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Infinitary Proof Theory

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1. Proof theoretic ordinals

1.1 Preliminaries

One of the aims of infinitary proof theory is the computation of the proof theoretical ordinal of axiom systems. We will indicate in these lectures that there are different types of proof theoretical ordinals for axiom systems.
Proof theory was launched by the consistency problem for axioms systems. Its original aim was to give finitary consistency proofs. However, according to GÖDEL's second incompleteness theorem, finitary consistency proofs are impossible for axiom systems which allow sufficiently much coding machinery.
Ordinals entered the stage when GENTZEN in [5] and [6] proved the consistency of the axioms of number theory using a transfinite induction. His proof is completely finitary except for the transfinite induction. The infinite content of the axioms for Number Theory is thus pinpointed in the transfinite induction used in the consistency proof. Therefore it seemed to be a good idea to regard the order–type of the least well–ordering which is needed in the consistency proof for an axioms system as characteristic for these axioms and to call it its proof theoretic ordinal. But as observed by KREISEL there is a serious obstacle.
To state KREISEL's theorem we use some obvious abbreviations. The system EA of *Elementary Arithmetic* is formulated in the language of arithmetic with the non-logical symbols

$$(0, 1, +, \cdot, 2^x, =, \leq)$$

together with their defining axioms among them

(exp) $2^0 = 1$ and $2^{x+1} = 2^x + 2^x$

($\leq$) $x \leq 0 \leftrightarrow x = 0$ and
$x \leq y + 1 \leftrightarrow x \leq y \vee x = y + 1.$

The scheme

(Ind) $\varphi(0) \wedge (\forall x)[\varphi(x) \to \varphi(x+1)] \;\; \to \;\; (\forall x)\varphi(x)$

of *Mathematical Induction* is restricted to Δ_0–formulas φ. A formula is Δ_0 iff it only contains bounded quantifiers $(\forall x < a)$ or $(\exists x < a)$. Mostly we use the system PRA which has constants for all primitive recursive functions and in which the scheme of mathematical induction is restricted to Σ^0_1–formulas. By $\bot$ we denote the false sentence $0 = 1$. We assume that there is an elementary coding for the language of arithmetic and that there is a predicate

$$Prf_{Ax}(i, v) \;\; :\Leftrightarrow \;\; \text{“}i \text{ codes a proof from } Ax \text{ of the formula coded by } v\text{”}.$$

For an axiom system Ax we obtain the *provability predicate* as

$$\Box_{Ax} x \;\; :\Leftrightarrow \;\; (\exists y) Prf_{Ax}(y, x).$$

By $PRWO(\prec)$ we denote that there are no primitive recursive infinite descending sequences in *field* $\prec$. By $TI(\prec)$ we denote the scheme of induction along $\prec$.

1.1.1 Theorem *(Kreisel) For any consistent axiom system Ax there is a primitive recursive well–ordering $\prec_{Ax}$ of order type ω such that*

$$PRA + PRWO(\prec_{Ax}) \vdash Con(Ax)$$

Sketch of the proof of Theorem 1.1.1 Define

$$x \prec_{Ax} y \quad :\Leftrightarrow \quad \begin{cases} x < y & \text{if } (\forall i < x)[\neg \mathit{Prf}_{Ax}(i, \ulcorner \bot \urcorner)] \\ y < x & \text{otherwise} \end{cases} \tag{i}$$

and let

$$F(x) \quad :\Leftrightarrow \quad (\forall i \leq x)[\neg \mathit{Prf}_{Ax}(i, \ulcorner \bot \urcorner)]. \tag{ii}$$

Now we obtain

$$PRA \vdash (\forall x \prec_{Ax} y) F(x) \quad \to \quad F(y) \tag{iii}$$

since if we assume $\neg F(x)$ we have $(\exists i \leq x)[\mathit{Prf}_{Ax}(i, \ulcorner \bot \urcorner)]$ and get $x + 1 \prec_{Ax} x$ and thus together with the premise of (iii) also $F(x+1)$. But this implies $F(x)$, a contradiction.
Since $F(x)$ is primitive recursive we obtain from (iii)

$$PRA + PRWO(\prec_{Ax}) \vdash (\forall x) F(x) \tag{iv}$$

and thus

$$PRA + PRWO(\prec_{Ax}) \vdash \mathit{Con}(Ax). \tag{v}$$

Since $\mathit{Con}(Ax)$ is true we have $\mathit{otyp}(\prec_{Ax}) = \omega$. ☐

Recall that ω_1^{CK} denotes the first ordinal which cannot be represented as the order–type of a recursive well–ordering. It is well-known that for every ordinal $\lambda < \omega_1^{CK}$ there is a primitive recursive (even elementary) well–ordering of order–type λ. There is a theorem recently observed by BEKLEMISHEV which points exactly in the opposite direction of Theorem 1.1.1.

1.1.2 Theorem *(Beklemishev) For any $\lambda < \omega_1^{CK}$ there is a primitive recursive well–ordering of order–type λ such that*

$$PRA + PRWO(\prec) \not\vdash \mathit{Con}(PA).$$

To show Theorem 1.1.2 we first observes two other facts.

1.1.3 Theorem *(Beklemishev) For every ordinal $\lambda < \omega_1^{CK}$ there is a primitive recursive well–ordering E of order type λ such that*

$$PA \vdash \mathit{Con}(PRA + TI(E)).$$

Sketch of the proof of Theorem. 1.1.3 Let R be a primitive recursive well–ordering such that

$$\mathit{otyp}(R) = \lambda. \tag{i}$$

Put

$$x \; R_z \; y \quad :\Leftrightarrow \quad x + y \leq z \land x \; R \; y. \tag{ii}$$

Then R_z is a finite ordering and we get a proof of $TI(R_z)$ primitive recursively from z. Hence

$$PRA \vdash (\forall z) \Box_{PRA} TI(R_{\dot z}). \tag{iii}$$

By the arithmetical fixed-point theorem we define a formula

$$x \; E \; y \quad :\Leftrightarrow \quad x \; R \; y \land (\forall u < x + y) \neg C(u, \ulcorner (x \; E \; y) \urcorner) \tag{iv}$$

where $C(u, v)$ is the primitive recursive predicate saying that v is a code for $x \; E \; y$ and u codes a proof of a contradiction from $PRA + TI(E)$. Then

$$
\begin{aligned}
PRA \vdash C(z, \ulcorner(x\ E\ y)\urcorner) \wedge (\forall u < z)\neg C(u, \ulcorner(x\ E\ y)\urcorner) & \\
\rightarrow\ & (x\ E\ y \leftrightarrow x\ R_z\ y) \\
\rightarrow\ & (\forall z)\Box_{PRA}(x\ E\ y \leftrightarrow x R_{\dot{z}}\ y) \\
\rightarrow\ & \Box_{PRA}\, TI(E)
\end{aligned} \tag{v}
$$

by (iii). Since PA proves $Con(PRA)$ and the least number principle we get from (v)

$$PA \vdash (\exists z)C(z, \ulcorner(x\ E\ y)\urcorner) \ \rightarrow\ Con(PRA + TI(E)). \tag{vi}$$

But (vi) means

$$PA \vdash Con(PRA + TI(E)). \tag{vii}$$

This, however, also entails that E and R coincide and we have $otyp(E) = otyp(R) = \lambda$. □

1.1.4 Theorem *Let Ax_1 and Ax_2 be theories which comprise PRA (either directly or via interpretation). Then $Ax_1 \vdash Con(Ax_2)$ implies $Ax_2 \subseteq_{\Pi^0_1} Ax_1$, i.e. Ax_1 is Π^0_1–conservative over Ax_2.*

Sketch of the proof of Thm.1.1.4 By formalized Σ^0_1 completeness we get for a Π^0_1 formula P

$$PRA \vdash \neg P \ \rightarrow\ \Box_{Ax_2}\neg P \tag{i}$$

and thus

$$PRA \vdash \neg\Box_{Ax_2}\neg P \ \rightarrow\ P. \tag{ii}$$

If $Ax_2 \vdash P$ we get $PRA \vdash \Box_{Ax_2} P$ and thus also

$$PRA \vdash \Box_{Ax_2}\neg\neg P. \tag{iii}$$

Hence

$$PRA \vdash \neg\Box_{Ax_2}\bot \ \rightarrow\ \neg\Box_{Ax_2}\neg P \tag{iv}$$

which is

$$PRA \vdash Con(Ax_2) \ \rightarrow\ \neg\Box_{Ax_2}\neg P. \tag{v}$$

Because of

$$Ax_1 \vdash Con(Ax_2) \tag{vi}$$

we obtain from (vi),(v) and (ii)

$$Ax_1 \vdash P$$

and are done. □

Now we obtain Theorem 1.1.2 from Theorems 1.1.4 and 1.1.3 by choosing $\prec$ to be the well–ordering E constructed in Theorem 1.1.3. □

It follows from Theorems 1.1.1 and 1.1.2 that the order–type of a well–ordering which suffices for a consistency proof by induction along this well–ordering is not a very intrinsic measure. The order relation constructed in proving both theorems, however, appear quite artificial. For "natural well–orderings" these pathological phenomena do not arise. But the real obstacle here is to find a mathematically sound definition of "naturalness" for well–orderings. Therefore one is looking for a more stable definition of the proof theretic ordinal of an axiom system.
Already GENTZEN in [7] observed that his consistency proof also entails the result that the axioms of Peano Arithmetic cannot prove the well–foundedness of primitive recursive well–orderings of order–types exceeding the order–type of the well–ordering which he used in his consistency proof.

On the other hand he could show that for any lower order–type λ there is a primitive recursive well–ordering of order–type λ whose well–foundedness can be derived from the axioms of Peano arithmetic. So his ordinal is characteristic for PA in that sense that it is the least upper bound for the order–types of primitiv recursive well–oderings whose well foundedness can be proved in PA. The well–foundedness of a relation $\prec$ can be expressed by the formula

$$TI(\prec, X) \quad :\Leftrightarrow \quad (\forall x)[(\forall y \prec x)(y \in X) \to x \in X] \quad \to \quad (\forall x)[x \in X].$$

Let PR denote the collection of primitive recursive relations. According to GENTZEN's observation we define

$$\|Ax\| := \sup \{ otyp(\prec) \mid \ \prec \in PR \wedge Ax \vdash TI(\prec, X) \} \tag{1.1}$$

and call $\|Ax\|$ the *proof–theoretic ordinal* of the axiom system Ax. For reasons which will become clear in the next sections we call $\|Ax\|$ the Π^1_1–ordinal of Ax and will later indicate that there are also other characteristic ordinals for a set Ax of axioms.

1.2 Some basic facts about ordinals

Ordinals are originally introduced as equivalence classes of well–orderings. From a set theoretical point of view this is problematic since these equivalence classes are not sets but proper classes. Ordinals in the set theoretical sense are therefore introduced as sets which are well–ordered by the $\in$–relation. This entails that an ordinal α the set of all ordinals $\beta < \alpha$. When we talk about ordinals we have the set theoretical meaning of ordinals in mind. But this is of no importance. All we have to know about ordinals are a few basic facts which we will describe shortly.

(On1) The class On of ordinals is a non void transitive class, which is well-ordered by the membership relation $\in$. We define $\alpha < \beta$ as $\alpha \in On \wedge \beta \in On \wedge \alpha \in \beta$.

In general we use lower case Greek letters as syntactical variables for ordinals. The well-foundedness of $\in$ on the class On implies the principle of *transfinite induction*

$$(\forall \xi \in On)[(\forall \eta < \xi) F(\eta) \Rightarrow F(\xi)] \Rightarrow (\forall \xi \in On) F(\xi)$$

and transfinite recursion which, for a given function g, allows the definition of a function f satisfying the recursion equation

$$f(\eta) = g(\{ f(\xi) \mid \ \xi < \eta \}).$$

(On2) The class On of ordinals is unbounded, i.e., $(\forall \xi \in On)(\exists \eta \in On))[\xi < \eta]$. The *cardinality* $|M|$ of a set M is the least ordinal α such that M can be mapped bijectively onto α. An ordinal α is a *cardinal* if $|\alpha| = \alpha$.

(On3) If $M \subseteq On$ and $|M| \in On$ then M is bounded in On, i.e., there is an $\alpha \in On$ such that $M \subseteq \alpha$.

For every ordinal α we have by *(On1)* and *(On2)* a least ordinal α' which is bigger than α. We call α' the *successor* of α. There are three types of ordinals:

- the least ordinal 0,
- successor ordinals, i.e., ordinals of the form α',
- ordinals which are neither 0 nor successor ordinals. Such ordinals are called *limit ordinals*. We denote the class of limit ordinals by Lim.

Considering these three types of ordinals we reformulate transfinite induction and recursion as follows:

Transfinite induction: If $F(0)$ *and* $(\forall\alpha\in On)[F(\alpha)\Rightarrow F(\alpha')]$ *as well as* $(\forall\xi<\lambda)F(\xi)\Rightarrow F(\lambda)$ *for* $\lambda\in Lim$ *then* $(\forall\xi\in On)F(\xi)$.

Transfinite recursion: For given $\alpha\in On$ *and functions* g, h *there is a function* f *satisfying the recursion equations*

$$\begin{aligned}&f(0)=\alpha\\&f(\xi')=g(f(\xi))\\&f(\lambda)=h(\{f(\eta)\mid\ \eta<\lambda\})\text{ for }\lambda\in Lim.\end{aligned}$$

An ordinal κ satisfying

(R1) $\quad\kappa\in Lim$

(R2) $\quad$ If $M\subseteq\kappa$ and $|M|<\kappa$ then M is bounded in κ, i.e., there is an $\alpha\in\kappa$ such that $M\subseteq\alpha$

is called *regular*. The class of regular ordinals is denoted by $\mathbf{Reg}$.

