THE DOMINATION MONOID IN HENSELIAN VALUED FIELDS

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ABSTRACT. We study the domination monoid in various classes of structures arising from the model theory of henselian valuations, including $\mathcal{RV}$-expansions of henselian valued fields of residue characteristic 0 (and, more generally, of benign valued fields), $\mathfrak{p}$-adically closed fields, monotone D-henselian differential valued fields with many constants, regular ordered abelian groups, and pure short exact sequences of abelian structures. We obtain Ax–Kochen–Ershov type reductions to suitable fully embedded families of sorts in quite general settings, and full computations in concrete ones.

In their seminal work [HHM08] on stable domination, Haskell, Hrushovski and Macpherson introduced the *domination monoid* $\widetilde{\mathrm{Inv}}(\mathfrak{U})$, and showed that in algebraically closed valued fields it decomposes as $\widetilde{\mathrm{Inv}}(\mathrm{k}(\mathfrak{U}))\times\widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U}))$, where k denotes the residue field and Γ the value group.[1] A similar result was proven in [EHM19] in the case of real closed fields with a convex valuation. This paper revolves around understanding $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in more general classes of valued fields, and expansions thereof. A special case of our results is the following.

Theorem A (Corollary 6.22)**.** *Let T be the theory of a henselian valued field of equicharacteristic 0, or algebraically maximal Kaplansky, possibly enriched on* k *and* Γ*. If all* $\mathrm{k}^\times/(\mathrm{k}^\times)^n$ *are finite,*

$$\widetilde{\mathrm{Inv}}(\mathfrak{U})\cong\widetilde{\mathrm{Inv}}(\mathrm{k}(\mathfrak{U}))\times\widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U}))$$

More generally, we obtain a two-step reduction, first to leading term structures, and then, using technology on pure short exact sequences recently developed in [ACGZ20], to k and Γ, albeit in a form which, in general, is (necessarily) slightly more involved. We also compute $\widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U}))$ when the theory of Γ has an archimedean model, and prove several accessory statements.

Before stating our results in more detail, let us give an informal account of the context (see Section 1 for the precise definitions). The starting point is the space $S^{\mathrm{inv}}(\mathfrak{U})$ of *invariant types* over a monster model $\mathfrak{U}$. It is a dense subspace of $S(\mathfrak{U})$, whose points may be canonically extended to larger parameter sets. Such extensions allow to define the *tensor product*, or *Morley product*, obtaining a semigroup $(S^{\mathrm{inv}}(\mathfrak{U}),\otimes)$. The space $S^{\mathrm{inv}}(\mathfrak{U})$ also comes with a preorder $\geq_{\mathrm{D}}$, called *domination*: roughly, $p\geq_{\mathrm{D}} q$ means that q is recoverable from p plus a small amount of information. The quotient by the induced equivalence relation, *domination-equivalence* $\sim_{\mathrm{D}}$, is then a poset, denoted by $(\widetilde{\mathrm{Inv}}(\mathfrak{U}),\geq_{\mathrm{D}})$. If $\otimes$ respects $\geq_{\mathrm{D}}$, i.e. if $(S^{\mathrm{inv}},\otimes,\geq_{\mathrm{D}})$ is a preordered semigroup, then $\sim_{\mathrm{D}}$ is a congruence with respect to $\otimes$ and we say that *the domination monoid is well-defined*, and equip $(\widetilde{\mathrm{Inv}}(\mathfrak{U}),\geq_{\mathrm{D}})$ with the operation induced by $\otimes$. Compatibility of $\otimes$ and $\geq_{\mathrm{D}}$ in a given theory can be shown by using certain sufficient criteria, isolated in [Men20b] and applied e.g. in [Menb], or by finding a nice system of representatives for $\sim_{\mathrm{D}}$-classes (cf. Proposition 1.6). Nevertheless, in general, $\otimes$ may fail to respect $\geq_{\mathrm{D}}$ [Men20b]. Hence, when dealing with $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in a given structure, one needs to understand whether it is well-defined as a monoid; and, when dealing with it in the abstract, the monoid structure cannot be taken for granted.

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[1]Strictly speaking, [HHM08] works with $\overline{\mathrm{Inv}}(\mathfrak{U})$, which is in general different, but coincides with $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in their setting. See [Men20a, Remark 2.1.14 and Theorem 5.2.22].

Recall that to each valued field K is associated a short exact sequence $\mathcal{RV}$ of abelian groups augmented by an absorbing element $1 \to (\mathrm{k}, \times) \to (\mathrm{K}, \times)/(1+\mathfrak{m}) \to \Gamma \cup \{\infty\} \to 0$. This sequence is interpretable in K, and this interpretation endows it with extra structure. The amount of induced structure clearly depends on whether K has extra structure itself, but at a bare minimum k will carry the language of fields and Γ that of ordered abelian groups. By [Fle11], henselian valued fields of residue characteristic 0 eliminate quantifiers relatively to $\mathcal{RV}$, and the latter is fully embedded with the structure described above. Actually, this holds resplendently, in the sense that it is still true after arbitrary expansions of $\mathcal{RV}$. The same result holds in the algebraically maximal Kaplansky case, by [HH19]. These are known after [Tou18] as classes of *benign* valued fields and, in several contexts, they turn out to be particularly amenable to model-theoretic investigation. One of our main results says that the context of domination is no exception.

Theorem B (Theorem 6.21). *In every $\mathcal{RV}$-expansion of a benign theory of valued fields there is an isomorphism of posets*

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathcal{RV}(\mathfrak{U}))$$

If $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathcal{RV}(\mathfrak{U})$, then $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathfrak{U}$, and the above is an isomorphism of monoids.

Having reduced $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ to the short exact sequence $\mathcal{RV}$, the next step is to reduce it to its kernel k and quotient Γ. If we now add an angular component map, the sequence $\mathcal{RV}$ splits and we obtain a product decomposition as in Theorem A (Remark 6.4). Without an angular component, a product decomposition is not always possible; nevertheless, k and Γ still exert a tight control on $\mathcal{RV}$. In fact, this behaviour turns out not to be peculiar of $\mathcal{RV}$, and to hold e.g. in every short exact sequences of abelian groups with torsion-free quotient. Even more generally, a decomposition theorem is possible in short exact sequences of abelian structures, provided they satisfy a purity assumption (Definition 4.5), using the relative quantifier elimination from [ACGZ20]. For reasons to be clarified later (Remark 4.19), in this context it is more natural to look at types in infinitely many variables, say κ, hence at the corresponding analogue $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ of the domination monoid.

Theorem C (Corollary 4.12). *Let $\mathfrak{U}$ be a pure short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ of L-abelian structures, where $\mathcal{A}$ and $\mathcal{C}$ may carry extra structure. Let $\kappa \geq |L|$ be a small cardinal. There is an expansion $\mathcal{A}_{\mathcal{F}}$ of $\mathcal{A}$ by imaginary sorts yielding an isomorphism of posets*

$$\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathcal{A}_{\mathcal{F}}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}_\kappa(\mathcal{C}(\mathfrak{U}))$$

If $\otimes$ respects $\geq_{\mathrm{D}}$ in both $\mathcal{A}_{\mathcal{F}}(\mathfrak{U})$ and $\mathcal{C}(\mathfrak{U})$, then $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathfrak{U}$, and the above is an isomorphism of monoids.

In algebraically or real closed valued fields, the reduction of the monoid $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ to $\widetilde{\mathrm{Inv}}(\mathrm{k}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U}))$ is complemented by an explicit computation of the factors, carried out in [HHM08, Mena]. In particular, if $\Gamma(\mathfrak{U})$ is a divisible ordered abelian group, then $\widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U}))$ is isomorphic to the upper semilattice of finite sets of invariant convex subgroups of $\Gamma(\mathfrak{U})$ (see Definition 3.16). A further contribution of the present work is the computation of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in the next simplest class of theories of ordered abelian groups, namely those with an archimedean model, known as *regular*.

Theorem D (Corollary 3.40). *Let T be the theory of a regular ordered abelian group and κ a small infinite cardinal. Denote by $\hat{\kappa}$ the ordered monoid of cardinals smaller or equal than κ with cardinal sum, by X the set of invariant convex subgroups of $\mathfrak{U}$, by $\mathscr{P}_{\leq\kappa}(X)$ the upper semilattice of its subsets of size at most κ, and by $\mathbb{P}_T$ the set of primes $\mathfrak{p}$ such that $\mathfrak{U}/\mathfrak{p}\mathfrak{U}$ is infinite. Then $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}^{\mathrm{eq}})$ is well-defined, and*

$$\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}^{\mathrm{eq}}) \cong \mathscr{P}_{\leq\kappa}(X) \times \prod_{\mathbb{P}_T} \hat{\kappa}$$

In particular, Theorem D applies to Presburger Arithmetic, the theory of $(\mathbb{Z}, +, <)$. Pairing this (or rather, its version for finitary types) with a suitable generalisation of Theorem B, we obtain the following.

Theorem E (Corollary 7.9)**.** *In the theory* $\mathrm{Th}(\mathbb{Q}_\mathfrak{p})$ *of* $\mathfrak{p}$*-adically closed fields,* $\otimes$ *respects* $\geq_\mathrm{D}$*. If* X *is the set of invariant convex subgroups of* $\Gamma(\mathfrak{U})$*, then* $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathscr{P}_{<\omega}(X)$*.*

We also obtain a similar result (Corollary 7.7) for Witt vectors over $\mathbb{F}_\mathfrak{p}^{\mathrm{alg}}$. Finally, we move to the context of monotone D-henselian differential valued fields with many constants. While Theorem B does not generalise to this context (Remark 8.6), we prove in Theorem 8.3 that its analogue for $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ does. In the model companion $\mathsf{VDF}_{\mathcal{EC}}$, we produce a full computation.

Theorem F (Theorem 8.5)**.** *In* $\mathsf{VDF}_{\mathcal{EC}}$*, for every small infinite cardinal* κ*, the monoid* $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ *is well-defined, and we have isomorphisms*

$$\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathrm{k}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}_\kappa(\Gamma(\mathfrak{U})) \cong \prod_{\delta(\mathfrak{U})}^{\leq\kappa} \hat{\kappa} \times \mathscr{P}_{\leq\kappa}(X)$$

where X *is the set of invariant convex subgroups of* $\Gamma(\mathfrak{U})$*,* $\delta(\mathfrak{U})$ *is a certain cardinal, and* $\prod_{\delta(\mathfrak{U})}^{\leq\kappa} \hat{\kappa}$ *denotes the submonoid of* $\prod_{\delta(\mathfrak{U})} \hat{\kappa}$ *consisting of* $\delta(\mathfrak{U})$*-sequences with support of size at most* κ*.*

Similar results hold in the setting of σ-henselian valued difference fields (Remark 8.7).

The paper is structured as follows. In the first two sections we recall some preliminary notions and facts, and deal with some easy observations about orthogonality of invariant types. In Section 3 we prove Theorem D, while in Section 4 we study expanded pure short exact sequences of abelian structures, proving Theorem C. The results from these two sections are then combined in Section 5 to deal with the case of ordered abelian groups with finitely many definable convex subgroups. In Section 6 we prove Theorem B, and illustrate how it may be combined with Theorem C to obtain statements such as Theorem A. Section 7 deals with finitely ramified mixed characteristic henselian valued fields and includes a proof of Theorem E, and Section 8 deals with the differential case, including a proof of Theorem F. Some open questions are listed in Section 9.

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1. Preliminaries

1.1. **Notation and conventions.** The set of natural numbers is denoted by ω and always contains 0. The set of prime natural numbers is denoted by $\mathbb{P}$.

We denote by L a possibly multi-sorted first-order language, by $|L|$ the cardinality of the set of its formulas, and by T a complete L-theory. Sorts are denoted by upright letters, as in $\mathrm{A}, \mathrm{K}, \mathrm{k}, \Gamma$, and families of sorts by calligraphic letters such as $\mathcal{A}$.

Lowercase Latin letters such as a, b, c and x, y, z denote *tuples* of parameters and variables respectively. Given a tuple a, we denote by a_i its i-th element, starting from 0, and by $|a|$ its length, so for example $a = (a_0, \dots, a_{|a|-1})$. If A is a set of parameters, we often write $a \in A$ in place of $a \in A^{|a|}$. Tuples of variables denoted by different letters, as in $p(x) \cup q(y)$, are assumed to be disjoint. Terms may contain parameters, as in $t(x, d)$. We write e.g. $t(x)$ if they do not.

Formally, a *monster model* for us will be a pair $(\mathfrak{U}, \kappa(\mathfrak{U}))$ such that $\kappa(\mathfrak{U})$ is a strong limit cardinal, i.e. closed under $\beth$, and of cofinality larger than $|L|$, and $\mathfrak{U}$ is a $\kappa(\mathfrak{U})$-saturated and $\kappa(\mathfrak{U})$-strongly homogeneous model of T. When a monster model is fixed, "small" means "of size $< \kappa(\mathfrak{U})$". In practice, we will usually just write $\mathfrak{U}$ and rarely refer to $\kappa(\mathfrak{U})$. We write $A \subset^+ \mathfrak{U}$ to say that A is a small subset of $\mathfrak{U}$, and $M \prec^+ \mathfrak{U}$ for small elementary substructures. *Definable* means $\mathfrak{U}$-definable unless otherwise stated.

If κ is a cardinal, $S_{<\kappa}(A)$ stands for the union of all spaces of types over A in fewer than κ variables. We also write $S(A)$ for $S_{<\omega}(A)$. If X is an A-definable set, we denote by $S_X(A)$ the space of types over A concentrating on X, and by $S_{X^{<\omega}}(A)$ the disjoint union of all $S_{X^n}(A)$. We use similar notations with X replaced by a formula φ defining it. If $\mathcal{C}$ is a family of sorts, $S_{\mathcal{C}^{<\omega}}(A)$ denotes the disjoint union of all spaces of types in finitely many variables, each with sort in $\mathcal{C}$.

If $p \in S(A)$ is a type and f an A-definable function with domain in p, we denote by f_*p the pushforward $\{\varphi(y) \in L(A) \mid p(x) \vdash \varphi(f(x))\}$. A *global type* is a type in $S(\mathfrak{U})$. Realisations of global types and sets $B \supseteq \mathfrak{U}$ live inside a larger monster model, denoted by $\mathfrak{U}_1$.

1.2. **Invariant types.** If $A \subseteq B$, a type $p \in S(B)$ is *A-invariant* iff, for all $d, d' \in B$, whenever $p \vdash \varphi(x, d)$ and $d' \equiv_A d$ then $p \vdash \varphi(x, d')$. An *invariant* type is a global type which is A-invariant for some small A. The space of global A-invariant [resp. invariant] types concentrating on X is denoted by $S^{\mathrm{inv}}_X(\mathfrak{U}, A)$ [resp. $S^{\mathrm{inv}}_X(\mathfrak{U})$]. We use conventions analogous to those we employ for usual type spaces, and write e.g. $S^{\mathrm{inv}}(\mathfrak{U})$.

If $p \in S^{\mathrm{inv}}(\mathfrak{U}, A)$ and $B \supseteq \mathfrak{U}$, there is a unique A-invariant extension of p to a type over B. It does not depend on A, and we denote it by $p \mid B$. If $p(x), q(y) \in S(\mathfrak{U})$, p is A-invariant, and $b \vDash q$, we define $\varphi(x, y, d) \in (p \otimes q)(x, y)$ iff $(p \mid \mathfrak{U}b) \vdash \varphi(x, b, d)$. This does not depend on b, nor on A, and yields a global type $p \otimes q$, which is A-invariant if and only if q is. The operation $\otimes$ is associative, and endows $S^{\mathrm{inv}}_{<\omega}(\mathfrak{U})$ with the structure of a monoid, the neutral element being the unique 0-type, i.e. the elementary diagram of $\mathfrak{U}$. For further details, see e.g. [Men20a, Subsection 2.1.2].

1.3. **Domination.** We briefly recall some definitions and facts about domination, and refer the reader to [Men20b, Men20a] for a more thorough treatment.

Definition 1.1.

1. If $p(x), q(y) \in S(\mathfrak{U})$, let $S_{pq}(A)$ be the set of types over A in variables xy extending $(p(x) \upharpoonright A) \cup (q(y) \upharpoonright A)$.
2. We say that $p(x) \in S(\mathfrak{U})$ *dominates* $q(y) \in S(\mathfrak{U})$, and write $p \geq_{\mathrm{D}} q$, iff there are a small $A \subset^+ \mathfrak{U}$ and $r \in S_{pq}(A)$ such that $p(x) \cup r(x, y) \vdash q(y)$.
3. We say that $p, q \in S(\mathfrak{U})$ are *domination-equivalent*, and write $p \sim_{\mathrm{D}} q$, iff $p \geq_{\mathrm{D}} q$ and $q \geq_{\mathrm{D}} p$. We denote the domination-equivalence class of p by $[\![p]\!]$.
4. The *domination poset* $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is the quotient of $S^{\mathrm{inv}}(\mathfrak{U})$ by $\sim_{\mathrm{D}}$, equipped with the partial order induced by $\geq_{\mathrm{D}}$, denoted by the same symbol.

In other words, domination is the semi-isolation counterpart to $\mathrm{F}^{\mathrm{s}}_{\kappa(\mathfrak{U})}$-isolation in the sense of [She90, Chapter IV]. For a theory where the two notions are distinct, see [Menb, Example 3.3].

In what follows we will be mostly concerned with domination on invariant types. When describing a witness to $p \geq_{\mathrm{D}} q$, we write e.g. "let r contain $\varphi(x, y)$" with the meaning "let $r \in S_{pq}(A)$ contain $\varphi(x, y)$, for an A such that $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, A)$".

Fact 1.2 ([Men20b, Lemma 1.14]). Let $p_0, p_1, q \in S(\mathfrak{U})$. If p_0, p_1 are invariant and $p_0 \geq_{\mathrm{D}} p_1$, then $p_0 \otimes q \geq_{\mathrm{D}} p_1 \otimes q$.

Definition 1.3. We say that $\otimes$ *respects* $\geq_{\mathrm{D}}$ iff $q_0 \geq_{\mathrm{D}} q_1$ implies $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$. If this is the case, the *domination monoid* is the expansion of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ by the operation induced by $\otimes$, denoted by the same symbol. We also say that $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ *is well-defined*[2] in place of "$\otimes$ respects $\geq_{\mathrm{D}}$".

Domination witnessed by algebraicity is always compatible with $\otimes$, as made precise below.

Fact 1.4 ([Men20b, Proposition 1.23])**.** Suppose that $p \in S^{\mathrm{inv}}(\mathfrak{U})$, that $q_0, q_1 \in S(\mathfrak{U})$, and that for $i < 2$ there are realisations $a_i \vDash q_i$ such that $a_1 \in \mathrm{acl}(\mathfrak{U}a_0)$. Then $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$.

Frequently, we will consider a family of sorts, say $\mathcal{A} = \{\mathrm{A}_s \mid s \in S\}$, as a standalone structure, by equipping it with the traces of some $\emptyset$-definable relations. Such an $\mathcal{A}$ is called *fully embedded* iff every subset of $(\mathrm{A}_{s_0} \times \ldots \times \mathrm{A}_{s_n})(\mathfrak{U})$ is definable[3] in $\mathfrak{U}$ if and only if it is definable in $\mathcal{A}(\mathfrak{U})$. For brevity, when we talk of a fully embedded $\mathcal{A}$ in the abstract, as in Fact 1.5 below, we tacitly consider a structure on $\mathcal{A}$ to be fixed.

Fact 1.5 ([Men20b, Proposition 2.3.31])**.** Let $\mathcal{A}$ be a fully embedded family of sorts, and let $\iota \colon S_{\mathcal{A}^{<\omega}}(\mathcal{A}(\mathfrak{U})) \to S(\mathfrak{U})$ send a type of $\mathcal{A}(\mathfrak{U})$ to the unique type of $\mathfrak{U}$ it entails.

1. p is invariant if and only if $\iota(p)$ is, and $\iota \upharpoonright S^{\mathrm{inv}}(\mathcal{A}(\mathfrak{U}))$ is an injective $\otimes$-homomorphism.
2. ι induces an embedding of posets $\bar{\iota} \colon \widetilde{\mathrm{Inv}}(\mathcal{A}(\mathfrak{U})) \hookrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{U})$.
3. If $\otimes$ respects $\geq_{\mathrm{D}}$ in both $\mathcal{A}(\mathfrak{U})$ and $\mathfrak{U}$, then $\bar{\iota}$ is also an embedding of monoids.

Proposition 1.6. Let $\mathcal{A}$ be fully embedded, and suppose that for each $p \in S^{\mathrm{inv}}(\mathfrak{U})$ there is a tuple τ^p of definable functions with codomains in $\mathcal{A}$ such that, for each $p, q \in S^{\mathrm{inv}}(\mathfrak{U})$, we have $p \sim_{\mathrm{D}} \tau^p_* p$ and $p \otimes q \sim_{\mathrm{D}} \tau^p_* p \otimes \tau^q_* q$. If $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathcal{A}(\mathfrak{U})$, then $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathfrak{U}$.

Proof. By Fact 1.2 it is enough to show that if $q_0 \geq_{\mathrm{D}} q_1$ then $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$. Note that τ induces an inverse of $\bar{\iota}$. By assumption, $p \otimes q_0 \sim_{\mathrm{D}} \tau^p_* p \otimes \tau^{q_0}_* q_0$ and $\tau^p_* p \otimes \tau^{q_1}_* q_1 \sim_{\mathrm{D}} p \otimes q_1$. Since $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathcal{A}(\mathfrak{U})$, we obtain $\tau^p_* p \otimes \tau^{q_0}_* q_0 \geq_{\mathrm{D}} \tau^p_* p \otimes \tau^{q_1}_* q_1$, and we are done. □

1.4. **A word on $*$-types.** In what follow, we will sometimes need to deal with types in a (small) infinite number of variables, also known in the literature as $*$-types. We define $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ as the quotient of $S_{<\kappa^+}(\mathfrak{U})$ by $\sim_{\mathrm{D}}$. Note that, by padding with realised coordinates and permuting variables, every $\sim_{\mathrm{D}}$-class has a representative with variables indexed by κ. We leave to the reader easy tasks such as defining the α-th power $p^{(\alpha)}$, for α an ordinal, or such as convincing themself that basic statements such as Fact 1.5 generalise. Nevertheless, it is not clear if well-definedness of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ implies well-definedness of $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ (the converse is easy). We leave this open as Question 9.1. In the rest of the paper we will say e.g. "$\otimes$ preserves $\geq_{\mathrm{D}}$" with the understanding that, whenever $*$-types are involved, this is to be read as "$\otimes$ preserves $\geq_{\mathrm{D}}$ on $*$-types".

