAG, the theory of *non-trivial divisible ordered abelian groups* by DOAG. The latter is axiomatised by

- OAG;
- $\exists x\, x \neq 0$, and
- for each $n \geq 1$, an axiom of the form $\forall x \exists y \underbrace{y + \cdots + y}_{n \text{ times}} = x$.

The following exercise is easy.

Exercise 2.1. *Let* $\Gamma \models \text{DOAG}$*. Then the order on* Γ *is dense without endpoints, i.e.,* $\Gamma \models \text{DLO}$.

We now give a useful criterion for quantifier elimination. For simplicity we only state the version for countable languages.

Fact 2.2. *Let* T *be an* $\mathcal{L}$*-theory, with* $\mathcal{L}$ *countable. The following are equivalent:*

(a) *T has* quantifier elimination *(QE):*
for every $\mathcal{L}$-formula $\varphi(\overline{x}) = \varphi(x_1, \ldots, x_n)$ (in $n \geq 1$ variables) there is a quantifier free formula $\psi(\overline{x})$ such that $T \models \forall \overline{x}\,(\varphi \leftrightarrow \psi)$.

(b) *Let $M, N \models T$, where M is countable and N is $\aleph_1$-saturated. Let $A \subseteq M$ be a substructure. Then every $\mathcal{L}$-embedding $f : A \hookrightarrow N$ extends to an $\mathcal{L}$-embedding $\tilde{f} : M \hookrightarrow N$.*

Proposition 2.3. *The theory* DOAG *is complete and has quantifier elimination.*

Proof. Since $\{0\}$ embeds as a substructure in any model of DOAG, quantifier elimination implies completeness.

To prove quantifier elimination, we first show the existence of a divisible hull for ordered abelian groups.

Lemma 2.4. *Any ordered abelian group $(A, 0, \leq, +)$ admits a* divisible hull*, i.e., there is a divisible ordered abelian group $B \geq A$ with the following property:*

($*$) *Every embedding $f : A \hookrightarrow C$ into a divisible ordered abelian group C extends uniquely to an embedding of B into C.*

Moreover, the property ($*$) determines B uniquely up to unique isomorphism over A. We denote it by $\mathrm{Div}(A)$*. Note that the factor group* $\mathrm{Div}(A)/A$ *is torsion.*

Proof of the lemma. Set $B := A \otimes_{\mathbb{Z}} \mathbb{Q}$ as an abelian group. Then we have $B = \{\frac{a}{n} \mid a \in A, n \in \mathbb{Z}, n \geq 1\}$, where $\frac{a}{n} = \frac{a'}{n'}$ iff $n'a = na'$. Now we set $\frac{a}{n} \leq \frac{a'}{n'} :\Leftrightarrow n'a \leq na'$.

It is straightforward to verify that B gets the structure of an ordered abelian group in this way and that $\iota : A \hookrightarrow B$, $a \mapsto \frac{a}{1}$ is an embedding which has the desired properties. The details are left to the reader. □

We now show quantifier elimination in DOAG, using Fact 2.2. Let $\Gamma, \Gamma' \models$ DOAG, where $|\Gamma| = \aleph_0$ and Γ' is $\aleph_1$-saturated. Let $A \leq \Gamma$ be a substructure and $f : A \hookrightarrow \Gamma'$ an $\mathcal{L}_{\mathrm{oag}}$-embedding.

Step 1: We may assume A is a group.
Indeed, define $\tilde{f}(-a) := -f(a)$. It is easy to see that $\tilde{f}$ defines an $\mathcal{L}_{\mathrm{oag}}$-embedding of the group generated by A into Γ'.

Step 2: We may assume that A is a divisible subgroup.
Indeed, by Lemma 2.4, f extends to an embedding of $\mathrm{Div}(A)$ into Γ'. This yields the result.

Step 3: Let $\gamma \in \Gamma \backslash A = \Gamma \backslash \mathrm{Div}(A)$. Then f extends to an embedding $\tilde{f} : \langle A, \gamma\rangle \hookrightarrow \Gamma'$.
Since Γ/A is torsion free, $\langle A, \gamma\rangle = A \oplus \mathbb{Z} \cdot \gamma$ as groups. The order on $\langle A, \gamma\rangle$ is determined by the *cut of γ over A*, i.e., by the couple $(L(\gamma/A), R(\gamma/A))$, where $L(\gamma/A) = \{a \in A \mid a < \gamma\}$ and $R(\gamma/A) = \{a \in A \mid a > \gamma\}$. Indeed, let $a + z\gamma, a' + z'\gamma \in \langle A, \gamma\rangle$. Assume that $z < z'$. (The other cases are similar.) Then

$$a + z\gamma \leq a' + z'\gamma \Longleftrightarrow a - a' \leq (z' - z)\gamma \Longleftrightarrow \frac{a - a'}{z' - z} < \gamma \Longleftrightarrow \frac{a - a'}{z' - z} \in L(\gamma/A).$$

As $\Gamma' \models$ DOAG, we have $\Gamma' \models$ DLO by Exercise 2.1. The image of the cut $(L(\gamma/A), R(\gamma/A))$ under f is a cut over $f(A)$ which is realised in Γ' by $\aleph_1$-saturation of Γ', say by $\gamma' \in \Gamma'$. Then $\gamma \mapsto \gamma'$ induces an embedding $\tilde{f} : \langle A, \gamma\rangle \hookrightarrow \Gamma'$ which extends f.

Now choose an enumeration $\{\gamma_n \mid n \in \omega\}$ of Γ. Repeating steps 2 and 3, we may construct an increasing sequence of embeddings $f_n : A_n \hookrightarrow \Gamma'$ extending f such that $\gamma_n \in A_n$ for all n. Then $\tilde{f} := \bigcup_{n \in \omega} f_n$ does the job. □

Corollary 2.5. DOAG *is o-minimal.*

Proof. Let $\Gamma \models$ DOAG. Any atomic formula $\varphi(x)$ (with parameters in Γ) in one variable is equivalent to either a tautology or the negation of a tautology or to a formula of the form $nx \leq \gamma$ or to one of the form $nx = \gamma$, where $n \in \mathbb{N}$ and $\gamma \in \Gamma$. These define intervals (possibly singletons or the empty set). We conclude by quantifier elimination (Proposition 2.3). □

Exercise 2.6. *Let* $\mathcal{L}_{\text{pres}} = \mathcal{L}_{\text{oag}} \cup \{1, \equiv_n, n \geq 2\}$. Presburger arithmetic *is the* $\mathcal{L}_{\text{pres}}$*-theory* PRES *which is given by the following axioms:*

(i) OAG*;*
(ii) "1 is the smallest positive element";
(iii) $\forall xy\,(x \equiv_n y \leftrightarrow \exists z\, x + nz = y)$ *(one axiom for each* $n \geq 2$*), and*
(iv) $\forall x\,(\bigvee_{i=0}^{n-1} x \equiv_n \underbrace{1 + \cdots + 1}_{i \text{ times}})$ *(one axiom for each* $n \geq 2$*).*

(1) Observe that $\mathcal{Z} = (\mathbb{Z}, 0, 1, +, \leq, \equiv_n) \models$ PRES *and that* $\mathcal{Z}$ *is a definitional expansion of the underlying ordered abelian group.*
(2) Show that PRES *is complete and eliminates quantifiers.*

Let us now mention (without proof) an important general model-theoretic result about ordered abelian groups.

Fact 2.7 (Gurevich-Schmitt [6])**.** *Every ordered abelian group is NIP.*

3. Preliminaries on the Model Theory of Valued Fields

Recall that a *valued field* (K, v) is a field K together with a (surjective) valuation map $v : K \to \Gamma \cup \{\infty\}$, where Γ is an ordered abelian group – the *value group* – and ∞ a new element $> \Gamma$, such that the following hold for all $x, y \in K$:

(i) $v(x) = \infty \iff x = 0$;
(ii) $v(x \cdot y) = v(x) + v(y)$;
(iii) $v(x + y) \geq \min\{v(x), v(y)\}$.

Notation and conventions 3.1. *Let* (K, v) *be a valued field.*

- *The value group is denoted by* Γ_v*.*
- $\mathcal{O}_v := \{x \in K \mid v(x) \geq 0\}$ *denotes the* valuation ring.
- $\mathfrak{m}_v := \{x \in K \mid v(x) > 0\}$ *denotes the (unique) maximal ideal of* $\mathcal{O}_v$*.*
- $\text{res} : \mathcal{O}_v \twoheadrightarrow k_v := \mathcal{O}_v/\mathfrak{m}_v$*,* $a \mapsto \text{res}(a) = \overline{a}$ *is the* residue map*, and* k_v *is the* residue field*.*

If the valuation map v *is clear from the context, we sometimes write* $\Gamma_K, \mathcal{O}_K, \mathfrak{m}_K, k_K$ *instead of* $\Gamma_v, \mathcal{O}_v, \mathfrak{m}_v, k_v$*.*

Various languages for valued fields. In order to treat valued fields as first order structures, one has to choose a language. We now present three languages (1-sorted, 2-sorted and 3-sorted), all having the same expressive power.

The 1-sorted language $\mathcal{L}_{\text{div}} = \mathcal{L}_{\text{Ring}} \cup \{|\}$, where $\mathcal{L}_{\text{Ring}} = \{0, 1, +, -, \times\}$ is the ring language and "$|$" is a binary relation symbol. It has one sort **VF** for the valued field. A valued field (K, v) gives rise to an $\mathcal{L}_{\text{div}}$-structure by setting $x|y :\iff v(x) \leq v(y)$.

The 2-sorted language $\mathcal{L}_\Gamma$ with sorts **VF** for the valued field and **Γ** for the value group (including ∞). It has $\mathcal{L}_{\text{Ring}}$ on the **VF**-sort, $\mathcal{L}_{\text{oag}} \cup \{\infty\}$ on the **Γ**-sort as well as a function symbol val : $\mathbf{VF} \to \mathbf{\Gamma}$. A valued field (K, v) gives rise to an $\mathcal{L}_\Gamma$-structure in the obvious way.

The 3-sorted language $\mathcal{L}_{\Gamma k}$ with sorts $\mathbf{VF}$, $\mathbf{\Gamma}$ and $\mathbf{k}$ (for the residue field). In addition to $\mathcal{L}_\Gamma$, it has $\mathcal{L}_{\text{ring}}$(a copy of $\mathcal{L}_{\text{Ring}}$) on sort $\mathbf{k}$ and a function symbol $\text{Res} : \mathbf{VF} \times \mathbf{VF} \to \mathbf{k}$. A valued field (K, v) gives rise to an $\mathcal{L}_{\Gamma k}$-structure by setting

$$\text{Res}(a, b) := \begin{cases} \text{res}(\frac{a}{b}) \text{ if } b \neq 0 \text{ and } \frac{a}{b} \in \mathcal{O}_v; \\ 0 \text{ otherwise.} \end{cases}$$

Remark 3.2. *Given a valued field (K, v), the structures associated to it in the languages $\mathcal{L}_{\text{div}}$, $\mathcal{L}_\Gamma$ and $\mathcal{L}_{\Gamma k}$ are mutually biinterpetable in each other, uniformly for all valued fields.*

As an example, let us indicate how to interpret the corresponding $\mathcal{L}_{\Gamma k}$-structure (K, Γ_v, k_v) in $(K, |)$. We have

- $x \in \mathcal{O}_v \iff 1|x$;
- $x \in \mathcal{O}_v^\times \iff (1|x \wedge x|1)$;
- $x \in \mathfrak{m}_v \iff (1|x \wedge x \not| 1$.

It follows that $k_v = \mathcal{O}_v/\mathfrak{m}_v$ and $\Gamma_v = K^\times/\mathcal{O}_v^\times$ are interpretable in $(K, |)$. Moreover, for $a, b \in K^\times$ one has $a|b$ if and only if $a/\mathcal{O}_v^\times \leq b/\mathcal{O}_v^\times$ in Γ_v. Thus $\mathcal{L}_{\text{oag}}$ is in this way interpretable in $(K, |)$. We leave the argument for the other parts of $\mathcal{L}_{\Gamma k}$ as an exercise.

4. Algebraically Closed Valued Fields

In this section, we treat the basic model theory of non-trivially valued algebraically closed fields. The corresponding theory is the model-completion of the theory of valued fields. Its study started with work of Abraham Robinson.

Definition 4.1. *In any of the languages $\mathcal{L}_{\text{div}}$, $\mathcal{L}_\Gamma$, $\mathcal{L}_{\Gamma k}$, let* ACVF *be the theory of* non-trivially valued algebraically closed fields.

Lemma 4.2. *Let (K, v) be a valued field and let v^{alg} be an extension of v to a valuation on K^{alg}, an algebraic closure of K. Then $k_v \subseteq k_{v^{alg}}$ and $\Gamma_v \subseteq \Gamma_{v^{alg}}$ canonically, and with these identifications we have $k_{v^{alg}} = (k_v)^{alg}$ and $\Gamma_{v^{alg}} = \text{Div}(\Gamma_v)$.*

In particular, if $(K, v) \models$ ACVF, then $k_v \models$ ACF and $\Gamma_v \models$ DOAG.