(On4) $\quad$ The class $\mathbf{Reg}$ is unbounded, i.e., $(\forall\xi\in On)(\exists\eta\in\mathbf{Reg})[\xi\le\eta]$.

We define

$$\sup M:=\min\{\xi\in On\mid\ (\forall\eta\in M)(\eta\le\xi)\}$$

as the least upper bound for a set $M\subseteq On$. In set theoretic terms it is $\sup M=\bigcup M$. It follows that $\sup M$ is either the biggest ordinal in M, i.e., $\sup M=\max M$, or $\sup M\in Lim$. By ω we denote the least limit ordinal. It exists according to *(O4)* and *(O1)*. The ordinal ω_1 denotes the first uncountable ordinal, i.e., the first ordinal whose cardinality is bigger than that of ω. It exists by *(On3)*.

For every class $M\subseteq On$ there is a uniquely determined transitive class $otyp(M)\subseteq On$ and an order preserving function $en_M\colon otyp(M)\xrightarrow{onto}M$. The function en_M enumerates the elements of M in increasing order. Since $otyp(M)$ is transitive it is either $otyp(M)=On$ or $otyp(M)\in On$. We call $otyp(M)$ the *order type* of M. In fact $otyp(M)$ is the MOSTOWSKI collapse of M and en_M the inverse of the collapsing function (usually denoted by π). By *(On3)* we have $otyp(M)\in On$ iff M is bounded in On. Unbounded, i.e., proper classes of ordinals have order type On. If M is a set of ordinals then $otyp(M)\in On$.

If M is a transitive class and $f\colon M\longrightarrow On$ an order preserving function then $\alpha\le f(\alpha)$ for all $\alpha\in M$.

A class M is *closed* (in a regular ordinal κ) iff $\sup N\in M$ holds for every class $N\subseteq M$ such that $|N|\in On$ ($|N|<\kappa$). We call M *club* (in κ) iff M is closed and unbounded (in κ).

We call an order preserving function $f\colon M\longrightarrow On$ (κ-) continuous iff M is (κ-) closed and f preserves suprema, i.e., $\sup\{f(\xi)\mid\ \xi\in N\}=f(\sup(N))$ for any $N\subseteq M$ such that $|N|\in On$ ($|N|<\kappa$).

A normal (κ-normal) function is an order-preserving continuous function

$$f\colon On\longrightarrow On\ \text{ or }\ f\colon\kappa\longrightarrow\kappa\ \text{ respectively.}$$

For $M\subseteq On$ ($M\subseteq\kappa$) the enumerating function en_M is a (κ-)normal function iff M is club (in κ).

Extending their primitive recursive definitions continuously into the transfinite we obtain the basic arithmetical functions $+$, $\cdot$ and exponentiation for all ordinals. The ordinal sum, for example, satisfies the recursion equations

$$\begin{aligned}&\alpha+0=\alpha\\&\alpha+\beta'=(\alpha+\beta)'\\&\alpha+\lambda=\sup_{\xi<\lambda}(\alpha+\xi)\ \text{ for }\ \lambda\in Lim.\end{aligned}$$

It is easy to see that the function $\lambda\xi\,.\,\alpha+\xi$ is the enumerating function of the class $\{\xi\in On\mid\ \alpha\le\xi\}$

which is club in all regular $\kappa > \alpha$. Hence $\lambda\xi\,.\,\alpha+\xi$ is a κ-normal function for all regular $\kappa > \alpha$. We define

$$\mathbb{H} := \{\alpha \in On \mid \alpha \neq 0 \wedge (\forall\xi < \alpha)(\forall\eta < \alpha)[\xi + \eta < \alpha]\}$$

and call the ordinals in $\mathbb{H}$ *additively indecomposable*. Then $\mathbb{H}$ is club (in any regular ordinal $> \omega$), $1 := 0' \in \mathbb{H}$, $\omega \in \mathbb{H}$ and $\omega \cap \mathbb{H} = \{1\}$. Hence $en_{\mathbb{H}}(0) = 1$ and $en_{\mathbb{H}}(1) = \omega$ which are the first two examples of the fact that

$$(\forall\xi \in On)[en_{\mathbb{H}}(\xi) = \omega^{\xi}]. \tag{1.2}$$

Thus $\lambda\xi\,.\,\omega^{\xi}$ is a (κ-)normal function (for all $\kappa \in \mathbf{Reg}$ bigger than ω). We have

$$\mathbb{H} \subseteq Lim \cup \{1\}$$

and obtain

$$\alpha \in \mathbb{H} \text{ iff } (\forall\xi < \alpha)[\xi + \alpha = \alpha].$$

Thus for a finite set $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{H}$ we get

$$\alpha_1 + \ldots + \alpha_n = \alpha_{k_1} + \ldots + \alpha_{k_m}$$

for $\{k_1, \ldots, k_m\} \subseteq \{1, \ldots, n\}$ such that $k_i < k_{i+1}$ and $\alpha_{k_i} \geq \alpha_{k_{i+1}}$. By induction on α we obtain thus ordinals $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{H}$ such that for $\alpha \neq 0$ we have

$$\alpha = \alpha_1 + \ldots + \alpha_n \text{ and } \alpha_1 \geq \ldots \geq \alpha_n. \tag{1.3}$$

This is obvious for $\alpha \in \mathbb{H}$ and immediate from the induction hypothesis and the above remark if $\alpha = \xi + \eta$ for $\xi, \eta < \alpha$. It follows by induction on n that the ordinals $\alpha_1, \ldots, \alpha_n$ in (1.3) are uniquely determined. We therefore define an *additive normal form*

$$\alpha =_{NF} \alpha_1 + \ldots + \alpha_n :\Leftrightarrow \alpha = \alpha_1 + \ldots + \alpha_n,\ \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{H} \text{ and } \alpha_1 \geq \ldots \geq \alpha_n.$$

We call $\{\alpha_1, \ldots, \alpha_n\}$ the set of *additive components* of α if $\alpha =_{NF} \alpha_1 + \ldots + \alpha_n$.
We use the additive components to define the *symmetric sum* of ordinals $\alpha =_{NF} \alpha_1 + \ldots + \alpha_n$ and $\beta =_{NF} \alpha_{n+1} + \ldots + \alpha_m$ by

$$\alpha \mathbin{\#} \beta := \alpha_{\pi(1)} + \ldots + \alpha_{\pi(m)}$$

where π is a permutation of the numbers $\{1, \ldots, m\}$ such that

$$1 \leq i < j \leq m \Rightarrow \alpha_{\pi(i)} \geq \alpha_{\pi(j)}.$$

In contrast to the "ordinary ordinal sum" the symmetric sum does not cancel additive components. By definition we have

$$\alpha \mathbin{\#} \beta = \beta \mathbin{\#} \alpha.$$

It is easy to check that the symmetric sum is order preserving in its both arguments.
As another consequence of (1.3) we obtain the CANTOR *normal form* for ordinals for the basis ω, which says that for every ordinal $\alpha \neq 0$ there are ordinals $\xi_1, \ldots, \xi_n$ such that

$$\alpha =_{NF} \omega^{\xi_1} + \ldots + \omega^{\xi_n}.$$

Since $\lambda\xi\,.\,\omega^{\xi}$ is a normal function we have $\alpha \leq \omega^{\alpha}$ for all ordinals α. We call α an ε-number if $\omega^{\alpha} = \alpha$ and define

$$\varepsilon_0 := \min\{\alpha \mid \omega^{\alpha} = \alpha\}.$$

more generally let $\lambda\xi\,.\,\varepsilon_{\xi}$ enumerate the fixed points of $\lambda\xi\,.\,\omega^{\xi}$. If we put

$$exp^0(\alpha,\beta) := \beta \text{ and } exp^{n+1}(\alpha,\beta) := \alpha^{exp^n(\alpha,\beta)}$$

we obtain

$$\varepsilon_0 := \sup_{n<\omega} exp^n(\omega, 0).$$

For $0 < \alpha < \varepsilon_0$ we have $\alpha < \omega^\alpha$ and obtain by the CANTOR Normal Form Theorem uniquely determined ordinals $\alpha_1, \ldots, \alpha_n < \alpha$ such that $\alpha =_{NF} \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$.
For a class $M \subseteq On$ we define its *derivative*

$$M' := \{\xi \in On |\ en_M(\xi) = \xi\}.$$

The derivative f' of a function f is defined by $f' := en_{Fix(f)}$, where

$$Fix(f) := \{\xi |\ f(\xi) = \xi\}.$$

Thus f' enumerates the fixed-points of f. If M is club (in some regular κ) then M' is also club (in κ). Thus if f is a normal function f' is a normal function, too.
If $\{M_\iota |\ \iota \in I\}$ is a collections of classes club (in some regular κ) and $|I| \in On$ ($|I| \in \kappa$) then $\bigcap_{\iota \in I} M_\iota$ is also club (in κ).
These facts give raise to a hierarchy of club classes. We define

$$\begin{aligned} &Cr(0) := \mathbb{H} \\ &Cr(\alpha') := Cr(\alpha)' \\ &Cr(\lambda) := \textstyle\bigcap_{\xi<\lambda} Cr(\xi) \ \text{ for } \ \lambda \in Lim. \end{aligned}$$

If we put

$$\varphi_\alpha := en_{Cr(\alpha)},$$

then all φ_α are normal functions and we have by definition

$$\alpha < \beta \Rightarrow \varphi_\alpha(\varphi_\beta(\gamma)) = \varphi_\beta(\gamma). \tag{1.4}$$

The function φ is commonly called VEBLEN function. From (1.4) we obtain immediately

$$\begin{aligned} \varphi_{\alpha_1}(\beta_1) \leq \varphi_{\alpha_2}(\beta_2) \ \text{ iff } \quad & \alpha_1 < \alpha_2 \ \text{ and } \ \beta_1 \leq \varphi_{\alpha_2}(\beta_2) \\ \text{or } \quad & \alpha_1 = \alpha_2 \ \text{ and } \ \beta_1 \leq \beta_2 \\ \text{or } \quad & \alpha_2 < \alpha_1 \ \text{ and } \ \varphi_{\alpha_1}(\beta_1) \leq \beta_2. \end{aligned} \tag{1.5}$$

We define the VEBLEN normal form for ordinals $\varphi_\xi(\eta)$ by

$$\alpha =_{NF} \varphi_\xi(\eta) :\Leftrightarrow \ \alpha = \varphi_\xi(\eta) \ \text{ and } \ \eta < \alpha.$$

Then $\alpha =_{NF} \varphi_{\xi_1}(\eta_1)$ and $\alpha =_{NF} \varphi_{\xi_2}(\eta_2) \Rightarrow \xi_1 = \xi_2$ and $\eta_1 = \eta_2$. Since $\xi < \alpha$ and $\eta < \beta \in Cr(\alpha)$ implies $\varphi_\xi(\eta) < \beta$ we call $Cr(\alpha)$ the class of α–critical ordinals.. If α is itself α–critical then $\xi, \eta < \alpha \Rightarrow \varphi_\xi(\eta) < \alpha$. Therefore we define the class SC of *strongly critical ordinals* by

$$SC := \{\alpha \in On |\ \alpha \in Cr(\alpha)\}.$$

The class SC is club (in all regular ordinals $\kappa > \omega$). Its enumerating function is denoted by $\lambda\xi \,.\, \Gamma_\xi$. Regarding that by (1.5) $\lambda\xi \,.\, \varphi_\xi(0)$ is order preserving one easily proves

$$SC = \{\alpha |\ \varphi_\alpha(0) = \alpha\}.$$

If we define $\gamma_0 := 0$ and $\gamma_{n+1} := \varphi_{\gamma_n}(0)$ then we obtain

$$\Gamma_0 = \sup_{n\,<\,\omega} \gamma_n.$$

We define the set of *strongly critical components* $SC(\alpha)$ of an ordinal α by

$$SC(\alpha) := \begin{cases} \{0\} & \text{if } \alpha = 0 \\ \{\alpha\} & \text{if } \alpha \in SC \\ SC(\xi) \cup SC(\eta) & \text{if } \alpha =_{NF} \varphi_\xi(\eta) \\ SC(\alpha_1) \cup \ldots SC(\alpha_n) & \text{if } \alpha =_{NF} \alpha_1 + \ldots + \alpha_n. \end{cases} \tag{1.6}$$

For every $\alpha < \Gamma_0$ there are uniquely determined ordinals $\xi_1, \ldots, \xi_n < \alpha$ and $\eta_1, \ldots, \eta_n < \alpha$ such that

$$\alpha =_{NF} \varphi_{\xi_1}(\eta_1) + \ldots + \varphi_{\xi_n}(\eta_n) \text{ and } \eta_i < \varphi_{\xi_i}(\eta_i) \text{ for } i \in \{1, \ldots, n\}. \tag{1.7}$$

Recall that a relation $\prec$ is well–founded if there is no infinite descending sequence $\cdots x_{n+1} \prec x_n \prec \cdots$ in $\mathit{field}(\prec)$. For $x \in \mathit{field}(\prec)$ we define

$$\mathit{otyp}_\prec(x) := \sup\left\{\mathit{otyp}_\prec(y) \,\middle|\, y \prec x\right\}$$

and

$$\mathit{otyp}(\prec) := \sup\left\{\mathit{otyp}_\prec(x) \,\middle|\, x \in \mathit{field}(\prec)\right\}.$$

We call $\mathit{otyp}(\prec)$ the ordertype of $\prec$. It is easy to see that $\mathit{otyp}_\prec(x)$ and $\mathit{otyp}(\prec)$ are ordinals. This is all we need to know about ordinals for the moment. We will have to come back to the theory later.