2. Orthogonality

Definition 2.1.

1. We say that $p, q \in S(A)$ are *weakly orthogonal*, and write $p \perp^{\mathrm{w}} q$, iff $p(x) \cup q(y)$ implies a complete xy-type over A.
2. We say that $p, q \in S^{\mathrm{inv}}(\mathfrak{U})$ are *orthogonal*, and write $p \perp q$, iff for every $B \supseteq \mathfrak{U}$ we have $(p \mid B) \perp^{\mathrm{w}} (q \mid B)$.
3. Two definable sets φ, ψ are *orthogonal* iff for every $n, m \in \omega$, every $p \in S_{\varphi^n}(\mathfrak{U})$, and every $q \in S_{\psi^m}(\mathfrak{U})$, we have $p \perp^{\mathrm{w}} q$.
4. Two families of sorts $\mathcal{A}$, $\mathcal{C}$ are *orthogonal* iff every cartesian product of sorts in $\mathcal{A}$ is orthogonal to every cartesian product of sorts in $\mathcal{C}$.

[2]As a partially ordered monoid. Because $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is always well-defined as a poset, this should cause no confusion.
[3]Recall that when we say "definable" we mean "definable with parameters".

Remark 2.2. The following statements are well-known (and easy to prove).

1. If $p \in S(A)$ is weakly orthogonal to itself, then it is realised in $\operatorname{dcl}(A)$.
2. If $p, q \in S^{\mathrm{inv}}(\mathfrak{U})$ and $p \perp^{\mathrm{w}} q$, then $p(x) \otimes q(y) = q(y) \otimes p(x)$: they both coincide with (the unique completion of) $p(x) \cup q(y)$.
3. Two definable sets φ, ψ are orthogonal if and only if every definable subset of $\varphi^m(x) \wedge \psi^n(y)$, where x, y are disjoint tuples of variables, is a finite union of rectangles, i.e. can be defined by a finite disjunction of formulas of the form $\theta(x) \wedge \eta(y)$.
4. If two definable sets are orthogonal and M is any model containing the parameters used to define them, then the definition of orthogonality still holds after replacing $\mathfrak{U}$ with M.
5. Adding imaginaries preserves orthogonality, in the following sense. Let $\mathcal{A}$ be a family of sorts, and let $\tilde{\mathcal{A}}$ be a larger family, consisting of $\mathcal{A}$ together with imaginary sorts obtained as definable quotients of products of elements of $\mathcal{A}$. Let $\tilde{\mathcal{C}}$ be obtained similarly from another family of sorts $\mathcal{C}$. If $\mathcal{A}$ and $\mathcal{C}$ are orthogonal, then so are $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{C}}$.

Fact 2.3 ([Men20b, Proposition 3.13])**.** Suppose that $p_0, p_1 \in S^{\mathrm{inv}}(\mathfrak{U})$ and $q \in S(\mathfrak{U})$. If $p_0 \geq_{\mathrm{D}} p_1$ and $p_0 \perp^{\mathrm{w}} q$, then $p_1 \perp^{\mathrm{w}} q$. In particular, if $p_0 \geq_{\mathrm{D}} q$ and $p_0 \perp^{\mathrm{w}} q$, then q is realised.

As a consequence, $\perp^{\mathrm{w}}$ induces a well-defined relation on the domination poset, which we may therefore expand to $(\widetilde{\operatorname{Inv}}(\mathfrak{U}), \geq_{\mathrm{D}}, \perp^{\mathrm{w}})$. By [Men20b, Proposition 2.3.31] the map $\bar{\iota}$ from Fact 1.5 is a homomorphism for both $\perp^{\mathrm{w}}$ and $\not\perp^{\mathrm{w}}$. We prove the analogous statements for orthogonality.

Proposition 2.4. Let $p_0, p_1, q \in S^{\mathrm{inv}}(\mathfrak{U})$. If $p_0 \perp q$ and $p_0 \geq_{\mathrm{D}} p_1$, then $p_1 \perp q$. In particular, $\perp$ induces a well-defined relation on $\widetilde{\operatorname{Inv}}(\mathfrak{U})$.

Proof. Fix r witnessing $p_0 \geq_{\mathrm{D}} p_1$ and let $B \supseteq \mathfrak{U}$. Let $b \vDash p_1 \mid B$ and $c \vDash q \mid B$. By [Men20b, Lemma 1.13] $(p_0 \mid B) \cup r \vdash (p_1 \mid B)$. Let a be such that $ab \vDash (p_0 \mid B) \cup r$. Since $p_0 \perp q$, we have $(p_0 \mid B) \perp^{\mathrm{w}} (q \mid B)$, hence $a \vDash p_0 \mid Bc$. By [Men20b, Lemma 1.13] we have $(p_0 \mid Bc) \cup r \vdash (p_1 \mid Bc)$. Therefore $b \vDash p_1 \mid Bc$, and we are done. □

Proposition 2.5. Let $\mathcal{A}$ be a fully embedded family of sorts, and let ι be the natural map $S_{\mathcal{A}^{<\omega}}(\mathcal{A}(\mathfrak{U})) \to S(\mathfrak{U})$. Its restriction $\iota \restriction S^{\mathrm{inv}}_{\mathcal{A}^{<\omega}}(\mathcal{A}(\mathfrak{U}))$ is both a $\perp$-homomorphism and a $\not\perp$-homomorphism. In particular, so is the induced map $\bar{\iota}\colon \widetilde{\operatorname{Inv}}(\mathcal{A}(\mathfrak{U})) \hookrightarrow \widetilde{\operatorname{Inv}}(\mathfrak{U})$.

Proof. Let $p, q \in S^{\mathrm{inv}}_{\mathcal{A}^{<\omega}}(\mathcal{A}(\mathfrak{U}))$ be orthogonal and let $\mathfrak{U}_1 \succ \mathfrak{U}$ be $|\mathfrak{U}|^+$-saturated and $|\mathfrak{U}|^+$-strongly homogeneous. We show that, for $\varphi(x, y, z) \in L(\mathfrak{U})$ and $d \in \mathfrak{U}_1$, if $(\iota p(x) \otimes \iota q(y)) \mid \mathfrak{U}_1 \vdash \varphi(x, y, d)$ then $(\iota p \mid \mathfrak{U}d)(x) \cup (\iota q \mid \mathfrak{U}d)(y) \vdash \varphi(x, y, d)$. By full embeddedness, there are $\chi(x, y, w) \in L_{\mathcal{A}}(\mathcal{A}(\mathfrak{U}))$ and $e \in \mathcal{A}(\mathfrak{U}_1)$ such that $\mathfrak{U}_1 \vDash \forall x, y\, (\chi(x, y, e) \leftrightarrow \varphi(x, y, d))$. Because $(p \mid \mathcal{A}(\mathfrak{U})e) \perp^{\mathrm{w}} (q \mid \mathcal{A}(\mathfrak{U})e)$, there are $\theta_p(x, w), \theta_q(y, w) \in L_{\mathcal{A}}(\mathcal{A}(\mathfrak{U}))$ such that $(p \mid \mathcal{A}(\mathfrak{U})e) \vdash \theta_p(x, e)$, $(q \mid \mathcal{A}(\mathfrak{U})e) \vdash \theta_q(y, e)$, and $\mathcal{A}(\mathfrak{U}_1) \vDash \forall x, y\, ((\theta_p(x, e) \wedge \theta_q(y, e)) \to \chi(x, y, e))$. Since p, q are invariant, we have inclusions

$$\pi_p(x) \coloneqq \{\theta_p(x, e') \mid e' \in \mathfrak{U}_1, e \equiv_{\mathfrak{U}d} e'\} \subseteq \iota p \mid \mathfrak{U}_1$$
$$\pi_q(y) \coloneqq \{\theta_q(y, e') \mid e' \in \mathfrak{U}_1, e \equiv_{\mathfrak{U}d} e'\} \subseteq \iota q \mid \mathfrak{U}_1$$

Hence π_p, π_q are consistent, and since they are fixed by $\operatorname{Aut}(\mathfrak{U}_1/\mathfrak{U}d)$ they are equivalent to partial types σ_p, σ_q over $\mathfrak{U}d$. But $\sigma_p \subseteq \iota p \mid \mathfrak{U}d$, $\sigma_q \subseteq \iota q \mid \mathfrak{U}d$, and by construction $\sigma_p(x) \cup \sigma_q(y) \vdash \varphi(x, y, d)$.

Suppose now that there is $B \subseteq \mathcal{A}(\mathfrak{U}_1)$ such that $(p \mid B) \not\perp^{\mathrm{w}} (q \mid B)$. Since the roles of p and q are symmetric, it is enough to show that $(p \mid B) \vdash (\iota p \mid B)$. Suppose $\varphi(x, w, t) \in L(\emptyset)$, $d \in \mathfrak{U}$, $e \in B$, and $\iota p(x) \mid B \vdash \varphi(x, d, e)$. Since x, t are $\mathcal{A}$-variables , and $d \in \mathfrak{U}$, full embeddedness yields an $L_{\mathcal{A}}(\mathcal{A}(\mathfrak{U}))$-formula $\psi(x, t)$ equivalent to $\varphi(x, d, t)$. So $\psi(x, e) \in p \mid B$ and we are done.

The "in particular" part follows from Proposition 2.4. □

Lemma 2.6. Let $p, q_0, q_1 \in S(\mathfrak{U})$, and assume that $p \perp^{\mathrm{w}} q_0$ and that $(p(x) \cup q_0(y)) \geq_{\mathrm{D}} q_1(z)$, witnessed by a small type $r \in S_{p\otimes q_0, q_1}(M)$. Suppose furthermore that

$$(r \restriction x) \perp^{\mathrm{w}} (r \restriction yz) \tag{1}$$

Then $q_0 \geq_{\mathrm{D}} q_1$, witnessed by $r \restriction yz$.

Proof. Let $\chi(z, d) \in q_1(z)$ be given. By assumption there is a formula $\rho(x, y, z) \in r$ such that $p(x) \cup q_0(y) \vdash \forall z\ (\rho(x, y, z) \to \chi(z, d))$. By (1) there are $\rho_0(x)$ and $\rho_1(y, z)$ in r such that $p(x) \cup q_0(y) \vdash \forall z\ ((\rho_0(x) \wedge \rho_1(y, z)) \to \chi(z, d))$. By spelling this out, there is $\varphi(x, e) \in p(x)$ such that $q_0(y) \vdash \forall x, z\ \big(\varphi(x, e) \wedge \rho_0(x) \wedge \rho_1(y, z)) \to \chi(z, d)\big)$, and in particular

$$q_0(y) \vdash \forall z\ \Big(\big(\rho_1(y, z) \wedge \exists x\ (\varphi(x, e) \wedge \rho_0(x))\big) \to \chi(z, d)\Big)$$

Since z does not appear in $\exists x\ (\varphi(x, e) \wedge \rho_0(x))$, and the latter is true in $\mathfrak{U}$ since $\varphi(x, e) \wedge \rho_0(x) \in p$, we obtain $q_0(y) \vdash \forall z\ \big(\rho_1(y, z) \to \chi(z, d)\big)$. As $\rho_1(y, z) \in r \restriction yz$, we are done. □

Corollary 2.7. Suppose that $\mathcal{A}, \mathcal{C}$ are orthogonal families of sorts. Let $p \in S^{\mathrm{inv}}_{\mathcal{A}^{<\omega}}(\mathfrak{U})$ and $q_0, q_1 \in S^{\mathrm{inv}}_{\mathcal{C}^{<\omega}}(\mathfrak{U})$. If $(p \cup q_0) \geq_{\mathrm{D}} q_1$, then $q_0 \geq_{\mathrm{D}} q_1$.

Proof. By point 4 of Remark 2.2 and Lemma 2.6. □

Corollary 2.8. Suppose that $\mathcal{A}, \mathcal{C}$ are orthogonal, fully embedded families of sorts. Assume that for every $p \in S^{\mathrm{inv}}(\mathfrak{U})$ there are some $p_{\mathcal{A}} \in S^{\mathrm{inv}}_{\mathcal{A}^{<\omega}}(\mathfrak{U})$ and $p_{\mathcal{C}} \in S^{\mathrm{inv}}_{\mathcal{C}^{<\omega}}(\mathfrak{U})$ such that $p \sim_{\mathrm{D}} p_{\mathcal{A}} \cup p_{\mathcal{C}}$. Then the map $[\![p]\!] \mapsto ([\![p_{\mathcal{A}}]\!], [\![p_{\mathcal{C}}]\!])$ is an isomorphism of posets $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \to \widetilde{\mathrm{Inv}}(\mathcal{A}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\mathcal{C}(\mathfrak{U}))$. Moreover, if $\otimes$ respects $\geq_{\mathrm{D}}$, then this is also an isomorphism of monoids.

Proof. By Fact 1.5 we have embeddings of posets $\widetilde{\mathrm{Inv}}(\mathcal{A}(\mathfrak{U})) \hookrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{U})$ and $\widetilde{\mathrm{Inv}}(\mathcal{C}(\mathfrak{U})) \hookrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{U})$, yielding an embedding $\widetilde{\mathrm{Inv}}(\mathcal{A}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\mathcal{C}(\mathfrak{U})) \hookrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{U})$. It is therefore enough to show that the natural candidate for its inverse, $[\![p]\!] \mapsto ([\![p_{\mathcal{A}}]\!], [\![p_{\mathcal{C}}]\!])$, is well-defined and a morphism of posets.

Suppose that $p \sim_{\mathrm{D}} p_{\mathcal{A}} \cup p_{\mathcal{C}}$ and $p \sim_{\mathrm{D}} q_{\mathcal{A}} \cup q_{\mathcal{C}}$. Then $(p_{\mathcal{A}} \cup p_{\mathcal{C}}) \sim_{\mathrm{D}} (q_{\mathcal{A}} \cup q_{\mathcal{C}})$, so in particular $(p_{\mathcal{A}} \cup p_{\mathcal{C}}) \geq_{\mathrm{D}} q_{\mathcal{A}}$. By Corollary 2.7, this implies $p_{\mathcal{A}} \geq_{\mathrm{D}} q_{\mathcal{A}}$, and arguing similarly we obtain $[\![p_{\mathcal{A}}]\!] = [\![q_{\mathcal{A}}]\!]$ and $[\![q_{\mathcal{A}}]\!] = [\![q_{\mathcal{C}}]\!]$. Suppose now that $(p_{\mathcal{A}} \cup p_{\mathcal{C}}) \sim_{\mathrm{D}} p \geq_{\mathrm{D}} q \sim_{\mathrm{D}} (q_{\mathcal{A}} \cup q_{\mathcal{C}})$. Again by Corollary 2.7 we must have $p_{\mathcal{A}} \geq_{\mathrm{D}} q_{\mathcal{A}}$ and $p_{\mathcal{C}} \geq_{\mathrm{D}} q_{\mathcal{C}}$.

The "moreover" part is immediate. □

Example 2.9. Let A, C be two structures in disjoint languages, and let T be the theory of their disjoint union, with corresponding families of sorts $\mathcal{A}$ and $\mathcal{C}$. It is easy to see that $\mathcal{A}$ and $\mathcal{C}$ are orthogonal, and that every invariant type from $\mathcal{A}$ is orthogonal to every invariant type from $\mathcal{C}$. Therefore, for $\mathfrak{U} \vDash T$, we have that $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is isomorphic as a poset to $\widetilde{\mathrm{Inv}}(\mathcal{A}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\mathcal{C}(\mathfrak{U}))$, and is well-defined as a monoid if and only if both factors are.

Orthogonality is preserved by products. The proof is folklore, and essentially the same as in the stable case, but we record it here for convenience.

Proposition 2.10. If $p_0, p_1, q \in S^{\mathrm{inv}}(\mathfrak{U})$ and for $i < 2$ we have $p_i \perp q$ then $p_0 \otimes p_1 \perp q$.

Proof. Let $ab \vDash p_0 \otimes p_1$ and $c \vDash q$. Because $p_1 \perp q$ we have $c \vDash q \mid \mathfrak{U}b$, and by definition of $\otimes$ we have $a \vDash p_0 \mid \mathfrak{U}b$. Since $p_0 \perp q$, this entails $c \vDash q \mid \mathfrak{U}ab$, and we are done. □

It is an easy exercise to show that if $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, M)$ are weakly orthogonal and $\mathfrak{U}_1 \succ \mathfrak{U}$ is $|M|^+$-saturated and $|M|^+$-strongly homogeneous, then $(p \mid \mathfrak{U}_1)$ and $(q \mid \mathfrak{U}_1)$ are still weakly orthogonal. Nevertheless, this can fail for arbitrary $B \supseteq \mathfrak{U}$; in other words, weak orthogonality is indeed weaker than orthogonality. While this is folklore,[4] we could not find any example in print, so we conclude this section by recording an instance of this phenomenon.

[4]The second author would like to thank E. Hrushovski for pointing this out.

Counterexample 2.11. There is a theory with invariant types p, q such that $p \perp^{\mathrm{w}} q$ but $p \not\perp q$.

Proof. Let L be a two-sorted language with sorts P, O (points, orders) and a single relation symbol $R^{(\mathrm{P}^2,\mathrm{O})}$. Let K be the class of finite L-structures where, for every $d \in \mathrm{O}$, the relation $R(x, y, d)$ is a linear order. We use the notation $x \leq_d y$. A routine argument shows that K is a (strong) amalgamation class. Let T be the theory of its Fraïssé limit. In models of T, every $\leq_d$ is a dense linear order without endpoints and, given any pairwise distinct $d_0, \dots, d_n \in \mathrm{O}(\mathfrak{U})$ and nonempty intervals $(a_i, b_i)_{d_i}$, the intersection $\bigcap_{i\leq n}(a_i, b_i)_{d_i}$ is infinite.

Fix a small $M \vDash T$, and let $p(x)$ be the 1-type of sort P saying that for every $d \in \mathrm{O}(\mathfrak{U})$, according to $\leq_d$, the point x is just right of M, that is, $p(x) = \{m <_d x <_d e \mid d \in \mathrm{O}(\mathfrak{U}), m \in M, e \in \mathrm{P}(\mathfrak{U}), e > M\}$. Let $q(y)$ be the 1-type of sort P saying that for every $d \in \mathrm{O}(\mathfrak{U})$, according to $\leq_d$, the point y is bigger than $\mathrm{P}(\mathfrak{U})$. By quantifier elimination, p is complete and M-invariant, in fact finitely satisfiable in M. Similarly, q is an $\emptyset$-definable, hence $\emptyset$-invariant type.

By quantifier elimination, $p \perp^{\mathrm{w}} q$. Let b be a point of sort O such that M is $\leq_b$-cofinal in $\mathfrak{U}$, and set $B \coloneqq \mathfrak{U}b$. Then $(q(y) \mid B) \vdash y \geq_b \mathrm{P}(\mathfrak{U})$ and $(p(x) \mid B) \vdash x \geq_b \mathrm{P}(\mathfrak{U})$, and both $x <_b y$ and $y <_b x$ are consistent with $(p(x) \mid B) \cup (q(y) \mid B)$, which is therefore not complete. □

3. Regular ordered abelian groups

In this section we study the domination monoid in certain theories of (linearly) ordered abelian groups, henceforth *oags*. Model-theoretically, the simplest oags are the divisible ones. Their theory is o-minimal and their domination monoid was one of the first ones to be computed (see [HHM08, Mena]). It is isomorphic to the finite powerset semilattice $(\mathscr{P}_{<\omega}(X), \cup, \subseteq)$, with X the set of invariant convex subgroups of $\mathfrak{U}$, and weakly orthogonal classes of types correspond to disjoint finite sets. Divisible oags eliminate quantifiers in the language $L_{\mathrm{oag}} \coloneqq \{+, 0, -, <\}$. In this section we compute the domination monoid in the next simplest case.

Definition 3.1.

1. A (nontrivial) oag is *discrete* iff it has a minimum positive element. Otherwise, it is *dense*.
2. The *Presburger language* is $L_{\mathrm{Pres}} \coloneqq \{+, 0, -, <, 1, \equiv_n \mid n \in \omega\}$.
3. An oag M is viewed as an L_{Pres}-structure by interpreting $+, 0, -, <$ in the natural way, 1 as the minimum positive element if M is discrete and as 0 if M is dense, and $\equiv_n$ as congruence modulo nM.
4. An oag is *regular* iff it eliminates quantifiers in L_{Pres}.

Fact 3.2 ([RZ60, Zak61, Con62, Wei86, CH11])**.** For an oag M, the following are equivalent.

1. M is regular.
2. The only definable convex subgroups of M are $\{0\}$ and M.
3. The theory of M has an archimedean model, i.e. one L_{oag}-embeddable in $(\mathbb{R}, +, 0, -, <)$.
4. For every $n > 1$, if the interval $[a, b]$ contains at least n elements, then it contains an element divisible by n.
5. Every quotient of M by a nontrivial convex subgroup is divisible.

Fact 3.3 ([RZ60, Zak61])**.** Every discrete regular oag is a model of *Presburger Arithmetic*, i.e. it is elementarily equivalent to $\mathbb{Z}$. If M, N are dense regular oags, then $M \equiv N$ if and only if, for each $\mathfrak{p}$ prime, $M/\mathfrak{p}M$ and $N/\mathfrak{p}N$ are either both infinite or have the same finite size.