Proof. The inclusions $k_{v^{\text{alg}}} \subseteq (k_v)^{\text{alg}}$ and $\Gamma_{v^{\text{alg}}} \subseteq \text{Div}(\Gamma_v)$ both follow from the weak version of the Fundamental Inequality [11, Theorem 2.8].

Next, we show that $k_{v^{\text{alg}}}$ is algebraically closed. For this, let $p(X) \in k_{v^{\text{alg}}}[X]$, $p(X) = X^d + \beta_{d-1}X^{d-1} + \cdots + \beta_0$, where $d \geq 1$. Now choose a lifting

$$\tilde{p}(X) = X^d + b_{d-1}X^{d-1} + \cdots + b_0 \in \mathcal{O}_{v^{\text{alg}}}[X],$$

i.e., $\overline{b_i} = \beta_i$ for all i. Since K^{alg} is algebraically closed, $\tilde{p}(X) = \prod_{i=1}^d (X - a_i)$, where $a_i \in K^{\text{alg}}$. As $\mathcal{O}_{v^{\text{alg}}}$ is integrally closed in K^{alg} ([11, Proposition 1.14]), $a_i \in \mathcal{O}_{v^{\text{alg}}}$ for all i, and so $p(X) = \prod_{i=1}^d (X - \alpha_i)$, where $\alpha_i = \overline{a_i}$.

To finish the proof, we need to show that $\Gamma_{v^{\text{alg}}}$ is divisible. Let $\gamma \in \text{Div}(\Gamma_{v^{\text{alg}}})$ and let $n \geq 1$ be minimal such that $n\gamma \in \Gamma_{v^{\text{alg}}}$. Choose $a \in K^{\text{alg}}$ such that $v^{alg}(a) = n\gamma$. We find $b \in K^{\text{alg}}$ such that $b^n = a$. It follows that $v^{\text{alg}}(b) = \gamma$. □

The following result is proved in [11, Example 2.5].

Proposition 4.3. *Let (K,v) be a valued field and $\tilde{\gamma} \in \tilde{\Gamma} \geq \Gamma_v$. Then there is a unique extension w of v to $K(X)$ such that for all polynomials $\sum_i a_i X^i \in K[X]$ one has $w(\sum_i a_i X^i) = \min\{v(a_i) + i\tilde{\gamma}\}$.*

(a) If $\tilde{\gamma} = 0$, then w is called the Gauss extension *of v. This extension is determined by $w(X) = 0$ and $\mathrm{res}(X) \notin k_v^{alg}$, and it satisfies $k_w = k_v(\mathrm{res}(X))$, $\Gamma_w = \Gamma_v$.*

(b) If $\tilde{\gamma} \notin \mathrm{Div}(\Gamma_v)$, the extension is determined by $w(X) = \tilde{\gamma}$, and it satisfies $k_w = k_v$ and $\Gamma_w = \Gamma_v \oplus \mathbb{Z} \cdot \tilde{\gamma}$ as pure groups.

Corollary 4.4. *Let (K,v) be a valued field with $K = K^{alg}$, and let w be an extension of v to $K(X)$. Then there are three mutually exclusive cases:*

(a) (inertial) $k_w \supsetneq k_v$. Then there are $a \in K^\times$ and $b \in K$ such that $t = aX + b$ has valuation 0 and $\bar{t} \notin k_v^{alg} = k_v$, i.e., w is the Gauss extension with respect to t. One has $k_w = k_v(\bar{t})$ and $\Gamma_w = \Gamma_v$.

(b) (ramified) $\Gamma_w \supsetneq \Gamma_v$. Then there exists $c \in K$ such that $\tilde{\gamma} = w(X - c) \notin \Gamma_v = \mathrm{Div}(\Gamma_v)$, and one has $k_w = k_v$ and $\Gamma_w = \Gamma_v \oplus \mathbb{Z} \cdot \tilde{\gamma}$.

(c) (immediate) $k_w = k_v$ and $\Gamma_w = \Gamma_v$. This is equivalent to:

(IM) *The set $I(X/K) := \{w(X - c) \mid c \in K\}$ has no maximal element.*

Let $(c_\alpha)_{\alpha<\lambda}$ be a sequence in K (λ a limit ordinal) such that $(\gamma_\alpha = w(X - c_\alpha))_{\alpha<\lambda}$ is cofinal in $I(X/K)$. Then the extension w is determined by $w(X - c_\alpha) = \gamma_\alpha$ for all $\alpha < \lambda$.

Proof. The three cases are clearly mutually exclusive.

In case (b), as K is algebraically closed, there must be a linear polynomial $aX - d \in K[X]$ such that $w(aX - d) \notin \Gamma_v$. But then $a \neq 0$ and $c := d/a$ works. We finish by Proposition 4.3(b).

Now consider case (a). Let $f(X) \in K(X)$ be a rational function such that $w(f(X)) = 0$ and $\mathrm{res}(f(X)) \notin k_v$. By the previous paragraph, we know that $\Gamma_w = \Gamma_v$ in this case. As $K = K^{\mathrm{alg}}$, we may thus write

$$f(X) = \frac{\prod_i (a_i X + b_i)}{\prod_i (a_i' X + b_i')},$$

with all linear polynomials involved of valuation 0. Thus, the residue of one of them is not in k_v.

It is easy to see that in cases (a) and (b) the set $I(X/K)$ contains a maximal element. We leave the proof as an exercise. Conversely, assume the extension $(K(X), w)/(K, v)$ is immediate (i.e., $\Gamma_w = \Gamma_v$ and $k_w = k_v$), and let $c \in K$. Then $w(X - c) \in \Gamma_v$. So there is $b \in K^\times$ such that $v(b) = w(X - c)$. As $k_w = k_v$, there is $d \in K$ such that $w\left(\frac{X-c}{b} - d\right) > 0$. It follows that $w(X - c - db) > w(X - c)$, showing that $w(X - c)$ is not maximal in $I(X/K)$. The fact that the extension is determined by $w(X - c_\alpha) = \gamma_\alpha$ for all $\alpha < \lambda$ is left to the reader (easy exercise). □

The following result was (essentially) proved by Robinson in [15].

Theorem 4.5 (Robinson)**.** *(1) The theory* ACVF *eliminates quantifiers in $\mathcal{L}_{\mathrm{div}}$.*

(2) Its completions are given by $\mathrm{ACVF}_{p,q}$, where (p,q) is the pair of characteristics $(\mathrm{char}(K), \mathrm{char}(k_v)) \in \{(0,0), (0,p), (p,p) \mid p$ prime$\}$.

Proof. Part (2) follows from (1). Indeed, $(\mathbb{Q}, v_{\mathrm{triv}})$, $(\mathbb{Q}, v_p)$ and $(\mathbb{F}_p, v_{\mathrm{triv}})$, respectively, are common substructures for all models of $\mathrm{ACVF}_{0,0}$, $\mathrm{ACVF}_{0,p}$ and $\mathrm{ACVF}_{p,p}$, respectively. Here we have used the classification of valuations on $\mathbb{Q}$ (see [11, Proposition 1.15]).

We now prove quantifier elimination (part (1)). Let $K, K' \models$ ACVF with K countable and K' $\aleph_1$-saturated. Let $A \leq K$ be a substructure and let $f : A \hookrightarrow K'$ be an $\mathcal{L}_{\text{div}}$-embedding. We need to extend f to the whole of K.

Step 1: We may assume A is a field.

Indeed, f extends to $\tilde{f} : Q(A) \hookrightarrow K'$ in $\mathcal{L}_{\text{Ring}}$. Moreover, as $\frac{a}{b} | \frac{c}{d} \iff ad|bc$, the map $\tilde{f}$ automatically preserves "$|$".

Step 2: We may assume that $A = A^{alg}$.

Indeed, f extends to an $\mathcal{L}_{\text{Ring}}$-embedding $\tilde{f} : A^{\text{alg}} \hookrightarrow K'$. Now, $\mathcal{O}_1 := \tilde{f}(\mathcal{O}_K \cap A^{\text{alg}})$ and $\mathcal{O}_2 := \tilde{f}(A^{\text{alg}}) \cap \mathcal{O}_{K'}$ are both extensions of the valuation ring $\mathcal{O}_{f(A)}$ to $f(A)^{\text{alg}}$. By the conjugation theorem[1] ([11, Theorem 2.13 and Fact 2.14]), there is $\sigma \in \text{Aut}(f(A)^{\text{alg}}/f(A))$ such that $\sigma(\mathcal{O}_1) = \mathcal{O}_2$. Then $\sigma \circ \tilde{f}$ is an $\mathcal{L}_{\text{div}}$-embedding of A^{alg} extending f.

Step 3: Let $b \in K \setminus A = K \setminus A^{alg}$. Then f extends to an embedding $\tilde{f} : A(b) \hookrightarrow K'$.

Case 1: $k_{A(b)} \supsetneq k_A$. Up to an affine change of coordinates (over A), we may assume that $b \in \mathcal{O}_K$ and $\bar{b} \notin k_A = k_A^{\text{alg}}$. So $A(b)/A$ is the Gauss extension. By $\aleph_1$-saturation of K' there is $b' \in K'$ such that $b' \in \mathcal{O}_{K'}$ and $\overline{b'} \notin k_{f(A)} = k_{f(A)}^{\text{alg}}$. Then $b \mapsto b'$ defines an extension of f to $A(b)$, by Proposition 4.3(a).

Case 2: $\Gamma_{A(b)} \supsetneq \Gamma_A$. Up to translation by an element of A, we may assume that $\gamma = v(b) \notin \Gamma_A = \text{Div}(\Gamma_A)$. Let $(L(\gamma/\Gamma_A), R(\gamma/\Gamma_A))$ be the cut of γ over $\Gamma(A)$. Its image in $\Gamma_{f(A)}$ under the induced map $f_{\mathbf{\Gamma}}$ is realised in $\Gamma_{K'}$, by $\aleph_1$-saturation of K', say by γ'. Now let $b' \in K'$ be such that $v'(b') = \gamma'$. Then $b \mapsto b'$ defines an extension of f to $A(b)$, by Proposition 4.3(b).

Case 3: *$A(b)/A$ is an immediate extension.* Choose a sequence $(c_\alpha)_{\alpha<\omega}$ in A such that $\gamma_\alpha = v(b - c_\alpha)$ is strictly increasing and cofinal in $I(b/A)$. Now let $\alpha_1, \ldots, \alpha_n < \omega$ be given. Let $\beta < \omega$ such that $\beta > \alpha_i$ for all i. It follows from the ultrametric triangular inequality that

$$v(c_\beta - c_{\alpha_i}) = v(b - c_{\alpha_i}) \text{ for all } i,$$

as $v(b - c_\beta) = \gamma_\beta > v(b - c_{\alpha_i})$. The same is true for the images of the c_α's under f. By $\aleph_1$-saturation of K' we thus find $b' \in K'$ such that $v'(b' - f(c_\alpha)) = f_{\mathbf{\Gamma}}(v(b - c_\alpha)) = f_{\mathbf{\Gamma}}(\gamma_\alpha)$ for all $\alpha < \lambda$. By Corollary 4.4(c), $b \mapsto b'$ defines an extension of f to $A(b)$. □

Definition 4.6. *Let (K, v) be a valued field, $a \in K$ and $\gamma \in \Gamma_K \setminus \{\infty\}$. Then*

- $B_{>\gamma}(a) := \{b \in K \,|\, v(b - a) > \gamma\}$ *is called the* open ball, *and*
- $B_{\geq\gamma}(a) := \{b \in K \,|\, v(b - a) \geq \gamma\}$ *the* closed ball

of (valuative) radius γ around a.

We omit the easy proof of the following fact.

Fact 4.7. *Let (K, v) be a valued field. The set of open balls then is an open basis for a topology τ_v (the* valuation topology*) which turns K into a topological field.*

Remark 4.8. *Let (K, v) be a valued field.*

(1) Both 'open' and 'closed' balls in K are open for the valuation topology.

[1] If one wants to avoid the conjugation theorem for infinite algebraic normal extensions, one may extend f to any finite normal extension and iterate this process.

(2) If B, B' are two balls (open or closed) in K, then either $B \cap B' = \emptyset$ or $B \subseteq B'$ or $B' \subseteq B$.