1.3 Truth complexity for Π^1_1–sentences

1.3.1 Definition The TAIT–language for arithmetic contains the following symbols

- Set variables X, Y, X_1,...
- The logical symbols $\wedge$, $\vee$, $\forall$, $\exists$
- The binary relation symbols $\in$,$\notin$, $=$,$\neq$.
- The constant $\underline{0}$.
- Symbols for all primitive recursive functions.

Terms and formulas are constructed in the usual way.
Since there is no negation symbol we define

- $\sim(s = t) :\equiv s \neq t; \quad \sim(s \neq t) :\equiv s = t$
- $\sim(s \in X) :\equiv s \notin X; \quad \sim(s \notin X) :\equiv s \in X$
- $\sim(A \wedge B) :\equiv {\sim}A \vee {\sim}B; \quad \sim(A \vee B) :\equiv {\sim}A \wedge {\sim}B$
- $\sim(\forall x)F(x) :\equiv (\exists x){\sim}F(x); \quad \sim(\exists x)F(x) :\equiv (\forall x){\sim}F(x)$.

We observe that for any assignment Φ of subsets of $\mathbb{N}$ to the set variables occurring in F we obtain

$$\mathbb{N} \models {\sim}F[\Phi] \;\Leftrightarrow\; \mathbb{N} \models \neg F[\Phi]. \tag{1.8}$$

Therefore we commonly write $\neg F$ instead of ${\sim}F$.

Let $\mathsf{D}(\mathbb{N})$ be the *diagram* of $\mathbb{N}$, i.e. the set of true atomic sentences.

1.3.2 Observation *The true arithmetical sentences can be characterized by the following types*

- *the sentences in* $\mathsf{D}(\mathbb{N})$
- *the sentences of the form* $(F_0 \vee F_1)$ *or* $(\exists x)F(x)$ *where* F_i *and* $F(k)$ *is true for* some $i \in \{0, 1\}$ *or* $k \in \omega$ *respectively*

- *the sentences of the form* $(F_0 \wedge F_1)$ *or* $(\forall x)F(x)$ *where* F_i *and* $F(k)$ *is true for* all $i \in \{0,1\}$ *or* $k \in \omega$ *respectively*

According to Observation 1.3.2 we divide the arithmetical sentences into two types.

1.3.3 Definition

$$\begin{aligned}\bigwedge\text{–type} :=\ & \mathsf{D}(\mathbb{N}) \cup \{\text{sentences of the form } (F_0 \wedge F_1)\} \cup \\ & \{\text{sentences of the form } (\forall x)F(x)\}\end{aligned}$$

$$\begin{aligned}\bigvee\text{–type} :=\ & \{\neg F \mid F \in \bigwedge\text{–type}\} = \\ & \neg\mathsf{D}(\mathbb{N}) \cup \{\text{sentences of the form } (F_0 \vee F_1)\} \\ & \cup \{\text{sentences of the form } (\exists x)F(x)\}\end{aligned}$$

and define a *characteristic sequence* $CS(F)$ of sub–sentences of F by

1.3.4 Definition

$$CS(F) := \begin{cases} \emptyset & \text{if } F \text{ is atomic} \\ (F_0, F_1) & \text{if } F \equiv (F_0 \circ F_1) \\ (F(k) \mid k \in \omega) & \text{if } F \equiv (\mathsf{Q}x)F(x) \end{cases}$$

for $\circ \in \{\wedge, \vee\}$ and $\mathsf{Q} \in \{\forall, \exists\}$. The *length of the type* of a sentence F is the length of its characteristic sequence $CS(F)$.

From Observation 1.3.2 and Definition 1.3.3 we get immediately

1.3.5 Observation

$$F \in \bigwedge\text{–}\mathit{type} \;\Rightarrow\; [\mathbb{N} \models F \;\Leftrightarrow\; (\forall G \in CS(F))(\mathbb{N} \models G)]$$

and

$$F \in \bigvee\text{–}\mathit{type} \;\Rightarrow\; [\mathbb{N} \models F \;\Leftrightarrow\; (\exists G \in CS(F))(\mathbb{N} \models G)]$$

We use Observation 1.3.5 to define the *truth complexity* of a sentence F.

1.3.6 Definition We define the validity relation $\models^{\alpha} F$ inductively by the following two clauses

$(\bigwedge)$ If $F \in \bigwedge$–type and $(\forall G \in CS(F))[\models^{\alpha_G} G \ \&\ \alpha_G < \alpha]$ then $\models^{\alpha} F$

$(\bigvee)$ If $F \in \bigvee$–type und $(\exists G \in CS(F))[\models^{\alpha_G} G \ \&\ \alpha_G < \alpha]$ then $\models^{\alpha} F$.

Finally we put

$$tc(F) := \min(\{\alpha \mid \ \models^{\alpha} F\} \cup \{\omega\})$$

and call $tc(F)$ the *truth complexity* of the sentence F.

The next theorem is obvious from Observation 1.3.5 and Definition 1.3.6.

1.3.7 Theorem $\models^{\alpha} F$ *implies* $\mathbb{N} \models F$.

1.3.8 Observation *Let $rnk(F)$ be the number of logical symbols accurring in F. Then we get*

$$\mathbb{N} \models F \quad \Rightarrow \quad tc(F) \leq rnk(F)$$

and

$$\mathbb{N} \models F \quad \Leftrightarrow \quad tc(F) < \omega.$$

According to Observation 1.3.8 the notion of truth complexity is not very exciting for arithmetical sentences. This, however, will change if we extend it to the class of formulas containing also free set variables.

1.3.9 Definition We call an arithmetical formula which does not contain free number variables but may contain free set parameters a *pseudo* Π^1_1*–sentence.* For pseudo Π^1_1–sentences $F(\vec{X})$ we define

$$\mathbb{N} \models F(\vec{X}) \quad :\Leftrightarrow \quad \mathbb{N} \models (\forall \vec{X}) F(\vec{X}).$$

For pseudo Π^1_1–sentences there is a third type of open atomic pseudo sentences which are the sentences of the form

$$(t \in X) \quad \text{and} \quad (s \notin X).$$

1.3.10 Definition For a finite set Δ of pseudo Π^1_1–sentences we define the validity relation $\models^{\alpha} \Delta$ inductively by the following clauses

(Ax) $\quad s^{\mathbb{N}} = t^{\mathbb{N}} \quad \Rightarrow \quad \models^{\alpha} \Delta\, , s \in X, t \notin X$

$(\bigwedge)$ $\quad$ If $F \in \bigwedge$–type $\cap\, \Delta$ and $(\forall G \in CS(F))\, \big[\models^{\alpha_G} \Delta\, , G \;\&\; \alpha_G < \alpha \big]$ then $\models^{\alpha} \Delta$

$(\bigvee)$ $\quad$ If $F \in \bigvee$–type $\cap\, \Delta$ and $(\exists G \in CS(F))\big[\models^{\alpha_G} \Delta\, , G \;\&\; \alpha_G < \alpha \big]$ then $\models^{\alpha} \Delta$

Observe that for finite sets of formulas we always write $F_1, \ldots, F_n$ instead of $\{F_1, \ldots, F_n\}$. We often also write Δ, Γ instead of $\Delta \cup \Gamma$.
The aim is now to extend the second claim in observation 1.3.8 to formulas also containing set parameters. We will do that using the method of *search trees* as introduced by SCHÜTTE. Therefore we order the formulas in Δ arbitrarily and obtain finite sequence $\langle \Delta \rangle$ of pseudo Π^1_1–sentences. The leftmost formula in a sequence $\langle \Delta \rangle$ which does not belong to $\bigwedge$–type $\cup \bigvee$–type is the *redex* $R(\langle \Delta \rangle)$ of $\langle \Delta \rangle$. The sequence $\langle \Delta \rangle^r$ is obtained from $\langle \Delta \rangle$ by canceling its redex $R(\langle \Delta \rangle)$. We put

$$Ax(\Delta) \quad :\Leftrightarrow \quad \exists s, t, X [s^{\mathbb{N}} = t^{\mathbb{N}} \wedge \{t \in X, s \notin X\} \subseteq \Delta].$$

For the definition of a tree cf. Definition 1.7.1. Two pseudo Π^1_1–sentences are *numerical equivalent* if they only differ in terms whose evaluation yield the same value.

1.3.11 Definition For a finite sequence $\langle \Delta \rangle$ of pseudo Π^1_1-sentences we define its *search tree* $S_{\langle \Delta \rangle}$ together with a *label function*

$$\delta \colon S_{\langle \Delta \rangle} \longrightarrow \text{finite sequences of pseudo } \Pi^1_1\text{–sentences}$$

inductively by the following clauses

$(S_{\langle\rangle})$ $\qquad \langle\rangle \in S_{\langle \Delta \rangle} \wedge \delta(\langle\rangle) = \langle \Delta \rangle$

For the following clauses assume $s \in S_{\langle \Delta \rangle}$ and $\neg Ax(\delta(s))$

(S_{Id}) $\qquad R(\delta(s)) = \emptyset \quad \Rightarrow \quad s^\frown\langle 0 \rangle \in S_{\langle \Delta \rangle} \wedge \delta(s^\frown\langle 0 \rangle) = \delta(s)$

$(S_{\bigwedge})$ $\quad R(\delta(s)) \in \bigwedge$–type $\Rightarrow$ $(\forall F_i \in CS(R(\delta(s))))[s^\frown\langle i\rangle \in S_{\langle\Delta\rangle}] \wedge \delta(s^\frown\langle i\rangle) = \delta(s)^r, F_i$

$(S_{\bigvee})$ $\quad R(\delta(s)) \in \bigvee$–type $\Rightarrow$ $s^\frown\langle 0\rangle \in S_{\langle\Delta\rangle} \wedge \delta(s^\frown\langle 0\rangle) = \delta(s)^r, F_i, R(\delta(s))$, where F_i is the first formula in $CS(F)$ which is not numerical equivalent to a formula in $\bigcup_{s_0 \subseteq s} \delta(s_0)$.

1.3.12 Remark The search tree $S_{\langle\Delta\rangle}$ and δ are primitive recursively constructed from $\langle\Delta\rangle$.

1.3.13 Lemma *(Syntactical Main Lemma)* *If $S_{\langle\Delta\rangle}$ is well–founded then $\models^{otyp(s)} \bigvee \delta(s)$ holds for all $s \in S_{\langle\Delta\rangle}$.*

Proof An easy induction on $otyp(s)$. ☐

1.3.14 Lemma *(Semantical Main Lemma)* *If $S_{\langle\Delta\rangle}$ is not well–founded then there is an assignment $S_1, \ldots, S_n$ to the set variables occurring in $\langle\Delta\rangle$ such that $\mathbb{N} \not\models F[S_1, \ldots, S_n]$ for all $F \in \langle\Delta\rangle$.*

Sketch of the proof of Lemma 1.3.14. Pick an infinite path f in $S_{\langle\Delta\rangle}$ and let

$$f[n] := \langle f(0), \ldots, f(n-1)\rangle.$$

Observe

$$F \quad \text{atomic} \wedge F \in \delta(f[n]) \Rightarrow (\forall m \geq n)[F \in \delta(f[m])] \tag{i}$$

$$F \in \delta(f[n]) \cap \bigwedge\text{–type} \Rightarrow (\exists m)(\exists G \in CS(F))[G \in \delta(f[m])] \tag{ii}$$

$$F \in \delta(f[n]) \cap \bigvee\text{–type} \Rightarrow (\forall G \in CS(F))(\exists m)[G \in \delta(f[m])]. \tag{iii}$$

Notice that we identify numerical equivalent formulas.
We define an assignment

$$\Phi(X) := \{t^{\mathbb{N}} \mid (\exists m)[(t \notin X) \in \delta(f[m])]\}$$

and show by induction on $rnk(F)$ that $\mathbb{N} \not\models F[\Phi]$ for all $F \in \bigcup_{m\in\omega} \delta(f[m])$ using (ii) and (iii). ☐

1.3.15 Theorem *(ω–completeness Theorem)* *For a Π^1_1–sentence $(\forall X_1)\ldots(\forall X_n)F(X_1,\ldots,X_n)$ we have*

$$\mathbb{N} \models (\forall X_1)\ldots(\forall X_n)F(X_1,\ldots,X_n) \Leftrightarrow (\exists a < \omega_1^{CK}) \models^{\alpha} F\ (X_1,\ldots,X_n).$$

Proof First we show by an straight forward induction on α

$$\models^{\alpha} \Delta \Rightarrow \mathbb{N} \models \bigvee \Delta[\Phi] \tag{i}$$

for any assignment of subsets of $\mathbb{N}$ to the set variables occurring in Δ. The direction from right to left follows from (i).
For the opposite direction we assume

$$\not\models^{\alpha} F\ (X_1,\ldots,X_n) \tag{ii}$$

for all $\alpha < \omega_1^{CK}$. Then $S_{F(X_1,\ldots,X_n)}$ cannot be well–founded by the Syntactical Main Lemma (Lemma 1.3.13). By the Semantical Main Lemma we thus obtain an assignment Φ to the set variables $X_1,\ldots,X_n$ such that $\mathbb{N} \not\models F(X_1,\ldots,X_n)[\Phi]$. ☐

1.3.16 Definition Let $(\forall\vec{X})F(\vec{X})$ be a Π^1_1 sentence. We put

$$tc((\forall\vec{X})F(\vec{X})) := \min(\{\alpha \mid \models^{\alpha} F(\vec{X})\} \cup \omega_1^{CK})$$

and call $tc(F)$ the *truth complexity* of F. For a pseudo Π^1_1–sentence $G(\vec{X})$ containing the free set parameters $\vec{X}$ we define

$$tc(G(\vec{X})) := tc((\forall\vec{X})G(\vec{X})).$$

1.3.17 Theorem *For any (pseudo) Π^1_1–sentence F we have*

$$\mathbb{N} \models F \quad \Leftrightarrow \quad tc(F) < \omega_1^{CK}.$$

1.4 Inductive definitions

In order to link truth complexities with the proof theoretic ordinal of Ax defined in (1.1) we make a quick excursion into the theory of inductively defined sets.