Notation 3.4. For the rest of the section we adopt the following (not entirely standard) conventions. Let M be an oag and $A \subseteq M$.

1. $A_{>0}$ denotes $\{a \in A \mid a > 0\}$.
2. $\langle A \rangle$ denotes the group generated by A.
3. $\operatorname{div}(M)$ denotes the divisible hull of M.

4. We allow *intervals* to have endpoints in the divisible hull. In other words, an interval in M is a set of the form $\{x \in M \mid a \sqsubset_0 x \sqsubset_1 b\}$, for suitable $a, b \in \operatorname{div}(M) \cup \{\pm\infty\}$ and $\{\sqsubset_0, \sqsubset_1\} \subseteq \{<, \leq\}$.
5. A *cut* (L, R) is given by subsets $L, R \subseteq M$ such that $L \leq R$ and $L \cup R = M$. We call such a cut *realised* iff $L \cap R \neq \emptyset$, and *nonrealised* otherwise.
6. The *cut* (L, R) of $p \in S_1(M)$ is given by $L = \{m \in M \mid p(x) \vdash x \geq m\}$ and $R = \{m \in M \mid p(x) \vdash x \leq m\}$. The *cut* of $c \in N \succ M$ in M is defined to be $\operatorname{tp}(c/M)$.
7. We say that $c \in N \succ M$ *fills* a cut if the latter equals the cut of c.
8. For $a \in M$, we denote by a^+ the cut (L, R) with $L = \{m \in M \mid m \leq a\}$ and $R = \{m \in M \mid a < m\}$, and similarly for a^-. Analogous notions are defined for $a \in \operatorname{div}(M)$.

Remark 3.5.

1. Intervals in our sense are still definable in the language of oags. For example, $(a/n, +\infty)$ is defined by $a < n \cdot x$.
2. If (L, R) is a cut then $|L \cap R| \leq 1$.
3. A type is realised if and only if the corresponding cut is.
4. By regularity, a 1-type over $M \vDash T$ is determined by a cut in M together with a choice of cosets modulo each nM (where if M/nM is infinite a type $p(x)$ may say that the coset $x + nM$ is new, i.e. not represented in M) which is consistent with the theory of M as a pure group.
5. By the Chinese Remainder Theorem, if $\mathfrak{p}, \mathfrak{q}$ are distinct primes and i, j positive integers, then the imaginary sorts $M/\mathfrak{p}^i M$ and $M/\mathfrak{q}^j M$ are orthogonal, and to specify a consistent choice of cosets it is enough to do so modulo prime powers.

Lemma 3.6. Let M be a dense oag. Then, for every $n > 0$, every coset of nM is dense in M. In particular, given any nonrealised $p \in S_1(M)$, and any choice of cosets of the nM (including possibly new ones if M/nM is infinite) which is consistent with the theory of M as a pure group, there is $q \in S_1(M)$ concentrating on these cosets and in the same cut as p.

Proof. By density and point 4 of Fact 3.2, every nM is dense, and since translations are homeomorphisms for the order topology we are done. □

3.1. **Imaginaries in regular ordered abelian groups.** In general, adding imaginary sorts to $\mathfrak{U}$ may result in an enlargement of $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ (see [Men20b, Corollary 3.8]). Nevertheless, it is easy to see that if T eliminates imaginaries, even weakly or geometrically, then the natural embedding $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \hookrightarrow \widetilde{\operatorname{Inv}}(\mathfrak{U}^{\mathrm{eq}})$ is an isomorphism. Therefore, in this subsection we present a language in which regular oags weakly eliminate imaginaries.

Recall that T has *weak elimination of imaginaries* iff for every imaginary e there is a real tuple a such that $e \in \operatorname{dcl}^{\mathrm{eq}}(a)$ and $a \in \operatorname{acl}^{\mathrm{eq}}(e)$. In this subsection, we give a natural language in which every regular ordered abelian group admits weak elimination of imaginaries. A more general version of this result was independently obtained by Mariana Vicaría in [Vic21].

We first recall some notions from [Hru14] (see also [Joh20]).

Definition 3.7. We say that T has *density of definable types* if for every nonempty $\mathfrak{U}$-definable set D there is an $\operatorname{acl}^{\mathrm{eq}}(\ulcorner D \urcorner)$-definable global type concentrating on D.

The following result is proven in [Joh20, proof of Claim 4.2].

Fact 3.8. Suppose T contains a home sort K, i.e. a sort K such that $\mathfrak{U}^{\mathrm{eq}} = \operatorname{dcl}^{\mathrm{eq}}(\mathrm{K}(\mathfrak{U}))$. Assume that for every nonempty $\mathfrak{U}$-definable subset $D \subseteq \mathrm{K}^1$ there is an $\operatorname{acl}^{\mathrm{eq}}(\ulcorner D \urcorner)$-definable global type concentrating on D. Then T has density of definable types.

The following is [Hru14, Lemma 2.9]. A proof may be inferred from [Joh20, proof of Theorem 4.1].

Fact 3.9. Suppose that T has density of definable types and that every global definable type admits a canonical base in the sorts of T. Then T has weak elimination of imaginaries.

Let $\mathfrak{p}$ be a prime number and $n \geq 1$. Let $T_{\mathfrak{p}^n}$ be the L_{ab}-theory of $\bigoplus_{i\in\omega} \mathbb{Z}/\mathfrak{p}^n\mathbb{Z}$, where $L_{\mathrm{ab}} = \{0, +, -\}$. The following is well known.

Fact 3.10.

1. Let A be an infinite abelian group. Then $A \vDash T_{\mathfrak{p}^n}$ if and only if $\mathfrak{p}A = \{a \in A \mid \mathfrak{p}^{n-1}a = 0\}$.
2. $T_{\mathfrak{p}^n}$ has quantifier elimination and is totally categorical.
3. If $A \vDash T_{\mathfrak{p}^n}$, then $\mathfrak{p}A$ is a model of $T_{\mathfrak{p}^{n-1}}$, and the induced structure on $\mathfrak{p}A$ is that of a pure abelian group.
4. $T_{\mathfrak{p}^n}$ has weak elimination of imaginaries.

Proof. We sketch the argument for part (4). As every stable theory has density of definable types, by Fact 3.9 it is enough to show that for every model A of $T_{\mathfrak{p}^n}$, the canonical base of every (definable) type $q \in S_N(A)$ is interdefinable with a real tuple.

Since A is in particular an ω-stable one-based group, such a type q is the generic type of a coset $a+B$, where $a \in A^N$ and B is an $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable connected subgroup of A^N. It is not hard to see, by quantifier elimination, that B is defined by $C \cdot x = 0$ for some matrix $C \in \mathrm{Mat}_{m\times N}(\mathbb{Z})$. By the Elementary Divisor Theorem, replacing C with CQ for some $Q \in \mathrm{GL}_N(\mathbb{Z}_{(\mathfrak{p})})$ if necessary, we may assume that $B = \mathfrak{p}^{m_1}A \oplus \ldots \oplus \mathfrak{p}^{m_k}A \oplus (0)^{N-k} \leq A^N$, where $0 \leq m_i < n$ for $1 \leq i \leq k$. We thus obtain an $\emptyset$-definable isomorphism

$$f_B \colon A^N/B \cong A/\mathfrak{p}^{m_1}A \oplus \ldots \oplus A/\mathfrak{p}^{m_k}A \oplus A^{N-k} \cong \mathfrak{p}^{n-m_1}A \oplus \ldots \oplus \mathfrak{p}^{n-m_k}A \oplus A^{N-k}$$

Let $a' := f_B(a+B) \in A^N$. Then $\mathrm{dcl}^{\mathrm{eq}}(\mathrm{Cb}(q)) = \mathrm{dcl}^{\mathrm{eq}}(\ulcorner a+B \urcorner) = \mathrm{dcl}^{\mathrm{eq}}(a')$. □

Let $T_{\mathfrak{p}^\infty}$ be the following multi-sorted theory:

- for every $n > 0$ there is a sort $\mathrm{Q}_{\mathfrak{p}^n}$, endowed with a copy of L_{ab};
- for every $n > 0$ there is a function symbol $\rho_{\mathfrak{p}^{n+1}} \colon \mathrm{Q}_{\mathfrak{p}^{n+1}} \to \mathrm{Q}_{\mathfrak{p}^n}$;
- $M \vDash T_{\mathfrak{p}^\infty}$ if and only if, for all $n > 0$, $\mathrm{Q}_{\mathfrak{p}^n}(M) \vDash T_{\mathfrak{p}^n}$ and $\rho_{\mathfrak{p}^{n+1}} \colon \mathrm{Q}_{\mathfrak{p}^{n+1}}(M) \to \mathrm{Q}_{\mathfrak{p}^n}(M)$ is a surjective group homomorphism with kernel $\mathfrak{p}^n\,\mathrm{Q}_{\mathfrak{p}^{n+1}}(M)$.

Corollary 3.11. The theory $T_{\mathfrak{p}^\infty}$ is complete, totally categorical, and has quantifier elimination and weak elimination of imaginaries.

Proof. Immediate from Fact 3.10. □

We now consider a regular oag M. Since it is well known that Presburger Arithmetic eliminates imaginaries, we may assume that M is dense.

We view M as a structure in the language with one sort for the oag itself, endowed with L_{oag}, one sort $\mathrm{Q}_{\mathfrak{p}^n}$ for each prime $\mathfrak{p}$ and each $n > 0$, endowed with L_{ab} and interpreted as the group $M/\mathfrak{p}^nM$, functions $\pi_{\mathfrak{p}^n}$ for the quotient map from M to $M/\mathfrak{p}^nM$ and functions $\rho_{\mathfrak{p}^{n+1}}$ for the canonical surjection $M/\mathfrak{p}^{n+1}M \to M/\mathfrak{p}^nM$. For every prime $\mathfrak{p}$, let $d_\mathfrak{p} \in \mathbb{N} \cup \{\infty\}$ be such that $(M : \mathfrak{p}M) = \mathfrak{p}^{d_\mathfrak{p}}$. Set $T := \mathrm{Th}(M)$. We leave the easy proof of the following lemma to the reader.

Lemma 3.12. The theory T has quantifier elimination. Letting $\mathfrak{U} \vDash T$, we have the following.

1. For every $\mathfrak{p}$ prime and $n > 0$, the sort $\mathrm{Q}_{\mathfrak{p}^n}(\mathfrak{U})$ equipped with the abelian group structure is fully embedded.
2. If $d_\mathfrak{p} = \infty$, the structure given by $(\mathrm{Q}_{\mathfrak{p}^n}(\mathfrak{U}))_{n>0}$ together with the maps $\rho_{\mathfrak{p}^{n+1}}$ is fully embedded and a model of $T_{\mathfrak{p}^\infty}$ with no extra structure. If $d_\mathfrak{p}$ is finite, every sort $\mathrm{Q}_{\mathfrak{p}^n}(\mathfrak{U})$ is finite.

Theorem 3.13. The theory T has weak elimination of imaginaries.

Proof. We will use Fact 3.9 together with Fact 3.8.

Let D be a nonempty definable subset of the home sort. By quantifier elimination, D is a finite union of singletons and (nonempty) definable sets of the form

$$D' = \{x \in \mathfrak{U} \mid a < x < b, \pi_{\mathfrak{p}_1^{n_1}}(x) \in X_1, \dots, \pi_{\mathfrak{p}_k^{n_k}}(x) \in X_k\}$$

where $a \in \operatorname{div}(\mathfrak{U}) \cup \{-\infty\}$, $b \in \operatorname{div}(\mathfrak{U}) \cup \{\infty\}$, $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ are pairwise distinct primes, $n_1, \dots, n_k \in \omega \setminus \{0\}$, and $X_i \subseteq \mathrm{Q}_{\mathfrak{p}_i^{n_i}}(\mathfrak{U})$ is a definable set for $i = 1, \dots, k$.

Setting $n := \prod_{i=1}^k \mathfrak{p}_i^{n_i}$, we have $\mathfrak{U}/n\mathfrak{U} \cong \prod_{i=1}^k \mathrm{Q}_{\mathfrak{p}_i^{n_i}}(\mathfrak{U})$. As every coset of $n\mathfrak{U}$ is dense in $\mathfrak{U}$, it thus follows that D' is dense in the interval (a, b). In particular, $\inf(D)$ exists in $\operatorname{div}(\mathfrak{U}) \cup \{-\infty\}$.

Claim. There is an $\operatorname{acl}^{\mathrm{eq}}(\ulcorner D \urcorner)$-definable global type q concentrating on D.

Proof of Claim. If D admits a minimum a, this is clear, as we may just take as q the realised type given by a. Otherwise, let $a := \inf(D)$. We then find $a < b \in \mathfrak{U}$, distinct primes $\mathfrak{p}_1, \dots, \mathfrak{p}_k$, natural numbers $n_1, \dots, n_k$ and definable sets $X_i \subseteq \mathrm{Q}_{\mathfrak{p}_i^{n_i}}$ for $i = 1, \dots, k$ such that

$$\{x \in D \mid x < b\} = \{x \in \mathfrak{U} \mid a < x < b \text{ and } \pi_{\mathfrak{p}_i^{n_i}}(x) \in X_i \text{ for } i = 1, \dots, k\}$$

Note that, for every b' such that $a < b' < b$, one has

$$\{x \in D \mid x < b'\} = \{x \in \mathfrak{U} \mid a < x < b' \text{ and } \pi_{\mathfrak{p}_i^{n_i}}(x) \in X_i \text{ for } i = 1, \dots, k\}$$

Therefore we have $\ulcorner X_i \urcorner \in \operatorname{dcl}^{\mathrm{eq}}(\ulcorner D \urcorner)$ for all i. As $T_{\mathfrak{p}_i^\infty}$ is stable, there is an inverse system of global $\operatorname{acl}^{\mathrm{eq}}(\ulcorner X_i \urcorner)$-definable types $q_{\mathfrak{p}_i^m}$, with $q_{\mathfrak{p}_i^{n_i}}$ concentrating on X_i. For every $m > 0$ and prime $\mathfrak{p}$ different from $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ we let $q_{\mathfrak{p}^m}$ be the realised type of $0 \in \mathfrak{U}/\mathfrak{p}^m\mathfrak{U}$, which is $\emptyset$-definable. Lemma 3.6 ensures consistency of the set of formulas

$$q(x) := \{a < x < e \mid e \in \mathfrak{U}, e > a\} \cup \bigcup_{\mathfrak{p} \in \mathbb{P}, m > 0} q_{\mathfrak{p}^m}(\pi_{\mathfrak{p}^m}(x))$$

By quantifier elimination, q is a global $\operatorname{acl}^{\mathrm{eq}}(\ulcorner D \urcorner)$-definable type concentrating on D. □ CLAIM

Suppose that $q(x_0, \dots, x_n)$ is a global definable type, with $x_0, \dots, x_n$ variables from the main sort. We need to show that $\operatorname{Cb}(q)$ is interdefinable with a tuple from the sorts of $\mathfrak{U}$. We may assume that if $(a_0, \dots, a_n) \vDash q$, then $(a_0, \dots, a_n)$ is $\mathbb{Q}$-linearly independent over $\mathfrak{U}$.

By quantifier elimination, q is then determined by the following data:

- the (definable) cuts C_k of $\sum_{i=0}^n k_i a_i$ in $(\mathfrak{U}, <)$, for each $k = (k_0 \dots, k_n) \in \mathbb{Z}^{n+1}$, and
- the global definable types $q_{\mathfrak{p}^n}$ for each prime $\mathfrak{p}$ and $\ell > 0$, where $q_{\mathfrak{p}^\ell} := (\pi_{\mathfrak{p}^\ell})_* q$.

By Fact 3.10, for every $\mathfrak{p}$ and ℓ the theory $T_{\mathfrak{p}^\ell}$ has weak elimination of imaginaries, hence $\operatorname{Cb}(q_{\mathfrak{p}^\ell})$ is interdefinable with a finite tuple in $\mathrm{Q}_{\mathfrak{p}^\ell}(\mathfrak{U})$. We may thus conclude, since $\operatorname{Cb}(q)$ is interdefinable with the tuple given by all the $\operatorname{Cb}(q_{\mathfrak{p}^\ell})$ together with all elements $b_k \in \mathfrak{U}$ such that C_k equals $(b_k/m)^+$ or $(b_k/m)^-$ for some $m > 0$. □

Remark 3.14. For the above to go through, we need to have in our language the sorts $\mathrm{Q}_{\mathfrak{p}^n}$ even when they are finite. An alternative is to name enough constants, e.g. by naming a model.

3.2. **Moving to the right of a convex subgroup.**

Assumption 3.15. Until the end of the section, T is the complete L_{Pres}-theory of a regular oag. Imaginary sorts are not in our language until further notice.

Definition 3.16. Let $B \subseteq M$.

1. A type $q(x) \in S_1(M)$ is *right of* B iff $q(x) \vdash \{x > d \mid d \in B\} \cup \{x < d \mid d \in M, d > B\}$.
2. An element of an elementary extension of M is *right of* B if its type over M is.
3. A convex subgroup H of $\mathfrak{U}$ is *[A-]invariant* iff there is an [A-]invariant type to its right.

Remark 3.17. Let p be an M-invariant type. If the corresponding cut (L, R) is definable, then it is M-definable. If it is not definable, then exactly one between the cofinality of L and the coinitiality of R is small, and M contains a set cofinal in L or coinitial in R respectively.

In particular, in a regular oag a nontrivial convex subgroup H of $\mathfrak{U}$ is invariant if and only if the cofinality of H or the coinitiality of $(\mathfrak{U} \setminus H)_{>0}$ is small, while the trivial subgroup $\{0\}$ is invariant if and only if $\mathfrak{U}$ is dense.

Example 3.18. Let M be a small model, and let p be the global M-invariant type right of M and divisible by every n. In other words, $p(x) \coloneqq \{x > m \mid m \in M\} \cup \{x < d \mid d > M\} \cup \{x \equiv_n 0 \mid n \in \omega \setminus \{0\}\}$. If (L, R) is the cut of p, then M is cofinal in L. It is easy to see that $p(x^1) \otimes p(x^0)$ is axiomatised by $p(x^1) \cup p(x^0)$ together with $\langle x_1 \rangle_{>0} < \langle x_0 \rangle_{>0}$.

Fact 3.19 ([CH11, Corollary 1.10])**.** Definable functions in oags are piecewise affine: if M is an oag and $f\colon M^n \to M$ is an A-definable function, then there is a partition of M in finitely many A-definable sets such that the restriction of f to each such set is affine, that is, of the form $x \mapsto \frac{1}{s}\left(a + \sum_i r_i x_i\right)$, for suitable $a \in \operatorname{dcl}(A)$ and $r_i, s \in \mathbb{Z}$.

Lemma 3.20. In the theory of a regular oag, suppose that $p \in S^{\mathrm{inv}}_1(\mathfrak{U})$ and f is a definable function such that f_*p is right of a convex subgroup. Then $p \sim_{\mathrm{D}} f_*p$.

Proof. Clearly $p \geq_{\mathrm{D}} f_*p$. By Fact 3.19, f is piecewise affine. Because f_*p is not realised, f cannot be constant at p, hence it is invertible at p, and we have $f_*p \geq_{\mathrm{D}} f^{-1}_*(f_*p) = p$. □

Lemma 3.21. In Presburger Arithmetic, for every nonrealised $p \in S^{\mathrm{inv}}_1(\mathfrak{U})$ there are $d \in \mathfrak{U}$ and $k \in \mathbb{Z} \setminus \{0\}$ such that, if $f(t) \coloneqq kt + d$, then f_*p is right of an invariant convex subgroup.

Proof. By Fact 3.2, $\mathfrak{U}/\mathbb{Z}$ is divisible. It is easy to see that $\mathfrak{U}/\mathbb{Z}$ inherits saturation and strong homogeneity from $\mathfrak{U}$, and the conclusion follows by lifting the analogous result [HHM08, Corollary 13.11] (see also [Mena, Proposition 4.8]) from $\mathfrak{U}/\mathbb{Z}$. □

From the previous lemmas we obtain the following.

Corollary 3.22. In Presburger Arithmetic, every invariant 1-type is domination-equivalent to a type right of an invariant convex subgroup.

The rest of the subsection is dedicated to generalising the corollary above to the regular case.

Assumption 3.23. Until the end of the subsection, T denotes the complete theory of a dense regular oag M.

Proposition 3.24. Let $b \in \mathfrak{U} \setminus M$ be divisible by every $n > 1$ and let $B \coloneqq \langle Mb \rangle = M + \mathbb{Q}b$. If $M_{>0}$ is coinitial in $B_{>0}$, then $M \prec B \prec \mathfrak{U}$.

Proof. The natural embedding of M in B is easily seen to be *pure*, i.e. for every $n > 1$ we have $nB \cap M = nM$. Moreover, if $c = a + \gamma b$, with $a \in M$ and $\gamma \in \mathbb{Q}$, then for every n we clearly have $c - a \in nB$, hence B/nB may be naturally identified with M/nM.

We now show that B is a dense regular oag. Because M is dense and $M_{>0}$ is coinitial in $B_{>0}$, it follows that B is as well dense. In particular, nonempty intervals in B are infinite, so by Fact 3.2, we need to show that every nonempty $(c, d) \subseteq B$ contains an element divisible by n. By assumption, $(0, d - c)$ intersects M, hence contains an interval I of M, hence represents all elements of M/nM by Lemma 3.6. These can be identified with the elements of B/nB, as observed above, so there is $e \in I$ such that $c + e \in nB$. Clearly, $c + e \in (c, d)$, completing the proof that B is regular.

By Fact 3.3 and the identification of M/nM with B/nB, we obtain $B \equiv M$. Since the embedding of M in B is pure, M is an L_{Pres}-substructure of B. By definition of regularity, i.e. quantifier elimination in L_{Pres}, we have the conclusion. □

The following notion was introduced in [HHM08] in the setting of divisible ordered abelian groups. Below, it will turn out to be useful also in the dense regular case.