Proof. It is a consequence of the ultrametric triangular inequality that every element of a ball is a center. Both parts of the remark follow from this. □

Definition and Remark 4.9. *Let (K, v) be a valued field.*

- *A* generalised ball *in K is a 'closed' ball or an 'open' ball, a singleton or K itself. Note that part (2) of Remark 4.8 is true for generalised balls.*
- *A* Swiss cheese *in K is a set $B \setminus \bigcup_{i=0}^{n-1} B_i$, where $n \in \mathbb{N}$ and B, B_i are generalised balls such that $B_i \subsetneq B$ for all i.*

Lemma 4.10. *Let $A \subseteq M \models T$, and let Φ be a set of formulas in variables $x_1, \ldots, x_n$, with parameters in A. Suppose that for $p \neq q$ in $S_n(A)$ there is $\varphi(\overline{x}) \in \Phi$ such that $p \vdash \varphi$ if and only if $q \vdash \neg\varphi$. Then every formula over A in $x_1, \ldots, x_n$ is equivalent to a (finite) Boolean combination of formulas in Φ.*

Proof. This follows easily from compactness and is left to the reader. □

Corollary 4.11. *Let $K \models$ ACVF, and let $D \subseteq K$ be a definable set (with parameters). Then D is a finite disjoint union of Swiss cheeses.*

Proof. Let us start with an easy observation whose proof we leave as an exercise:

Claim. *Any finite Boolean combination of generalised balls is a finite disjoint union of Swiss cheeses.*

By Lemma 4.10, it suffices to show that if $K \models$ ACVF, then the 1-types over K are separated by generalised balls. This follows from our analysis of the extension of v to $K(X)$ (Corollary 4.4). For convenience, we will give the arguments in detail.

Let $p(x) \in S_1(K)$. By quantifier elimination (Theorem 4.5), p is determined by the quantifier free formulas it contains. Let $a \models p$. If p is realised, then $p \vdash x = a$, and all other q's imply $x \neq a$.

From now on, assume that a is transcendental over K.

If $k_{K(a)} \supsetneq k_K$, there are $c \in K^\times$ and $d \in K$ such that $t = ca + d$ gives the Gauss extension. Then $a \in B_{\geq v(1/c)}(-d/c)$ and $a \notin B_{>v(1/c)}(e)$ for all $e \in K$. This determines the isomorphism type of $K(a)/K$ (and thus the quantifier free part of p) completely.

If the value group grows, i.e., $\Gamma_{K(a)} \supsetneq \Gamma_K$, there is $c \in K$ such that $v(a - c) = \tilde{\gamma} \notin \Gamma_K$. Then for $\gamma \in L(\tilde{\gamma}/\Gamma_K)$ we have

$$p(x) \vdash x \in B_{\geq\gamma}(c)$$

and for $\gamma \in R(\tilde{\gamma}/\Gamma_K)$ we have $p(x) \vdash x \notin B_{\geq\gamma}(e)$ for all $e \in K$. This determines the isomorphism type of $K(a)/K$ completely.

Finally, if $K(a)/K$ is immediate, choose a sequence $(c_\alpha)_{\alpha<\lambda}$ in K such that $(\gamma_\alpha)_{\alpha<\lambda}$ is cofinal in $I(a/K)$. Then $p(x) \vdash x \in B_{\geq\gamma_\alpha}(c_\alpha)$ for all $\alpha < \lambda$, which determines the isomorphism type of $K(a)/K$ completely. □

Exercise 4.12 (Holly's Theorem)**.** *Say that two Swiss cheeses $C = B \setminus \bigcup_{i=0}^{n-1} B_i$ and $C' = B' \setminus \bigcup_{i=0}^{n'-1} B'_i$ are* trivially nested *if $B = B'_i$ or $B' = B_i$ for some i.*

Let $K \models$ ACVF, and let $D \subseteq K$ be a definable subset. Show that D is a finite union of disjoint non-trivially nested Swiss cheeses $C_i = B_i \setminus \bigcup_{j=0}^{n_i-1} B_{ij}$ and that *this decomposition is unique, up to permutation (both of the Swiss cheeses and of the 'holes' inside a given cheese).*

Corollary 4.13. *Let $A \subseteq K \models \mathrm{ACVF}$. Then the following holds:*

(1) $\mathrm{acl}(A) = Q(\langle A\rangle)^{alg}$
(2) $\mathrm{dcl}(A) = \big(Q(\langle A\rangle)^{perf}\big)^h$

Here, $\langle A\rangle$ denotes the subring generated by the set A, $Q(B)$ the field of fractions of the ring B, L^{perf} the perfect closure of a field L, and L^h the henselization of a valued field L. (See [11, Section 4.2] for a discussion of the henselization.)

Proof. The inclusion $Q(\langle A\rangle)^{\mathrm{alg}} \subseteq \mathrm{acl}(A)$ is clear. Now let $t \in K \setminus Q(\langle A\rangle)^{\mathrm{alg}}$, and let $K' \succcurlyeq K$ be an $|A|^+$-saturated elementary extension. Let $\epsilon \in K' \setminus \{0\}$ such that $v(\epsilon) > \Gamma_K$. Then $t \mapsto t+\epsilon$ defines an isomorphism $Q(\langle A\rangle)(t) \cong_A Q(\langle A\rangle)(t+\epsilon)$. Indeed, whenever $p(t) \neq 0$ for a polynomial $p(X) \in K[X]$, there is an open ball B containing t such that $v(p(t')) = v(p(t))$ for all $t' \in B$. (This may be seen using the fact that the function defined by p is continuous.) By quantifier elimination, it follows that $\mathrm{tp}(t/A) = \mathrm{tp}(t+\epsilon/A)$. As there are infinitely many such ϵ's, we conclude that $t \notin \mathrm{acl}(A)$ which shows (1).

We now prove (2). Note that the inclusion $Q(\langle A\rangle)^{\mathrm{perf}} \subseteq \mathrm{dcl}(A)$ is clear. It is a general fact that if (K,v) is henselian and $K_0 \subseteq K$ is a subfield, then $K_0^h \subseteq \mathrm{dcl}(K_0)$, by the universal property of the henselization ([11, Theorem 4.8]). Thus we know $\big(Q(\langle A\rangle)^{\mathrm{perf}}\big)^h \subseteq \mathrm{dcl}(A)$. Moreover, by the first part we have $\mathrm{dcl}(A) \subseteq Q(\langle A\rangle)^{\mathrm{alg}}$. The extension $K_1 := Q(\langle A\rangle)^{\mathrm{alg}} / \big(Q(\langle A\rangle)^{\mathrm{perf}}\big)^h =: K_0$ is a Galois extension. Let $a \in K_1 \setminus K_0$. By Galois theory there is $\sigma \in \mathrm{Gal}(K_1/K_0)$ such that $\sigma(a) \neq a$. As K_0 is henselian and K_1/K_0 is algebraic, σ preserves $\mathcal{O}_{K_1}$, so is an elementary map by quantifier elimination. This shows $a \notin \mathrm{dcl}(A)$. □

Theorem 4.14. *Every completion of* ACVF *is NIP.*

Proof. It is enough to show that all formulas $\varphi(x;\overline{y})$ with x a single variable are NIP (see [17, Proposition 3.7]).

By Corollary 4.11 every definable set of a model is a finite Boolean combination of generalised balls. Given $\varphi(x;\overline{y})$, by compactness there is an integer $N = N(\varphi)$ such that any instance $\varphi(x;\overline{b})$ of φ defines a set $D_{\overline{b}}$ which is a Boolean combination of $\leq N$ generalised balls. The (definable) family of generalised balls is of VC-dimension 2, so in particular is NIP. (Indeed, $B \cap B' \neq \emptyset$ implies $B \subseteq B'$ or $B' \subseteq B$.) As any Boolean combination of NIP formulas is NIP, φ is NIP. □

Exercise 4.15 (Dimension for definable sets in ACVF)**.** *Let $K \models$ ACVF, and let $\varphi(K) = D \subseteq K^n$ be a definable set. Let $K' \succcurlyeq K$ be sufficiently saturated. Let*

- $\mathrm{alg.dim}(D) := \max\{\mathrm{tr.deg}(K(\overline{a})/K) \mid K' \models \varphi(\overline{a})\}$, *called the* algebraic dimension *of D, and*
- $\mathrm{top.dim}(D)$, *the* topological dimension *of D, be equal to the maximal m such that there exists a projection $\pi : K^n \to K^m$ with non-empty interior, i.e., such that*

$$\mathrm{int}(\pi(D)) = \{b \in \pi(D) \mid \text{ there is } \Omega \subseteq K^m \text{ open s.t. } b \in \Omega \subseteq \pi(D)\} \neq \emptyset.$$

Show the following properties:

(1) $\mathrm{alg.dim}(D \cup D') = \max\{\mathrm{alg.dim}(D), \mathrm{alg.dim}(D')\}$.

(2) *If* $f : D \twoheadrightarrow D'$ *is a definable surjection, then* $\operatorname{alg.dim}(D) \geq \operatorname{alg.dim}(D')$.
(3) *If* $D \subseteq K^n$, *then its algebraic dimension is given by* $\max\{m \mid \exists$ *projection* $\pi : K^n \to K^m$ *such that* $\operatorname{alg.dim}(\pi(D)) = m\}$.
(4) $\operatorname{top.dim}(\mathcal{O}^d) = d = \operatorname{alg.dim}(\mathcal{O}^d)$. *More generally, if* $D \subseteq K^d$ *has non-empty interior, then* $\operatorname{top.dim}(D) = d = \operatorname{alg.dim}(D)$.
(5) *If* $D \subseteq K^d$ *for* $d > 0$ *and* $\operatorname{alg.dim}(D) = d$, *then* $\operatorname{int}(D) \neq \emptyset$.
(6) *Conclude that* $\operatorname{top.dim}(D) = \operatorname{alg.dim}(D)$ *for every definable set* D. *It follows in particular that the topological dimension is invariant under definable bijections and that the algebraic dimension is definable in parameters.*

Exercise 4.16 (The prime models of the completions of ACVF)**.** *Show that the followong structures are the (unique) prime models of their respective theory:*

- $(\mathbb{Q}(X)^{alg}, v_X)$, *where* v_X *is the* X*-adic valuation (characteristic* $(0,0)$*);*
- $(\mathbb{Q}^{alg}, v_p)$, *where* v_p *is the* p*-adic valuation (characteristic* $(0,p)$*), and*
- $(\mathbb{F}_p(X)^{alg}, v_X)$, *where* v_X *is the* X*-adic valuation (characteristic* (p,p)*).*

Theorem 4.17. *The theory* ACVF *eliminates quantifiers in* $\mathcal{L}_\Gamma$ *as well as in* $\mathcal{L}_{\Gamma k}$.

Proof. We prove the result for $\mathcal{L}_{\Gamma k}$, using Fact 2.2. The proof of the other result is similar, and it is a formal consequence.

Let $\mathcal{K} = (K, \Gamma_K, k_K)$ and $\mathcal{K}' = (K', \Gamma_{K'}, k_{K'})$ be models of ACVF such that $|K| = \aleph_0$ and $\mathcal{K}'$ is $\aleph_1$-saturated. Let

$$A = (\mathbf{VF}(A), \mathbf{\Gamma}(A), \mathbf{k}(A)) \leq \mathcal{K}$$

be a substructure. Observe that $\operatorname{val}(A) \subsetneq \mathbf{\Gamma}(A)$ and $\operatorname{Res}(A^2) \subsetneq \mathbf{k}(A)$ are possible. Let $f = (f_{\mathbf{VF}}, f_{\mathbf{\Gamma}}, f_{\mathbf{k}}) : A \hookrightarrow \mathcal{K}'$ be an $\mathcal{L}_{\Gamma k}$-embedding. We need to extend f to the whole of $\mathcal{K}$.

Step 1: We may assume $\mathbf{k}(A) = k_K$.

Indeed, this follows from the fact that the theory ACF of algebraically closed fields eliminates quantifiers.

Step 2: We may assume that $\mathbf{\Gamma}(A) = \Gamma_K$.

This is a consequence of quantifier elimination in DOAG (Proposition 2.3).

Step 3: We may assume that $\mathbf{VF}(A)$ *is a field.*

Clearly, we may extend $f_{\mathbf{VF}}$ to $Q(\mathbf{VF}(A))$ by putting $\operatorname{val}(a/b) := \operatorname{val}(a) - \operatorname{val}(b)$ and $\operatorname{Res}(a/b, c/d) := \operatorname{Res}(ad, bc)$.

Step 4: We may assume that $k_{\mathbf{VF}(A)} = k_K$.

To see this, let $\alpha \in k_K \setminus k_{\mathbf{VF}(A)}$. Enumerating k_K and iterating if necessary, it is enough to show that f may be extended to an embedding $\tilde{f}$ which is defined on some $a \in K$ with $\operatorname{res}(a) = \alpha$.

Case 1: $\alpha \in k_{\mathbf{VF}(A)}^{alg}$. Let $\overline{p}(X) = X^n + \beta_{n-1}X^{n-1} + \cdots + \beta_0$ be the minimal polynomial of α over $k_{\mathbf{VF}(A)}$, and let $p(X) = X^n + b_{n-1} + \cdots b_0$ be a lift of $\overline{p}$ to $\mathcal{O}_{\mathbf{VF}(A)}$. Then

$$p(X) = \prod_{i=1}^{n}(X - a_i) \text{ with } a_i \in \mathcal{O}_{\mathbf{VF}(A)^{\text{alg}}} \text{ for all } i,$$

and so there must be some i such that $\overline{a_i} = \alpha$.