1.4.1 Definition An n–ary *clause* on an infinite set N has the form

(C) $\quad P \longrightarrow c,$

where $P \subseteq N^n$ is the set of *premises* and $c \in N^n$ is the *conclusion* of the clause (C).
A set $S \subseteq N^n$ *satisfies* (C) if $P \subseteq S$ implies $c \in S$.
An *inductive definition* on N s a set $\mathbf{\Phi} := \{P_\iota \longrightarrow c_\iota \mid \iota \in I\}$ of clauses on N.
The least (with respect to set inclusion) set $I \subseteq N^n$ which simultaneously satisfies all clauses in an inductive definition $\mathbf{\Phi}$ is called *inductively defined* by $\mathbf{\Phi}$.

The special thing about inductive definition is the principle of *proof by induction on the definition* which says:

1.4.2 Theorem *If $I \subseteq N^n$ is inductively defined by an inductive definition $\mathbf{\Phi}$ and φ is a "property" which is preserved by all clauses in $\mathbf{\Phi}$, i.e.*

$$P_\iota \longrightarrow c_\iota \in \mathbf{\Phi} \wedge (\forall s \in P_\iota)\varphi(s) \quad\Rightarrow\quad \varphi(c_\iota),$$

then all elements of the set I have the property φ, i.e.

$$(\forall s \in I)[\varphi(s)].$$

Proof Obvious. □

1.4.3 Observation *An inductive definition $\mathbf{\Phi}$ induces an operator*

$$\Gamma_{\mathbf{\Phi}}: Pow(N^n) \longrightarrow Pow(N^n)$$

by defining

$$\Gamma_{\mathbf{\Phi}}(S) := \{c \mid (\exists P)[P \longrightarrow c \in \mathbf{\Phi} \wedge P \subseteq S]\}$$

which is monotonic, i.e.

$$S \subseteq T \subseteq N^n \quad\Rightarrow\quad \Gamma_{\mathbf{\Phi}}(S) \subseteq \Gamma_{\mathbf{\Phi}}(T).$$

Generalizing the situation in Observation 1.4.3 we make the following definition.

1.4.4 Definition Let N be a set. An n–ary *generalized monotone inductive definition* on N is a monotone operator

$$\Gamma\colon Pow(N^n) \longrightarrow Pow(N^n).$$

A set $S \subseteq N^n$ is *closed under* Γ, if $\Gamma(S) \subseteq S$.
A set $F \subseteq N^n$ is a *fixed–point* of Γ iff $\Gamma(F) = F$.
The least fixed–point (with respect to set–inclusion) of an operator Γ is called *the* fixed–point of Γ.

1.4.5 Observation *Every generalized monotone inductive definition Γ on a set N possesses a least fixed–point I_Γ which is the intersection of all Γ–closed sets.*

Proof Let

$$\mathfrak{M}_\Gamma := \{S \mid \Gamma(S) \subseteq S\}$$

and

$$I_\Gamma := \bigcap \mathfrak{M}_\Gamma.$$

For $S \in \mathfrak{M}_\Gamma$ we have $I_\Gamma \subseteq S$ and thus $\Gamma(I_\Gamma) \subseteq \Gamma(S) \subseteq S$ by monotonicity. Thus

$$\Gamma(I_\Gamma) \subseteq \bigcap \mathfrak{M}_\Gamma = I_\Gamma. \tag{i}$$

From (i) we obtain

$$\Gamma(\Gamma(I_\Gamma)) \subseteq \Gamma(I_\Gamma) \tag{ii}$$

again by monotonicity. Hence $\Gamma(I_\Gamma) \in \mathfrak{M}_\Gamma$ which entails

$$I_\Gamma \subseteq \Gamma(I_\Gamma). \tag{iii}$$

But (i) and (iii) show that I_Γ is a fixed–point and by definition of I_Γ this has to be the least one.
$\square$

1.5 The stages of an inductive definition

1.5.1 Definition For an arbitrary operator $\Gamma\colon Pow(N^n) \longrightarrow Pow(N^n)$ we define its α–th iteration Γ^α by

$$\Gamma^0(S) := S$$
$$\Gamma^{\alpha+1}(S) := \Gamma(\Gamma^\alpha(S))$$
$$\Gamma^\lambda(S) := \Gamma(\bigcup \{\Gamma^\xi(S) \mid \xi < \lambda\}) \text{ for } \lambda \in Lim.$$

We will frequently use the shorthand

$$\Gamma^{<\alpha}(S) := \bigcup_{\xi<\alpha} \Gamma^\xi(S).$$

We define

$$I_\Gamma^\alpha := \Gamma^{<\alpha}(\emptyset) \cup \Gamma(\Gamma^{<\alpha}(\emptyset))$$

and use the shorthand

$$I_\Gamma^{<\alpha} := \bigcup_{\xi<\alpha} I_\Gamma^\xi.$$

1.5.2 Lemma *For a monotone operator* Γ *we have*

$$I_\Gamma^\alpha := \Gamma(I_\Gamma^{<\alpha}).$$

1.5.3 Lemma *Let* $\Gamma\colon Pow(N^n) \longrightarrow Pow(N^n)$ *be an operator. Then there is a least ordinal* $|\Gamma| < |N|^+$ *such that*

$$I_\Gamma^{|\Gamma|} = I_\Gamma^{<|\Gamma|}.$$

We call $|\Gamma|$ the *closure ordinal* of the operator Γ.
Proof This is obvious for cardinality reasons. □

1.5.4 Theorem *Let* Γ *be a generalized monotone inductive definition. Then*

$$I_\Gamma = I_\Gamma^{|\Gamma|} = I_\Gamma^{<|\Gamma|}.$$

Proof Since $\Gamma(I_\Gamma^{<|\Gamma|}) = I_\Gamma^{|\Gamma|} = I_\Gamma^{<|\Gamma|}$ we have $I_\Gamma^{|\Gamma|} \in \mathfrak{M}_\Gamma$ and thus $I_\Gamma \subseteq I_\Gamma^{|\Gamma|}$. For the opposite inclusion we prove

$$I_\Gamma^\xi \subseteq I_\Gamma \tag{i}$$

by induction on $\xi \leq |\Gamma|$. By induction hypothesis we have $I_\Gamma^{<\xi} \subseteq I_\Gamma$ which by monotonicity entails $I_\Gamma^\xi = \Gamma(I_\Gamma^{<\xi}) \subseteq \Gamma(I_\Gamma) = I_\Gamma$. □

The following definition is an obvious generalization of Theorem 1.5.4.

1.5.5 Definition A generalized inductive definition on a set N is an operator

$$\Gamma\colon Pow(N^n) \longrightarrow Pow(N^n).$$

The fixed–point of a generalized inductive definition Γ is the set $I_\Gamma := I_\Gamma^{|\Gamma|}$.

1.5.6 Definition For a generalized inductive definition Γ and $n \in I_\Gamma$ we define

$$|n|_\Gamma := \min\left\{\xi \,\middle|\, n \in I_\Gamma^\xi\right\}.$$

1.5.7 Theorem *Let* Γ *be an generalized inductive definition on a set* N*. Then*

$$|\Gamma| = \sup\left\{|n|_\Gamma + 1 \,\middle|\, n \in I_\Gamma\right\}.$$

Proof By definition we have

$$\sigma := \sup\left\{|n|_\Gamma + 1 \,\middle|\, n \in I_\Gamma\right\} \leq |\Gamma|. \tag{i}$$

Assuming $\sigma < |\Gamma|$ we get $I_\Gamma^{<\sigma} \subsetneq I_\Gamma^\sigma$ and find some $x \in I_\Gamma$ such that $\sigma \leq |x|_\Gamma < |x|_\Gamma + 1 \leq \sigma$. A contradiction. □

1.6 Positively definable inductive definitions

1.6.1 Definition Let $\mathfrak{S} = (S, \cdots)$ be some infinite structure and $\mathcal{F}$ a class of $\mathcal{L}(\mathfrak{S})$–formulas. We will now and for ever assume that $\mathcal{F}$ contains all atomic formulas and is closed under the positive boolean operations $\vee$ and $\wedge$ and substitution with relations definable by formulas in $\mathcal{F}$. An operator

$$\Gamma\colon S^n \longrightarrow S^n$$

is *$\mathcal{F}$–definable* on the structure $\mathfrak{S}$ iff there is an $\mathcal{F}$–formula $\varphi(\vec{x}, X, \vec{y})$ and a tuple $\vec{a}$ of elements of S such that

$$\Gamma(M) = \{\vec{y} \in S^n \mid \ \mathfrak{S} \models \varphi[\vec{y}, M, \vec{a}]\}.$$

If $\mathcal{F}$ is the class of first order formulas we call Γ first order or – synonymously – *elementarily definable.*
We denote the operator defined by a formula $\varphi(X, \vec{x})$ by Γ_φ and the fixed–point of Γ_φ by I_φ. Anaologously we write shortly I^α_φ instead of $I^\alpha_{\Gamma_\varphi}$, $|\varphi|$ instead of $|\Gamma_\varphi|$ and $|x|_\varphi$ instead of $|x|_{\Gamma_\varphi}$.

1.6.2 Definition The class of X–positive $\mathcal{L}(\mathfrak{S})$–formulas is the least class containing all atomic formulas without occurrences of X and all atomic formulas of the shape $\vec{t} \in X$ which is closed under the positive boolean operations $\vee$ and $\wedge$ and under arbitrary quantifications.

1.6.3 Observation *Any operator Γ_φ which is defined by an X–positive formula is monotone. We call such operators* positive.

Proof Show

$$M \subseteq N \wedge \mathfrak{S} \models \varphi[M, \vec{n}] \ \Rightarrow \ \mathfrak{S} \models \varphi[N, \vec{n}]$$

for all $\vec{n} \in S^k$ by induction on the length of the X–positive formula $\varphi(X, \vec{x})$. $\square$

1.6.4 Definition Let $\mathcal{F}$ be a class of $\mathcal{L}(\mathfrak{S})$–formulas. A relation $R \subseteq S^n$ is called *positively $\mathcal{F}$–inductive* on the structure $\mathfrak{S} = (S, \cdots)$ if there is an X–positive formula $\varphi(X, \vec{x}, \vec{y})$ in $\mathcal{F}$ and a tuple $\vec{s} \in S^m$ such that

$$\vec{x} \in R \ \Leftrightarrow \ (\vec{x}, \vec{s}) \in I_\varphi.$$

In the case that $\mathcal{F}$ is the class of first order formulas we talk about *positively inductive* relations.

1.6.5 Theorem *Every positively inductive relation on a structure $\mathfrak{S}$ is Π^1_1-definable.*

Proof This follows immediately from Observations 1.6.3 and 1.4.5. $\square$

1.6.6 Definition The ordinal

$$\kappa^{\mathfrak{S}} := \sup\{|\varphi| \mid \ \varphi(X, \vec{x}) \text{ is an } X\text{–positive elementary } \mathcal{L}(\mathfrak{S})\text{–formula}\}$$

is called the *closure ordinal* of the structure $\mathfrak{S}$.

1.7 Well–founded trees and positive inductive definitions

We now leave the general situation and return to the structure $\mathbb{N}$ of arithmetic.