Definition 3.25. An extension $M < N$ of oags is an *i-extension* iff there is no $b \in N_{>0}$ such that the set $\{m \in M \mid m < b\}$ is closed under sum.

Lemma 3.26. Let H be a convex subgroup of M and $N > M$. The set of elements of N right of H is closed under sum.

Proof. If $H = M$, the statement is trivial. If $H = \{0\}$, let $0 < c, d < M_{>0}$ and pick $a \in M_{>0}$. By density, there is $b \in M$ with $0 < b < a$, and since b and $a - b$ are both in $M_{>0}$ we conclude $c+d < b+a-b = a$. If H is proper nontrivial, by Fact 3.2 the quotient M/H is divisible, and the conclusion follows from the previous case applied to M/H viewed as a subgroup of the quotient of N by the convex hull of H. □

Corollary 3.27. An extension $M < N$ of oags, with M dense regular, is an i-extension if and only if the map $H \mapsto H \cap M$ is a bijection between the convex subgroups of N and M.

Proof. If b witnesses that $M < N$ is not an i-extension, by Lemma 3.26 the convex subgroups generated by b and by $\{m \in M \mid |m| < b\}$ restrict to the same convex subgroup of M. Conversely, if $H_0 \subseteq H_1$ are convex subgroups of N with the same restriction to M, and $b \in (H_1 \setminus H_0)_{>0}$, then $\{m \in M \mid |m| < b\}$ is closed under sum. □

Proposition 3.28. Every $M \vDash T$ has a maximal elementary i-extension.

Proof. The size of an i-extension is bounded by [HHM08, proof of Lemma 13.9], and both i-extensions and elementary extensions are transitive and closed under unions of chains (for a proof, see [Men20a, Lemma 4.2.16]). Now apply Zorn's Lemma. □

Proposition 3.29. Suppose $M \vDash T$ has no proper elementary i-extension and let $p \in S_1(M)$ be nonrealised. Then there are $a \in M$ and $\beta \in \mathbb{Z} \setminus \{0\}$ such that, if $f(t) = a + \beta t$, then the pushforward f_*p is right of a convex subgroup.

Proof. Let $b \vDash p$, and suppose first that b is divisible by every n. Consider $B := \langle Mb \rangle = M + \mathbb{Q}b$. We have two subcases.

1. There are $a' \in M$ and $\beta' \in \mathbb{Q}$ such that $0 < a' + \beta' b < M_{>0}$. By Lemma 3.26, we may multiply by the denominator of β' and still obtain a positive element smaller than $M_{>0}$, so we have the conclusion with the convex subgroup $\{0\}$.
2. If instead there is no such $a' + \beta' b$, then $M_{>0}$ is coinitial in $B_{>0}$, and by Proposition 3.24 $B \succ M$. By maximality of M, there must be convex subgroups $H_0 \subsetneq H_1$ of B such that $H_0 \cap M = H_1 \cap M$. Hence any positive $a + \beta b \in H_1 \setminus H_0$ is right of $H_0 \cap M$. We conclude again by clearing the denominator of β and using Lemma 3.26.

This shows the conclusion when b is divisible by all n. In the general case, by Lemma 3.6, there is $c \in \mathfrak{U}$ with the same cut in M as b which is divisible by every n. As we just proved, there is $f(t) := a + \beta t$, with $\beta \in \mathbb{Z}$ and $a \in M$, such that the cut of $f(c)$ in M is that of a convex subgroup. Because $f(t)$ sends intervals to intervals, it sends cuts to cuts, hence the cut of $f(b)$ equals that of $f(c)$. □

Corollary 3.30. Every nonrealised $p \in S_1^{\mathrm{inv}}(\mathfrak{U})$ is domination-equivalent to a type right of an invariant convex subgroup.

Proof. If p is M-invariant, up to enlarging M we may assume that it has no proper elementary i-extension. Let $f(t)$ be an M-definable function given by Proposition 3.29 applied to $p \upharpoonright M$. Then f_*p is M-invariant, and it is routine to check that its cut is that of a convex subgroup of $\mathfrak{U}$. Now apply Lemma 3.20. □

3.3. Computing the domination monoid.

Lemma 3.31. Let $H_0 \subsetneq H_1$ be convex subgroups of $M \vDash T$, and for $i < 2$ let $q_i(x^i) \in S_1(M)$ be right of H_i. Suppose that there is no prime $\mathfrak{p} \in \mathbb{P}$ such that both $q_i(x^i)$ prove that x^i is in a new coset modulo some $\mathfrak{p}^{\ell_i}$ Then $q_0 \perp^{\mathrm{w}} q_1$.

Proof. By assumption and quantifier elimination we only need to show that the cut of every $k_0x^0 + k_1x^1$ is determined by $q_0(x^0) \cup q_1(x^1)$. If $k_1 = 0$ we are done. But, if $k_1 \neq 0$, it follows from Lemma 3.26 that $k_0x^0 + k_1x^1$ and k_1x^1 have the same cut. □

Proposition 3.32. Suppose that $q_H(x) \in S_1^{\mathrm{inv}}(\mathfrak{U})$ is right of the convex subgroup H and prescribes realised cosets modulo every n for x. For an invariant $*$-type q, the following are equivalent.

1. For every (equivalently, some) $b \vDash q$, no type right of H in realised in $\langle \mathfrak{U}b \rangle$.
2. $q_H \perp^{\mathrm{w}} q$.
3. q_H commutes with q.
4. $q_H \perp q$.

Proof. To show $1 \Rightarrow 2$, consider $q_H(x) \cup q(y)$. By assumption on q_H we only need to deal with inequalities of the form $kx + \sum_{i<|y|} k_iy_i + d \geq 0$, but 1 gives immediately that the cut of kx in $\langle \mathfrak{U}b \rangle$ is determined. If 1 fails, as witnessed by $f(b)$, say, then $q_H(x) \otimes q(y)$ and $q(y) \otimes q_H(x)$ disagree on the formula $f(y) < x$, proving $3 \Rightarrow 1$, and $2 \Rightarrow 3$ holds for every type in every theory.

We are left to show $2 \Rightarrow 4$, the converse being trivial. Suppose that $B \supseteq \mathfrak{U}$ is such that $(q_H \mid B) \not\perp^{\mathrm{w}} (q \mid B)$. Because the cosets modulo every n of a realisation of q_H are all realised in $\mathfrak{U}$, this can only happen if some inequality of the form $kx + \sum_{i<|y|} k_iy_i + d \geq 0$, with $k_i \in \mathbb{Z}$ and $d \in \langle B \rangle$, is not decided. Hence, if 4 fails, it fails for the pushforward of q under the map $y \mapsto \sum_{i<|y|} k_iy_i$, and we may therefore assume that q is a 1-type. By Corollary 3.30 and Proposition 2.4, we may furthermore assume that q is right of a convex subgroup. We have thus reduced to the case of two 1-types $q_0(x)$, $q_1(y)$, to the right of distinct (by 2) convex subgroups $H_0 \subsetneq H_1$, where for some $i < 2$ the cosets of a realisation of q_i are realised. Clearly, $q_0 \mid B$ and $q_1 \mid B$ concentrate right of convex subgroups of $\langle B \rangle$, and the cosets of q_i are still realised. To conclude, observe that since $H_0 \subsetneq H_1$, by Lemma 3.26 the cut of $k_xx + k_yy$ in $\langle B \rangle$ must coincide with that of k_yy if $k_y \neq 0$, and with that of k_xx otherwise. □

Lemma 3.33. Let c be a possibly infinite tuple, of cardinality λ. Then, in $\langle Mc \rangle$, at most λ cuts of convex subgroups of M are filled. Moreover, for every $\mathfrak{p} \in \mathbb{P}$ and $k \in \omega \setminus \{0\}$, at most λ new cosets modulo $\mathfrak{p}^k$ are represented in $\langle Mc \rangle$.

Proof. The first statement is obtained by arguing with $\mathbb{Q}$-linear dimension in the respective divisible hulls. The "moreover" statement is similarly clear since, for fixed $\mathfrak{p}^k$, a single element can contribute at most one new coset modulo $\mathfrak{p}^k$. □

Note that if, say, c_0, is in a new coset modulo $\mathfrak{p}^2$ but in a realised one modulo $\mathfrak{p}$, then there is a point in $\operatorname{dcl}(\mathfrak{U}c_0)$ which is in a new coset modulo $\mathfrak{p}$.

Definition 3.34. Let q be an invariant global $*$-type, and $c \vDash q$. Let $\mathcal{H}(q)$ be the set of cuts of convex subgroups of $\mathfrak{U}$ filled in $\langle \mathfrak{U}c \rangle$ and, for $\mathfrak{p} \in \mathbb{P}$, let $\kappa_{\mathfrak{p}}(q) := \dim_{\mathbb{F}_{\mathfrak{p}}}((\operatorname{dcl}(\mathfrak{U}c)/(\mathfrak{p}\operatorname{dcl}(\mathfrak{U}c)))/(\mathfrak{U}/\mathfrak{p}\mathfrak{U}))$.

Theorem 3.35. Let p, q be invariant $*$-types. Then $p \geq_{\mathrm{D}} q$ if and only if

1. $\mathcal{H}(p) \supseteq \mathcal{H}(q)$, and
2. for every $\mathfrak{p} \in \mathbb{P}$ we have $\kappa_{\mathfrak{p}}(p) \geq \kappa_{\mathfrak{p}}(q)$.

Hence, the $\sim_{\mathrm{D}}$-class of q is determined by $\mathcal{H}(q)$ and the function $\mathfrak{p} \mapsto \kappa_{\mathfrak{p}}(q)$.

Proof. Most of the proof will consist in finding a nice representative q' for the $\sim_{\mathrm{D}}$-class of q. Let $c \vDash q$, and use Lemma 3.33 to index on a suitable cardinal κ, bounded by the cardinality of c, the (necessarily invariant) convex subgroups H_j whose cuts are filled in $\langle \mathfrak{U}c\rangle$. Since we may assume that q is not realised, Corollary 3.30 tells us that $\kappa \neq 0$. For $j < \kappa$, let q_j be the type right of H_j divisible by every nonzero integer. For each $\mathfrak{p} \in \mathbb{P}$, let $\kappa_\mathfrak{p} \coloneqq \kappa_\mathfrak{p}(q)$, which is bounded by the cardinality of c by Lemma 3.33, and set $\mu \coloneqq \sup_{\mathfrak{p}\in\mathbb{P}} \kappa_\mathfrak{p}$. Choose types $(q'_j \mid j < \mu)$ in the same cut as q_0 and whose product represents these new cosets, say by taking as q'_j the 1-type of a new element in the cut of q_0 which is in a new coset modulo $\mathfrak{p}$ if $j < \kappa_\mathfrak{p}$, and divisible by $\mathfrak{p}$ otherwise.

Let $q' \coloneqq \bigotimes_{j<\mu} q'_j \otimes \bigotimes_{j<\kappa} q_j$. We show that $q'(y) \geq_{\mathrm{D}} q(x)$. Partition y as $y^\mu y^\kappa$ according to the definition of q'. Let $b \in \mathrm{dcl}(\mathfrak{U}c)$ be a maximal tuple amongst those with these properties:

1. each b_k falls in the cut of an invariant convex subgroup, and
2. if $b_k < b_{k'}$ then $\langle b_j\rangle_{>0} < \langle b_{k'}\rangle_{>0}$.

Note that, since $\kappa \neq \emptyset$, a point in the cut of an invariant convex subgroup does exist, hence so does such a tuple b, and a maximal one exists because the size of b is at most that of c, by looking at $\mathbb{Q}$-linear dimension over $\mathfrak{U}$ in the divisible hull. By Fact 3.19 definable functions are piecewise affine and, by clearing denominators using Lemma 3.26, we may assume that $b \in \langle \mathfrak{U}c\rangle$.

Write $b_k = f_k(c)$, for suitable affine functions f_k. Let $M \prec^+ \mathfrak{U}$ be large enough to contain the parameters of the f_k, such that q and q' are M-invariant, and such that M has no proper elementary i-extension. Let $r \in S_{qq'}(M)$ contain the following formulas.

1. If the cut of b_k has small cofinality on the right, by choice of q' there is $j < \kappa$ such that y^κ_j is in the same cut as b_k according to q'. In this case, put in r the formula $f_k(x) > y^\kappa_j$.
2. If the cut of b_k has small cofinality on the left, choose j as in the previous point and put in r the formula $f_k(x) < y^\kappa_j$.
3. Suppose that for some $d \in \mathfrak{U}$ the type q proves that $x_i - d$ is divisible by $\mathfrak{p}^\ell$ but x_i is in a new coset modulo $\mathfrak{p}^{\ell+1}$. Since q is M-invariant, we may find such a d in M. Put in r the formula $\mathfrak{p}^{\ell+1} \mid ((x_i - d) - \mathfrak{p}^\ell \cdot y^\mu_j)$, for a suitable $j < \mu$, making sure to use different j's for different new cosets.

Note that if, say, $\mathfrak{p}^\ell \mid (x_i - x_{i'})$, this information is contained in $q \restriction \emptyset \subseteq r$. This, together with point 3 above, ensures that the restriction of q to the language $L_{\mathrm{Pres}} \setminus \{<\}$ (with parameters from $\mathfrak{U}$) is recovered by $q' \cup r$.

Claim. $q' \cup r$ entails the quantifier-free $|b|$-type of the $f_k(x)$ over $\mathfrak{U}$ in the language $\{+, 0, 1, <\}$.

Proof of Claim. It is enough to show that the cut of every $\sum_k \beta_k f_k(x)$ in $\mathfrak{U}$ is decided, where only finitely many $\beta_k \in \mathbb{Z}$ are nonzero. By choice of r and Remark 3.17, $q' \cup r$ determines the cut of each $f_k(x)$ over $\mathfrak{U}$. Moreover, r contains the information that $\langle f_k(x)\rangle_{>0} < \langle f_{k'}(x)\rangle_{>0}$ (for $b_k < b_{k'}$). By this, the fact that the $f_k(x)$ are right of convex subgroups, and Lemma 3.26, the cut of $\sum_k \beta_k f_k(x)$ must be that of $\mathrm{sign}(\beta_k) f_k(x)$, with k the largest such that $\beta_k \neq 0$. □ CLAIM

We are left to show that $q' \cup r$ decides the cut of every $\sum_i \gamma_i x_i$ in $\mathfrak{U}$. After possibly composing with an M-definable injective affine function, by i-completeness of M we may assume that we are dealing with a term of the form $\sum_i \gamma_i x_i + d$, with $d \in M$, whose cut in M is right of a convex subgroup of M. Since $\mathrm{tp}(\sum_i \gamma_i x_i + d/\mathfrak{U})$ is M-invariant, $\sum_i \gamma_i x_i + d$ is in the cut of a(n M-invariant) convex subgroup of $\mathfrak{U}$. By maximality of b, there must be k and positive integers n, m such that $nb_k \leq m\left(\sum_i \gamma_i x_i + d\right) \leq (n+1)b_k$. Therefore we have

$$r \vdash nf_k(x) \leq m\left(\sum_i \gamma_i x_i + d\right) \leq (n+1)f_k(x)$$

and we conclude, by applying the Claim, that $q' \geq_{\mathrm{D}} q$.

Similar arguments show that $q \geq_{\mathrm{D}} q'$. It is moreover clear that, if p satisfies 1 and 2 from the conclusion, and p' is defined analogously to q', then $p' \geq_{\mathrm{D}} q'$. That 1 is necessary to have $p \geq_{\mathrm{D}} q$ follows from Proposition 3.32 and Fact 2.3. As for 2, suppose that for some $\mathfrak{p} \in \mathbb{P}$ we have $\kappa_{\mathfrak{p}}(q) > \kappa_{\mathfrak{p}}(p)$. We reach a contradiction by finding a type dominated by q but not by p. We may assume that 1 holds. Choose any cut in $\mathcal{H}(q) = \mathcal{H}(p)$, and let p_0 be the 1-type in that cut in a new coset modulo $\mathfrak{p}$ but divisible by every other prime. It is enough to show that if $\kappa > \kappa_{\mathfrak{p}}(p)$ then $p(x)$ does not dominate $p_0^{(\kappa)}(y)$. Suppose $r(x, y) \in S_{pp_0^{(\kappa)}}(M)$ witnesses domination. Because $\kappa > \kappa_{\mathfrak{p}}(p)$, there must be j such that, for every $\mathfrak{U}$-definable function f, we have $p \cup r \not\vdash f(x) = y_j$. It follows that, if $d \in \mathfrak{U}$ is in a coset modulo $\mathfrak{p}$ not represented in M, then $p \cup r$ is consistent with $y_j \equiv_{\mathfrak{p}} d$, hence $p \cup r \not\vdash p_0^{(\kappa)}$. □

Proposition 3.36. For all invariant $*$-types p, q and $\mathfrak{p} \in \mathbb{P}$, we have $\mathcal{H}(p \otimes q) = \mathcal{H}(p) \cup \mathcal{H}(q)$ and $\kappa_{\mathfrak{p}}(p \otimes q) = \kappa_{\mathfrak{p}}(p) + \kappa_{\mathfrak{p}}(q)$.

Proof. By Proposition 3.32, in its notation, $\mathcal{H}_i$ is precisely the set of convex invariant subgroups H such that $q_i \not\perp q_H$. By Proposition 2.10, we therefore have the first statement. The second one is an easy consequence of the definition of $\otimes$. □

Definition 3.37. Let X be the set of invariant convex subgroups of $\mathfrak{U}$ and let $K = \mathscr{P}_{<\omega}(X) \times \prod_{\mathbb{P}_T}^{\mathrm{bdd}} \omega$, where $\mathbb{P}_T$ is the set of primes $\mathfrak{p}$ such that if $M \vDash T$ then $\mathfrak{p}M$ has infinite index, and $\prod_{\mathbb{P}_T}^{\mathrm{bdd}} \omega$ denotes the bounded $\mathbb{P}_T$-indexed sequences of natural numbers. Equip K with the operation $(\cup, +)$, with $+$ denoting pointwise addition.

Corollary 3.38. The monoid $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined and isomorphic to

$$K \setminus \{(a, b) \in K \mid a = \emptyset, b \neq 0\}$$

Proof. Send $[\![p]\!]$ to $(\mathcal{H}(p), \mathfrak{p} \mapsto \kappa_{\mathfrak{p}}(p))$. By Theorem 3.35 this is well-defined and an embedding of posets and, by Proposition 3.36, it is also a morphism of monoids. Surjectivity can be shown by building a suitable q' as in the proof of Theorem 3.35. □

By using Theorem 3.13 and adapting the arguments of this subsection to work in the language with the sorts $\mathrm{Q}_{\mathfrak{p}^n}$, which by Lemma 3.12 are fully embedded with the structure of an abelian group, it is possible to compute $\widetilde{\mathrm{Inv}}(\mathfrak{U}^{\mathrm{eq}})$. We leave the details to the reader.

Corollary 3.39. The monoid $\widetilde{\mathrm{Inv}}(\mathfrak{U}^{\mathrm{eq}})$ is well-defined and isomorphic to

$$K \setminus \{(a, b) \in K \mid a = \emptyset, \mathrm{supp}(b) \text{ infinite}\}$$

Since Theorem 3.35 was proven for $*$-types, we similarly obtain the following.

Corollary 3.40 (Theorem D)**.** Let T be the theory of a regular oag and κ a small infinite cardinal. Denote by $\hat{\kappa}$ the ordered monoid of cardinals smaller or equal than κ with cardinal sum, and by X be the set of invariant convex subgroups of $\mathfrak{U}$. Then $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}^{\mathrm{eq}})$ is well-defined, and

$$\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}^{\mathrm{eq}}) \cong \mathscr{P}_{\leq\kappa}(X) \times \prod_{\mathbb{P}_T} \hat{\kappa}$$

If we denote the right hand side by K, then the composition of the embedding $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \hookrightarrow \widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}^{\mathrm{eq}})$ with the isomorphism above has image $K \setminus \{(a, b) \in K \mid a = \emptyset, b \neq 0\}$.

4. Pure short exact sequences

In this section we study short exact sequences of abelian structures $0 \to \mathcal{A} \xrightarrow{\iota} \mathcal{B} \xrightarrow{\nu} \mathcal{C} \to 0$ which satisfy a *purity* assumption, and where $\mathcal{A}$ and $\mathcal{C}$ may be equipped with extra structure. We view them as multi-sorted structures, and use the relative quantifier elimination results from [ACGZ20] to describe the domination poset in terms of $\mathcal{A}$ and $\mathcal{C}$. A decomposition of the form $\widetilde{\mathrm{Inv}}(\mathcal{A}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\mathcal{C}(\mathfrak{U}))$ only holds in special cases, while in general we will need to look at $*$-types and introduce a family of imaginaries of $\mathcal{A}$ which depends on $\mathcal{B}$.

Let us start by recalling the setting of [ACGZ20]. For the sake of readability, we will use notations such as $t(x)$ for a tuple of terms, 0 for a tuple of zeroes of the appropriate length, etc. Tuples of the same length may be added, and tuples of appropriate lengths used as arguments, as in $f(t(x,0)-d)=0$.

Definition 4.1. Let L be a language with sorts indexed by a set S, relation symbols (R_i) and function symbols (f_j). An *L-abelian structure* is an L-structure $A = ((\mathrm{A}_s);(R_i),(f_j))$ with the following properties.

1. The sorts A_s are abelian groups, equipped with pairwise disjoint copies of the language $L_{\mathrm{ab}} = \{+,0,-\}$ of abelian groups.
2. Each R_i is a subgroup of $\mathrm{A}_{s_0} \times \ldots \times \mathrm{A}_{s_m}$, for certain $s_0,\ldots,s_m \in S$.
3. Each f_j is a group homomorphism $\mathrm{A}_{s_0} \times \ldots \times \mathrm{A}_{s_n} \to \mathrm{A}_s$, for certain $s_0,\ldots,s_n,s \in S$.