Similarly, find a zero a' of the polynomial $f_{\mathbf{VF}}(p(X))$ such that $\mathrm{res}(a') = f_{\mathbf{k}}(\alpha)$. We now claim that $a \mapsto a'$ defines an extension of f. To see this, note that $[k_{\mathbf{VF}(A)}(\alpha) : k_{\mathbf{VF}(A)}] = n \geq [\mathbf{VF}(A)(a) : \mathbf{VF}(A)]$. By the weak fundamental inequality ([11, Theorem 2.8]), it follows that $[\mathbf{VF}(A)(a) : \mathbf{VF}(A)] = n$ (in particular $p(X)$ is irreducible) and that the valuation $v\restriction_{\mathbf{VF}(A)}$ extends uniquely to $\mathbf{VF}(A)(a)$. The same is true for the corresponding images under f in $\mathcal{K}'$, and so the isomorphism of fields given by $a \mapsto a'$ respects the valuation.

Case 2: α is transcendental over $k_{\mathbf{VF}(A)}$. Choose any $a \in \mathcal{O}_K$ and $a' \in \mathcal{O}_{K'}$ with $\overline{a} = \alpha$ and $\overline{a'} = f_{\mathbf{k}}(\alpha)$. Then $a \mapsto a'$ defines an extension of the embedding, by uniqueness of the Gauss extension.

Step 5: We may assume that $\mathrm{val}(\mathbf{VF}(A)) = \Gamma_K$.

Let $\gamma \in \Gamma_K \setminus \Gamma_{\mathbf{VF}(A)}$. It is enough to show that we may extend f to an embedding $\tilde{f}$ which is defined on some $a \in K$ with $\mathrm{val}(a) = \gamma$.

Case 1: $\gamma \in \mathrm{Div}(\Gamma_{\mathbf{VF}(A)})$. Let $n \geq 2$ be minimal such that $\delta = n\gamma \in \Gamma_{\mathbf{VF}(A)}$. Choose $c \in \mathbf{VF}(A)$ such that $\mathrm{val}(c) = \delta$, then choose $a \in K$ with $a^n = c$. Similarly, choose $a' \in K'$ with $a'^n = f_{\mathbf{VF}}(c)$. As in Step 4, Case 1, by the fundamental inequality, $a \mapsto a'$ works.

Case 2: $\gamma \notin \mathrm{Div}(\Gamma_{\mathbf{VF}(A)})$. Choose $a \in K$ and $a' \in K'$ such that $\mathrm{val}(a) = \gamma$ and $\mathrm{val}(a') = f_{\mathbf{\Gamma}}(\gamma)$. Then $a \mapsto a'$ works (exercise).

Step 6: Extend f to the whole of $\mathcal{K}$, when the valued field extension $K/\mathbf{VF}(A)$ is immediate. This is possible by quantifier elimination in $\mathcal{L}_{\mathrm{div}}$ (Theorem 4.5), as $x|y \Longleftrightarrow \mathrm{val}(x) \leq \mathrm{val}(y)$, i.e., "|" is quantifier-free definable in $\mathcal{L}_{\Gamma k}$. □

Definition 4.18. *An $\emptyset$-definable set $D \subseteq M^n$ is called* stably embedded *in M if any M-definable subset of D^k (for all k) is definable with parameters from D.*

Corollary 4.19. *Let $\mathcal{K} = (K, \Gamma_K, k_K) \models$ ACVF. Then the following hold:*

(1) *The value group Γ_K is stably embedded in $\mathcal{K}$, and the structure induced on Γ_K by $\mathcal{K}$ is that of a pure model of* DOAG *(with a constant named for* $\mathrm{val}(p)$ *in case of mixed characteristic* $(0, p)$*).*
(2) *The residue field k_K is stably embedded in $\mathcal{K}$, and the structure induced on k_K by $\mathcal{K}$ is that of a pure algebraically closed field.*
(3) *$k_K \perp \Gamma_K$, i.e., every definable subset of $k_K^n \times \Gamma_K^m$ (for all n, m) is a finite union of sets of the form $D \times E$, where $D \subseteq k_K^n$ and $E \subseteq \Gamma_K^m$ are definable.*

Proof. We prove (2). Parts (1) and (3) are left as exercises.

By quantifier elimination (Theorem 4.17), every definable set $X \subseteq k_K^n$ is a finite union of sets which are defined by formulas $\varphi(\overline{x})$ of the form

$$\varphi_{\mathbf{VF}}(\overline{a}, \overline{b}) \wedge \varphi_{\mathbf{\Gamma}}(\mathrm{val}(F_1(\overline{a}, \overline{b}), \ldots, \mathrm{val}(F_l(\overline{a}, \overline{b}), \overline{\gamma}) \wedge \varphi_{\mathbf{k}}(\mathrm{Res}(a_1, b_1), \ldots, \mathrm{Res}(a_n, b_n), \overline{\alpha}, \overline{x}),$$

where $\varphi_{\mathbf{VF}}(\overline{y}, \overline{z})$ is a quantifier free formula in $\mathcal{L}_{\mathrm{Ring}}$, $\varphi_{\mathbf{\Gamma}}(\overline{u}, \overline{v})$ is a quantifier free formula in $\mathcal{L}_{\mathrm{oag}}$, $\varphi_{\mathbf{k}}(\overline{s}, \overline{t}, \overline{x})$ is a quantifier free $\mathcal{L}_{\mathrm{ring}}$-formula, $F_1, \ldots, F_l$ are polynomials over $\mathbb{Z}$ or $\mathbb{F}_p$, $\overline{a}, \overline{b}$ are parameters from K, $\overline{\gamma}$ from Γ_K and $\overline{\alpha}$ from k_K.

Such a formula $\varphi(\overline{x})$ defines either $\emptyset$ or the set given by $\varphi_{\mathbf{k}}(\beta_1, \ldots, \beta_n, \overline{\alpha}, \overline{x})$, where $\beta_i = \mathrm{Res}(a_i, b_i) \in k_K$. The result follows. □

Exercise 4.20. *Let $\mathcal{K} = (K, \Gamma_K, k_K) \models$ ACVF. The aim of this exercise is to show that there are no definable functions $f : k_K^n \to K^m$ or $f : \Gamma_K^n \to K^m$ with infinite image. Here are some hints:*

(1) *Show that one may assume* $n = 1$.
(2) *Show that any infinite definable set* $D \subseteq K$ *embeds definably* $\mathcal{O}_K$.
(3) *Observe that both* $\mathrm{val} : \mathcal{O}_K \to \Gamma_K$ *and* $\mathrm{res} : \mathcal{O}_K \to k_K$ *have infinite image.*
(4) *Use* $k_K \perp \Gamma_K$ *to conclude.*

More advanced topics in the model theory of algebraically closed valued fields. In these notes, we only presented the basic layer of the model theory of ACVF. However, there have been important developments in the last decade, and we would like to mention some of these briefly, also to suggest further reading.

Through their work on ACVF ([8, 9]), Haskell, Hrushovski and Macpherson have made available methods of stability theory, and more generally of geometric model theory, for the study of valued fields. They classify the imaginaries in ACVF, and they develop a theory of stable domination. The more advanced model theory of ACVF has already had important applications, e.g., Hrushovski-Kazhdan's new kind of motivic integration (see [7]) or Hrushovski-Loeser's work on non-archimedean geometry.

Classification of imaginaries. By the quantifier elimination result we proved (Theorem 4.5), we know exactly the definable sets in models of ACVF. Classifying the imaginaries in this setting, i.e., the definable quotients, is a much more involved task, and it is essential for geometric model theory. In [8] such a classification is given. It is enough to add to the valued field sort **VF** certain natural definable quotients, called the geometric sorts, so that all definable quotients in ACVF may be embedded into finite products of sorts. For any $n \geq 1$, one adds

- the set S_n of $\mathcal{O}_K$-lattices[2] in K^n, given by $\mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O}_K)$, and
- the union T_n of all $s/\mathfrak{m}s$, where s is an $\mathcal{O}_K$-lattice in K^n.

Note that $\Gamma_K \cong S_1(K)$ and $k_K \subseteq T_1(K)$, canonically, so the geometric sorts may be seen as higher dimensional analogs of the value group and the residue field.

Stable domination. If B is a C-definable ball, an element $a \in B$ is called *generic* in B over C if it is not contained in any C-definable finite union of proper subballs of B. By quantifier elimination and compactness, there is a unique type $p_B|C$ of a generic element in B over C. Now, if $a \models p_{\mathcal{O}}|C$ and $C \subseteq C' \subseteq K \models \mathrm{ACVF}$, then $a \models p_{\mathcal{O}}|C'$ if and only if $\mathrm{res}(a) \notin k_{C'}^{\mathrm{alg}}$, i.e., if $\mathrm{res}(a)$ is generic over $k_{C'}$ in the residue field $\mathbf{k}$, which is a stable stably embedded definable set in ACVF. This means that $p_{\mathcal{O}}$ is *stably dominated*: its generic extension is controlled by the image of a realisation under definable maps to stable stably embedded sets. Over a model, the stable stably embedded sets are all in the definable closure of $\mathbf{k}$, but over arbitrary parameter sets, more general ones have to be considered. It is not hard to see that the generic type p_B of a ball B is stably dominated if and only if B is a closed ball.

In [9], it is shown that if $M \models \mathrm{ACVF}$ is maximally valued (cf. the following section) and $\overline{a}$ is any tuple, $\mathrm{tp}(\overline{a}/M, \mathrm{dcl}(M\overline{a}) \cap \mathbf{\Gamma})$ is stably dominated. Thus, types in ACVF are controlled in a strong sense by the value group and the residue field.

A model-theoretic approach to non-archimedean geometry. Using the full machinery of geometric model theory in ACVF, in recent celebrated work ([10], see also [3]), Hrushovski and Loeser have developed a model-theoretic approach to non-archimedean geometry. In particular, they have obtained very general topological

[2]An $\mathcal{O}_K$-lattice in K^n is a free $\mathcal{O}_K$-submodule of rank n.

tameness results for the (Berkovich) analytification V^{an} of an algebraic variety V defined over a valued field, without any smoothness assumption on V.

5. Kaplansky Theory

In this section, we give a brief introduction to Kaplansky theory. In his beautiful 1942 paper [12], building on earlier work of Ostrowski, Kaplansky developed a theory of pseudo-convergence which turned out to be crucial for a good understanding of maximally valued fields, i.e., valued fields which do not admit proper immediate extensions. These play a prominent role in the model theory of valued fields, e.g., in the proof of the Ax-Kochen-Ershov theorem as we will see in Section 6. Maybe due to its 'analytic' flavor, Kaplansky theory tends to be neglected by algebraists.

Definition 5.1. *Let (K, v) be a valued field.*

(1) *A sequence $(c_\alpha)_{\alpha<\lambda}$ in K (where λ is some limit ordinal) is called* pseudo-Cauchy (PC) *if there is $\alpha_0 < \lambda$ such that $v(c_{\alpha_3} - c_{\alpha_2}) > v(c_{\alpha_2} - c_{\alpha_1})$ for all $\alpha_0 \leq \alpha_1 < \alpha_2 < \alpha_3 < \lambda$.*

Note that for any $\alpha \geq \alpha_0$, one has

$$\gamma_\alpha := v(c_\alpha - c_{\alpha+1}) = v(c_\alpha - c_{\alpha'}) \text{ for all } \alpha < \alpha' < \lambda.$$

(2) *Let $(c_\alpha)_{\alpha<\lambda}$ be a PC sequence in K, with notations from (1). An element $a \in K$ is a* pseudo-limit (PL) *of (c_α) if $v(a - c_\alpha) = \gamma_\alpha$ for all $\alpha \geq \alpha_0$. We will denote this by $(c_\alpha) \Rightarrow a$.*

(3) *The valued field (K, v) is called* pseudo-complete[3] *if every PC sequence in K has a PL in K.*

The following are useful observations. Their easy proofs are left to the reader.

Remark 5.2. (1) *If $(c_\alpha) \Rightarrow a$ and b is such that $v(a - b) > \gamma_\alpha$ for all $\alpha \geq \alpha_0$, then $(c_\alpha) \Rightarrow b$. In particular, pseudo-limits are not necessarily unique.*

(2) *If $(c_\alpha)_{\alpha<\lambda}$ is PC, either $v(c_\alpha)$ is eventually strictly increasing (which happens iff $(c_\alpha) \Rightarrow 0$) or $v(c_\alpha)$ is eventually constant.*

Lemma 5.3. *Let $(L, w)/(K, v)$ be an immediate extension of valued fields, and let $a \in L \setminus K$. Then there is a PC sequence $(c_\alpha)_{\alpha<\lambda}$ in K without PL in K and such that $(c_\alpha) \Rightarrow a$.*

In particular, any pseudo-complete valued field is maximally valued and thus henselian.