1.7.1 Definition A *tree* is a set of (codes for) finite number sequences which is closed under initial sequences. I.e.

$$T \text{ is a tree} \quad :\Leftrightarrow \quad T \subseteq Seq \wedge (\forall t \in T)[s \subseteq t \rightarrow s \in T],$$

where $s \subseteq t$ stands for $lh(s) \leq lh(t) \wedge (\forall i < lh(s))[(s)_i = (t)_i]$.
A path in a tree T is a subset $f \subseteq T$ which is linearly ordered by and closed under $\subseteq$.
A tree is well–founded if it has no infinite path.
For a *node* $s \in T$ in a well–founded tree we define

$$otyp_T(s) := \sup \left\{ otyp_T(s^\frown\langle y \rangle) \,\middle|\, s^\frown\langle y \rangle \in T \right\}$$

and

$$otyp(T) := otyp_T(\langle\, \rangle).$$

1.7.2 Definition Let T be a tree. We define the X–positive formula

$$\varphi_T(X, x) \quad :\Leftrightarrow \quad (\forall y)[x^\frown\langle y \rangle \in T \quad \rightarrow \quad x^\frown\langle y \rangle \in X].$$

1.7.3 Lemma *Let T be a well–founded tree and $s \in T$. Then $s \in I_{\varphi_T}^{otyp_T(s)}$.*

Proof We induct on $otyp_T(s)$. If $otyp_T(s) = 0$ then there is no $x \in S$ such that $s^\frown\langle x \rangle \in T$. Hence $\varphi_T(\emptyset, s)$ which entails $s \in I_{\varphi_T}^0$. Now let $otyp_T(s) > 0$. For every $s^\frown\langle x \rangle \in T$ we have $\alpha := otyp_T(s^\frown\langle x \rangle) < otyp_T(s)$. By induction hypothesis we therefore obtain $s^\frown\langle x \rangle \in I_{\varphi_T}^\alpha \subseteq I_{\varphi_T}^{<otyp_T(s)}$. Hence $\varphi_T(I_{\varphi_T}^{<otyp_T(s)}, s)$ which entails $s \in I_{\varphi_T}^{otyp_T(s)}$. ☐

1.7.4 Corollary *For a well–founded tree T we have $|s|_{\varphi_T} \leq otyp_T(s)$ for all $s \in T$. Hence $|\varphi_T| \leq otyp(T)$.*

For a tree T and a node $s \in T$ we define the restriction of T above s as

$$T{\restriction}s := \left\{ t \in Seq \,\middle|\, s^\frown t \in T \right\}.$$

Apparently $T{\restriction}s$ is again a tree. If $T{\restriction}s$ possesses an infinite path P then there is an $s^\frown\langle y \rangle \in T$ such that the tail of P above s belongs to $T{\restriction}s^\frown\langle y \rangle$. This shows that $T{\restriction}s$ is well-founded if $T{\restriction}s^\frown\langle y \rangle$ is well–founded for all $s^\frown\langle y \rangle \in T$.

1.7.5 Lemma *Let T be a tree and $s \in T$. If $s \in I_{\varphi_T}$ then $T{\restriction}s$ is well–founded and $otyp(T{\restriction}s) \leq |s|_{\varphi_T}$.*

Proof The proof is by induction on $|s|_{\varphi_T}$. If $|s|_{\varphi_T} = 0$ then we have $\varphi_T(\emptyset, s)$, i.e. $(\forall x)[s^\frown\langle x \rangle \notin T]$. Hence $T{\restriction}s = \langle\, \rangle$ and $otyp(T{\restriction}s) = 0$. If $|s|_{\varphi_T} > 0$ we have $(\forall x)[s^\frown\langle x \rangle \in T \Rightarrow s^\frown\langle x \rangle \in I_{\varphi_T}^{<|s|_{\varphi_T}}]$. Then by induction hypothesis $T{\restriction}s^\frown\langle x \rangle$ is well–founded for all $s^\frown\langle x \rangle \in T$ and $otyp(T{\restriction}s^\frown\langle x \rangle) < |s|_\varphi$. This implies that $T{\restriction}s$ is well-founded, too and $otyp(T{\restriction}s) \leq |s|_\varphi$. ☐

As a consequence of Corollary 1.7.4 and Lemma 1.7.5 we obtain

1.7.6 Theorem *A tree T is well-founded iff $\langle\, \rangle \in I_\varphi$ and for well-founded trees T we have $otyp(T) + 1 = |\varphi|$.*

Proof Let T be well–founded. Then $\langle\ \rangle \in I_\varphi$ by Lemma 1.7.3. If conversely $\langle\ \rangle \in I_\varphi$ then $T = T{\restriction}\langle\ \rangle$ is well–founded by Lemma 1.7.5. For a well-founded tree T we get by Corollary 1.7.4 and Lemma 1.7.5

$$otyp_T(s) = otyp(T{\restriction}s) \leq |s|_{\varphi_T} \leq otyp_T(s).$$

Hence

$$T \text{ well–founded} \wedge s \in T \;\Rightarrow\; otyp_T(s) = |s|_{\varphi_T} \tag{1.9}$$

and

$$otyp(T) = otyp(T{\restriction}\langle\ \rangle) = |\langle\ \rangle|_{\varphi_T} < |s|_{\varphi_T}$$

for all $s \in T$. But $|\varphi_T| = \sup\left\{|s|_{\varphi_T} + 1 \,\middle|\, s \in I_\varphi\right\} = |\langle\ \rangle|_{\varphi_T} + 1 = otyp(T) + 1.$ $\square$

1.7.7 Theorem *The Π^1_1–relations on $\mathbb{N}$ are exactly the positively inductive relations .*

Proof By Theorem 1.6.5 we know that all positively inductive relations are Π^1_1 definable. Conversely let R be a Π^1_1–relation. Then there is a Π^1_1–formula $(\forall\vec{X})\phi(\vec{X},\vec{x})$ such that by Theorem 1.7.6

$$\begin{aligned} \vec{s} \in R \;&\Leftrightarrow\; \mathbb{N} \models (\forall\vec{X})\phi(\vec{X},\vec{s}) \\ &\Leftrightarrow\; S_{\phi(\vec{X},\vec{s})} \text{ is well–founded} \\ &\Leftrightarrow\; \langle\ \rangle \in I_{\varphi_{S_{\phi(\vec{X},\vec{s})}}}. \end{aligned} \tag{i}$$

$\square$

1.7.8 Theorem *(Stage Theorem) If $\models^{\!\alpha} (\exists x)[\varphi(X,\vec{x}) \wedge \vec{x} \notin X], \Delta(X,Y)$ for a finite set $\Delta(X,Y)$ of X–positive formulas then $\mathbb{N} \models \bigvee \Delta[I_\varphi^{<2^\alpha}, S]$ for any set $S \subseteq \mathbb{N}$.*

Proof To show the theorem by induction on α we need a more general statement. For an X–positive formula $\varphi(X,\vec{x})$ and a tuple $\vec{t} = (\vec{t}_1,\ldots,\vec{t}_n)$ of terms we introduce the formula

$$\varphi_{\vec{t}_1,\ldots,\vec{t}_n}(X,\vec{x}) \;:\Leftrightarrow\; \varphi(X,\vec{x}) \vee \vec{x} = \vec{t}_1^{\,\mathbb{N}} \vee \ldots \vee \vec{x} = \vec{t}_n^{\,\mathbb{N}}. \tag{i}$$

We claim

$$s \in I_\varphi^\alpha \;\Rightarrow\; I^\beta_{\varphi_s} \subseteq I_\varphi^{\alpha+\beta}. \tag{ii}$$

We prove (ii) by induction on β. Let $x \in I^\beta_{\varphi_s}$. Then $\varphi(I^{<\beta}_{\varphi_s}, x) \vee x = s$ which implies by induction hypothesis implies $\varphi(I_\varphi^{<\alpha+\beta}, x) \vee x = s$. Together with the hypothesis $s \in I_\varphi^\alpha \subseteq I_\varphi^{\alpha+\beta}$ this yields $x \in I_\varphi^{\alpha+\beta}$.
Let S be an arbitrary subset of $\mathbb{N}$. We show

$$\models^{\!\alpha} (\exists x)[\varphi(X,\vec{x}) \wedge \vec{x} \notin X], \vec{t}_1 \notin X, \ldots, \vec{t}_n \notin X, \Delta[X,Y] \;\Rightarrow\; \mathbb{N} \models \Delta[I^{<2^\alpha}_{\varphi_{\vec{t}_1,\ldots,\vec{t}_n}}, S] \tag{iii}$$

for a finite set $\Delta[X,Y]$ of X–positive formulas by induction on α. If (iii) holds by (Ax) then $\Delta[X,Y]$ contains a formula $\vec{s} \in X$ such that $\vec{s}^{\,\mathbb{N}} = \vec{t}_i^{\,\mathbb{N}}$ for some $i \in \{1,\ldots,n\}$. Since $\vec{t}_i^{\,\mathbb{N}} \in I^{<2^\alpha}_{\varphi_{\vec{t}_1,\ldots,\vec{t}_n}}$ we obtain $\mathbb{N} \models \bigvee \Delta[I^{<2^\alpha}_{\varphi_{\vec{t}_1,\ldots,\vec{t}_n}}, S]$.
If the last inference is

$$\begin{aligned} &\left\{ \models^{\!\alpha_\iota} (\exists x)[\varphi(X,\vec{x}) \wedge \vec{x} \notin X], \vec{t}_1 \notin X, \ldots, \vec{t}_n \notin X, \Delta_\iota[X,Y] \,\middle|\, \iota \in J \right\} \;\Rightarrow \\ &\qquad\qquad \models^{\!\alpha} (\exists x)[\varphi(X,\vec{x}) \wedge \vec{x} \notin X], \vec{t}_1 \notin X, \ldots, \vec{t}_n \notin X, \Delta[X,Y] \end{aligned}$$

then we have by induction hypothesis

$$\mathbb{N} \models \bigvee \Delta_\iota[I^{<2^{\alpha_0}}_{\varphi_{\vec{t}_1,\ldots,\vec{t}_n}}, S] \tag{iv}$$

for all $\iota \in J$. Hence $\mathbb{N} \models \bigvee \Delta\iota[I^{<2^\alpha}_{\varphi_{\vec{t}_1,\dots,\vec{t}_n}}, S]$ for all $\iota \in J$ which entails $\mathbb{N} \models \bigvee \Delta[I^{<2^\alpha}_{\varphi_{\vec{t}_1,\dots,\vec{t}_n}}, S]$ by the soundness of the inferences of the infinitary calculus.
The really interesting case is

$$\models^{\alpha_0} (\exists x)[\varphi(X,\vec{x}) \wedge \vec{x} \notin X], \vec{t}_1 \notin X, \dots, \vec{t}_n \notin X, \varphi(X,\vec{s}) \wedge \vec{s} \notin X, \Delta[X,Y] \;\Rightarrow\; \models^{\alpha} (\exists x)[\varphi(X,\vec{x}) \wedge \vec{x} \notin X], \vec{t}_1 \notin X, \dots, \vec{t}_n \notin X, \Delta[X,Y]. \tag{v}$$

From the premise in (v) we obtain

$$\models^{\alpha_0} (\exists x)[\varphi(X,\vec{x}) \wedge \vec{x} \notin X], \vec{t}_1 \notin X, \dots, \vec{t}_n \notin X, \varphi(X,\vec{s}), \Delta[X,Y] \tag{vi}$$

and

$$\models^{\alpha_0} (\exists x)[\varphi(X,\vec{x}) \wedge \vec{x} \notin X], \vec{t}_1 \notin X, \dots, \vec{t}_n \notin X, \vec{s} \notin X, \Delta[X,Y]. \tag{vii}$$

From the induction hypothesis for (vii) we obtain

$$\mathbb{N} \models \bigvee \Delta[I^{<2^{\alpha_0}}_{\varphi_{\vec{t}_1,\dots,\vec{t}_n,\vec{s}}}, S]. \tag{viii}$$

Assuming

$$\mathbb{N} \not\models \bigvee \Delta[I^{<2^{\alpha}}_{\varphi_{\vec{t}_1,\dots,\vec{t}_n}}, S] \tag{ix}$$

we also have

$$\mathbb{N} \not\models \bigvee \Delta[I^{<2^{\alpha_0}}_{\varphi_{\vec{t}_1,\dots,\vec{t}_n}}, S] \tag{x}$$

which together with the induction hypothesis for (vi) imply

$$\mathbb{N} \models \varphi(I^{<2^{\alpha_0}}_{\varphi_{\vec{t}_1,\dots,\vec{t}_n}}, \vec{s}). \tag{xi}$$

Hence

$$\vec{s} \in I^{2^{\alpha_0}}_{\varphi_{\vec{t}_1,\dots,\vec{t}_n}} \tag{xii}$$

which by (ii) implies

$$I^{<2^{\alpha_0}}_{\varphi_{\vec{t}_1,\dots,\vec{t}_n,\vec{s}}} \subseteq I^{<2^{\alpha_0}\cdot 2}_{\varphi_{\vec{t}_1,\dots,\vec{t}_n}} \subseteq I^{<2^{\alpha}}_{\varphi_{\vec{t}_1,\dots,\vec{t}_n}} \tag{xiii}$$

By (xiii) and (viii) we finally obtain

$$\mathbb{N} \models \bigvee \Delta[I^{<2^{\alpha}}_{\varphi_{\vec{t}_1,\dots,\vec{t}_n}}, S]$$

contradicting (ix). So we have (iii). The theorem, however, is a special case of (iii). $\square$

In a special situation we can sharpen the Stage Theorem.

1.7.9 Definition For an order relation $\prec$ let

$$\varphi_{\prec}(X,x) \quad :\Leftrightarrow \quad (\forall y \prec x)[y \in X] \tag{1.10}$$

and $Acc(\prec) := I_{\varphi_\prec}$. We call $Acc(\prec)$ the *accessible part* of $\prec$. By $Acc^\alpha(\prec) := I^\alpha_{\varphi_\prec}$ we denote the αth stage of the accessible part.

1.7.10 Observation *For a well-founded relation $\prec$ we have*

$$x \in Acc^\alpha(\prec) \quad \Leftrightarrow \quad otyp_\prec(x) \le \alpha.$$

More precisely we have $|x|_{\varphi_\prec} = otyp_\prec(x)$ for all $x \in field(\prec)$.