For example, a chain complex of modules may be viewed as an abelian structure.

Recall that the class of *pp formulas* is obtained from the class of atomic formulas over $\emptyset$ by closing under conjunction and existential quantification. In an L-abelian structure, each pp formula defines a subgroup of a suitable product of sorts.

Definition 4.2. A *fundamental family of pp formulas* for an L-abelian structure A is a family $\mathcal{F}$ of pp formulas such that, in A, every pp formula is equivalent to a conjunction of formulas of the form $\varphi(t(x))$, with $\varphi(w) \in \mathcal{F}$ and $t(x)$ a tuple of L-terms.

Example 4.3. In the simplest possible example of abelian structure, namely an abelian group, the family $\mathcal{F} := \{\exists y\ x = n \cdot y \mid n \in \omega\}$ is always fundamental (see [Hod93, Lemma A.2.1]). In an arbitrary abelian structure, one may always resort to taking as $\mathcal{F}$ the set of all pp formulas, which is trivially fundamental.

Remark 4.4. In an L-abelian structure, each L-term $t(x)$ is built from homomorphisms f_j of abelian groups by taking $\mathbb{Z}$-linear combinations and compositions. Hence, $t(x)$ is itself a homomorphism of abelian groups.

Definition 4.5. Fix a language L of abelian structures.

1. We define the language $L_{\mathrm{abc}} := L_{\mathrm{a}} \cup L_{\mathrm{b}} \cup L_{\mathrm{c}} \cup \{\iota_s, \nu_s \mid s \in S\}$, where $L_{\mathrm{a}}, L_{\mathrm{b}}, L_{\mathrm{c}}$ are pairwise disjoint copies of L, with families of sorts $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ respectively, while $\iota_s \colon \mathrm{A}_s \to \mathrm{B}_s$ and $\nu_s \colon \mathrm{B}_s \to \mathrm{C}_s$ are function symbols. We use notations such as $\iota\colon \mathcal{A} \to \mathcal{B}$, $\nu\colon \mathcal{B} \to \mathcal{C}$ with the obvious meaning. An *$\mathcal{A}$-sort* is simply a sort in $\mathcal{A}$, and similarly for other families of sorts. Juxtaposition denotes union, so if we speak e.g. of the *$\mathcal{AC}$-sorts* we mean the collection of all $\mathcal{A}$-sorts and $\mathcal{C}$-sorts. If L is a single-sorted language, we simply write e.g. A in place of $\mathcal{A}$.
2. A *pure short exact sequence of abelian L-structures*, denoted by
$$0 \to \mathcal{A} \xrightarrow{\iota} \mathcal{B} \xrightarrow{\nu} \mathcal{C} \to 0$$
is an L_{abc}-structure where the following hold.[5]

[5] In fact, 2e) follows from the other conditions. Analogously for condition 2d). See [ACGZ20, Lemma 4.14].

   a) $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are L-abelian structures.
   b) $\iota\colon \mathcal{A} \to \mathcal{B}$, $\nu\colon \mathcal{B} \to \mathcal{C}$ are homomorphisms of L-abelian structures.
   c) ι is injective, ν is surjective, and $\operatorname{im}(\iota) = \ker(\nu)$.
   d) $\varphi(\mathcal{A}) = \iota^{-1}(\varphi(\mathcal{B}))$ holds for each pp L-formula φ.
   e) $\varphi(\mathcal{C}) = \nu(\varphi(\mathcal{B}))$ holds for each pp L-formula φ.
3. Given a pure short exact sequence of abelian structures as above, and a family of pp formulas $\mathcal{F}$ fundamental for $\mathcal{B}$, define the following.
   a) For each $\varphi(x) \in \mathcal{F}$, denote by A_x the product of $\mathcal{A}$-sorts corresponding to the tuple x, and let A_φ be the quotient group $\mathrm{A}_x/\varphi(\mathcal{A})$ and $\pi_\varphi\colon \mathrm{A}_x \to \mathrm{A}_\varphi$ the associated projection map.
   b) For each $\varphi(x) \in \mathcal{F}$, define $\rho_\varphi\colon \mathrm{B}_x \to \mathrm{A}_\varphi$ as follows. Set $\rho_\varphi = 0$ outside $\nu^{-1}(\varphi(\mathcal{C}))$. On $\nu^{-1}(\varphi(\mathcal{C})) = \varphi(\mathcal{B}) + \iota(\mathrm{A}_x)$, define it as the composition of the group homomorphisms
   $$\varphi(\mathcal{B}) + \iota(\mathrm{A}_x) \to (\varphi(\mathcal{B}) + \iota(\mathrm{A}_x))/\varphi(\mathcal{B}) \cong \iota(\mathrm{A}_x)/(\varphi(\mathcal{B}) \cap \iota(\mathrm{A}_x)) \cong \mathrm{A}_\varphi$$
   c) The language L_{abcq} is the expansion of L_{abc} by the family of sorts $\mathcal{A}_\mathcal{F} := (\mathrm{A}_\varphi)_{\varphi \in \mathcal{F}}$ and maps ρ_φ, π_φ.
   d) The language L_{acq} is the restriction of L_{abc} to the $\mathcal{A}\mathcal{C}\mathcal{A}_\mathcal{F}$-sorts, with symbols $L_\mathrm{a} \cup L_\mathrm{c} \cup \{\pi_\varphi \mid \varphi \in \mathcal{F}\}$.
   e) A *special term* is one of the form $\nu_s(x)$, for x a variable from a $\mathcal{B}$-sort, or of the form $\rho_\varphi(t(x))$, with x a tuple of variables from the $\mathcal{B}$-sorts and t a tuple of L_b-terms.
4. Let L^*_{ac} be an expansion of $L_\mathrm{a} \cup L_\mathrm{c}$ with the same sorts.
   a) Denote by L^*_{abc}, L^*_{abcq}, L^*_{acq} the corresponding expansions of L_{abc}, L_{abcq}, L_{acq}.
   b) A pure short exact sequence as above, with a fixed fundamental family $\mathcal{F}$ for $\mathcal{B}$ and with arbitrary L^*_{ac}-structure on the $\mathcal{A}\mathcal{C}$-sorts, is viewed as an L^*_{abcq}-structure in the natural way. We call such a structure an *expanded pure short exact sequence* of abelian L-structures.
   c) The reducts of an expanded pure short exact sequence to L^*_{abc} and L^*_{acq} are defined by restricting to the sorts from L_{abc} and L_{acq} respectively.

Example 4.6. A short exact sequence of abelian groups $0 \to \mathrm{A} \to \mathrm{B} \to \mathrm{C} \to 0$ is pure if and only if, for each n, we have $n\mathrm{B} \cap \mathrm{A} = n\mathrm{A}$. This holds, for example, if C is torsion-free, and in particular in the special cases below. We may take as $\mathcal{F}$ that of Example 4.3.

1. Suppose that the expansion L^*_{ac} endows A, C with the structure of ordered abelian groups. Note that one then recovers, definably, an ordered abelian group structure on B, induced by declaring that $\iota(\mathrm{A})$ is convex. Because of this, and of fact that the kernel of a morphism of oags is convex, this setting is equivalent to that of a short exact sequence of oags. This will be used in Section 5, with B an oag and A a suitably chosen convex subgroup. The sorts A_φ coincide with the quotients $\mathrm{A}/n\mathrm{A}$.
2. In Section 6 we will deal, in the valued field context, with the sequence
   $$1 \to \mathrm{k}^\times \to \mathrm{RV} \setminus \{0\} \to \Gamma \to 0$$
   In this case, the extra structure in L^*_{ac} is induced by the field structure on k and the order on Γ. The sorts A_φ are in this case $\mathrm{k}^\times/(\mathrm{k}^\times)^n$.

Without loss of generality, we may assume that, for each variable x from an $\mathcal{A}$-sort A_s, the formula $\varphi := x = 0$ is in $\mathcal{F}$, and identify A_s with $\mathrm{A}_\varphi = \mathrm{A}_s/0\mathrm{A}_s$. In other words, we may and will assume that $\mathcal{A} \subseteq \mathcal{A}_\mathcal{F}$.

Remark 4.7. Since pp formulas commute with cartesian products, every split short exact sequence is pure. Note that, since purity is a first-order property, in order to show that a short exact sequence is pure it is enough to show that some elementarily equivalent structure splits.

Remark 4.8. Even if a short exact sequence splits, it need not do so definably. Note that, in point 4 of Definition 4.5, we are *not* allowed to add a splitting map. If we do, then matters simplify considerably. For instance, if in L^*_{ac} there is no symbol involving $\mathcal{A}$ and $\mathcal{C}$ jointly, after adding a splitting map the short exact sequence becomes interdefinable with the disjoint union of $\mathcal{A}$ and $\mathcal{C}$ and, by Example 2.9, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ decomposes as a product.

Fact 4.9. Let $\varphi(x^{\mathrm{a}}, x^{\mathrm{b}}, x^{\mathrm{c}})$ be an L^*_{abcq}-formula with x^{a} a tuple of variables from the $\mathcal{A}_{\mathcal{F}}$-sorts, while x^{b}, x^{c} are tuples of $\mathcal{B}$-sorts and $\mathcal{C}$-sorts variables respectively. There are an L^*_{acq}-formula ψ and special terms σ_i such that, in the common L^*_{abcq}-theory of all expanded pure short exact sequences, $\varphi(x^{\mathrm{a}}, x^{\mathrm{b}}, x^{\mathrm{c}}) \leftrightarrow \psi(x^{\mathrm{a}}, \sigma_1(x^{\mathrm{b}}), \dots, \sigma_m(x^{\mathrm{b}}), x^{\mathrm{c}})$.

Proof. For φ an L^*_{abc}-formula and x^{a} a tuple of $\mathcal{A}$-sorts variables, this is [ACGZ20, Corollary 4.20]. In general, let y be a tuple of $\mathcal{A}$-sorts variables compatible with writing $x^{\mathrm{a}} = \pi y$, where π is the tuple of projections $(\pi_i \mid i < |x^{\mathrm{a}}|)$ from a suitable cartesian product of $\mathcal{A}$-sorts to the sort of x^{a}_i. By using the natural interpretation of L^*_{abcq} in L^*_{abc}, we find an L^*_{abc}-formula θ with

$$\varphi(x^{\mathrm{a}}, x^{\mathrm{b}}, x^{\mathrm{c}}) \leftrightarrow \exists y \left(\theta(y, x^{\mathrm{b}}, x^{\mathrm{c}}) \wedge \pi(y) = x^{\mathrm{a}}\right)$$

and the conclusion follows easily by applying [ACGZ20, Corollary 4.20] to θ. □

Before using this fact to study domination in short exact sequences let us note one of its immediate consequences.

Corollary 4.10. The L^*_{acq}-reduct is fully embedded. In particular, $\mathcal{A}$ and $\mathcal{C}$ are orthogonal if and only if they are such in the L^*_{acq}-reduct. Even more in particular, if L^*_{ac} does not contain any symbol involving simultaneously $\mathcal{A}$ and $\mathcal{C}$, then the corresponding expansions of $\mathcal{A}$ and $\mathcal{C}$ are fully embedded and orthogonal.

Given Fact 4.9, we could hope that an expanded pure short exact sequence is controlled, domination-wise, by its L^*_{acq}-part. This is indeed true, provided we are allowed to pass to $*$-types. This is a necessity since, in general, there are finite tuples from $\mathcal{B}$ that cannot be domination-equivalent to any finitary tuple from the L^*_{acq}-reduct; see Remark 4.19. Therefore, in what follows, we will freely use types in infinitely many variables. See Subsection 1.4.

Proposition 4.11. Consider an expanded pure short exact sequence of L-abelian structures, let $\mathcal{F}$ be a fundamental family for $\mathcal{B}$, and let $\kappa \geq |L|$ be a small cardinal. There is a family of κ-tuples of definable functions $\{\tau^p \mid p \in S_\kappa(\mathfrak{U})\}$ with the following properties.

1. The domain of the functions in each tuple τ^p is a product of sorts appearing in the sorts of the variables of p, possibly with repetitions.
2. Each τ^p is partitioned as (ρ^p, ν^p) in such a way that:
   a) Each function in ρ^p is either an identity map on one of the A_φ, or has domain a cartesian product of $\mathcal{B}$-sorts and codomain one of the A_φ.
   b) Each function in ν^p is either the identity map on a $\mathcal{C}$-sort, or is one of the ν_s.
3. For each $p \in S_\kappa(\mathfrak{U})$ we have $p \sim_{\mathrm{D}} \tau^p_* p$.
4. For each $p, q \in S^{\mathrm{inv}}_\kappa(\mathfrak{U})$ we have $p \otimes q \sim_{\mathrm{D}} \tau^p_* p \otimes \tau^q_* q$.

Proof. Let $abc \vDash p(x^{\mathrm{a}}, x^{\mathrm{b}}, x^{\mathrm{c}})$, where the tuples a, b, c realise the variables $x^{\mathrm{a}}, x^{\mathrm{b}}, x^{\mathrm{c}}$ of p from the $\mathcal{A}_{\mathcal{F}}$-sorts, $\mathcal{B}$-sorts, and $\mathcal{C}$-sorts respectively. We define the tuples ν^p and ρ^p as follows.

1. For each coordinate in x^{c} of sort C_s, put in ν^p the corresponding identity map on C_s.
2. For each coordinate in x^{b} of sort B_s, put in ν^p the corresponding map $\nu_s \colon \mathrm{B}_s \to \mathrm{C}_s$.
3. For each coordinate in x^{a} of sort A_φ, put in ρ^p the corresponding identity map on A_φ.
4. For each finite tuple of L_{b}-terms $t(x^{\mathrm{b}}, w)$ and $\varphi \in \mathcal{F}$, if there is $d \in \mathfrak{U}$ such that $p \vdash t(x^{\mathrm{b}}, 0) - d \in \nu^{-1}(\varphi(\mathcal{C}))$, choose such a d, call it $d_{p,\varphi,t}$, and put in ρ^p the map $\rho_\varphi(t(x^{\mathrm{b}}, 0) - d_{p,\varphi,t})$.

Let τ^p be the concatenation of ρ^p and ν^p, let $q(y) := \tau^p_* p(x)$, let D_p be the set of all $d_{p,\varphi,t}$ as above, and let $r(x, y) \in S_{pq}(D_p)$ be be the small type saying "$q(y) = \tau_* p(x)$". Clearly $p \cup r \vdash q$, and we show below that $q \cup r \vdash p$.

By Fact 4.9, we only need to show that $q \cup r$ recovers the formulas

$$\varphi(x^{\mathrm{a}}, d^{\mathrm{a}}, \sigma_1(x^{\mathrm{b}}, d^1), \dots, \sigma_m(x^{\mathrm{b}}, d^m), x^{\mathrm{c}}, d^{\mathrm{c}})$$

implied by p, where the σ_i are special terms, φ is an $\mathcal{L}^*_{\mathrm{acq}}$-formula, and the d^- are tuples of parameters from the appropriate sorts of $\mathfrak{U}$. Let us say that $q \cup r$ *has access to* the term (with parameters) $\sigma(x^{\mathrm{b}}, d)$ iff for some $\mathfrak{U}$-definable function f we have $q(y) \cup r(x, y) \vdash f(y) = \sigma(x^{\mathrm{b}}, d)$. We show below that $q \cup r$ has access to all special terms with parameters, which clearly implies that $q \cup r \vdash p$.

By construction, $q \cup r$ has access to each $\nu_s(x^{\mathrm{b}}_i)$. Because ν is a homomorphism of L-structures, $q \cup r$ also has access to each $\nu(t_0(x^{\mathrm{b}}, d))$, for t_0 an L_{b}-term. In particular, $q \cup r$ decides whether a given tuple $t(x^{\mathrm{b}}, d)$ of L_{b}-terms is in $\nu^{-1}(\varphi(\mathcal{C}))$ or not. If not, then $q \cup r$ entails $\rho_\varphi(t(x^{\mathrm{b}}, d)) = 0$.

Suppose instead that $q \cup r \vdash t(x^{\mathrm{b}}, d) \in \nu^{-1}(\varphi(\mathcal{C}))$. By Remark 4.4, we have $t(x^{\mathrm{b}}, d) = t(x^{\mathrm{b}}, 0) + t(0, d)$, and by construction and the fact that p is consistent with $q \cup r$ we have that p entails $t(x^{\mathrm{b}}, 0) - d_{p,\varphi,t} \in \nu^{-1}(\varphi(\mathcal{C}))$. Note that this formula is over D_p, hence is in r. It follows that

$$q \cup r \vdash t(0, d) + d_{p,\varphi,t} = t(x^{\mathrm{b}}, 0) + t(0, d) - (t(x^{\mathrm{b}}, 0) - d_{p,\varphi,t}) \in \nu^{-1}(\varphi(\mathcal{C}))$$

But $t(0, d) + d_{p,\varphi,t} \in \mathfrak{U}$, and ρ_φ is a homomorphism of L-structures when restricted to $\nu^{-1}(\varphi(\mathcal{C}))$. Because of this, and because $q \cup r$ has access, by construction, to $\rho_\varphi(t(x^{\mathrm{b}}, 0) - d_{p,\varphi,t})$, it also has access to $\rho_\varphi(t(x^{\mathrm{b}}, 0) - d_{p,\varphi,t}) + \rho_\varphi(t(0, d) + d_{p,\varphi,t}) = \rho_\varphi(t(x^{\mathrm{b}}, d))$. Therefore $q \cup r$ has access to all special terms with parameters, hence $q \sim_{\mathrm{D}} p$ and we are only left to prove point 4.

By definition of $\otimes$, if $p(x) \otimes q(y) \vdash t(x, y, d) \in \nu^{-1}(\varphi(\mathcal{C}))$, then there is $\tilde{b} \in \mathfrak{U}$ with $p(x) \vdash t(x, \tilde{b}, d) \in \nu^{-1}(\varphi(\mathcal{C}))$. Hence, by arguing as above, $p \vdash t(x, 0, 0) - d_{p,\varphi,t} \in \nu^{-1}(\varphi(\mathcal{C}))$. So

$$p(x) \otimes q(y) \vdash \nu^{-1}(\varphi(\mathcal{C})) \ni t(x, y, d) - t(x, 0, 0) + d_{p,\varphi,t} = t(0, y, 0) + t(0, 0, d) + d_{p,\varphi,t}$$

and because $t(0, 0, d) + d_{p,\varphi,t} \in \mathfrak{U}$, by construction we have $q(y) \vdash t(0, y, 0) - d_{q,\varphi,t} \in \nu^{-1}(\varphi(\mathcal{C}))$. Similar arguments as above show that, in order to have access to $\rho_\varphi(t(x, y, d))$, it is enough to have access to $\rho_\varphi(t(x, 0, 0) - d_{p,\varphi,t})$ and to $\rho_\varphi(t(0, y, 0) - d_{q,\varphi,t})$, and the conclusion follows. ☐

Corollary 4.12 (Theorem C)**.** Suppose that $\mathfrak{U}$ is an expanded pure short exact sequences of L-abelian structures and $\kappa \geq |L|$ is a small cardinal.

1. There is an isomorphism of posets $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U} \restriction L^*_{\mathrm{acq}})$.
2. If $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathfrak{U} \restriction L^*_{\mathrm{acq}}$, then the same is true in $\mathfrak{U}$, and the above is also an isomorphism of monoids.
3. If $\mathcal{A}$ and $\mathcal{C}$ are orthogonal, then there is an isomorphism of posets
$$\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathcal{A}_{\mathcal{F}}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}_\kappa(\mathcal{C}(\mathfrak{U}))$$
Moreover, if $\otimes$ respects $\geq_{\mathrm{D}}$ in both $\mathcal{A}_{\mathcal{F}}(\mathfrak{U})$ and $\mathcal{C}(\mathfrak{U})$, then the same is true in $\mathfrak{U}$, and the above is also an isomorphism of monoids.

Proof.

1. By Fact 1.5 we have an embedding of posets $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U} \restriction L^*_{\mathrm{acq}}) \hookrightarrow \widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$. This embedding is surjective by Proposition 4.11, its inverse being induced by the maps τ.
2. By Proposition 4.11 we may apply Proposition 1.6 to the family of sorts $\mathcal{A}_{\mathcal{F}}\mathcal{C}$.
3. By combining the previous point with Corollary 2.8. ☐

Remark 4.13. The reader may find in [ACGZ20, Section 4] some variants of Fact 4.9 for settings such as abelian groups augmented by an absorbing element. These in turn yield variants of Proposition 4.11 and its consequences, with no significant difference in the proofs.

When specialised to abelian groups, the results above enjoy a form of local finiteness.

Notation 4.14. From now until the end of the section, L is just the language of abelian groups, and $\mathcal{F}$ the family of formulas $\{\exists y\ x = n \cdot y \mid n \in \omega\}$. We will write for instance $\rho_n \colon \mathrm{B} \to \mathrm{A}/n\mathrm{A}$ in place of $\rho_\varphi \colon \mathrm{B} \to \mathrm{A}_\varphi$, and identify A with A/0A for notational convenience.

Definition 4.15. A $*$-type $p(x)$ is *locally finitary* iff x has finitely many coordinates of each sort.

Proposition 4.16. Let L be the language of abelian groups, and consider a pure short exact sequence equipped with an L^*_{abcq}-structure. Let $p(x)$ be a locally finitary global type. Then, in Proposition 4.11, we may choose τ^p in such a way that $\tau^p_* p$ is locally finitary.