Proof. By the proof of Corollary 4.4(c), the set $I(a/K)$ has no maximal element. Now choose inductively elements $c_\alpha \in K$ such that the sequence $(\gamma_\alpha = v(a - c_\alpha))$ is strictly increasing and cofinal in $I(a/K)$. Then $(c_\alpha) \Rightarrow a$ by construction. If $b \in K$, by cofinality of the sequence, we have $\gamma_\alpha > v(a - b)$ for all sufficiently large α, and so $(c_\alpha) \not\Rightarrow b$. This proves the lemma. (Note that maximally valued fields are henselian, since the henselisation (K^h, v^h) is an immediate extension of (K, v) which is proper precisely in case (K, v) is not henselian.) □

We will see later that the converse of the lemma is true, so that the pseudo-complete valued fields are just the maximally valued ones.

[3] Valued fields with this property are sometimes called *spherically complete*.

Lemma 5.4. *Let λ be a limit ordinal and let $(c_\alpha)_{\alpha<\lambda}$ be a sequence in (K, v). Let $a \in K$ such that $v(a - c_\alpha)$ is eventually strictly increasing. Then (c_α) is pseudo-Cauchy and $(c_\alpha) \Rightarrow a$.*

Proof. This follows more or less from the definitions and is left as an exercise. □

The following lemma is very useful in practice.

Lemma 5.5. *Let $(c_\alpha)_{\alpha<\lambda}$ be a PC sequence in (K, v). Then there is an elementary extension $(K', v') \succcurlyeq (K, v)$ and an element $a' \in K'$ such that $(c_\alpha) \Rightarrow a'$.*

Proof. The set of formulas $\pi(x) := \{v(x - c_\alpha) = \gamma_\alpha \mid \alpha_0 \leq \alpha < \lambda)\}$ is finitely satisfiable in (K, v). Indeed, given finitely many $\alpha_1, \ldots, \alpha_n < \lambda$, let $\beta > \alpha_i$ for all i. Then $v(c_\beta - c_{\alpha_i}) = \gamma_{\alpha_i}$ for all i. By compactness there is $(K', v') \succcurlyeq (K, v)$ and an element $a' \in K'$ such that $\models \pi(a')$, i.e., $(c_\alpha) \Rightarrow a'$. □

Proposition 5.6. *The Hahn series field $k((t^\Gamma))$ is pseudo-complete, hence also maximally valued and henselian.*

Proof. Recall that the Hahn series field $K = k((t^\Gamma))$ is given by

$$k((t^\Gamma)) = \left\{ a = \sum_{\gamma\in\Gamma} a_\gamma t^\gamma \mid a_\gamma \in k, \text{supp}(a) \text{ is well-ordered} \right\},$$

with valuation $v(a) = \min(\text{supp}(a))$, where $\text{supp}(a) = \{\gamma \in \Gamma \mid a_\gamma \neq 0\}$. One has $k_K = k$ and $\Gamma_K = \Gamma$.

To prove that K is pseudo-complete, we use a diagonalization argument. Let $(a_\alpha)_{\alpha<\lambda}$ be a PC sequence in K, say $a_\alpha = \sum_{\gamma\in\Gamma} a_{\alpha,\gamma} t^\gamma$. Let α_0 and γ_α be as in the definition of a PC sequence. For all $\alpha' > \alpha \geq \alpha_0$ and $\gamma < \gamma_\alpha$, one has $\gamma_\alpha = v(a_{\alpha'} - a_\alpha)$, so $a_{\alpha,\gamma} = a_{\alpha',\gamma}$. Let

$$b_\gamma = \begin{cases} a_{\gamma,\alpha} \text{ if } \gamma < \gamma_\alpha \text{ for some } \alpha < \lambda; \\ 0 \text{ otherwise.} \end{cases}$$

This is well-defined by the above. Put $b = \sum b_\gamma t^\gamma$. Clearly, $(a_\alpha) \Rightarrow b$. We need to check that $b \in K$, and in particular that $\text{supp}(b)$ is well-ordered. Let $\gamma \in \text{supp}(b)$. Then there exists α such that $\gamma_\alpha > \gamma$, and so $(-\infty, \gamma] \cap \text{supp}(b) = (-\infty, \gamma] \cap \text{supp}(a_\alpha)$. The right hand side is well-ordered, which means every initial segment of $\text{supp}(b)$ is well-ordered, and hence $\text{supp}(b)$ is well-ordered. □

Lemma 5.7. *Let $\Gamma \models$ OAG and $(\gamma_\alpha)_{\alpha<\lambda}$ a strictly increasing sequence in Γ with λ a limit ordinal. Let $\delta_1, \ldots, \delta_n \in \Gamma$ and $k_1, \ldots, k_n \in \mathbb{N}$ pairwise distinct integers. Define, for $i = 1, \ldots, n$,*

$$f_i(x) = k_i x + \delta_i.$$

Then there exists $i_0 \in \{1, \ldots, n\}$ such that $f_{i_0}(\gamma_\alpha) < f_j(\gamma_\alpha)$ for all $j \neq i_0$, eventually.

Proof. Easy exercise by induction on n. □

Below, we will make use of the following formal version of the Taylor expansion.

Lemma 5.8. *Let $P(X) \in K[X]$, $\deg(P) \leq n$. Then there exist $P_0, \ldots, P_n \in K[X]$ such that $P(X + Y) = \sum_{i=0}^n P_i(X)Y^i$. One has $P_0(X) = P(X)$, $P_1(X) = P'(X)$, and $\deg(P_i) \leq n - i$.*

Moreover, if $P(X) \in \mathcal{O}_K[X]$, then $P_i \in \mathcal{O}_K[X]$ for all i.

Proof. For $P(X) = X^n$, one has $P_i(X) = \binom{n}{i} X^{n-i}$, so the result is clear in this case. Extend to arbitrary P by K-linearity. $\square$

Proposition 5.9 (Pseudo-continuity of polynomial functions)**.** *Let $(a_\alpha)_{\alpha<\lambda}$ be a PC sequence in (K, v), and let $P(X) \in K[X] \smallsetminus K$. Then $(P(a_\alpha))_{\alpha<\lambda}$ is PC and if $(a_\alpha) \Rightarrow a$, then $(P(a_\alpha)) \Rightarrow P(a)$.*

Proof. By Lemma 5.5 we may assume there is some PL $a \in K$, i.e., $(a_\alpha) \Rightarrow a$. Let $X = a$ and $Y = a_\alpha - a$. By Lemma 5.8, we get

$$P(a_\alpha) - P(a) = P_1(a)(a_\alpha - a) + P_2(a)(a_\alpha - a)^2 + \ldots + P_n(a)(a_\alpha - a)^n.$$

Now, $v\left(P_i(a)(a_\alpha - a)^i\right) = v(P_i(a)) + i\gamma_\alpha$; by Lemma 5.7 (taking $\delta_i = v(P_i(a))$, $x = \gamma_\alpha$ and $k_i = i$) there exists i_0 such that eventually

$$v\left(P_{i_0}(a)(a_\alpha - a)^{i_0}\right) < v\left(P_j(a)(a_\alpha - a)^j\right)$$

for all $j \neq i_0$. Then $v\left(P(a_\alpha) - P(a)\right) = \delta_{i_0} + i_0\gamma_\alpha$ is eventually strictly increasing in α. By Lemma 5.4, $(P(a_\alpha))_{\alpha<\lambda}$ is PC and has $P(a)$ as a PL. $\square$

Definition 5.10. *Let $(a_\alpha)_{\alpha<\lambda}$ be a PC sequence in (K, v). We say that $(a_\alpha)_{\alpha<\lambda}$ is of* transcendental type *over K if $(P(a_\alpha)) \not\Rightarrow 0$ for any $P(X) \in K[X] \smallsetminus K$. Otherwise, it is of* algebraic type *over K.*

Theorem 5.11. *Let $(a_\alpha)_{\alpha<\lambda}$ be PC in (K, v) of transcendental type over K. Then v admits a unique extension w to $K(X)$ such that $w(P(X))$ is the eventual value of $v(P(a_\alpha))$ for all $P(X) \in K[X] \smallsetminus K$. This extension is immediate and satisfies $(a_\alpha) \Rightarrow X$. Moreover, if (L, w) is an extension of (K, v) with $a \in L$ such that $(a_\alpha) \Rightarrow a$, then $X \mapsto a$ defines an isomorphism over K. In particular, all pseudo-limits of (a_α) are transcendental over K.*

Proof. It is easy to see that w defines a valuation. E.g., for axiom (iii), let $f, g \in K[X] \smallsetminus K$. Then $w(f + g)$ equals the eventual value of $v(f(a_\alpha) + g(a_\alpha))$. But we have $v(f(a_\alpha) + g(a_\alpha)) \geq \min\{v(f(a_\alpha)), v(g(a_\alpha))\}$ for all α, and the right hand side is eventually equal to $\min\{w(f(X)), w(g(X))\}$, so we may conclude.

Clearly, $\Gamma_w = \Gamma_v$ holds. To see that the extension is immediate, let $w(f/g) = 0$. As $\Gamma_w = \Gamma_v$, we may assume $w(f) = w(g) = 0$. It is enough to check that $\mathrm{res}(f(X)) \in k_K$.

We know from pseudo-continuity that $f(a_\alpha) \Rightarrow f(X)$, as $(a_\alpha) \Rightarrow X$ by construction. Thus, $w(f(X) - f(a_\alpha))$ is eventually strictly increasing. On the other hand, $v(f(a_\alpha)) = 0$ eventually. It follows that $w(f(X) - f(a_\alpha)) > 0$ eventually, and so $\mathrm{res}(f(X)) = \mathrm{res}(f(a_\alpha))$. Thus $(K(X), w)/(K, v)$ is immediate. The moreover part is left as an exercise. $\square$

Theorem 5.12. *Let $(a_\alpha)_{\alpha<\lambda}$ be a PC sequence in (K, v), without PL in K, of algebraic type over K. Let $\mu(X) \in K[X] \smallsetminus K$ be a non-constqant polynomial of minimal degree such that $(\mu(a_\alpha)) \Rightarrow 0$. Then μ is irreducible and* $\deg(\mu) \geq 2$. *There is an immediate algebraic extension $K(a)$ of K which is completely determined by the following two properties:*

(1) $(a_\alpha) \Rightarrow a$
(2) $\mu(a) = 0$

Proof. The proof of this theorem is similar to the previous one, albeit slightly more complicated. It uses Euclidean division. We refer to Kaplansky's original paper [12] for the details. $\square$

Remark 5.13. *Note that μ is not unique in Theorem 5.12, and that the isomorphism type of the extension $K(a)$ depends on the choice of μ.*

Corollary 5.14. *Let (K,v) be a valued field.*

(1) (K,v) is maximally valued if and only if it is pseudo-complete.
(2) (K,v) is algebraically maximal *(i.e., does not admit a proper immediate algebraic extension) if and only if every PC sequence of algebraic type over K has a PL in K*

Proof. ad (1): "$\rightarrow$" is by Lemma 5.3, and "$\leftarrow$" follows from Theorem 5.11 together with Theorem 5.12.
ad (2): "$\rightarrow$" follows from Theorem 5.12. For "$\leftarrow$", use that if L/K immediate, then any $a \in L \setminus K$ is a PL of a PC sequence in K without PL in K (Lemma 5.3). If a is algebraic and $(a_\alpha) \Rightarrow a$, with $a_\alpha \in K$, then the sequence (a_α) is of algebraic type over K, by the last part of Theorem 5.11. □

Corollary 5.15. *If (K,v) is of residue characteristic 0 then K^h is the (unique) maximal algebraic immediate extension of K. I.e., (K,v) is henselian if and only if it is algebraically maximal.*

Proof. If (K,v) is henselian and $\mathrm{char}(k_K)=0$, then by the fundamental equality ([11, Fact 2.17]), for every finite extension L/K we get $[L:K] = e\cdot f$, where $e = (\Gamma_L : \Gamma_K)$ and $f = [k_L : k_K]$. Hence L cannot be a proper immediate extension. □

The next result, dealing with the existence of maximally valued immediate extensions, is due to Krull. The issue of uniqueness of such extensions is more delicate. Kaplansky addresses it in his paper [12]. We do not present this here, as we will not need it in our course.

Fact 5.16 (Krull)**.** *(1) Let (K,v) be a valued field. Then $|K| \leq |k_K|^{|\Gamma_K|}$.*
(2) Any valued field admits a maximally valued immediate extension.

Proof. We give a short proof due to Gravett (see [5]). For $\gamma \in \Gamma_K$, consider the subgroups $\gamma\mathfrak{m}_K := \{x \in K \,|\, v(x) > \gamma\}$ and $\gamma\mathcal{O}_K := \{x \in K \,|\, v(x) \geq \gamma\}$ of the additive group of K. Then $\gamma\mathfrak{m}_K \subseteq \gamma\mathcal{O}_K$. Each coset of $\gamma\mathcal{O}_K$ contains exactly $|k_K|$ cosets of $\gamma\mathfrak{m}_K$. Using the axiom of choice, we find a family of maps $f_\gamma : K/\gamma\mathfrak{m}_K \to k_K$ such that for all γ and all $a+\gamma\mathfrak{m}_K, b+\gamma\mathfrak{m}_K \in K/\gamma\mathfrak{m}_K$, if $a+\gamma\mathcal{O}_K = b+\gamma\mathcal{O}_K$ and $f_\gamma(a+\gamma\mathfrak{m}_K) = f_\gamma(b+\gamma\mathfrak{m}_K)$, then $a+\gamma\mathfrak{m}_K = b+\gamma\mathfrak{m}_K$.