To sharpen the Stage Theorem in the case of an accessibility definition, we need some additional notions. For a transitive relation $\prec$ let

$$\Gamma_\prec(X) := X \cup \{x \mid (\forall y \prec x)[y \in X]\} \tag{1.11}$$

and

$$\Gamma^\alpha_\prec(X) := \Gamma_\prec(X \cup \Gamma^{<\alpha}_\prec(X)) \tag{1.12}$$

where

$$\Gamma^{<\alpha}_\prec(X) := \bigcup_{\xi<\alpha} \Gamma^\xi_\prec(X).$$

Then we obviously have

$$Acc^\alpha(\prec) = \Gamma^\alpha_\prec(\emptyset). \tag{1.13}$$

For a set $M \subseteq Acc(\prec)$ let $\overline{en_M}$ enumerate the set $\{|n|_{\varphi_\prec} \mid n \in Acc(\prec) \setminus M\}$. We define a new operator

$$R^\alpha_\prec(X) := X \cup \{n \in Acc(\prec) \mid |n|_{\varphi_\prec} \leq \overline{en_X}(\alpha)\}. \tag{1.14}$$

Since $\overline{en_{X\cup\{s\}}}(\alpha) \leq \overline{en_X}(\alpha+1)$ we obviously have

$$R^\alpha_\prec(X \cup \{x\}) \subseteq R^{\alpha+1}_\prec(X) \cup \{x\} \tag{1.15}$$

For $R^{<\alpha}_\prec(X) := \bigcup_{\xi<\alpha} R^\xi_\prec(X)$ we claim

$$R^\alpha_\prec(X) = \Gamma_\prec(R^{<\alpha}_\prec(X)). \tag{1.16}$$

To prove the inclusion from left to right in (1.16) let $n \in R^\alpha_\prec(X)$. If $n \in X$, we are done because $X \subseteq \Gamma_\prec(R^{<\alpha}_\prec(X))$. Otherwise we have $|n|_{\varphi_\prec} \leq \overline{en_X}(\alpha)$. Let $m \prec n$. If $\overline{en_X}(\beta) < |m|_{\varphi_\prec} < |n|_{\varphi_\prec} \leq \overline{en_X}(\alpha)$ for all $\beta < \alpha$ we have $m \in X \subseteq R^{<\alpha}_\prec(X)$. Otherwise we have $|m|_{\varphi_\prec} \leq \overline{en_X}(\beta)$ for some $\beta < \alpha$. This shows

$$(\forall m \prec n)[m \in R^{<\alpha}_\prec(X)],$$

i.e. $n \in \Gamma_\prec(R^{<\alpha}_\prec(X))$.
For the opposite direction assume $n \in \Gamma_\prec(R^{<\alpha}_\prec(X))$. Again we are done if $n \in X$. Otherwise we have

$$(\forall m \prec n)[m \notin X \Rightarrow |m|_{\varphi_\prec} < \overline{en_X}(\alpha)].$$

Pick an $\prec$–minimal $m \prec n$ such that $\{k \mid m \prec k \prec n\} \subseteq X$ and $m \notin X$. Then $|m|_{\varphi_\prec} = \overline{en_X}(\beta)$ for some $\beta < \alpha$ and therefore $|n|_{\varphi_\prec} = \overline{en_X}(\beta+1) \leq \overline{en_X}(\alpha)$. If such an m does not exist we have either $|n|_{\varphi_\prec} = \overline{en_X}(0) \leq \overline{en_X}(\alpha)$ or $(\forall m \prec n)(\exists k \prec n)[m \prec k \prec n \wedge k \notin X]$ which implies $(\forall m \prec n)[|m|_{\varphi_\prec} < |k|_{\varphi_\prec} < \overline{en_X}(\alpha)]$. So we have $|n|_{\varphi_\prec} \leq \overline{en_X}(\alpha)$ in any case which implies $n \in R^\alpha_\prec(X)$. ☐

Since $X \cup R^{<\alpha}_\prec(X) = R^{<\alpha}_\prec(X)$ we obtain from (1.16)

$$\Gamma^\alpha_\prec(X) = R^\alpha_\prec(X) \tag{1.17}$$

immediately by induction on α.

1.7.11 Lemma *(Boundedness Lemma) Let* $Prog(X, \prec) :\equiv (\forall x)[(\forall y \prec x)(y \in X) \to x \in X]$ *and assume*

$$\models^{\alpha} \neg Prog(\prec, X), t_1 \notin X, \ldots, t_n \notin X, , \Delta(X, Y)$$

for a transitive relation $\prec$ *and a finite set* $\Delta(X, Y)$ *of* X*–positive formulas. Then*

$$\mathbb{N} \models \bigvee \Delta[\Gamma^{\alpha}_{\prec}(\{t_1^{\mathbb{N}},\dots,t_n^{\mathbb{N}}\}),S]$$

holds for any set $S \subseteq \mathbb{N}$.

Proof We use (1.17) and prove

$$\models^{\alpha} \neg Prog(\prec,X), t_1 \notin X,\dots,t_n \notin X,,\Delta(X,Y) \;\Rightarrow\; \mathbb{N} \models \bigvee \Delta[R^{\alpha}_{\prec}(\{t_1^{\mathbb{N}},\dots,t_n^{\mathbb{N}}\}),S]. \quad \text{(i)}$$

The proof parallels that of Theorem 1.7.8 but due to (1.15) with a sharper bound.
If (i) holds by (Ax) then $\Delta[X,Y]$ contains a formula $s \in X$ such that $s^{\mathbb{N}} = t_i^{\mathbb{N}}$ for some $i \in \{1,\dots,n\}$. Since $t_i^{\mathbb{N}} \in R^{\alpha}_{\prec}(\vec{t}_1,\dots,\vec{t}_n)$ we obtain $\mathbb{N} \models \bigvee \Delta[R^{\alpha}_{\prec}(\vec{t}_1,\dots,\vec{t}_n),S]$.
If the last inference is

$$\{\models^{\alpha_\iota} \neg Prog(\prec,X), t_1 \notin X,\dots,t_n \notin X, \Delta_\iota[X,Y] \mid \iota \in J\} \;\Rightarrow\; \models^{\alpha} \neg Prog(\prec,X), t_1 \notin X,\dots,t_n \notin X, \Delta[X,Y]$$

then we have by induction hypothesis

$$\mathbb{N} \models \bigvee \Delta_\iota[R^{\alpha_0}_{\prec}(\vec{t}_1,\dots,\vec{t}_n),S] \quad \text{(ii)}$$

for all $\iota \in J$. Hence $\mathbb{N} \models \bigvee \Delta\iota[R^{\alpha}_{\prec}(\vec{t}_1,\dots,\vec{t}_n),S]$ for all $\iota \in J$ which by the soundness of the inferences entails $\mathbb{N} \models \bigvee \Delta[R^{\alpha}_{\prec}(\vec{t}_1,\dots,\vec{t}_n),S]$.
The really interesting case is

$$\models^{\alpha_0} \neg Prog(\prec,X), t_1 \notin X,\dots,t_n \notin X, (\forall y \prec s)[y \in X] \wedge s \notin X, \Delta[X,Y] \;\Rightarrow\; \models^{\alpha} \neg Prog(\prec,X), t_1 \notin X,\dots,t_n \notin X, \Delta[X,Y]. \quad \text{(iii)}$$

From the premise in (iii) we obtain

$$\models^{\alpha_0} \neg Prog(\prec,X), t_1 \notin X,\dots,t_n \notin X, (\forall y \prec s)[y \in X], \Delta[X,Y] \quad \text{(iv)}$$

and

$$\models^{\alpha_0} \neg Prog(\prec,X), t_1 \notin X,\dots,t_n \notin X, s \notin X, \Delta[X,Y]. \quad \text{(v)}$$

From the induction hypothesis for (v) we obtain

$$\mathbb{N} \models \bigvee \Delta[R^{\alpha_0}_{\prec}(\vec{t}_1,\dots,\vec{t}_n,\vec{s}),S] \quad \text{(vi)}$$

which together with (1.15) imply

$$\mathbb{N} \models \bigvee \Delta[R^{\alpha_0+1}_{\prec}(\vec{t}_1,\dots,\vec{t}_n) \cup \{\vec{s}\},S]. \quad \text{(vii)}$$

Assuming

$$\mathbb{N} \not\models \bigvee \Delta[R^{\alpha}_{\prec}(\vec{t}_1,\dots,\vec{t}_n),S] \quad \text{(viii)}$$

we also have

$$\mathbb{N} \not\models \bigvee \Delta[R^{\alpha_0}_{\prec}(\vec{t}_1,\dots,\vec{t}_n),S] \quad \text{(ix)}$$

which together with the induction hypothesis for (iv) imply

$$\mathbb{N} \models (\forall y \prec s)[y \in R^{\alpha_0}_{\prec}(\vec{t}_1,\dots,\vec{t}_n)] \quad \text{i.e.} \quad s \in \Gamma_{\prec}(R^{\alpha_0}_{\prec}(\{t_1,\dots,t_n\})). \quad \text{(x)}$$

Hence

$$s \in R^{\alpha_0+1}_{\prec}(\vec{t}_1,\dots,\vec{t}_n) \subseteq R^{\alpha}_{\prec}(\vec{t}_1,\dots,\vec{t}_n). \quad \text{(xi)}$$

By (xi) and (vii) we finally obtain

$$\mathbb{N} \models \bigvee \Delta[R^{\alpha}_{\prec}(t_1, \ldots, t_n), S].$$

So we have (i). The Lemma, however, follows from (i) and (1.16). $\square$

From the Boundedness Lemma together with (1.13) we obtain the next theorem.

1.7.12 Theorem *(Boundedness Theorem) For any arithmetical definable transitive relation $\prec$ and a finite set of X–positive arithemetical formulas we have*

$$\models^{\alpha} \neg Prog(\prec, X), \Delta[X, Y] \quad \Rightarrow \quad \mathbb{N} \models (\forall Y) \left(\bigvee \Delta[Acc^{\alpha}(\prec), Y] \right).$$

1.7.13 Theorem *It is $\kappa^{\mathbb{N}} = \omega_1^{CK}$.*

Proof If $\prec$ is a recursive well–ordering then by Observation 1.7.10 we obtain $otyp(\prec) \leq |\varphi_{\prec}| \leq \kappa^{\mathbb{N}}$. Since $\omega_1^{CK} = \sup \{ otyp(\prec) | \prec \text{ is recursive} \}$ this implies $\omega_1^{CK} \leq \kappa^{\mathbb{N}}$.
For an X– positive formula $\varphi(X, \vec{x})$ we have

$$\begin{aligned} \vec{s} \in I_{\varphi} &\Leftrightarrow \mathbb{N} \models (\forall X)[(\forall \vec{x})(\varphi(X, \vec{x}) \to \vec{x} \in X) \to \vec{s} \in X] \\ &\Leftrightarrow (\exists \alpha < \omega_1^{CK}) \left[\models^{\alpha} \neg (\forall \vec{x})(\varphi(X, \vec{x}) \to \vec{x} \in X), \vec{s} \in X \right] \\ &\Rightarrow (\exists \alpha < \omega_1^{CK}) \left[\mathbb{N} \models \vec{s} \in I_{\varphi}^{2^{\alpha}} \right]. \end{aligned} \tag{i}$$

Since $\alpha < \omega_1^{CK}$ implies $2^{\alpha} < \omega_1^{CK}$ we have $|\varphi| \leq \omega_1^{CK}$ for all positive elementary inductive definitions. Hence $\kappa^{\mathbb{N}} \leq \omega_1^{CK}$. $\square$

1.8 The Π^1_1–ordinal of an axiom system

1.8.1 Definition For a theory Ax in the language of (2^{nd}–order) arithmetic we define

$$\|Ax\|_{\Pi^1_1} := \sup \{ tc(F) | \; F \in \Pi^1_1 \wedge Ax \vdash F \}.$$

We call $\|Ax\|_{\Pi^1_1}$ the Π^1_1–ordinal of Ax.

We are going to show that the Π^1_1–ordinal and the proof theoretic ordinal defined in (1.1) coincide.

1.8.2 Lemma *For a well–ordering $\prec$ we have*

$$otyp(\prec) \leq tc(TI(\prec, X)).$$

Proof Apply the Boundedness Theorem (Theorem 1.7.12) and Observation 1.7.10. $\square$

For a primitive recursive well-ordering $\prec$ and $s \in field(\prec)$ we obtain by an easy induction on $otyp_{\prec}(s)$

$$\models^{5 \cdot (otyp_{\prec}(s)+1)} \neg (\forall x)[(\forall y \prec x)(y \in X) \to x \in X], s \in X. \tag{1.18}$$

From (1.18) and Lemma 1.8.2 we obtain the following theorem.

1.8.3 Theorem *For an arithmetical definable well–ordering $\prec$ we have*

$$otyp(\prec) = tc(TI(\prec, X)).$$

We just want to remark that this can be extended to Σ^1_1–definable well–orderings. Details are in [2].