Proof. Write $p(x) = p(x^{\mathrm{a}}, x^{\mathrm{b}}, x^{\mathrm{c}})$ as in the proof of Proposition 4.11, and recall that a term in the language of abelian groups is just a $\mathbb{Z}$-linear combination. If $k = (k_i)_{i<|x^{\mathrm{b}}|} \in \mathbb{Z}^{|x^{\mathrm{b}}|}$, denote $k \cdot x^{\mathrm{b}} \coloneqq \sum_{i<|x^{\mathrm{b}}|} k_i x^{\mathrm{b}}_i$. For each $n \in \omega$, consider the set

$$K^p_n \coloneqq \{k \in \mathbb{Z}^{|x^{\mathrm{b}}|} \mid \exists d \in \mathrm{B}(\mathfrak{U})\ p \vdash k \cdot x^{\mathrm{b}} - d \in \nu^{-1}(n\mathrm{C})\}$$

It is easy to see that K^p_n is a subgroup of $\mathbb{Z}^{|x^{\mathrm{b}}|}$, hence is finitely generated, say by $k^n_0, \dots, k^n_{m(n)}$. Choose witnesses $d_{p,n,i}$ of the fact that $k^n_i \in K^p_n$. Proceed as in the proof of Proposition 4.11 but, instead of putting in ρ^p each $\rho_\varphi(t(x^{\mathrm{b}}, 0) - d_{p,\varphi,t})$, just ensure that ρ^p extends the tuple

$$(\rho_n(k^n_i \cdot x^{\mathrm{b}} - d_{p,n,i}))_{n\in\omega, i\le m(n)}$$

So τ^p consists of the tuple above, a finite tuple of identity maps on sorts $\mathrm{A}/n\mathrm{A}$ or C, and finitely many copies of ν hence, in its codomain, each sort appears only finitely many times. Therefore, $\tau^p_* p$ is locally finitary.

The proof of Proposition 4.11 now goes through, with a pair of modifications which we now spell out. The first one concerns proving access to each $\rho_n(t(x^{\mathrm{b}}, d))$. Fix n and $t(x^{\mathrm{b}}, d)$. Without loss of generality d is a singleton and $t(x^{\mathrm{b}}, d) = \ell \cdot x^{\mathrm{b}} - d$. If $p \vdash t(x^{\mathrm{b}}, d) \in \nu^{-1}(n\mathrm{C})$, by definition we have $\ell \in K^p_n$, so we may write $\ell = \sum_{i\le m(n)} e_i k^n_i$ for suitable $e_i \in \mathbb{Z}$. This allows us to rewrite

$$t(x^{\mathrm{b}}, d) = \ell \cdot x^{\mathrm{b}} - d = \left(\sum_{i\le m(n)} e_i k^n_i\right) \cdot x^{\mathrm{b}} - d = \sum_{i\le m(n)} e_i (k^n_i \cdot x^{\mathrm{b}} - d_{p,n,i}) + \sum_{i\le m(n)} e_i d_{p,n,i} - d$$

Since $\ell \cdot x^{\mathrm{b}} - d$ and all $k^n_i \cdot x^{\mathrm{b}} - d_{p,n,i}$ are in $\nu^{-1}(n\mathrm{C})$, so is $\sum_{i\le m(n)} e_i d_{p,n,i} - d$. Since $\rho_n \restriction \nu^{-1}(n\mathrm{C})$ is a homomorphism and $\sum_{i\le m(n)} e_i d_{p,n,i} - d \in \mathfrak{U}$, we have that $q \cup r$ has access to $\rho_n(t(x^{\mathrm{b}}, d))$.

The only remaining details concern point 4 of Proposition 4.11, and boil down to proving $K^{p\otimes q}_n = K^p_n \times K^q_n$, where we identify e.g. K^p_n with $K^p_n \times \{0\}$. By construction $K^p_n \cap K^q_n = \{0\}$, so we only need to show generation. Suppose that $(k, \ell) \in K^{p\otimes q}_n$, i.e. there is $d \in \mathrm{B}(\mathfrak{U})$ such that $p(x) \otimes q(y) \vdash k \cdot x + \ell \cdot y - d \in \nu^{-1}(n\mathrm{C})$. By definition of $\otimes$, there is $\tilde{b} \in \mathrm{B}(\mathfrak{U})$ with $p(x) \vdash k \cdot x + \ell \cdot \tilde{b} - d \in \nu^{-1}(n\mathrm{C})$. In particular, $k \in K^p_n$. Moreover, $p(x) \otimes q(y) \vdash k \cdot x + \ell \cdot y - d - k \cdot x - \ell \cdot \tilde{b} + d \in \nu^{-1}(n\mathrm{C})$, so $q(y) \vdash \ell \cdot y - (\ell \cdot \tilde{b} - d) \in \nu^{-1}(n\mathrm{C})$, hence $\ell \in K^q_n$, as required. □

Remark 4.17. In the case of abelian groups, we therefore have an analogue of Corollary 4.12 where κ-types are replaced by locally finitary ω-types.

We leave to the reader the easy task to explicitly state the analogue mentioned above (and to define the correct analogue of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$), but let us point out the following special case.

Corollary 4.18. Let $\mathfrak{U}$ be an expanded pure short exact sequences of abelian groups where, for all $n > 0$, the sort $\mathrm{A}/n\mathrm{A}$ is finite. If A and C are orthogonal, there is an isomorphism of posets

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathrm{A}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\mathrm{C}(\mathfrak{U})) \tag{2}$$

If $\otimes$ respects $\ge_{\mathrm{D}}$ in A and C, then $\otimes$ respects $\ge_{\mathrm{D}}$, and the above is an isomorphism of monoids.

Proof. Use Proposition 4.16 and observe that for each p we may replace τ^p by its composition with the projection on the nonrealised coordinates of $\tau^p_* p$ and still have the same results. If for all positive n the sort $\mathrm{A}/n\mathrm{A}$ is finite, and p is a finitary type, this yields another finitary type. The conclusion now follows as in the proof of Corollary 4.12. □

Remark 4.19. The product decomposition fails in general if we insist on using only A instead of all the $\mathrm{A}/n\mathrm{A}$. For example, let A be a regular oag divisible by all primes except 2, and with $[\mathrm{A} : 2\mathrm{A}]$ infinite, and let $\mathrm{C} \models \mathsf{DOAG}$. Consider the expanded short exact sequence $0 \to \mathrm{A} \to \mathrm{B} \to \mathrm{C} \to 0$. As pointed out in Example 4.6, the sequence induces a group ordering on B. Consider the type $p(y)$ concentrating on B, at $+\infty$, and in a new coset modulo 2B. If q is any nonrealised 1-type of an element of sort A divisible by all n, then $p \perp^{\mathrm{w}} q$. It follows that p cannot dominate any nonrealised type in a cartesian power of A: such a type must have a coordinate in a nonrealised cut, and hence dominate a type q as above. So, if we had a product decomposition as in (2), then p would be domination-equivalent to a type in a cartesian power of C. This is a contradiction, because C is orthogonal to $(\mathrm{A}/n\mathrm{A})_{n<\omega}$, while p dominates a nonrealised type in $\mathrm{A}/2\mathrm{A}$.

Remark 4.20. Analogously, one sees that ω-types are a necessity: let A be a regular oag with each $[\mathrm{A} : n\mathrm{A}]$ infinite, $\mathrm{C} \models \mathsf{DOAG}$, and take as $p \in S_{\mathrm{B}}(\mathfrak{U})$ the type at $+\infty$ in a new coset of each $n\mathrm{A}$. For each $n > 1$, there is a nonrealised 1-type q_n of sort $\mathrm{A}/n\mathrm{A}$ such that $p \geq_{\mathrm{D}} q_n$. One shows that the only way for a finitary type in $((\mathrm{A}/n\mathrm{A})_{n\in\omega}, \mathrm{C})$ to dominate all of the q_n is to have a nonrealised coordinate in the sort A, hence to dominate a type orthogonal to p.

5. Finitely many definable convex subgroups

The results of the previous two sections may be combined to describe domination in oags with only finitely many definable convex subgroups. While we are primarily interested in the case where such subgroups are already L_{oag}-definable, our proofs work also if the subgroups are definable "by fiat" using additional unary predicates, hence we work in this more general setting.

Assumption 5.1. In this section we let G be an oag equipped with extra unary predicates $H_0, \ldots, H_s$. We assume that each H_i defines a convex subgroup, that $H_i \subsetneq H_{i+1}$, that $H_0 = \{0\}$, that $H_s = G$, and that the H_i exhaust the list of definable convex subgroups of G.

So the situation is $0 = H_0 \subsetneq H_1 \subsetneq \ldots \subsetneq H_{s-1} \subsetneq H_s = G$. Since the H_i are convex, each G/H_i is still an oag, and in particular torsion free; hence, for each $1 \leq i < s$, the short exact sequence $0 \to H_i/H_{i-1} \to G/H_{i-1} \to G/H_i \to 0$ is pure. As pointed out in Example 4.6, since the order on G/H_{i-1} is definable from the orders on H_i/H_{i-1} and G/H_i, this is (interdefinable with) an expanded pure short exact sequence of abelian groups.

Lemma 5.2. All the oags H_{i+1}/H_i are regular.

Proof. By Fact 3.2 it suffices to check that H_{i+1}/H_i has no proper nontrivial definable convex subgroups. Any such subgroup would yield one in G, against Assumption 5.1. □

Definition 5.3. We work with the following sorts.

1. For $0 \leq i < s$ we have a sort S_i for G/H_i carrying the language of oags together with predicates for H_j/H_i, for $i < j < s$.
2. For $1 \leq i < s$ and $n \in \omega$ we also have sorts $\mathrm{Q}_{i,n}$ for $H_i/(nH_i + H_{i-1})$ carrying the language of abelian groups. We denote by $\mathcal{Q}_i$ the family of sorts $(\mathrm{Q}_{i,n})_{n<\omega}$.

We add to our language the canonical projection and inclusion maps together with, for each $n \in \omega$ and $1 \leq i \leq s-1$, the maps $\rho_{n,i} \colon \mathrm{S}_{i-1} \to \mathrm{Q}_{i,n}$ as in Notation 4.14, relative to the short exact sequence $0 \to \mathrm{Q}_{i,0} \to \mathrm{S}_{i-1} \to \mathrm{S}_i \to 0$.

Lemma 5.4. For each i the following statements hold.

1. The family of sorts $\mathcal{Q}_i$ is fully embedded.
2. The sort S_i is fully embedded.
3. The family of sorts $\mathcal{Q}_i$ is orthogonal to S_i.
4. For each $j \neq i$, the families $\mathcal{Q}_i$ and $\mathcal{Q}_j$ are orthogonal.

Proof. By Corollary 4.10 and point 5 of Remark 2.2. □

Theorem 5.5. Let G be as in Assumption 5.1, viewed as a structure in the language of Definition 5.3, and let κ be a small infinite cardinal. Then $\otimes$ respects $\geq_{\mathrm{D}}$, and

$$\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathrm{S}_{s-1}(\mathfrak{U})) \times \prod_{i=1}^{s-1} \widetilde{\mathrm{Inv}}_\kappa(\mathcal{Q}_i(\mathfrak{U}))$$

Proof. By the lemmas above, Corollary 4.12, Corollary 3.40, and induction. □

If the H_i are already definable in L_{oag},[6] a result of Mariana Vicaría [Vic21] yields weak elimination of imaginaries in the language with sorts $\mathrm{S}_i/n\mathrm{S}_i$ for $0 \leq i < s$ and $n \in \omega$. The special cases of $\mathbb{Z}^n$ and $\mathbb{Z}^n \times \mathbb{Q}$ with the lexicographic ordering have been independently obtained by Martina Liccardo in [Lic21] in a slightly different language.

Corollary 5.6. Let G be a pure oag with only finitely many definable convex subgroups $0 = H_0 \subsetneq H_1 \subsetneq \ldots \subsetneq H_{s-1} \subsetneq H_s = G$, and κ a small infinite cardinal. Then $\otimes$ respects $\geq_{\mathrm{D}}$, and

$$\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}^{\mathrm{eq}}) \cong \prod_{i=1}^{s} \widetilde{\mathrm{Inv}}_\kappa(\mathcal{Q}_i(\mathfrak{U}))$$

Proof Sketch. After adding the sorts from Vicaría's result and, for $1 \leq i \leq s$ and $n \in \omega$, the $\mathrm{Q}_{i,n}$ (note that the sorts $\mathrm{Q}_{s,n} = \mathrm{S}_{s-1}/n\mathrm{S}_{s-1}$ were not in our previous language), the family of sorts used in Theorem 5.5 is fully embedded, and so are the short exact sequences $0 \to \mathrm{Q}_{i,n} \to \mathrm{S}_{i-1}/n\mathrm{S}_{i-1} \to \mathrm{S}_i/n\mathrm{S}_i \to 0$, to which Corollary 4.12 may be applied. From this, we obtain an embedding $\prod_{i=1}^{s} \widetilde{\mathrm{Inv}}_\kappa(\mathcal{Q}_i(\mathfrak{U})) \hookrightarrow \widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}^{\mathrm{eq}})$. We leave to the reader to check surjectivity and transfer of compatibility of $\otimes$ and $\geq_{\mathrm{D}}$, by showing that every $*$-type is dominated by its image among a suitable tuple of definable maps. □

We believe that Vicaría's proof, hence also that of the previous corollary, should go through also in the case where the H_i are explicitly named by predicates, i.e. not necessarily L_{oag}-definable.

Remark 5.7. It follows from Theorem 5.5 and Corollary 3.40 that, if all the G/nG are finite, then every element of $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ is idempotent.

Remark 5.8. It is possible to obtain a suitable version of Theorem 5.5 for locally finitary types. As we saw in Remark 4.19, for these objects the product decomposition fails in general. In fact, it is possible to build examples where there are 1-types in the home sort S_0 whose "cut part" is for example in S_{j-1}, but with "coset parts" in sorts $\mathrm{Q}_{i,n}$ for multiple $i \leq j$.

6. Benign valued fields

In this section we prove, under suitable assumptions, the existence of an isomorphism $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathcal{RV}(\mathfrak{U}))$, where $\mathcal{RV}$ is a certain expanded pure short exact sequence associated to a valued field. We will take an axiomatic approach which, in the next sections, will allow us to generalise our results to wider settings with minimal modifications. In a nutshell, we will assume that T is a complete $\mathcal{RV}$-expansion of a theory of henselian valued fields with elimination of K-quantifiers

[6] Oags with finitely many definable convex subgroups are known as *polyregular*. Note that every H_i must be fixed by every automorphism, and is therefore $\emptyset$-definable.

and "enough maximal saturated models"[7] (see below for the precise definitions). In particular, our results hold in every *benign* valued field in the sense of [Tou18][8], namely, henselian valued fields which are either: of equicharacteristic 0, or algebraically maximal Kaplansky.

We briefly recall the definition and main properties of the *leading term structure* RV, and refer the reader to [Fle11] for further details. To a valued field K one may associate the short exact sequence $1 \to \mathrm{k}^\times \to \mathrm{K}^\times/(1+\mathfrak{m}) \to \Gamma \to 0$. After adding absorbing elements $0, 0, \infty$ to the three middle terms, this may be viewed as a short exact sequence of abelian monoids $1 \to \mathrm{k} \to \mathrm{K}/(1+\mathfrak{m}) \to \Gamma \cup \{\infty\} \to 0$. As pointed out in Remark 4.13, we may harmlessly conflate the two settings. The middle term $\mathrm{K}/(1+\mathfrak{m})$ is called the *leading term structure* RV, and comes with a natural map $\mathrm{rv}\colon \mathrm{K} \to \mathrm{K}/(1+\mathfrak{m}) = \mathrm{RV}$ through which the valuation $v\colon \mathrm{K} \to \Gamma \cup \{\infty\}$ factors. It is common to abuse the notation and still denote by Γ the monoid $\Gamma \cup \{\infty\}$, and by $v\colon \mathrm{RV} \to \Gamma$ the map in the short exact sequence above.

Besides the structure of a (multiplicatively written) monoid, RV is equipped with a "partially defined sum". More precisely, it is equipped with a ternary relation $\oplus(x_0, x_1, x_2)$, defined as

$$\oplus(x_0, x_1, x_2) \overset{\text{def}}{\iff} \exists y_0, y_1, y_2 \in \mathrm{K} \left(y_2 = y_0 + y_1 \wedge \bigwedge_{i<3} \mathrm{rv}(y_i) = x_i \right)$$

When there is a unique x_2 such that $\oplus(x_0, x_1, x_2)$, we write $x_0 \oplus x_1 = x_2$, and say that $x_0 \oplus x_1$ *is well-defined*. It turns out that $\mathrm{rv}(x) \oplus \mathrm{rv}(y)$ is well-defined if and only if $v(x+y) = \min\{v(x), v(y)\}$. If we say that $\bigoplus_{i<\ell} x_i$ *is well-defined*, we mean that, regardless of the choice of parentheses and order of the summands, the "sum" is well-defined and always yields the same result.

Let $\mathcal{RV}$ be the expansion of the short exact sequence $1 \to \mathrm{k} \xrightarrow{\iota} \mathrm{RV} \xrightarrow{v} \Gamma \to 0$ by the field structure on k and the order on Γ.

Remark 6.1. The structure $\mathcal{RV}$ induces an expansion of RV, which turns out to be precisely that given by multiplication and the ternary relation $\oplus$ (see e.g. [ACGZ20, Lemma 5.17]).

This expansion is biinterpretable with $\mathcal{RV}$, and can be axiomatised independently, see [Tou18, Appendix B]. Hence, we may view RV as a standalone structure $(\mathrm{RV}, \cdot, \oplus)$, and as such it is fully embedded in the structure $(\mathrm{K}, \mathrm{RV}, \mathrm{rv})$, and in the expanded pure short exact sequence $\mathcal{RV}$.

Remark 6.2. By the Five Lemma, an extension of valued fields is immediate, i.e. does not change k nor Γ, if and only if it does not change $\mathcal{RV}$.

Definition 6.3. Let L be a language as follows.

1. The sorts are $\mathrm{K}, \mathrm{k}, \mathrm{RV}, \Gamma$.
2. There are function symbols $\mathrm{rv}\colon \mathrm{K} \to \mathrm{RV}$, $\iota\colon \mathrm{k} \to \mathrm{RV}$, $v\colon \mathrm{RV} \to \Gamma$.
3. K and k carry disjoint copies of the language of rings.
4. $\Gamma = \Gamma \cup \{\infty\}$ carries the (additive) language of ordered groups, together with an absorbing element ∞ and an extra constant symbol $v(\mathrm{Char}(\mathrm{K}))$.
5. RV carries the (multiplicative) language of groups, together with an absorbing element 0 and a ternary relation symbol $\oplus$.
6. We denote by $\mathcal{RV}$ the reduct to the sorts $\mathrm{k}, \mathrm{RV}, \Gamma$.
7. There may be other arbitrary symbols on $\mathcal{RV}$, i.e., as long as they do not involve K.

We say that T is an *$\mathcal{RV}$-expansion* of a theory T' of valued fields, iff T is a complete L-theory, $\mathrm{K} \models T'$, and the sorts and symbols from (1)–(5) above are interpreted in the natural way.

[7] The latter follows if maximal immediate extensions of models of T are unique and elementary. See Corollary 6.14.

[8] [Tou18, Definition 1.57] allows $\{k\}$-$\{\Gamma\}$-expansions in the definition of benign. Since we are shortly going to allow more general expansions, the difference is immaterial for our purposes.

In the rest of the section, T is a theory as above. For notational simplicity, we will freely confuse the sort k with the image of its embedding ι in RV.

Remark 6.4. Angular components factor through the map rv, yielding a splitting of $\mathcal{RV}$. Therefore, the Denef–Pas language (and in fact each of its $\{\mathrm{k},\Gamma\}$-expansions[9]) may be seen as an $\mathcal{RV}$-expansion. Note that in that case $\mathcal{RV}$ is definably isomorphic to $\mathrm{k}\times\Gamma$.

Fact 6.5. Fix a language L as in Definition 6.3 and a prime $\mathfrak{p}$. Each of the following incomplete theories eliminates K-sorted quantifiers.

1. The theory of all $\mathcal{RV}$-expansions of henselian valued fields of residue characteristic 0.
2. The theory of all $\mathcal{RV}$-expansions of algebraically maximal Kaplansky valued fields of residue characteristic $\mathfrak{p}$.

Proof. See [Fle11, Proposition 4.3] for residue characteristic 0 case and [HH19, Corollary A.3] for the algebraically maximal Kaplansky case. □

Remark 6.6. Suppose that T eliminates K-sorted quantifiers. Then every formula is equivalent to one of the form $\varphi(x, \mathrm{rv}(f_0(y)), \dots, \mathrm{rv}(f_m(y)))$, where $\varphi(x, z_0, \dots, z_m)$ is a formula in $\mathcal{RV}$, x and z tuples of $\mathcal{RV}$-variables, y a tuple of K-variables, and the f_i polynomials with integer coefficients. In particular, $\mathcal{RV}$ is fully embedded.

Proof. By inspecting the formulas without K-sort quantifiers and observing that, for example, if y is of sort K then $T \vdash y = 0 \leftrightarrow \mathrm{rv}(y) = 0$. □

Definition 6.7. Let $K_0 \subseteq K_1$ be an extension of valued fields. A basis $(a_i)_i$ of a K_0-vector subspace of K_1 is *separating*[10] iff for all finite tuples d from K_0^ℓ and pairwise distinct indices i_j,

$$v\left(\sum_{j<\ell} d_j a_{i_j}\right) = \min_{j<\ell}\bigl(v(d_j) + v(a_{i_j})\bigr)$$

Fact 6.8. A basis $(a_i)_i$ is separating if and only if each sum $\bigoplus_{j<\ell} \mathrm{rv}(d_j)\,\mathrm{rv}(a_{i_j})$ is well-defined. If this is the case, it equals $\mathrm{rv}\left(\sum_{j<\ell} d_j a_{i_j}\right)$.