Now associate to each $a \in K$ the function $g_a : \Gamma_K \to k_K$ given by $\gamma \mapsto f_\gamma(a+\gamma\mathfrak{m}_K)$. It is easy to see that $a \mapsto g_a$ is injective. Indeed, if $a \neq b$, then for $\gamma = v(a-b)$ we get $g_a(\gamma) \neq g_b(\gamma)$. This proves (1).

Part (2) follows from (1) by Zorn's Lemma. □

6. Pas' Theorem and the Ax-Kochen-Ershov Principle

In this section we prove Pas' theorem (from [14]) and deduce from it the celebrated results due to Ax-Kochen and Ershov on the model theory of henselian valued fields in residue characteristic 0.

The notion of an angular component we introduce now was originally used in the context of cell decomposition in p-adic fields. We will not treat cell decomposition in these notes, but we will use angular components in Pas' theorem which is our route to the Ax-Kochen Ershov principle.

Definition 6.1. *Let (K,v) be a valued field. An* angular component map *for (K,v) is a map $\mathrm{ac}: K \to k_K$ satisfying*

1. $\mathrm{ac}(x)=0$ *iff* $x=0$;
2. $\mathrm{ac}|_{K^\times}: K^\times \to k_K^\times$ *is a group homomorphism, and*
3. $\mathrm{ac}|_{\mathcal{O}_K^\times} = \mathrm{res}|_{\mathcal{O}_K^\times}$.

We call (K,v,ac) an ac-valued field.

Examples 6.2.
1. *Let $K=k((t^\Gamma))$ be the Hahn series field. Then sending any non-zero element $a=\sum a_\gamma t^\gamma$ to the first non-zero coefficient, i.e., $a \mapsto a_{v(a)}$, defines an* ac *map.*
2. *In $(\mathbb{Q}_p, v_p)$, sending any $a \in \mathbb{Q}_p^\times$ with p-adic expansion $a=\sum a_i p^i$ (where $a_i \in \{0,\dots,p-1\}$) to $\mathrm{ac}(a)=a_{v_p} \bmod p$ defines an* ac *map.*

Remark 6.3. *There exist henselian valued fields that do not admit an angular component map.*

Lemma 6.4. *Let $s:\Gamma_K \to K^\times$ be a* cross-section*, i.e., a group homomorphism such that $\mathrm{val}\circ s = \mathrm{id}_{\Gamma_K}$. Then*

$$\mathrm{ac}(x)=\begin{cases}\mathrm{res}\left(\frac{x}{s(\mathrm{val}(x))}\right), & \text{if } x\neq 0;\\ 0, & \text{otherwise}\end{cases}$$

is an angular component map.

Proof. This follows from the definitions. We leave the proof to the reader. □

Note that the two examples in 6.2 come from the cross-sections $\gamma \mapsto t^\gamma$, and $i \mapsto p^i$, respectively.

Definition 6.5. *The* Denef-Pas language $\mathcal{L}_{\mathrm{DP}}$ *is the three-sorted language with sorts* $\mathbf{VF}$, $\mathbf{\Gamma}$, $\mathbf{k}$, *the usual languages on each sort,* $\mathrm{val}:\mathbf{VF}\to\mathbf{\Gamma}$, *and* $\mathrm{ac}:\mathbf{VF}\to\mathbf{k}$.

We consider ac-valued fields in $\mathcal{L}_{\mathrm{DP}}$. Observe that $\mathrm{res}:\mathbf{VF}\to\mathbf{k}$ is (quantifier free) definable in $\mathcal{L}_{\mathrm{DP}}$, as $\mathrm{res}(x)=\mathrm{ac}(x)$ for $x\in\mathcal{O}^\times$ and $\mathrm{res}(x)=0$ otherwise.

Definition 6.6. *The theory of henselian ac-valued fields of equicharacteristic 0 is denoted by T_{Pas}.*

Note that $k((t^\Gamma)) \models T_{\mathrm{Pas}}$ for any ordered abelian group Γ and any field k of characteristic 0.

Theorem 6.7 (Pas' Theorem [14])**.** *The theory T_{Pas} eliminates* $\mathbf{VF}$*-quantifiers.*

Remark 6.8. *The language $\mathcal{L}_{\Gamma k}$ would not suffice to eliminate* $\mathbf{VF}$*-quantifiers. Indeed, consider for example $K=\mathbb{Q}((t))$. The substructures $A=(\mathbb{Q}((t^2)),\mathbb{Z},\mathbb{Q})$ and $B=(\mathbb{Q}((2t^2)),\mathbb{Z},\mathbb{Q})$ contain the full value group and the full residue field of K. The map $t^2\mapsto 2t^2$ extends to an $\mathcal{L}_{\Gamma k}$-isomorphism $A\cong B$, but this is not an elementary map in the sense of K, as in K, t^2 is a square, but $2t^2$ is not.*

Before we start the proof of Pas' theorem, we show two lemmas.

Lemma 6.9. *Let (K,v,ac_K) be an ac-valued field.*

1. *If $x\in K^\times$ and $y\in B_{>v(x)}(x)$ then $\mathrm{ac}_K(y)=\mathrm{ac}_K(x)$.*
2. *If $(L,w)\supseteq(K,v)$ is unramified, i.e., if $\Gamma_w=\Gamma_v$, then ac_K uniquely extends to an* ac *map on L.*

(3) *If $(L, w, \mathrm{ac}_L) \supseteq (K, v, \mathrm{ac}_K)$, then the extension ac_L is determined by its values on representatives of generators of the quotient group Γ_L/Γ_K.*

Proof. ad (1): If $y \in B_{>v(x)}(x)$ then $y = ax$ for some $a \in 1+\mathfrak{m}_K$, and so by axioms (2) and (3) of an ac map we get $\mathrm{ac}(a) = \mathrm{res}(1) = 1$ and $\mathrm{ac}(x) = \mathrm{ac}(y)$..

ad (2): For $b \in L^\times$ let $a \in K^\times$ be such that $v(a) = w(b)$. Then $\mathrm{ac}_L(b) = \mathrm{ac}_K(a)\,\mathrm{res}_L(b/a)$ is well-defined, and is the only option that works.

ad (3): Similar to (2). □

Lemma 6.10. *Let $(L, w) \supseteq (K, v)$ be valued fields of residue characteristic 0 with (L, w) henselian and $k_L = k_K$. Then the following holds for any $n > 0$.*

(1) *For any $a \in \mathfrak{m}_L$ there is $b \in L$ such that $b^n = 1 + a$*
(2) *Let $\gamma \in \Gamma_L \cap \mathrm{Div}(\Gamma_K)$ and let $n > 0$ be minimal such that $n\gamma \in \Gamma_K$. Then there is $b \in L$ such that $w(b) = \gamma$ and $b^n \in K$.*

Proof. Part (1) is by Hensel's Lemma. Now let γ and n be as in (2). Choose $c \in K$ such that $v(c) = n\gamma$. Let $a \in L$ be such that $w(a) = \gamma$. As $k_L = k_K$, multiplying c by some unit in $\mathcal{O}_K^\times$, we may assume that $\mathrm{res}(a^n c^{-1}) = 1$. Then $a^n = c(1+\varepsilon)$ for some $\varepsilon \in \mathfrak{m}_L$, and by (1), $1 + \varepsilon = d^n$ for some $d \in L$. Thus, $c = (a/d)^n$, and so $b = a/d$ works. □

Proof of Theorem 6.7. Let $\mathcal{K} = (K, \Gamma_K, k_K, \mathrm{ac})$ and $\mathcal{L} = (L, \Gamma_L, k_L, \mathrm{ac})$ be models of T_{Pas}. Assume that K is countable and that $\mathcal{L}$ is $\aleph_1$-saturated. Now let $A = (\mathbf{VF}(A), \mathbf{\Gamma}(A), \mathbf{k}(A)) \leq \mathcal{K}$. Assume $f = (f_{\mathbf{VF}}, f_{\mathbf{\Gamma}}, f_{\mathbf{k}}) : A \hookrightarrow \mathcal{L}$ is an $\mathcal{L}_{\mathrm{DP}}$-embedding such that $f_{\mathbf{\Gamma}}$ is $\mathcal{L}_{\mathrm{oag}} \cup \{\infty\}$-elementary and $f_{\mathbf{k}}$ is $\mathcal{L}_{\mathrm{ring}}$-elementary. By Fact 2.2 it is enough to show that f extends to an embedding of $\mathcal{K}$.

Step 0: We may assume that $\mathbf{\Gamma}(A) = \Gamma_K$ and $\mathbf{k}(A) = k_K$.

Indeed, this follows from the fact that the respective maps $f_{\mathbf{\Gamma}}$ and $f_{\mathbf{k}}$ are elementary and $\mathcal{L}$ is $\aleph_1$-saturated.

Step 1: We may assume that $\mathbf{VF}(A)$ is a field.

Indeed, as a map of rings, $f_{\mathbf{VF}}$ extends to an embedding $\widetilde{f_{\mathbf{VF}}}$ defined on the field generated by $\mathbf{VF}(A)$. As $v(a/b) = v(a) - v(b)$ and $\mathrm{ac}(a/b) = \mathrm{ac}(a)/\mathrm{ac}(b)$, the map $\tilde{f} = (\widetilde{f_{\mathbf{VF}}}, f_{\mathbf{\Gamma}}, f_{\mathbf{k}})$ is an $\mathcal{L}_{\mathrm{DP}}$-embedding.

Step 2: We may extend f so that $\mathrm{res}(\mathbf{VF}(A)) = k_K$.

To see this, let $\alpha \in k_K \setminus \mathrm{res}(\mathbf{VF}(A))$. If $\alpha \notin k_{\mathbf{VF}(A)}^{\mathrm{alg}}$ then use the Gauss extension together with Lemma 6.9(2). (We leave the details as an exercise.) Otherwise, consider the minimal polynomial $\overline{P}(X) = X^d + \beta_{d-1}X^{d-1} + \ldots + \beta_0$ for α over $k_{\mathbf{VF}(A)}$. Take any lift $P(X) = X^d + b_{d-1}X^{d-1} + \ldots + b_0$ of $\overline{P}$ to $\mathcal{O}_{\mathbf{VF}(A)}$. Since the residue characteristic is 0, α is a simple root of $\overline{P}(X)$, and so by Hensel's lemma, there is $a \in \mathcal{O}_{\mathbf{VF}(A)^{\mathrm{alg}}}$ with $P(a) = 0$ and $\mathrm{res}(a) = \alpha$. Find $b \in L$ such that b is a root of $f_{\mathbf{VF}}(P(X))$ and $\mathrm{res}(b) = f_{\mathbf{k}}(\alpha)$; then $a \mapsto b$ works. Indeed, by the fundamental inequality the valuation extends uniquely in this field extension, and the value group does not grow. From the latter, we deduce that ac extends uniquely, using Lemma 6.9(2).

Step 3: We may extend f so that $\Gamma_{\mathbf{VF}(A)} = \Gamma_K$.

Indeed, let $\gamma \in \Gamma_K \setminus \Gamma_{\mathbf{VF}(A)}$. If $\gamma \notin \mathrm{Div}(\Gamma_{\mathbf{VF}(A)})$ then extending the valued field map is easy, but we also need to take care of ac. Let $a \in K$ such that $v(a) = \gamma$.

As $k_{\mathbf{VF}(A)} = k_K$ by step 2, we have $\mathrm{ac}(a) = \mathrm{ac}(c)$ for some $c \in \mathcal{O}^{\times}_{\mathbf{VF}(A)} \subseteq \mathbf{VF}(A)$. Thus, replacing a by a/c, we may assume $\mathrm{ac}(a) = 1$. Similarly, we find $b \in L$ such that $v(b) = f_{\mathbf{\Gamma}}(\gamma)$ and $\mathrm{ac}(b) = 1$. Lemma 6.9(3) together with Proposition 4.3(b) imply that $a \mapsto b$ defines an extension.