1.8.4 Lemma *For any axiom system Ax in the language of (2^{nd}–order) arithmetic we have*

$$\|Ax\| \leq \|Ax\|_{\Pi^1_1}.$$

Proof This is an immediate consequence of Theorem 1.8.3. □

1.8.5 Lemma *If Ax is an axiom system comprising PA then $\|Ax\| = \|Ax\|_{\Pi^1_1}$.*

Sketch of the proof Assume that Ax is a theory comprising PA and let $(\forall\vec{Y})F(\vec{Y})$ be a Π^1_1-sentence. Denote by $\prec_{S_{F(\vec{Y})}}$ the KLEENE–BROUWER ordering in the search tree $S_{F(\vec{Y})}$ for $F(\vec{Y})$ and assume that $Ax \not\vdash TI(\prec_{S_{F(\vec{Y})}}, X)$. Then there is a model $\mathfrak{M} \models Ax$ and an assignment $T \subseteq \mathfrak{M}$ for X such that $\mathfrak{M} \not\models TI(\prec_{S_{F(\vec{Y})}}, X)[T]$. Therefore there is an infinite path, say $P \subseteq \mathfrak{M}$, through $S_{F(\vec{Y})}$ which is definable by an first order formula with parameter T. According to the Semantical Main Lemma we get assignments $\Phi(Y_i) \subseteq \mathfrak{M}$ for all Y_i belonging to $\vec{Y}$ which are definable by first order formulas with parameter T. Since we have induction in $\mathfrak{M}$ for first order formulas we obtain $\mathfrak{M} \not\models F(\vec{Y})[\Phi]$ as in the proof of the Semantical Main Lemma using a local truth predicate. Hence $Ax \not\vdash F(\vec{Y})$ and we have shown

$$Ax \vdash F(\vec{Y}) \Rightarrow Ax \vdash TI(\prec_{S_{F(\vec{Y})}}, X).$$

Since $\prec_{S_{F(\vec{Y})}}$ is primitive recursively definable and we have $tc(F(\vec{Y})) \leq otyp(\prec_{S_{F(\vec{Y})}}) \leq \|Ax\|$ for $Ax \vdash F(\vec{Y})$ this implies $\|Ax\|_{\Pi^1_1} \leq \|Ax\|$. □

1.8.6 Theorem *Let Ax be a Σ^1_1–set of arithmetical sentences. The theory Ax is Π^1_1–sound iff $\|Ax\| < \omega_1^{CK}$.*

Proof If $\|Ax\| < \omega_1^{CK}$ we have $\|Ax\|_{\Pi^1_1} < \omega_1^{CK}$ and thus $\mathbb{N} \models F$ for all F such that $Ax \vdash F$ by Theorem 1.3.17. If conversely Ax is Π^1_1–sound then $\{tc(F) \mid Ax \vdash F\}$ is a Σ^1_1–definable subset of ω_1^{CK}. Hence $\|Ax\| = \|Ax\|_{\Pi^1_1} = \sup\{tc(F) \mid Ax \vdash F\} < \omega_1^{CK}$. □

The following theorem is an immediate consequence of Theorem 1.8.3.

1.8.7 Theorem

$$\begin{aligned}\|Ax\| &\leq \sup\{otyp(\prec) \mid \prec \in PR \wedge Ax \vdash TI(\prec)\}\\ &\leq \sup\{otyp(\prec) \mid \prec \in \Pi^\infty_0 \wedge Ax \vdash TI(\prec)\}\\ &\leq \sup\{otyp(\prec) \mid \prec \in \Sigma^1_1 \wedge Ax \vdash TI(\prec)\}\\ &\leq \|Ax\|_{\Pi^1_1} = \|Ax\|\end{aligned}$$

1.8.8 Theorem *(Kreisel)* *Let Ax be a theory which contains PA. Then*

$$\|Ax\|_{\Pi^1_1} = \|Ax + F\|_{\Pi^1_1}$$

holds for every true Σ^1_1–sentence F.

Proof Assume

$$Ax + F \vdash TI(\prec, X) \tag{i}$$

for a primitive recursive ordering $\prec$. Then

$$Ax \vdash \neg F \vee TI(\prec, X) \tag{ii}$$

which implies

$$\models^{\alpha} \neg\, F, TI(\prec, X) \tag{iii}$$

for some $\alpha < \|Ax\|_{\Pi^1_1}$. For $F \equiv (\exists Y)F_0(Y)$ we obtain from (iii)

$$\models^{\alpha} \neg\, Prog(\prec, X), \neg F_0(Y), (\forall x \in \mathit{field}(\prec))[x \in X] \tag{iv}$$

which by the Boundedness Theorem (Theorem 1.7.12) implies

$$\mathbb{N} \models \neg F_0[S] \vee (\forall x \in \mathit{field}(\prec))[x \in Acc^{\alpha}(\prec)] \tag{v}$$

for every set $S \subseteq \mathbb{N}$. Since $\mathbb{N} \models (\exists Y)F_0(Y)$ there is a set $S \subseteq N$ such that $\mathbb{N} \models F_0[S]$ and we obtain from (v)

$$\mathbb{N} \models (\forall x \in \mathit{field}(\prec))[otyp_{\prec}(x) \leq \alpha]. \tag{vi}$$

Hence

$$\|Ax + F\|_{\Pi^1_1} = \|Ax + F\| \leq \|Ax\|_{\Pi^1_1}.$$

The opposite inequality holds obviously. $\square$

It follows from KREISEL's theorem that the Π^1_1–ordinal of an axiom system does not characterize its arithmetical power. Therefore more refined notions of proof theoretic ordinals have been developed (e.g. in [12]). Most recently BEKLEMISHEV could define the Π^0_n–ordinal of a theory for all levels of the arithmetical hierarchy using iterated reflection principles. All these notions, however, need a representation of ordinals either by notation systems or by elementarily definable order relations on $\mathbb{N}$. But it can be shown that different representations satisfying mild conditions yield the same proof theoretic ordinals.

In this lecture we will concentrate on the computation of the Π^1_1 ordinals. In the second part Weierman will say something about the Π^0_2–ordinal of PA which characterizes its provably recursive functions. We just want to mention that the computations we are going to show are *profound ordinal analyses* in the sense of [11] and [12] and thus also comprise a computation of the the Π^0_2 ordinals. But we don't want to give details about that here.

2. The ordinal analysis for PA

2.1 Logic

To fix the logical frame we introduce a formal system for first order logic (without identity) which is based on a one sided sequent calculus à la TAIT.

2.1.1 Definition

(AxL) $\vdash^m \Delta, A, \neg A$ for any m, if A is an atomic formula

($\vee$) If $\vdash^{m_0} \Delta, A_i$ for some $i \in \{1, 2\}$, then $\vdash^m \Delta, A_1 \vee A_2$ for all $m > m_0$

($\wedge$) If $\vdash^{m_i} \Delta, A_i$ and $m_i < m$ for all $i \in \{1, 2\}$, then $\vdash^m \Delta, A_1 \wedge A_2$

($\exists$) If $\vdash^{m_0} \Delta, A(t)$, then $\vdash^m \Delta, (\exists x)A(x)$ for all $m > m_0$

($\forall$) If $\vdash^{m_0} \Delta, A(u)$ and u not free in $\Delta, (\forall x)A(x)$, then $\vdash^m \Delta, (\forall x)A(x)$ for all $m > m_0$.

One should observe the similarity of this calculus to the truth definition given in Definition 1.3.10. By an easy induction on m we obtain

2.1.2 Lemma *If* $\vdash^m \Delta$ *then* $\models \bigvee \Delta$.

Using the technique of search trees one can also prove the completeness of this calculus. I.e. we have

2.1.3 Theorem *A formula of first order predicate calculus is logically valid iff there is a natural number m such that $\vdash^m F$.*

We will omit the proof which is very similar to the proof of the ω–completeness theorem. One has to modify the definition of search tree in the obvious way. The Syntactical Main Lemma then follows as before. To show the Semantical Main Lemma one assumes that the search tree contains an infinite path and constructs a term model together with an assignment of terms to the free variables such that all formulas occurring in the infinite path become false under this assignment.
One of the consequences of the completeness theorem for the TAIT–calculus is the admissibility of the cut rule. We obtain

2.1.4 Theorem *If $\vdash^m \Delta, F$ and $\vdash^n \Delta, \neg F$ then there is a k such that $\vdash^k \Delta$.*

But Theorem 2.1.5 does not say anything about the size of k. Therefore one augments the clauses in Definition 2.1.1 by a cut rule

(Cut) If $rnk(F) < r$, $\vdash^m_r \Delta, F$ and $\vdash^m_r , \Delta \neg F$ then $\vdash^n_r \Delta$ for all $n > m$

and replaces $\vdash^m \Delta, \ldots$ in all clauses by $\vdash^m_r \Delta, \ldots$. The subscript r is thus a measure for the complexity of all cut formulas occurring in the derivation. Obviously we have $\vdash^m \Delta \;\Leftrightarrow\; \vdash^m_0 \Delta$.

2.1.5 Theorem *(Gentzen's Hauptsatz) If $\vdash^m_r \Delta$ then $\vdash^{2_r(m)}_0 \Delta$ where $2_r(x)$ is defined by $2_0(x) = x$ and $2_{n+1}(x) = 2^{2_n(x)}$*

We will not prove the Hauptsatz but leave it as an exercise which should be solved after having seen the cut–elimination for the semi–formal calculus which we are going to introduce in 2.3.3.

2.1.6 Theorem *Let $\Delta(\vec{x})$ be a finite set of formulas in the the language of arithmetic with all number variables shown. Then $\overset{m}{\vdash} \Delta(\vec{x})$ implies $\overset{m}{\models} \Delta(\underline{\vec{n}})$ for all tuples $\underline{\vec{n}}$ of numerals.*

The *proof* of Theorem 2.1.6 is straightforward by induction on m using the obvious property

$$s^{\mathbb{N}} = t^{\mathbb{N}} \text{ and } \overset{\alpha}{\models} \Delta(s) \;\Rightarrow\; \overset{\alpha}{\models} \Delta(t). \tag{2.1}$$

2.2 The theory NT

Instead of analyzing the axioms in PA we do that for a richer language which has constants for all primitive recursive functions and relations.
The language $\mathcal{L}(NT)$ is a first order language which contains set parameters denoted by capital Latin letters X, Y, Z, X_1, ... and constants for 0 and all primitive recursive functions and relations. We assume that the symbols for primitive recursive functions are built up from the symbols C_k^n for the constant function, P_k^n for the projection on the n-th component, S for the successor function by a substitution operator Sub and the recursion operator Rec.
The theory NT comprises the universal closure of the following formulas:

The successor axioms

$$(\forall x)[\neg \underline{0} = Sx]$$
$$(\forall x)(\forall y)[S(x) = S(y) \Rightarrow x = y]$$

The defining axioms for function and relation symbols which are the universal closures of the following formulas

$$C_k^n(x_1, \dots, x_n) = \underline{k}$$
$$P_k^n(x_1, \dots, x_n) = x_k$$
$$Sub(g, h_1, \dots, h_m)(x_1, \dots, x_n) = g(h_1(x_1, \dots, x_n)) \dots (h_m(x_1, \dots, x_n))$$
$$Rec(g, h)(0, x_1, \dots, x_n) = g(x_1, \dots, x_n)$$
$$Rec(g, h)(Sy, x_1, \dots, x_n) = h(y, Rec(g, h)(y, x_1, \dots, x_n), x_1, \dots, x_n)$$
$$(x_1, \dots, x_n) \in R \leftrightarrow \chi_R(x_1, \dots, x_n) = 0$$

The scheme of Mathematical Induction

$$F(\underline{0}) \wedge (\forall x)[F(x) \to F(S(x))] \to (\forall x)F(x)$$

for all $\mathcal{L}(NT)$-formulas $F(u)$.

The identity axioms

$$(\forall x)[x = x]$$
$$(\forall x)(\forall y)[x = y \to y = x]$$
$$(\forall x)(\forall y)(\forall z)[x = y \wedge y = z \to x = z]$$
$$(\forall \vec{x})(\forall \vec{y})[x_1 = y_1 \wedge \dots \wedge x_n = y_n \;\to\; f(x_1, \dots, x_n) = f(y_1, \dots, y_n)]$$
$$(\forall \vec{x})(\forall \vec{y})[x_1 = y_1 \wedge \dots \wedge x_n = y_n \;\to\; (R(x_1, \dots, x_n) \to R(y_1, \dots, y_n))]$$
$$(\forall x)(\forall y)[x = y \;\to\; (x \in X \to y \in X)].$$

If $NT \vdash F$ there are finitely many axioms $A_1, \dots, A_n$ of NT such that $\neg A_1 \vee \cdots \vee \neg A_n \vee F$ is logically valid. Due to the completeness of the TAIT–calculus (cf. Theorem 2.1.3) we therefore have the following theorem.

2.2.1 Theorem *Let F be a formula which is provable in NT. Then there are finitely many axioms $A_1, \dots, A_n$ and an $m < \omega$ such that $\models^{m} \neg A_1, \dots, \neg A_n, F$.*

2.3 The upper bound

It follows from Theorems 2.1.3 and 2.1.6 that we have

$$\models^{m} \neg A_1, \dots, \neg A_n, F \tag{2.2}$$

for the provable pseudo Π^1_1–sentences of NT. In order to determine the Π^1_1–ordinal of NT we have to compute $tc(F)$. Our strategy will be the following. First we compute an upper bound, say α, for the truth complexities of all axioms in NT. This gives

$$\models^{\alpha} A_{\ i} \tag{2.3}$$

for all axioms A_i. Then we extend the infinitary calculus for the truth definition to an infinitary calculus with cut and use the cut rule to get rid of all the axioms. Then we eliminate the cuts. If we succeed in controlling the length of an infinite derivation during the cut elimination procedure we will obtain an upper bound for the truth complexity of F.
We start with the computation of the truth complexities of the axioms of NT.
All numerical instances of the defining axioms for primitive recursive function and relations belong to the diagram $\mathsf{D}(\mathbb{N})$. Therefore we obtain their universal closure by a finite number of applications of the $\bigwedge$–rule. The same is true for all identity axioms except the last one. But there we observe

$$\models^{5} (\forall x)(\forall y)[x = y \quad \to \quad (x \in X \to y \in X)]\,.$$

So we have

$$tc(F) < \omega \tag{2.4}$$

for all mathematical and identity axioms except induction. What really needs checking is the truth complexity of the scheme of Mathematical Induction. Here we need the following lemmas.