Lemma 6.9. Let $p \in S^{\mathrm{inv}}_{\mathrm{K}^{\le\omega}}(\mathfrak{U}, M_0)$, let $M_0 \preceq M \prec^+ \mathfrak{U} \subseteq B$, and let $a \vDash p \mid B$. Let $(f_i)_{i\in I}$ be a family of M-definable functions $\mathrm{K}^\omega \to \mathrm{K}$ such that $(f_i(a))_{i\in I}$ is a separating basis of the $\mathrm{K}(M)$-vector space they generate. Assume that one of the following conditions hold:

1. M is $|M_0|^+$-saturated, or
2. p is definable.

Then $(f_i(a))_{i\in I}$ is a separating basis of the $\mathrm{K}(B)$-vector space they generate.

Proof. Towards a contradiction, suppose there are an $L(M)$-formula

$$\varphi(x,w) := v\left(\sum_{i<\ell} w_i f_i(x)\right) > \min_{i<\ell}\{v(w_i) + v(f_i(x))\}$$

and a tuple $d \in B$ such that $a \vDash \varphi(x,d)$. Let H be the set of parameters appearing in $f_0, \dots, f_{k-1}$, i.e. appearing in $\varphi(x,w)$. If M is $|M_0|^+$-saturated, let $\tilde{d} \in M$ be such that $\tilde{d} \equiv_{M_0H} d$, while if p is definable let $\tilde{d} \in M$ be a realisation of $d_p\varphi$. By choice of $\tilde{d}$, we have $a \vDash \varphi(x,\tilde{d})$, contradicting that $(f_i(a))_{i\in I}$ is separating over M. □

[9] A $\{\mathrm{k},\Gamma\}$-expansion is an expansion where the new symbols only involve the sorts k and Γ. Symbols involving both at the same time are allowed; if we want to exclude this possibility, we speak of $\{\mathrm{k}\}$-$\{\Gamma\}$-expansions.

[10] In the literature, the terms *separated basis*, or *valuation basis* are also used. If such a basis exists for every finite dimensional K_0-vector subspace, the extension is called *vs-defectless* or *separated*.

By Lemma 6.9, saturation of M allows to lift separating bases. Since maximality of M guarantees the existence of enough such bases (see Lemma 6.16 below), we give the following definition.

Definition 6.10. We say that T has *enough saturated maximal models* iff for every $\kappa > |L|$, for every $M_0 \vDash T$ of size at most κ there is $M \succ M_0$ of size at most $\beth_2(\kappa)$ which is maximally complete and $|M_0|^+$-saturated.

Remark 6.11. If we restrict the attention to definable types, saturation is not necessary to lift separating bases (cf. Lemma 6.9), and it is enough to assume only "enough maximal models" for weak versions of the results of this section to go through.

The following proposition is folklore; we include a proof for convenience.

Proposition 6.12. Let T be an $\mathcal{RV}$-expansion of a theory of henselian valued fields eliminating K-quantifiers, where every $M \vDash T$ has a unique maximal immediate extension up to isomorphism over M. If $M' \vDash T$ is maximal, $\kappa > |L|$, and $\mathcal{RV}(M')$ is κ-saturated, then M' is κ-saturated.

Proof. We may assume that κ is a successor, hence regular, because if κ is limit then κ-saturation is the same as λ-saturation for all $\lambda < \kappa$. It is enough to prove the following: if $M \equiv M'$ is κ-saturated, then the set $\mathcal{S}$ of partial elementary maps between M and M' with domain of size less than κ has the back-and-forth property. In fact, we only need the "forth" part (and at any rate, the "back" part is true by κ-saturation of M). So assume $f \in \mathcal{S}$, with

$$f\colon A = (\mathrm{K}(A), \mathcal{RV}(A)) \to A' = (\mathrm{K}(A'), \mathcal{RV}(A'))$$

and suppose that $A \subseteq B \subseteq M$, with $|B| < \kappa$. In order to extend f to some $g \in \mathcal{S}$ with domain containing B, consider the following two constructions.

Construction 1: Enlarge A to an elementary substructure. That is, there are $A_1 \supseteq A$ and $f_1\colon A_1 \to A_1'$ extending f such that $f_1 \in \mathcal{S}$ and $A_1 \preceq M$. To do this, we find A_1' with $A' \subseteq A_1' \preceq M'$ and $|A_1'| < \kappa$ using Löwenheim-Skolem, and invoke κ-saturation of M to obtain the desired A_1, f_1.

Construction 2: For a given $\hat{B}$ such that $A \subseteq \hat{B} \subseteq M$ and $|\hat{B}| < \kappa$, enlarge $\mathcal{RV}(A)$ so that it contains $\mathcal{RV}(\hat{B})$. That is, there are $A_1 \supseteq A$ and $f_1\colon A_1 \to A_1'$ extending f such that $f_1 \in \mathcal{S}$ and $\mathcal{RV}(A_1) \supseteq \mathcal{RV}(\hat{B})$. This is done by simply extending f on $\mathcal{RV}$ using κ-saturation of $\mathcal{RV}(M')$, and then using elimination of K-quantifiers to obtain that the extension is still an elementary map.

By repeated applications of the construction above, we find an elementary chain $(M_n)_{n\in\omega}$ of elementary submodels of M, with $A \subseteq M_0$, and $f_n \in \mathcal{S}$ with domain M_n such that $f_0 \supseteq f$, $f_{n+1} \supseteq f_n$, and that if B_n is the structure generated by M_nB then $\mathcal{RV}(B_n) \subseteq \mathcal{RV}(M_{n+1})$. Let $M_\omega \coloneqq \bigcup_{n\in\omega} M_n$ and let $f_\omega \coloneqq \bigcup_{n\in\omega} f_n$. Since κ is regular and uncountable we have $f \in \mathcal{S}$, and by construction the structure B_ω generated by $M_\omega B$ is K-generated and an immediate extension of M_ω. Since M' is maximal and the maximal immediate extension of M_ω is uniquely determined up to M_ω-isomorphism, we may extend f_ω to a map $g \in \mathcal{S}$ with domain $B_\omega \supseteq B$. ☐

Remark 6.13. If k and Γ are orthogonal, then it is enough to assume that $\mathrm{k}(M')$ and $\Gamma(M')$ are κ-saturated. This also applies to the case of $\{\mathrm{k}\}$-$\{\Gamma\}$-expansion of the Denef–Pas language.

Proof. For the first part, use $\aleph_1$-saturation of $\mathrm{k}(M')$ to obtain angular component maps and pass to a $\{\mathrm{k}\}$-$\{\Gamma\}$-expansion of the Denef–Pas language, then observe that saturation is inherited by reducts. The second part is by Remark 6.4. ☐

Corollary 6.14. Suppose that T satisfies the assumptions of Proposition 6.12, and furthermore that every maximal immediate extension of every $M \vDash T$ is an elementary extension. Then T has enough saturated maximal models.

Proof. Given $\kappa > |L|$ and $M_0 \models T$ of size $|M_0| \leq \kappa$, find $M_1 \succ M_0$ which is $|M_0|^+$-saturated of size $|M_1| \leq \beth(|M_0|)$. Let M be a maximal immediate extension of M_1. Then $\mathcal{RV}(M) = \mathcal{RV}(M_1)$, and the latter is $|M_0|^+$-saturated because M_1 is. By assumption, $M \succ M_1$, and by Proposition 6.12 M is $|M_0|^+$-saturated. To conclude, observe that, since by Krull's inequality (see [Dri14, Proposition 3.6]) we have $|K| \leq \mathrm{k}^\Gamma$, we obtain

$$|M| \leq |\mathrm{k}(M)|^{|\Gamma(M)|} = |\mathrm{k}(M_1)|^{|\Gamma(M_1)|} \leq \beth(|M_0|)^{\beth(|M_0|)} = \beth_2(|M_0|) \qquad \square$$

Corollary 6.15. Every $\mathcal{RV}$-expansion of a benign T has enough saturated maximal models.

Proof. Maximal immediate extensions are unique by [Kap42, Theorem 5]. It easy to see that the assumptions of Fact 6.5 are preserved by taking maximal immediate extensions, hence elementarity follows from elimination of K-quantifiers. The conclusion then follows by Corollary 6.14. $\square$

Lemma 6.16. Let $a \models p \in S^{\mathrm{inv}}_{\mathrm{K}^n}(\mathfrak{U}, M_0)$ and $M_0 \prec M \prec^+ \mathfrak{U}$.

1. If M is maximally complete, then there is a sequence $(f_i)_{i<\omega}$ of polynomials in $\mathrm{K}(M)[x]$ such that $\{f_i(a) \mid i < \omega\}$ is a separating basis of $\mathrm{K}(M)[a]$.
2. If M is $|M_0|^+$-saturated then, for each $(f_i)_{i<\omega}$ as above, $\{f_i(a) \mid i < \omega\}$ is a separating basis of $\mathrm{K}(\mathfrak{U})[a]$.
3. If $q \in S_{\mathrm{K}^{<\omega}}(\mathfrak{U})$, $(a, b) \models p \otimes q$, and $(f^p_i(a))_{i<\omega}$, $(f^q_j(b))_{j<\omega}$ are separating bases of $\mathrm{K}(\mathfrak{U})[a]$ and $\mathrm{K}(\mathfrak{U})[b]$, then $(f^p_i(a) \cdot f^q_j(b))_{i<\omega,j<\omega}$ is a separating basis of $\mathrm{K}(\mathfrak{U})[ab]$.

Proof. The first part is by [Bau82, Lemma 3] (and does not require saturation, see also [HHM08, Lemma 12.2]) and the second one by Lemma 6.9 applied to $(f_i)_{i<\omega}$, and we only need to prove the last part. The fact that the tuple $(f^p_i(a) \cdot f^q_j(b))_{i<\omega,j<\omega}$ is linearly independent follows from the definition of $\otimes$, and clearly it generates $\mathrm{K}(\mathfrak{U})[ab]$ as a $\mathrm{K}(\mathfrak{U})$-vector space. Let us check that this basis is separating. Let B be the structure generated by $\mathfrak{U}b$. By Lemma 6.9, $(f^p_i(a))_{i<\omega}$ is a separating basis of the $\mathrm{K}(B)$-vector space $\mathrm{K}(B)[a]$. Therefore we have

$$\begin{aligned} v\Big(\sum_{i,j} d_{ij} f^p_i(a) f^q_j(b)\Big) &= v\Big(\sum_i \Big(\sum_j d_{ij} f^q_j(b)\Big) f^p_i(a)\Big) \\ &= \min_i \Big(v\Big(\sum_j d_{ij} f^q_j(b)\Big) + v(f^p_i(a))\Big) = \min_i \Big(\min_j (v(d_{ij}) + v(f^q_j(b))) + v(f^p_i(a))\Big) \\ &= \min_{i,j} \Big(v(d_{ij}) + v(f^q_j(b)) + v(f^p_i(a))\Big) = \min_{i,j} \Big(v(d_{ij}) + v(f^q_j(b) \cdot f^p_i(a))\Big) \qquad \square \end{aligned}$$

Proposition 6.17. Suppose that T eliminates K-quantifiers and has enough saturated maximal models. For every $p \in S^{\mathrm{inv}}(\mathfrak{U})$ there is $q \in S^{\mathrm{inv}}_{\mathcal{RV}^\omega}(\mathfrak{U})$ such that $p \sim_{\mathrm{D}} q$. More precisely, let $p(x, z) \in S^{\mathrm{inv}}(\mathfrak{U}, M_0)$, where x is a tuple of K-variables and z a tuple of $\mathcal{RV}$-variables. Let $(a, c) \models p(x, z)$, let $M \succ M_0$ be $|M_0|^+$-saturated and maximally complete, and let $(f_i)_{i<\omega}$ be given by Lemma 6.16 applied to a and M. Then p is domination-equivalent to the $*$-type

$$q(y, t) \coloneqq \mathrm{tp}(\mathrm{rv}(f_i(a))_{i<\omega}, c/\mathfrak{U})$$

witnessed by

$$r(x, z, y, t) \coloneqq \mathrm{tp}(a, c, \mathrm{rv}(f_i(a))_{i<\omega}, c/M)$$

Proof. That $p \cup r \vdash q$ is trivial, so let us show $q \cup r \vdash p$.

By Fact 6.5, it is enough to show that $q \cup r$ has access to every $\mathrm{rv}(f(x))$, that is, that for every $f \in \mathrm{K}(\mathfrak{U})[x]$, there is a $\mathfrak{U}$-definable function g such that $q \cup r \vdash \mathrm{rv}(f(x)) = g(y)$. Write $f(x) = \sum_{i<\ell} d_i f_i(x)$. By Fact 6.8, we have $\mathrm{rv}(f(a)) = \bigoplus_{i<\ell} \mathrm{rv}(d_i)\,\mathrm{rv}(f_i(a))$, and we only need to ensure that $q \cup r$ "knows this", i.e. that $q \cup r \vdash \mathrm{rv}(f(x)) = \bigoplus_{i<\ell} \mathrm{rv}(d_i)\,\mathrm{rv}(f_i(x))$. But by Fact 6.8 whether the $(f_i(a))_{i<\omega}$ form a separating basis or not only depends on the type of their images in RV, which is part of q by definition. $\square$

The work done so far is enough to obtain an infinitary version of Theorem B. After stating such a version, we will proceed to finitise it.

Remark 6.18. Separating bases of vector spaces of uncountable dimension are not guaranteed to exist. Nevertheless, a $*$-type version of Lemma 6.16 still holds, with the $f_i(a)$ now enumerating separating bases of all countable dimensional subspaces of $\mathrm{K}(M)[a]$.

Corollary 6.19. If κ is a small infinite cardinal, there is an isomorphism of posets $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathcal{RV}(\mathfrak{U}))$. If $\otimes$ respects $\geq_\mathrm{D}$ on $*$-types in $\mathcal{RV}(\mathfrak{U})$, then the same holds in $\mathfrak{U}$, and the above is also an isomorphism of monoids.

Proof. By the $*$-types versions of Proposition 6.17, Lemma 6.16, and Proposition 1.6. ☐

Lemma 6.20. Let $M_0 \prec^+ M \prec^+ \mathfrak{U}$, let $e \vDash q \in S^{\mathrm{inv}}_{\mathrm{RV}^\omega}(\mathfrak{U}, M_0)$. Let $I \subseteq \omega$ be such that $(v(e_i))_{i\in I}$ generates $\mathbb{Q}\langle\Gamma(\mathfrak{U})v(e)\rangle$ over $\mathbb{Q}\,\Gamma(\mathfrak{U})$ as $\mathbb{Q}$-vector spaces. Let $G \subseteq \mathrm{RV}$ be the closure of $\mathrm{RV}(\mathfrak{U})e$ under the RV product, the corresponding inverse function, and well-defined sums $\oplus$. Let $(g_j)_{j\in J} \subseteq \mathrm{dcl}(\mathfrak{U}e) \cap \mathrm{k}$ be such that $\mathrm{k} \cap G \subseteq \mathrm{acl}(\mathfrak{U}(g_j)_{j\in J})$. Let $b := (e_i, g_j \mid i \in I, j \in J)$. Then there is $M \prec N \prec^+ \mathfrak{U}$ such that e and b are interalgebraic over N.

Proof. Fix $\ell \in \omega \setminus I$. By assumption, there are $n_\ell > 0$, $d_\ell \in \mathfrak{U}$, a finite $I_0 \subseteq I$ and, for $i \in I_0$, integers $n_{\ell,i} \in \mathbb{Z}$, such that

$$n_\ell v(e_\ell) = v(d_\ell) + \sum_{i\in I_0} n_{\ell,i} v(e_i)$$

By M_0-invariance, we may assume that $d_\ell \in M$. Let $h_\ell(x)$ be the M-definable function

$$h_\ell(y) := \frac{y_\ell^{n_\ell}}{d_\ell \prod_{i\in I_0} y_i^{n_{\ell,i}}}$$

By construction, $v(h_\ell(e)) = 0$, hence $h_\ell(e) \in \mathrm{k}^\times$ so by assumption on $(g_j)_{j\in J}$ we have $h_\ell(e) \in \mathrm{acl}(\mathfrak{U}(g_j)_{j\in J})$. Let $N \succ M$ be a small model such that $\{h_\ell(e) \mid \ell \in \omega \setminus I\} \subseteq \mathrm{acl}(N(g_j)_{j\in J})$ and $\{g_j \mid j \in J\} \subseteq \mathrm{dcl}(Ne)$.

By definition of h_ℓ, for each $\ell \in \omega \setminus I$, we therefore have $e_\ell^{n_\ell} \in \mathrm{acl}(Mb)$. Moreover, since Γ is ordered and the kernel of $v\colon \mathrm{RV} \to \Gamma$ is the multiplicative group of a field, RV has finite torsion, hence e_ℓ is algebraic over $e_\ell^{n_\ell}$. It follows that $e \in \mathrm{acl}(Mb)$, and we are done. ☐

Theorem 6.21 (Theorem B)**.** Let T be an $\mathcal{RV}$-expansion of a benign theory of valued fields, or more generally an $\mathcal{RV}$-expansion of a theory of valued fields with enough saturated maximal models eliminating K-quantifiers. There is an isomorphism of posets $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathcal{RV}(\mathfrak{U}))$. If $\otimes$ respects $\geq_\mathrm{D}$ in $\mathcal{RV}(\mathfrak{U})$, then $\otimes$ respects $\geq_\mathrm{D}$ in $\mathfrak{U}$, and the above is an isomorphism of monoids.

Proof. Fix $p(x, z) \in S^{\mathrm{inv}}(\mathfrak{U})$ and $ac \vDash p$, where x is a tuple of K-variables and z a tuple of $\mathcal{RV}$-variables. Let $(f_i)_{i<\omega}$ be given by Lemma 6.16. As usual, denote by $\mathfrak{U}(a)$ the field generated by a over $\mathfrak{U}$. Since $\mathrm{trdeg}(\mathfrak{U}(a)/\mathfrak{U})$ is finite, by the Abhyankar inequality so is $\dim_\mathbb{Q}(\mathbb{Q}\,\Gamma(\mathfrak{U}(a))/\mathbb{Q}\,\Gamma(\mathfrak{U}))$. By rearranging the f_i we may assume that m is such that $v(f_i(a))_{i<m}$ generates $\mathbb{Q}\,\Gamma(\mathfrak{U}(a))$ over $\mathbb{Q}\,\Gamma(\mathfrak{U})$. Again by the Abhyankar inequality, $\mathrm{trdeg}(\mathrm{k}(\mathfrak{U}(a))/\,\mathrm{k}(\mathfrak{U}))$ is finite. Let $(g_j \mid j < n)$ be a transcendence basis of $\mathrm{k}(\mathfrak{U}(a))$ over $\mathrm{k}(\mathfrak{U})$. By choice of the f_j and Fact 6.8, each g_j is in the closure of $\mathrm{RV}(\mathfrak{U})(\mathrm{rv}(f_i(a)))_{i<\omega}$ under the group operation, inverse, and well-defined sums $\oplus$, so we may write $g_j = h_j(a)$ for suitable definable functions h_j. We are now in a position to apply Lemma 6.20 with $e = (\mathrm{rv}(f_i(a)))_{i<\omega}$, the g_j defined above, and $I = \{i \in \omega \mid i < m\}$. Together with Proposition 6.17, we obtain

$$p \sim_\mathrm{D} p' := \mathrm{tp}(\mathrm{rv}(f_i(a))_{i<m}, (h_j(a))_{j<n}, c/\mathfrak{U}) \tag{3}$$

Therefore, every (finitary) type is equivalent to one in $\mathcal{RV}$. By full embeddedness of $\mathcal{RV}$, and Fact 1.5, we obtain the required isomorphism of posets.

By Proposition 1.6 it is enough to show that if p', q' are obtained from p, q as in (3) above, then $p \otimes q \sim_{\mathrm{D}} p' \otimes q'$. Denote by $\rho^p(x,z) := (\mathrm{rv}(f_i^p(x))_{i<m_p}, (h_j^p(x))_{j<n_p}, \mathrm{id}^p(z))$ the tuple of definable functions from (3), and similarly for q and $p \otimes q$. By point 3 of Lemma 6.16 we may take as $(f_i^{p\otimes q})_{i<\omega}$ (a reindexing on ω of) the concatenation of $(f_i^p)_{i<\omega}$ with $(f_i^q)_{i<\omega}$. By the properties of $\otimes$, the concatenation of $(f_i^p(a))_{i<m_p}$ and $(f_i^q(b))_{i<m_q}$ is a basis of the vector space $\mathbb{Q}\langle\Gamma(\mathfrak{U})(v(f_i^p(a)))_{i<\omega}(v(f_i^q(b)))_{i<\omega}\rangle$ over $\mathbb{Q}\,\Gamma(\mathfrak{U})$, and it follows that as $(f_i^{p\otimes q})_{i<m_{p\otimes q}}$ we may take the concatenation of $(f_i^p)_{i<m_p}$ with $(f_i^q)_{i<m_q}$. Similarly, as $(h_j^{p\otimes q})_{j<n_{p\otimes q}}$ and $\mathrm{id}_{p\otimes q}$ we may take the concatenation of the respective tuples for p and q, and ultimately we obtain that as $\rho^{p\otimes q}$ we may take the concatenation of ρ^p with ρ^q. By (3), we have $p \otimes q \sim_{\mathrm{D}} p' \otimes q'$ and we are done. □

In the case of $\{\mathrm{k}, \Gamma\}$-expansions, we are in the setting of Section 4, therefore Corollary 6.19 and Theorem 6.21 may be combined with the results from Section 4. We spell out a particularly nice case (in particular, we assume to be dealing with $\{\mathrm{k}\}$-$\{\Gamma\}$-expansions), and leave to the reader the task of stating the more general versions.

Corollary 6.22 (Theorem A)**.** Let T be a complete $\{\mathrm{k}\}$-$\{\Gamma\}$-expansion of a benign theory of valued fields where, for all $n > 1$, the group $\mathrm{k}^\times/(\mathrm{k}^\times)^n$ is finite. There is an isomorphism of posets

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathrm{k}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U}))$$

If $\otimes$ respects $\geq_{\mathrm{D}}$ in k and Γ, then $\otimes$ respects $\geq_{\mathrm{D}}$, and the above is an isomorphism of monoids.