If $\gamma \in \mathrm{Div}(\Gamma_{\mathbf{VF}(A)})$, let $n \geq 2$ be minimal such that $n\gamma \in \Gamma_{\mathbf{VF}(A)}$. By Lemma 6.10(2), we find $a \in K$ such that $v(a) = \gamma$ and $a^n \in \mathbf{VF}(A)$. By the fundamental inequality, the valued field extension is completely determined (since $[\mathbf{VF}(A)(a) : \mathbf{VF}(A)] = n = (\Gamma_{\mathbf{VF}(A)(a)} : \Gamma_{\mathbf{VF}(A)})$). To preserve ac, we need to find $b \in L$ such that $\mathrm{val}(b) = f_{\mathbf{\Gamma}}(\gamma)$ and $\mathrm{ac}(b) = f_{\mathbf{k}}(\mathrm{ac}(a))$; by Lemma 6.9(3), this is enough. Now, by Lemma 6.10(2) again, there exists $c \in L$ such that $c^n \in f_{\mathbf{VF}}(\mathbf{VF}(A))$ and $\mathrm{val}(c) = f_{\mathbf{\Gamma}}(\gamma)$. We have $f_{\mathbf{k}}(k_K) \preccurlyeq k_L$ by assumption, hence $\mathrm{ac}(c) \in f_{\mathbf{k}}(k_K) = k_{f_{\mathbf{VF}}(\mathbf{VF}(A))}$, since by elementarity, $f_{\mathbf{k}}(k_K)$ is relatively algebraically closed in k_L. Thus there exists $d \in f_{\mathbf{VF}}(\mathcal{O}^{\times}_{\mathbf{VF}(A)})$ such that $\mathrm{res}(d) = f_{\mathbf{k}}(\mathrm{ac}(a))\,\mathrm{ac}(c^{-1})$. The element $b := cd$ is as wanted.

We now have $f : (\mathbf{VF}(A), \mathbf{\Gamma}(A), \mathbf{k}(A)) \hookrightarrow \mathcal{L}$ with $\mathbf{\Gamma}(A) = \Gamma_K = \Gamma_{\mathbf{VF}(A)}$ and $\mathbf{k}(A) = k_K = k_{\mathbf{VF}(A)}$, so $K/\mathbf{VF}(A)$ is an immediate extension. By Lemma 6.9(2), any further extensions of f which preserve val and res will preserve ac, as the value group remains unchanged.

Step 4: We may assume $\mathbf{VF}(A)$ *is henselian.*

Indeed, f extends uniquely to $\mathbf{VF}(A)^h$ by the universal property of the henselization.

Step 5: Let $\mathbf{VF}(A) = \mathbf{VF}(A)^h$*, and let* $a \in K \smallsetminus \mathbf{VF}(A)$*. Then* f *extends to an embedding of* $\mathbf{VF}(A)(a)$*.*

By Lemma 5.3, there exists a PC sequence $(a_\alpha)_{\alpha<\omega}$ in $\mathbf{VF}(A)$ without PL in $\mathbf{VF}(A)$ such that $(a_\alpha) \Rightarrow a$. By Corollary 5.15, $\mathbf{VF}(A)$ is algebraically maximal. But then Corollary 5.14(2) tells us that $(a_\alpha)_{\alpha<\omega}$ is of transcendental type over $\mathbf{VF}(A)$. Theorem 5.11 then implies that the extension $\mathbf{VF}(A)(a)/\mathbf{VF}(A)$ is completely determined by $(a_\alpha) \Rightarrow a$. By saturation of $\mathcal{L}$, we can find $b \in L$ such that $(f_{\mathbf{VF}}(a_\alpha)) \Rightarrow b$, and thus $a \mapsto b$ works.

Iterating the last two steps, we may extend f to the whole of $\mathcal{K}$. □

Corollary 6.11. *(1) The completions of* T_{Pas} *are given by* $T_{\mathrm{Pas}} \cup \mathrm{Th}(k) \cup \mathrm{Th}(\Gamma)$*, where* k *is a field of characteristic zero and* Γ *an ordered abelian group. In particular,* $k((t^\Gamma)) \models T_{\mathrm{Pas}} \cup \mathrm{Th}(k) \cup \mathrm{Th}(\Gamma)$*.*

(2) Let $\mathcal{K}, \mathcal{L} \models T_{\mathrm{Pas}}$*. Then*
  (a) $\mathcal{K} \equiv \mathcal{L}$ *iff* $\left[k_K \equiv_{\mathcal{L}_{\mathrm{ring}}} k_L \text{ and } \Gamma_K \equiv_{\mathcal{L}_{\mathrm{oag}}} \Gamma_L\right]$*, and*
  (b) If $\mathcal{K} \subseteq \mathcal{L}$ *then* $\mathcal{K} \preceq \mathcal{L}$ *iff* $[k_K \preceq k_L \text{ and } \Gamma_K \preceq \Gamma_L]$*.*

Proof. In (2b), "$\Rightarrow$" is clear, and "$\Leftarrow$" is given by Pas' Theorem.

In (2a), "$\Rightarrow$" is clear. To show "$\Leftarrow$", consider $\mathbb{Q}$ as a common substructure of k_K and k_L, and (0) as a common substructure of Γ_K, Γ_L. The corresponding map $f : \mathcal{K} \supseteq (\mathbb{Q}, (0), \mathbb{Q}, \mathrm{ac}_K) \cong (\mathbb{Q}, (0), \mathbb{Q}, \mathrm{ac}_L) \subseteq \mathcal{L}$ is an $\mathcal{L}_{\mathrm{DP}}$-isomorphism which is elementary on the level of residue fields (as $k_K \equiv k_L$ by assumption) and on the level of value groups (as $\Gamma_K \equiv \Gamma_L$ by assumption). It follows from Pas' Theorem that f is elementary.

Finally, (1) follows from (2b). □

Corollary 6.12. *Let $\mathcal{K} = (K, \Gamma_K, k_K) \models T_{\mathrm{Pas}}$. Then the following holds:*

(1) *Γ_K is stably embedded in $\mathcal{K}$, and the induced structure is that of a pure ordered abelian group.*
(2) *k_K is stably embedded in $\mathcal{K}$, and the induced structure is that of a pure field.*
(3) *$k_K \perp \Gamma_K$*

Proof. The proof is more or less the same as in ACVF (Corollary 4.19). □

Corollary 6.13 (The famous AKE transfer principle)**.** *Let φ be a sentence in $\mathcal{L}_{\mathrm{DP}}$. Then there is $N = N(\varphi) \in \mathbb{N}$ such that for all primes $p > N$ we have $\mathbb{Q}_p \models \varphi$ iff $\mathbb{F}_p((t)) \models \varphi$.*

Proof. Suppose not, and let φ be a counterexample. Let X be an infinite set of primes such that $\mathbb{Q}_p \models \neg\varphi$ iff $\mathbb{F}_p((t)) \models \varphi$ for all $p \in X$. Let $\mathcal{U}$ be a non-principal ultrafilter on the set of primes such that $X \in \mathcal{U}$.

For p prime, let $K_p = \mathbb{F}_p((t))$ if $\mathbb{F}_p((t)) \models \varphi$, and $K_p = \mathbb{Q}_p$ otherwise; similarly, let $L_p = \mathbb{Q}_p$ if $\mathbb{Q}_p \models \varphi$ and $L_p = \mathbb{F}_p((t))$ otherwise. Then $\mathcal{K} = \prod_{\mathcal{U}} K_p$ and $\mathcal{L} = \prod_{\mathcal{U}} L_p$ are models of T_{Pas}. (In particular they have equicharacteristic 0 and are henselian.) We have $\Gamma_K = \mathbb{Z}^{\mathcal{U}} = \Gamma_L$ and $k_K = \prod_{\mathcal{U}} \mathbb{F}_p = k_L$. By Corollary 6.11(2a), we deduce from this that $\mathcal{K} \equiv \mathcal{L}$.

But $X \subseteq \{p \text{ prime} \mid K_p \models \varphi\} \cap \{p \text{ prime} \mid L_p \models \neg\varphi\}$; since $X \in \mathcal{U}$, we must have $\mathcal{K} \models \varphi$ and $\mathcal{L} \models \neg\varphi$, which is a contradiction. □

Example 6.14 (An application: the approximate solution of Artin's Conjecture)**.** *For $i, d \geq 1$, say that a field K is $C_i(d)$ if every homogeneous polynomial of degree d in more than d^i variables with coefficients in K has a non-trivial zero in K. We say that K is C_i if it is $C_i(d)$ for all d.*

A couple of facts:

(1) *Finite fields are C_1 (Chevalley).*
(2) *If k is C_i, then $k((t))$ is C_{i+1} (Greenberg). In particular, $\mathbb{F}_p((t))$ is C_2.*

Artin's Conjecture asserts that $\mathbb{Q}_p$ is C_2 for all p. This conjecture is not true; counterexamples were found by Terjanian in 1966. But it is approximately true, in the following sense:

Theorem. *For all d there exists $N(d) \in \mathbb{N}$ such that for all $p > N(d)$, $\mathbb{Q}_p$ is $C_2(d)$.*

Indeed, for each d there is an $\mathcal{L}_{\mathrm{Ring}}$-sentence φ_d expressing that a field is $C_2(d)$. The result then follows from the transfer principle, with $N(d) = N(\varphi_d)$.

Exercise 6.15. *Let $\mathcal{K} \models T_{\mathrm{Pas}}$. Then $\mathrm{Th}(\mathcal{K})$ is decidable iff $\mathrm{Th}(k_K)$ and $\mathrm{Th}(\Gamma_K)$ are decidable. For example, $\mathbb{C}((t))$ has a decidable theory as an $\mathcal{L}_{\mathrm{DP}}$-structure (and hence also as an $\mathcal{L}_{\Gamma k}$-structure and as a field).*

We now quote a result due to Delon. A rather direct proof of it may be found in [16].

Theorem 6.16 (Delon)**.** *Let $\mathcal{K} \models T_{\mathrm{Pas}}$. Then $\mathcal{K}$ is NIP iff k_K is NIP.*

Delon's theorem initially contained the condition that the value group be NIP as well. But this condition is always satisfied, by the Theorem of Gurevich-Schmitt (Fact 2.7).

Theorem 6.17 (Ax-Kochen-Ershov principle). *Let $\mathcal{K}$ and $\mathcal{L}$ be two henselian valued fields of equicharacterisic 0 in $\mathcal{L}_{\Gamma k}$. Then*

(1) $\mathcal{K} \equiv \mathcal{L}$ iff $[k_K \equiv k_L$ and $\Gamma_K \equiv \Gamma_L]$.
(2) If $\mathcal{K} \subseteq \mathcal{L}$ then $\mathcal{K} \preceq \mathcal{L}$ iff $[k_K \preceq k_L$ and $\Gamma_K \preceq \Gamma_L]$.

Remark 6.18. *The main difference between this theorem and Corollary 6.12 is that AKE does not require an angular component map. We will reduce it to Corollary 6.12 by observing that although there are henselian valued fields without angular component maps, all $\aleph_1$-saturated valued fields do admit an* ac *map.*

In order to perform the reduction to the language with angular components, we need a result from the model theory of modules, namely that $\aleph_1$-saturated modules are 'pure-injective'. The following fact states the result in the case of abelian groups. The proof is not difficult and may be found, e.g., in [2, Chapter II, Theorem 27].

Recall that if $A \leq B$ are abelian groups, A is called a *pure* subgroup of B if for every $n \geq 2$, whenever an element $a \in A$ is divisible by n in B, it is already divisible by n in A. This condition is in particular satisfied when B/A is torsion-free.

Fact 6.19. *Let A, B, U be abelian groups with U $\aleph_1$-saturated. Assume that A is a pure subgroup of B. Then every homomorphism $f : A \to U$ extends to a homomorphism $\tilde{f} : B \to U$.*

Proof of Theorem 6.17. Let $\mathcal{K}' \succeq \mathcal{K}$ and $\mathcal{L}' \succeq \mathcal{L}$ be $\aleph_1$-saturated elementary extensions. By Fact 6.19, the short exact sequence of abelian groups given by the valuation map

$$1 \longrightarrow \mathcal{O}_{K'}^{\times} \longrightarrow K'^{\times} \longrightarrow \Gamma_{K'} \longrightarrow 0$$

is split. Indeed, 6.19 applied to $U = A = \mathcal{O}_{K'}^{\times} \leq B = K'^{\times}$ and $f = \mathrm{id}_U$ yields a homomorphism $\tilde{f} : K'^{\times} \to \mathcal{O}_{K'}^{\times}$ such that $\tilde{f}\restriction_{\mathcal{O}_{K'}^{\times}} = \mathrm{id}$, since $\Gamma_{K'}$ is torsion free. The valuation map restricted to $\ker(\tilde{f})$ is then an isomorphism, so its inverse $s_{K'} : \Gamma_{K'} \to K'^{\times}$ is a cross-section. Similarly, we find a cross-section $s_{L'}$ in $\mathcal{L}'$, and by Lemma 6.4 these cross-sections give rise to ac maps so that $(\mathcal{K}', \mathrm{ac}_{K'}), (\mathcal{L}', \mathrm{ac}_{L'}) \models T_{\mathrm{Pas}}$. Now by Corollary 6.11(2a), we get $\mathcal{K}' \equiv \mathcal{L}'$ in $\mathcal{L}_{\mathrm{DP}}$, and hence also in the reduct $L_{\Gamma k}$. Thus $\mathcal{K} \equiv \mathcal{K}' \equiv \mathcal{L}' \equiv \mathcal{L}$. This proves part (1).