Proof. Apply Theorem 6.21. By Fact 6.5, if the extra structure on $\mathcal{RV}$ involves only k and Γ, and never both at the same time, then the sorts k and Γ are orthogonal. By Remark 6.1, $\mathcal{RV}$ is an expanded pure short exact sequence, and we may therefore conclude by Corollary 4.18. □

Note that, by full embeddedness, the conclusion remains true if we remove the sort RV and introduce symbols for the valuation $v\colon \mathrm{K} \to \Gamma$ and the modified residue map $\mathrm{Res}\colon \mathrm{K}^2 \to \mathrm{k}$. The algebraically closed and real closed cases of the corollary above (without extra structure on $\mathcal{RV}$) were obtained in [HHM08] and [EHM19] respectively. To see how the results about stable domination, or domination by a family of sorts in the sense of [EHM19, Definition 1.7], can be translated in our context, see [Mena, Section 6].

We leave the following remark, for the benefit of the reader interested in computing the image of the embedding $\widetilde{\mathrm{Inv}}(\mathrm{K}(\mathfrak{U})) \hookrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{U})$.

Remark 6.23. The proof of Theorem 6.21, with trivial changes, yields the analogous statement in the language with only two sorts, K and RV.

7. Mixed characteristic henselian valued fields

In this section, K is a henselian valued field of characteristic $(0, \mathfrak{p})$, for $\mathfrak{p}$ a prime. An analogue of Fact 6.5 holds, in a language which we are now going to describe.

For $n \in \omega$, let $\mathfrak{m}_n := \{x \in \mathrm{K} \mid v(x) > v(\mathfrak{p}^n)\}$. Define RV_n to be the multiplicative monoid $\mathrm{RV}_n := \mathrm{K}/(1 + \mathfrak{m}_n)$, and $\mathrm{RV}_n^\times := \mathrm{RV}_n \setminus \{0\}$. For each n, denote by $\mathrm{rv}_n\colon \mathrm{K} \to \mathrm{RV}_n$ the quotient map. Additionally, for $m > n$, we have natural maps $\mathrm{rv}_{m,n}\colon \mathrm{RV}_m \to \mathrm{RV}_n$. Moreover, the valuation $v\colon \mathrm{K} \to \Gamma$ induces maps $\mathrm{RV}_n \to \Gamma$, which we still denote by v. Let k_n be the kernel of this map, yielding a short exact sequence $1 \to \mathrm{k}_n \to \mathrm{RV}_n \xrightarrow{v} \Gamma \to 0$. We also have ternary relations $\oplus_n$, defined analogously to $\oplus$, and again $x \oplus_n y$ is well-defined precisely when $v(x+y) = \min\{v(x), v(y)\}$. Note that for $n = 0$ we recover the notions from the previous section. Moreover, we have the following more general version of Fact 6.8.

Fact 7.1. A basis $(a_i)_i$ is separating if and only if, for each $n \in \omega$, each sum $\mathrm{rv}_n(d_0)\,\mathrm{rv}_n(a_{i_0}) \oplus_n \ldots \oplus_n \mathrm{rv}_n(d_\ell)\,\mathrm{rv}_n(a_{i_\ell})$ is well-defined, if and only if this happens for $n = 0$. If this is the case, it equals $\mathrm{rv}_n\left(\sum_{j\leq\ell} d_j a_{i_j}\right)$.

Definition 7.2. Let L be a language as follows.

1. We have sorts K, Γ and, for each $n \in \omega$, sorts $\mathrm{k}_n, \mathrm{RV}_n$.
2. There are function symbols $\mathrm{rv}_n \colon \mathrm{K} \to \mathrm{RV}_n$, $\iota \colon \mathrm{k}_n \to \mathrm{RV}_n$, $v \colon \mathrm{RV}_n \to \Gamma$.
3. K carries a copy of the language of rings.
4. $\Gamma = \Gamma \cup \{\infty\}$ carries the (additive) language of ordered groups, together with an absorbing element ∞ and an extra constant symbol $v(\mathrm{Char}(\mathrm{K}))$.
5. Each RV_n and k_n carries the (multiplicative) language of groups, together with a ternary relation symbol $\oplus_n$.
6. We denote by $\mathcal{RV}_*$ the reduct to the sorts $\mathrm{k}_n, \mathrm{RV}_n, \Gamma$.
7. There may be other arbitrary symbols on $\mathcal{RV}_*$, i.e., as long as they do not involve K.

We say that T is an $\mathcal{RV}_*$*-expansion* of a theory T' of valued fields, iff T is a complete L-theory, $\mathrm{K} \vDash T'$, and the sorts and symbols from (1)–(5) above are interpreted in the natural way.

In the rest of the section, we work in a T as above, with T' a theory of henselian valued fields of characteristic $(0, \mathfrak{p})$. For notational simplicity, we will freely confuse the sort k_n with the image of its embedding in RV_n.

Fact 7.3 ([Fle11, Proposition 4.3])**.** $\mathcal{RV}_*$-expansions of theories of henselian valued fields of mixed characteristic eliminate K-quantifiers. In particular, every formula is equivalent to one of the form $\varphi(x, \mathrm{rv}_{n_0}(f_0(y)), \dots, \mathrm{rv}_{n_m}(f_m(y)))$, for $\varphi(x, z_0, \dots, z_m)$ a formula in $\mathcal{RV}_*$, x a tuple of $\mathcal{RV}_*$-variables, z a tuple of $\mathcal{RV}_*$-variables, y a tuple of K-variables, and the f_i polynomials with integer coefficients. In particular, $\mathcal{RV}_*$ is fully embedded.

Proposition 7.4. Suppose that T eliminates K-quantifiers and has enough saturated maximal models. For every $p \in S^{\mathrm{inv}}(\mathfrak{U})$ there is $q \in S^{\mathrm{inv}}_{\mathcal{RV}_*^\omega}(\mathfrak{U})$ such that $p \sim_{\mathrm{D}} q$. More precisely, let $p(x, z) \in S^{\mathrm{inv}}(\mathfrak{U}, M_0)$, where x is a tuple of K-variables and z a tuple of $\mathcal{RV}_*$-variables. Let $(a, c) \vDash p(x, z)$, let $M \succ M_0$ be $|M_0|^+$-saturated and maximally complete, and let $(f_i)_{i<\omega}$ be given by the $*$-type version of Lemma 6.16 applied to a and M (cf. Remark 6.18). Then p is domination-equivalent to $q(y, t) \coloneqq \mathrm{tp}(\mathrm{rv}_n(f_i(a))_{i,n<\omega}, c/\mathfrak{U})$, witnessed by $r(x, z, y, t) \coloneqq \mathrm{tp}(a, c, \mathrm{rv}_n(f_i(a))_{i,n<\omega}, c/M)$. If $\kappa \geq |L|$ is a small cardinal, there is an isomorphism of posets $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathcal{RV}_*(\mathfrak{U}))$. If $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathcal{RV}_*(\mathfrak{U})$, then the same holds in $\mathfrak{U}$, and the above is an isomorphism of monoids.

Proof. Adapt the proofs of Lemma 6.16, Proposition 6.17 and Corollary 6.19, replacing Fact 6.5 and Fact 6.8 by Fact 7.3 and Fact 7.1 respectively. □

The assumptions of Proposition 7.4 are satisfied in a number of cases of interest. Besides the algebraically closed case (where the computation of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is already known from [HHM08]), we note the following.

Remark 7.5. Every $\mathcal{RV}_*$-expansion of a finitely ramified henselian valued field has enough saturated models.

Proof. Finite ramification ensures that immediate extensions are precisely those where $\mathcal{RV}_*$ does not change. Together with Fact 7.3, this ensures that maximal immediate extensions are elementary, and by [Dri14, Corollary 4.29] they are also unique. We may therefore adapt the proof of Proposition 6.12, replacing $\mathcal{RV}$ with $\mathcal{RV}_*$. □

Remark 7.6. $\mathcal{RV}_*$ may be viewed as a short exact sequence of abelian structures, each consisting of an inverse system of abelian groups. It is well known that, in a sufficiently saturated elementary extension, there exists a compatible system of angular components or, in other words, a splitting of this short exact sequence, which is therefore pure. Hence, the results from Section 4 apply to this setting, for example by taking as $\mathcal{F}$ the family of all pp formulas.

If the residue field has elimination of imaginaries, e.g. if it is algebraically closed, we can then get rid of the imaginaries arising from $\mathcal{F}$ and obtain a product decomposition. We state a special case as an example application of the results above.

Corollary 7.7. In the theory of the Witt vectors over $\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}$, the domination monoid is well defined. If X is the set of invariant convex subgroups of $\Gamma(\mathfrak{U})$ and κ is a small infinite cardinal,

$$\widetilde{\operatorname{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\operatorname{Inv}}_\kappa(\mathrm{k}(\mathfrak{U})) \times \widetilde{\operatorname{Inv}}_\kappa(\Gamma(\mathfrak{U})) \cong \hat{\kappa} \times \mathscr{P}_{\le\kappa}(X)$$

Remark 7.8. The product decomposition does not hold for finitary types: using discreteness of the value group, it is possible to build a pro-definable surjection $\mathrm{K} \to \mathrm{k}^\omega$ (see [Tou18, proof of Remark 3.23]), hence a 1-type in K dominating the type of an infinite independent tuple in k.

However, finitisation is possible in the case of the $\mathfrak{p}$-adics.

Corollary 7.9 (Theorem E)**.** Let T be a complete $\{\Gamma\}$-expansion of the theory $\operatorname{Th}(\mathbb{Q}_{\mathfrak{p}})$ of $\mathfrak{p}$-adically closed fields. There is an isomorphism of posets $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \widetilde{\operatorname{Inv}}(\Gamma(\mathfrak{U}))$. If $\otimes$ respects $\ge_{\mathrm{D}}$ in $\Gamma(\mathfrak{U})$, then the same holds in $\mathfrak{U}$, and the above is also an isomorphism of monoids. In particular, in $\operatorname{Th}(\mathbb{Q}_{\mathfrak{p}})$, $\otimes$ respects $\ge_{\mathrm{D}}$ and, if X is the set of invariant convex subgroups of $\Gamma(\mathfrak{U})$, then

$$(\widetilde{\operatorname{Inv}}(\mathfrak{U}), \otimes, \ge_{\mathrm{D}}) \cong (\mathscr{P}_{<\omega}(X), \cup, \supseteq)$$

Proof. By Remark 7.5 we may apply Proposition 7.4. Since each k_n is finite, each RV_n is a finite cover of Γ, and it follows that each type in RV_n is algebraic over a type in Γ. We conclude by using the Abhyankar inequality and Proposition 1.6. The "in particular" part then follows from Corollary 3.38. □

8. D-henselian valued fields with many constants

In this section we deal with certain differential valued fields. Since the proofs are adaptations of those in Section 6, we will give sketches and leave it to the reader to fill in the details.

We let T be a complete theory subject to the following requirements.

1. The sorts are $\mathrm{K}, \mathrm{k}, \Gamma, \mathrm{RV}$, as in Section 6, naturally interpreted. We use the notation $\mathcal{RV}$.
2. k (hence K) has characteristic 0.
3. K and k carry a derivation[11] ∂, commuting with the residue map.
4. K is *monotone*, i.e. $v(\partial x) \ge v(x)$.
5. K has *many constants*[12], i.e. for every $\gamma \in \Gamma$ there is $x \in \mathrm{K}$ with $\partial x = 0$ and $v(x) = \gamma$.
6. K is *D-henselian*, i.e. the following holds. If $P(X) \in \mathcal{O}\{X\} = \mathcal{O}[\partial^i X]_{i\in\omega}$ is a differential polynomial over the valuation ring $\mathcal{O}$, and $a \in \mathcal{O}$ is such that $v(P(a)) > 0$ and for some i we have $v(\mathrm{d}\, P/\, \mathrm{d}(\partial^i X))(a) = 0$, then there is $b \in \mathcal{O}$ such that $P(b) = 0$ and $v(a-b) > 0$.
7. $\mathcal{RV}$ may carry additional structure.

Fact 8.1. The common theory of all the T as above (in a fixed language) eliminates K-quantifiers.

Proof. This is [Sca03, Theorem 6.4 and Corollary 5.8]. See also [ADH17, Corollary 8.3.3]. □

Proposition 8.2. *T has enough saturated maximal models.*

Proof sketch. By [Sca03, Remark 6.2], k is, in the terminology of [ADH17], linearly surjective, so by [ADH17, Theorem 7.4.3] T has uniqueness of maximal immediate extensions. Let N be a maximal immediate extension of M. Then N is monotone by [ADH17, Lemma 6.3.5], D-henselian by [ADH17, Theorem 7.4.3], and clearly has many constants. By elimination of K-quantifiers, $M \prec N$. Therefore, the proofs of Proposition 6.12 and Corollary 6.14 may be adapted. □

[11] We use the same symbol for both derivations.

[12] Here we are following the terminology of [ADH17]. In [Sca03], this condition is called having *enough constants*.

Theorem 8.3. Let κ be a small infinite cardinal. There is an isomorphism of posets $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathcal{RV}(\mathfrak{U}))$. If $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathcal{RV}(\mathfrak{U})$, then the same holds in $\mathfrak{U}$, and the above is also an isomorphism of monoids.

Proof sketch. By elimination of K-quantifiers, $\mathcal{RV}(M)$ is fully embedded in M. If we replace "polynomial" by "differential polynomial", $\mathrm{K}(M)[a]$ by $\mathrm{K}(M)\{a\}$, and so on, in the statements of Lemma 6.16 and Proposition 6.17, essentially the same proofs go through. We can then conclude as in the proof of Corollary 6.19. □

The derivation ∂ on K induces a map ∂_{RV} on RV which, for all $\gamma \in \Gamma$, fixes $v^{-1}(\gamma) \cup \{0\}$ setwise, defined by $\partial_{\mathrm{RV}}(\mathrm{rv}(x)) = \mathrm{rv}(\partial(x))$ iff $v(\partial(x)) = v(x)$, and $\partial_{\mathrm{RV}}(\mathrm{rv}(x)) = 0$ otherwise, which extends the derivation ∂ on k.

Lemma 8.4. ∂_{RV} is definable from the the short exact sequence structure, the differential field structure on k, and a predicate for $C \coloneqq \{c \in \mathrm{RV} \mid \partial_{\mathrm{RV}}(c) = 0\}$.

Proof. Suppose that $a \in \mathrm{RV}$ and $v(a) \notin \{0, \infty\}$. Since K has many constants, there is $c \in \mathrm{RV}(M)$ with $\partial_{\mathrm{RV}}(c) = 0$ and $v(c) = v(a)$. Then $a/c \in \mathrm{k}(\mathfrak{U})$, and we have $\partial_{\mathrm{RV}}(a) = c\partial(a/c)$. Because this does not depend on the choice of c, the function $y = \partial_{\mathrm{RV}}(x)$ is $\emptyset$-definable by the formula

$$\varphi(x, y) \coloneqq \exists z \in C \left(\big(v(z) = v(x)\big) \wedge \big(y = z\partial(x/z)\big)\right)$$
□

If our language had a section of the valuation, or an angular component compatible with ∂, we could recover C from the constant field of k, and conclude by using (the $*$-types version of) Remark 4.8. Yet, the absence of definable splitting is not a serious obstacle. For simplicity, we only give a result in the model companion $\mathsf{VDF}_{\mathcal{EC}}$, related to [Rid19, Proposition 3.2].

Theorem 8.5 (Theorem F)**.** In $\mathsf{VDF}_{\mathcal{EC}}$, for every small infinite cardinal κ, the monoid $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ is well-defined, and we have isomorphisms

$$\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathrm{k}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}_\kappa(\Gamma(\mathfrak{U})) \cong \prod_{\delta(\mathfrak{U})}^{\leq\kappa} \hat{\kappa} \times \mathscr{P}_{\leq\kappa}(X)$$

where X is the set of invariant convex subgroups of $\Gamma(\mathfrak{U})$, $\delta(\mathfrak{U})$ is a cardinal, and $\prod_{\delta(\mathfrak{U})}^{\leq\kappa} \hat{\kappa}$ denotes the submonoid of $\prod_{\delta(\mathfrak{U})} \hat{\kappa}$ consisting of $\delta(\mathfrak{U})$-sequences with support of size at most κ.

Proof. By Theorem 8.3 we reduce to $\mathcal{RV}$. Let $L_C \coloneqq L_{\mathrm{ab}} \cup \{C\}$, with C a unary predicate. Expand the language of $\mathcal{RV}$ by a predicate C on each sort, interpreted as the constants in both k and RV and as the full Γ in Γ, obtaining a short exact sequence of L_C-abelian structures[13], expanded by the differential field structure on k and the order on Γ. By virtue of Lemma 8.4, we may apply the material from Section 4, say by taking as a fundamental family that of all pp L_C-formulas, provided we show that $\mathcal{RV}$ is pure. If $M \vDash \mathrm{VDF}_{\mathcal{EC}}$ is $\aleph_1$-saturated then, since M has many constants, we may find a section $s\colon \Gamma(M) \to \mathrm{RV}(M)$ of the valuation with image included in $C(\mathrm{RV}(M))$. Hence $\mathcal{RV}(M)$ splits as a short exact sequence of L_C-abelian structures, so purity follows by Remark 4.7. Since k is a model of DCF_0, which eliminates imaginaries, we may get rid of the auxiliary sorts A_φ. We conclude by Corollary 3.40 and the fact that DCF_0 is ω-stable multidimensional (see [Men20b, Section 5] for the relation between our setting and that of domination via forking independence in stable theories). □

Even in the existentially closed case $\mathsf{VDF}_{\mathcal{EC}}$, finitisation is not only not to be expected (for instance because of [Rid19, Proposition 4.2]), but in fact not possible at all.

[13]To be precise, of abelian structures augmented by an absorbing element. See Remark 4.13.

Remark 8.6. In $\mathsf{VDF}_{\mathcal{EC}}$, it is possible to construct a 1-type $p \in S^{\mathrm{inv}}_{\mathrm{K}}(\mathfrak{U})$ such that the types $((v \circ \partial^n)_* p)_{n\in\omega}$ are pairwise weakly orthogonal, and in particular not domination-equivalent.

Computing the image of the home sort in finitely many variables seems difficult.

Remark 8.7. Most arguments in this section may be adapted to theories of σ-henselian valued difference fields of residue characteristic 0. An analogue of Theorem 8.3 goes through because, by [DO15, Theorem 5.8 and Theorem 7.3], there is still a quantifier reduction to $\mathcal{RV}$ and a σ-Kaplansky theory yielding uniqueness and elementarity of maximal immediate extensions. In (every completion of) the model companion of the *isometric* case (see [BMS07]), in sufficiently saturated models there is a section of the valuation with values in the fixed field. Hence, it is also possible to obtain the decomposition $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathrm{k}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}_\kappa(\Gamma(\mathfrak{U}))$, by regarding $\mathcal{RV}$ as a pure short exact sequence of $\mathbb{Z}[\sigma]$-modules, and using elimination of imaginaries in ACFA_0. The same goes through if we move from the isometric to the *multiplicative* setting, provided that, in the notation of [Pal12], ρ is transcendental. This applies for example to the model companion of the *contractive* (or *ω-increasing*) case (see [Azg10]).

9. Open questions

Our first question concerns transfer of compatibility of $\otimes$ with $\geq_{\mathrm{D}}$.

Question 9.1. If $\otimes$ respects $\geq_{\mathrm{D}}$ on finitary types, does $\otimes$ necessarily respect $\geq_{\mathrm{D}}$ on $*$-types?

In algebraically closed or real closed valued fields, the decomposition $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathrm{k}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U}))$ remains valid after passing to T^{eq}, as can be shown using resolutions (see [HHM08, EHM19, Mena]). A natural question is whether Theorem 6.21 generalises to T^{eq}, or at least to $T^{\mathcal{G}}$, the expansion of T by the geometric sorts of [HHM06].

Question 9.2. Let T be an $\mathcal{RV}$-expansion of a theory of valued fields with enough saturated maximal models eliminating K-quantifiers. Are there conditions guaranteeing that the isomorphism $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathcal{RV}(\mathfrak{U}))$ holds in $T^{\mathcal{G}}$, or even in T^{eq}? Does compatibility of $\geq_{\mathrm{D}}$ with $\otimes$ transfer?

By Corollary 7.9, in the theory of the $\mathfrak{p}$-adics, every element of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is idempotent. That of $\mathbb{Q}_{\mathfrak{p}}$ is notoriously a distal theory. Moreover, by [ACGZ20, Theorem 3.13], the theories from Remark 5.7 are distal, too. To this date, in every distal theory in which $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ has been computed, it has turned out to be well-defined, and each of its elements idempotent. In [Sim15, Chapter 9], the notion of a distal *type* is defined, and a theory is distal if and only if every invariant type is distal (distal types are automatically invariant). It follows easily from the definition of "distal type" that if p is distal then $p^{(\omega)} \sim_{\mathrm{D}} p^{(\omega+1)}$. All of this motivates the following question.

Question 9.3. Let p be a distal type. Is it true that $p \sim_{\mathrm{D}} p^{(2)}$?

By [Mena, Lemma 2.3] the answer is positive for 1-types in o-minimal theories.

While idempotents may also arise in the stable case (see [Men20a, Subsection 3.2.4]), in a stable theory $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ necessarily contains a copy of $(\omega, +)$ (see [Men20b, Proposition 5.20]). Moreover, idempotency seems hardly compatible with the Independence Property. Hence, while idempotency modulo domination-equivalence does not characterise distal *types*, it is not excluded that it characterise distal *theories*.

Question 9.4. Is it true that T is distal if and only if, for every $p \in S^{\mathrm{inv}}(\mathfrak{U})$, we have $p \sim_{\mathrm{D}} p^{(2)}$?

In light of this question, and of [ACGZ20, Main Theorem], it would be interesting to have a computation of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in an infinitely ramified mixed characteristic henselian valued field with distal value group and distal, or even finite, residue field.

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