For part (2), let $(\mathcal{L}', \mathcal{K}') \succeq (\mathcal{L}, \mathcal{K})$ be an $\aleph_1$-saturated extension of the pairs. We have already shown that there is a cross-section $s_{K'} : \Gamma_{K'} \to K'^{\times}$. By Fact 6.19 again, $s_{K'}$ extends to a cross-section $s_{L'} : \Gamma_{L'} \to L'^{\times}$ which gives rise to a compatible couple of ac maps. Indeed, let $\tilde{f} : K'^{\times} \to \mathcal{O}_{K'}^{\times}$ be the homomorphism found in the proof of (1). Since $\mathcal{O}_{L'}^{\times} \cap K'^{\times} = \mathcal{O}_{K'}^{\times}$, there is a (unique) homomorphism $g : \mathcal{O}_{L'}^{\times} K'^{\times} \to \mathcal{O}_{L'}^{\times}$ extending $\mathrm{id}_{\mathcal{O}_{L'}^{\times}} \cup \tilde{f}$. We apply Fact 6.19 to $U = \mathcal{O}_{L'}^{\times}$, $A = \mathcal{O}_{L'}^{\times} K'^{\times} \leq B = L'^{\times}$ and g, purity of $\mathcal{O}_{L'}^{\times} K'^{\times}$ in $L'^{\times}$ being a consequence of the fact that $\mathcal{K}'$ is an elementary substructure of $\mathcal{L}'$, as then $L'^{\times} / \left(\mathcal{O}_{L'}^{\times} K'^{\times}\right) \cong \Gamma_{L'}/\Gamma_{K'}$ is torsion free. We conclude by Corollary 6.11(2b). □

7. A Glimpse on p-adic Model Theory

In this last section, we give a glimpse on the basic model theory of p-adic fields, stressing the analogies with what we have seen so far. In particular we present some results by Macintyre from [13].

Definition 7.1. *Let p be a prime number. The theory pCF (of p-*adically closed fields*) is given by:*

- *(K, v) is a henselian valued field of characteristic $(0, p)$;*
- *$k_K = \mathbb{F}_p$, and*
- *$\Gamma_K \models$ PRES, with $1 = \mathrm{val}(p)$.*

Lemma 7.2. *In $\mathbb{Q}_p$, the valuation ring $\mathbb{Z}_p$ is $\mathcal{L}_{\mathrm{Ring}}(\emptyset)$-definable. If $p \neq 2$ then*

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \exists y(1 + px^2 = y^2)\}$$

and if $p = 2$ then

$$\mathbb{Z}_2 = \{x \in \mathbb{Q}_2 : \exists y(1 + 2x^3 = y^3)\}.$$

Indeed, these definitions of the valuation ring work in all models of pCF.

Proof. If $p \neq 2$, by Hensel's lemma, we get "$\subseteq$" as if $f(X) = X^2 - (1 + \varepsilon)$ then $f'(X) = 2X$. For the reverse direction, if $x \notin \mathbb{Z}_p$, then $v(x) < 0$, so

$$v(1 + px^2) = v(px^2) = 1 + 2v(x) \notin 2\mathbb{Z}$$

and in particular $1 + px^2$ is not a square. The argument for $p = 2$ is similar. □

Remark 7.3. *Ax found a formula that defines the valuation ring in a way that does not depend on p and indeed defines the valuation ring in any henselian valued field with value group $\mathbb{Z}$.*

Corollary 7.4. *As the valuation ring in models of pCF is uniformly definable in $\mathcal{L}_{\mathrm{Ring}}$, we may view pCF as a theory in $\mathcal{L}_{Rings}$.*

Definition 7.5. Macintyre's language $\mathcal{L}_{\mathrm{Mac}}$ *is equal to $\mathcal{L}_{\mathrm{Ring}} \cup \{P_n : n \geq 2\}$, where each P_n is a unary predicate.*

For $K \models p$CF, we interpret $P_n(K) = \{x^n : x \in K\}$. The proof of the following result is quite similar to that of Pas' Theorem.

Theorem 7.6 (Macintyre)**.** *The theory pCF is complete and has quantifier elimination in $\mathcal{L}_{\mathrm{Mac}}$.*

Definition and Remark 7.7. *Let $K \models p$CF and $m \geq 1$. Then*

$$U_m(K) = 1 + p^m \mathcal{O}_K \leq (\mathcal{O}_K^\times, \cdot)$$

is an open subgroup, and we have $U_m(K)/U_{m+1}(K) \cong \mathbb{F}_p$ and $\mathcal{O}_K^\times / U_1(K) \cong \mathbb{F}_p^\times$. Thus, $U_m(K) \leq \mathcal{O}_K^\times$ is of finite index for all m.

For $n \geq 2$, set $P_n^*(K) = P_n(K) \setminus \{0\}$.

Lemma 7.8. *Let $n \geq 2$ and $K \models p$CF.*

(1) If $m > 2v_p(n)$ then $U_m(K) \subseteq P_n^*(K)$
(2) $P_n^*(K) \leq (K^*, \cdot)$ is an open subgroup of finite index and all cosets are represented by integers.

Proof. Part (1) follows from Hensel's Lemma.

For part (2), note that $C_n = \{x \in K^\times : v(x) \in n\Gamma_K\}$ is a subgroup of $K^\times$ of index n, with all cosets represented by integers $(1, p, p^2, \ldots, p^{n-1})$. Now,

$$C_n / P_n^*(K) = \mathcal{O}_K^\times / P_n^*(K) \cap \mathcal{O}_K^\times,$$

as one easily shows (exercise). Thus, we are done by (1), as $U_m(K) \leq \mathcal{O}_K^\times$ is of finite index, and it is not difficult to see that all cosets of it are represented by integers. □

It follows that in pCF we have $\models P_n(x)$ iff $(x = 0 \vee P_n^*(x))$, and $\neg P_n(x)$ iff $\bigvee_{i=1}^{l_n} P_n^*(z_i x)$, for certain integers l_n and $z_1, \ldots, z_{l_n}$. Below, we write $\mathcal{L}_{\text{Mac}}^*$ for $\mathcal{L}_{\text{Ring}} \cup \{P_n^* : n \geq 2\}$.

Corollary 7.9. *Every formula $\varphi(x_1, \ldots, x_n)$ in $\mathcal{L}_{\text{Mac}}^*$ is equivalent in pCF to a finite positive boolean combination of formulas of the following forms (for $f \in \mathbb{Z}[\overline{X}]$, $n \geq 2$):*

(1) $f(\overline{x}) = 0$,
(2) $f(\overline{x}) \neq 0$,
(3) $P_n^*(f(\overline{x}))$.

Corollary 7.10. *The theory pCF is model-complete in $\mathcal{L}_{\text{Ring}}$.*

Proof. The formulas of the first two kinds in Corollary 7.9 are quantifier free in $\mathcal{L}_{\text{Ring}}$, those of the third kind are (equivalent in pCF to) existential $\mathcal{L}_{\text{Ring}}$-formulas. By Corollary 7.9, any $\mathcal{L}_{\text{Ring}}$-formula is thus equivalent to an existential one in pCF, yielding model-completeness. □

Corollary 7.11. *Let $K \models p$CF, and let $D \subseteq K$ be a definable subset. If D is infinite then D has non-empty interior. More generally,* alg. dim = top. dim *for all definable sets $D \subseteq K^m$.*

Proof. Let $D \subseteq K$ be definable and infinite. By Corollary 7.9, D is a finite union of sets given by formulas of the form

$$\left(\bigwedge_{i=1}^{r} f_i(x) = 0\right) \wedge \left(\bigwedge_{i=1}^{s} g_i(x) \neq 0\right) \wedge \left(\bigwedge_{i=1}^{t} P_{n_i}^*(h_i(x))\right).$$

We may thus assume that D is defined by a single formula of this form. As it is infinite, the polynomials f_i must be identically 0 for all i. But then what remains is open with respect to the valuation topology, and hence has non-empty interior.

The proof that alg. dim = top. dim is left as an exercise, and follows the same lines as in ACVF (cf. Exercise 4.15). □

Corollary 7.12. *Let $A \subseteq K \models p$CF. Then $\operatorname{acl}(A) = Q(\langle A \rangle)^{alg} \cap K \preceq K$.*

Proof. The inclusion "$\supseteq$" is clear. For the reverse, it is enough to show that $K_0 = Q(\langle A \rangle)^{\text{alg}} \cap K \models p$CF; then by model-completeness, we get $K_0 \preceq K$, so K_0 is algebraically closed in K.

Clearly, $k_{K_0} = \mathbb{F}_p$ and $v(p)$ is the minimal positive element in Γ_{K_0}. As any relatively algebraically closed subfield of a henselian field is henselian (easy exercise), K_0 is henselian. It remains to show that $\Gamma_{K_0} \models$ PRES. To do this, it is enough to show that $\operatorname{Div}(\Gamma_{K_0}) \cap \Gamma_K = \Gamma_{K_0}$. This follows from the following slight generalization of Lemma 6.10, the proof of which we leave to the reader. □

Lemma 7.13. *Let $(L, w) \supseteq (K, v)$ be valued fields of mixed characteristic $(0, p)$. Assume (L, w) is henselian, $k_L = k_K$ and $w(p)$ is the minimal positive element in Γ_L. Then the following holds.*

(1) Let $n \geq 1$ and $a \in L$ such that $w(a) > 2w(n)$. Then there is $b \in L$ such that $b^n = 1 + a$ and $w(b - 1) > w(n)$.
(2) Let $\gamma \in \Gamma_L \cap \operatorname{Div}(\Gamma_K)$. Assume that $n \geq 1$ is minimal such that $n\gamma \in \Gamma_K$. Then there is $b \in L$ such that $w(b) = \gamma$ and $b^n \in K$.

Remark 7.14. *In fact, a much stronger result holds in* pCF*, by a theorem of van den Dries, namely that the theory* pCF *admits definable Skolem functions, i.e., for all* $A \subseteq K \models p\mathrm{CF}$, $\mathrm{dcl}(A) \preceq K$. *In particular,* $\mathrm{acl} = \mathrm{dcl}$ *in* pCF.

We mention one more result (without proof) which is in the spirit of what we have seen in the earlier sections.

Theorem 7.15. *The theory* pCF *is NIP (Bélair), and in fact is dp-minimal (Dolich-Lippel-Goodrick).*

Acknowledgements

I thank all the participants for their comments and remarks during the course, and the referee for the careful reading of the notes. Special thanks go to the note takers, in particular to Peter Sinclair.

References

[1] James Ax. *Shorter Notes: On the Undecidability of Power Series Fields.* Proc. Amer. Math. Soc. **16** (1965), p. 846.

[2] Greg Cherlin. *Model theoretic algebra—selected topics.* Lecture Notes in Mathematics, vol. **521**. Springer-Verlag, Berlin-New York, 1976.

[3] Antoine Ducros. *Les espaces de Berkovich sont modérés (d'après Ehud Hrushovski et François Loeser).* Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043–1058. Astérisque **352** (2013), pp. 459–507.

[4] Antonio Engler and Alexander Prestel. *Valued Fields.* Springer Monographs in Mathematics. Springer-Verlag, Berlin-New York, 2005.

[5] Kenneth A. H. Gravett. *Note on a result of Krull.* Proc. Cambridge Philos. Soc. **52**(1956), p. 379.

[6] Yuri Gurevich and Peter H. Schmitt. *The theory of ordered abelian groups does not have the independence property.* Trans. Amer. Math. Soc. **284** (1984), pp. 171–182.

[7] Immanuel Halupczok. *An introduction to motivic integration.* This volume.

[8] Deirdre Haskell, Ehud Hrushovski and Dugald Macpherson. *Definable sets in algebraically closed valued fields. Part I: elimination of imaginaries.* J. Reine Angew. Math. **597** (2006), pp. 175–236.

[9] Deirdre Haskell, Ehud Hrushovski and Dugald Macpherson. *Stable domination and independence in algebraically closed valued fields.* Lecture Notes in Logic, vol. **30**. Association for Symbolic Logic, Chicago, IL, 2008.

[10] Ehud Hrushovski and François Loeser. *Non-archimedean Tame Topology and Stably Dominated Types.* Annals of Mathematics Studies, vol. **192**. Princeton University Press, Princeton-Oxford, 2016.

[11] Franziska Jahnke. *An introduction to valued fields.* This volume.

[12] Irving Kaplansky. *Maximal fields with valuations.* Duke Math. J. **9** (1942), pp. 303–321.

[13] Angus Macintyre. *On definable subsets of p-adic fields*, J. Symbolic Logic **41** (1976), pp. 605–610.

[14] Johan Pas. *Uniform p-adic cell decomposition and local zeta functions.* J. Reine Angew. Math. **399** (1989), pp. 137–172.

[15] Abraham Robinson. *Complete theories.* North-Holland Publishing Co., Amsterdam, 1956.

[16] Pierre Simon. *A guide to NIP theories.* Lecture Notes in Logic vol. **44**. Association for Symbolic Logic, Chicago, IL; Cambridge University Press, Cambridge, 2015.

[17] Pierre Simon. *NIP and definably amenable groups.* This volume.

Institut für Mathematische Logik und Grundlagenforschung, University of Münster, Einsteinstr. 62, 48149 Münster, Germany

E-mail address: `hils@uni-muenster.de`