2.3.1 Lemma *(Tautology Lemma) For every $\mathcal{L}(NT)$-formula we have* $\stackrel{2\cdot rnk(F)}{\models\!\!=} \Delta, \neg F, F$.

The proof is by induction on $rnk(F)$.

2.3.2 Lemma *(Induction Lemma) For any natural number n and any $\mathcal{L}(NT)$-sentence $F(\underline{n})$ we have*

$$\stackrel{2\cdot[rnkF(\underline{n}))+n]}{\models\!\!=} \neg F(\underline{0}), \neg(\forall x)[F(x) \to F(S(x))], F(\underline{n})\,.$$

The proof by induction on n is very similar to that of (1.18). For $n = 0$ this is an instance of the Tautology Lemma. For the induction step we have

$$\stackrel{2\cdot[rnkF(\underline{n}))+n]}{\models\!\!=} \neg F(\underline{0}), \neg(\forall x)[F(x) \to F(S(x))], F(\underline{n}) \tag{i}$$

by the induction hypothesis and obtain

$$\stackrel{2\cdot rnkF(\underline{n}))}{\models\!\!=} \neg F(\underline{0}), \neg(\forall x)[F(x) \to F(S(x))], \neg F(\underline{S(n)}), F(\underline{S(n)}) \tag{ii}$$

by the Tautology Lemma. From (i) and (ii) we get by ($\bigwedge$)

$$\stackrel{2\cdot[rnkF(\underline{n}))+n]+1}{\models\!\!=} \neg F(\underline{0}), \neg(\forall x)[F(x) \to F(S(x))], F(\underline{n}) \wedge \neg F(\underline{S(n)}), F(\underline{S(n)})\,. \tag{iii}$$

By a clause ($\exists$) we finally obtain

$$\models^{2\cdot[rnk F(\underline{n}))+n]+2} \neg F(\underline{0}), \neg(\forall x)[F(x) \to F(S(x))], F(\underline{S(n)})\ . \qquad \square$$

By Lemma 2.3.2 we have $tc(G) \leq \omega + 4$ for all instances G of the Mathematical Induction Scheme. Together with (2.4) we get

$$\models^{\omega+4} A_i \text{ i.e. } tc(A_i) \leq \omega + 4 \tag{2.5}$$

for all identity and non-logical axioms A_i of NT.

2.3.3 Definition For a finite set Δ of pseudo Π^1_1–sentences we define the semi–formal provability relation $\vdash^{\alpha}_{\rho} \Delta$ inductively by the following clauses

(Ax) $s^{\mathbb{N}} = t^{\mathbb{N}} \ \Rightarrow\ \vdash^{\alpha}_{\rho} \Delta, s \in X, t \notin X$

$(\bigwedge)$ If $F \in \bigwedge$–type $\cap\ \Delta$ and $(\forall G \in CS(F))\left[\vdash^{\alpha_G}_{\rho} \Delta, G\ \&\ \alpha_G < \alpha\right]$ then $\vdash^{\alpha}_{\rho} \Delta$

$(\bigvee)$ If $F \in \bigvee$–type $\cap\ \Delta$ and $(\exists G \in CS(F))[\vdash^{\alpha_G}_{\rho} \Delta, G\ \&\ \alpha_G < \alpha]$ then $\vdash^{\alpha}_{\rho} \Delta$

(cut) If $\vdash^{\alpha_0}_{\rho} \Delta, F$; $\vdash^{\alpha_0}_{\rho} \Delta, \neg F$ and $rnk(F) < \rho$ then $\vdash^{\alpha}_{\rho} \Delta$ for all $\alpha > \alpha_0$.

We call F the *main formula* of the clauses $(\bigwedge)$ and $(\bigvee)$. The main formulas of an axiom (Ax) are $s \in X$ and $t \notin X$. A cut possesses no main formula.
Observe that we have

$$\vdash^{\alpha}_{0} \Delta \ \Leftrightarrow\ \models^{\alpha} \Delta\ . \tag{2.6}$$

Hence

$$\vdash^{m} \Delta \ \Rightarrow\ \vdash^{m}_{0} \Delta \tag{2.7}$$

by Theorem 2.1.6. There are some obvious properties of $\vdash^{\alpha}_{\rho} \Delta$ which are proved by induction on α.

2.3.4 Lemma *(Soundness)* *If $\vdash^{\alpha}_{\rho} F_1, \ldots, F_n$ then $\mathbb{N} \models (F_1 \vee \cdots \vee F_n)[\Phi]$ for every assignment Φ of subsets of $\mathbb{N}$ to the set parameters in $F_1, \ldots, F_n$.*

2.3.5 Lemma *(Structural Lemma)* *If $\vdash^{\alpha}_{\rho} \Delta$ then $\vdash^{\beta}_{\sigma} \Gamma$ holds for all $\beta \geq \alpha$, $\sigma \geq \rho$ and $\Gamma \supseteq \Delta$.*

2.3.6 Lemma *(Inversion Lemma)* *If $F \in \bigwedge$–type and $\vdash^{\alpha}_{\rho} \Delta, F$ then $\vdash^{\alpha}_{\rho} \Delta, G$ for all $G \in CS(F)$.*

2.3.7 Lemma *($\vee$–Exportation)* *If $\vdash^{\alpha}_{\rho} \Delta, F_1 \vee \cdots \vee F_n$ then $\vdash^{\alpha}_{\rho} \Delta, F_1, \ldots, F_n$.*

2.3.8 Lemma *If $F \in \mathsf{D}(\mathbb{N})$ and $\vdash^{\alpha}_{\rho} \Delta, \neg F$ then $\vdash^{\alpha}_{\rho} \Delta$.*

2.3.9 Lemma *(Reduction Lemma)* *Let $\rho = rnk(F)$ for $F \in \bigwedge$–type, $F \equiv (s \in X)$ or $F \equiv (s \notin X)$. If $\vdash^{\alpha}_{\rho} \Delta, F$ and $\vdash^{\beta}_{\rho} \Gamma, \neg F$ then $\vdash^{\alpha+\beta}_{\rho} \Delta, \Gamma$.*

Proof The proof is by induction on β. If $\neg F$ is not the main formula in $\frac{\beta}{\rho}\, \Gamma, \neg F$ then we have the premises $\frac{\beta_\iota}{\rho}\, \Gamma_\iota, F$ for $\iota \in I$. If $I = \emptyset$ then $\Gamma \cap \mathsf{D}(\mathbb{N}) \neq \emptyset$ which entails $\Delta, \Gamma \cap \mathsf{D}(\mathbb{N}) \neq \emptyset$ and we obtain $\frac{\alpha+\beta}{\rho}\, \Delta, \Gamma$ by an inference $(\bigwedge)$ with empty premise. Otherwise we get

$$\frac{\alpha+\beta_\iota}{\rho}\, \Delta, \Gamma \tag{i}$$

by the induction hypothesis and obtain $\frac{\alpha+\beta}{\rho}\, \Delta, \Gamma$ from (i) by the same inference.
Now assume that $\neg F$ is the main formula. If $\rho = 0$ then $\neg F$ is atomic. If $F \in \bigwedge$–type we have $F \in \mathsf{D}(\mathbb{N})$ and obtain $\frac{\alpha+\beta}{\rho}\, \Delta, \Gamma$ by Lemma 2.3.8 and Lemma 2.3.4. If $F \equiv (s \in X)$ we show

$$\frac{\alpha}{\rho}\, \Delta, \Gamma \tag{ii}$$

by a side induction on α. First we observe that there is a formula $t \in X$ with $t^{\mathbb{N}} = s^{\mathbb{N}}$ in Γ since $\frac{\beta}{\rho}\, \Gamma, \neg F$ holds by (Ax) . If F is not the main formula of $\frac{\alpha}{\rho}\, \Delta, F$ then we have the premises $\frac{\alpha_\iota}{\rho}\, \Delta_\iota, F$ for $\iota \in I$. If $I = \emptyset$ we get $\frac{\alpha}{\rho}\, \Delta, \Gamma$ directly and for $I \neq \emptyset$ from the induction hypothesis by the same inference. If F is the main formula we are in the case of (Ax) which entails that there is a formula $r \notin X$ in Δ with $r^{\mathbb{N}} = s^{\mathbb{N}} = t^{\mathbb{N}}$. But then we obtain $\frac{\alpha}{\rho}\, \Delta, \Gamma$ by (Ax). The case $F \equiv (s \notin X)$ is symmetrical. From (ii) we get $\frac{\alpha+\beta}{\rho}\, \Delta, \Gamma$ by the Structural Lemma.
Now assume $\rho > 0$. Then $\neg F \in \bigvee$–type and we have the premise

$$\frac{\beta_0}{\rho}\, \Gamma, \neg F, \neg G \tag{iii}$$

for some $G \in CS(F)$. Then we obtain

$$\frac{\alpha+\beta_0}{\rho}\, \Delta, \Gamma, \neg G \tag{iv}$$

by induction hypothesis. From $\frac{\alpha}{\rho}\, \Delta, F$ we obtain

$$\frac{\alpha+\beta_0}{\rho}\, \Delta, \Gamma, G \tag{v}$$

by the Inversion and the Structural Lemma. Since $rnk(G) < rnk(F) = \rho$ we obtain the claim from (iv) and (v) by (cut). ☐

2.3.10 Lemma *(Basic Elimination Lemma)* *If* $\frac{\alpha}{\rho+1}\, \Delta$ *then* $\frac{2^\alpha}{\rho}\, \Delta$.

Proof Induction on α. If the last inference is not a cut of complexity ρ we obtain the claim immediately from the induction hypothesis and the fact that $\lambda\xi\,.\,2^\xi$ is order preserving. The critical case is a cut $\frac{\alpha_0}{\rho+1}\, \Delta, F$; $\frac{\alpha_0}{\rho+1}\, \Delta, \neg F \Rightarrow \frac{\alpha}{\rho+1}\, \Delta$ with $rnk(F) = \rho$. By the induction hypothesis and the Reduction Lemma we obtain $\frac{2^{\alpha_0}+2^{\alpha_0}}{\rho}\, \Delta$ and we have $2^{\alpha_0} + 2^{\alpha_0} = 2^{\alpha_0+1} \leq 2^\alpha$. ☐

Observe that our language so far only comprises formulas of finite rank. But we have designed the semi–formal calculus in such a way that it will also work for languages with formulas of complexities $\geq \omega$. The following results masters also this situation.

2.3.11 Lemma *(Predicative Elimination Lemma)* *If* $\frac{\alpha}{\beta+\omega^\rho}\, \Delta$ *then* $\frac{\varphi_\rho(\alpha)}{\beta}\, \Delta$.

Proof Induction on ρ with side induction on α. For $\rho = 0$ we obtain $\frac{2^\alpha}{\beta}\, \Delta$ by the Basic Elimination Lemma which, since $2^\alpha \leq \omega^\alpha = \varphi_0(\alpha)$, entails the claim. Now assume $\rho > 0$. If the last clause was not a cut of rank $\geq \beta$ we obtain the claim from the induction hypotheses and the fact that the functions φ_ρ are order preserving. Therefore assume that the last inference is

$$\left|\frac{\alpha_0}{\beta+\omega^\rho}\right. \Delta, F \quad \left|\frac{\alpha_0}{\beta+\omega^\rho}\right. \Delta, \neg F \quad \Rightarrow \quad \left|\frac{\alpha}{\beta+\omega^\rho}\right. \Delta$$

such that $\beta \leq rnk(F) < \beta+\omega^\rho$. But then there is an ordinal ϕ such that $rnk(F) = \beta + \phi$ which, writing ϕ in CANTOR normal form, means $rnk(F) = \beta+\omega^{\sigma_1}+\ldots+\omega^{\sigma_n} < \beta+\omega^\rho$. Hence $\sigma_1 < \rho$ and, putting $\sigma := \sigma_1$, we get $rnk(F) < \beta+\omega^\sigma\cdot(n+1)$. By the side induction hypothesis we have $\left|\frac{\varphi_\rho(\alpha_0)}{\beta}\right. \Delta, F$ and $\left|\frac{\varphi_\rho(\alpha_0)}{\beta}\right. \Delta, \neg F$. By a cut it follows $\left|\frac{\varphi_\rho(\alpha_0)+1}{\beta+\omega^\sigma\cdot(n+1)}\right. \Delta$. If we define $\varphi_\sigma^0(\xi) := \xi$ and $\varphi_\sigma^{n+1}(\xi) := \varphi_\sigma(\varphi_\sigma^n(\xi))$ we obtain from $\sigma < \rho$ by $n+1$–fold application of the main induction hypothesis $\left|\frac{\varphi_\sigma^{n+1}(\varphi_\rho(\alpha_0)+1)}{\beta}\right. \Delta$. Finally we show $\varphi_\sigma^n(\varphi_\rho(\alpha_0)+1) < \varphi_\rho(\alpha)$ by induction on n. For $n = 0$ we have $\varphi_\sigma^0(\varphi_\rho(\alpha_0)+1) = \varphi_\rho(\alpha_0)+1 < \varphi_\rho(\alpha)$ since $\alpha_0 < \alpha$ and $\varphi_\rho(\alpha) \in Cr(0)$. For the induction step we have $\varphi_\sigma^{n+1}(\varphi_\rho(\alpha_0)+1) = \varphi_\sigma(\varphi_\sigma^n(\varphi_\rho(\alpha_0)+1)) < \varphi_\rho(\alpha)$ since $\sigma < \rho$ and $\varphi_\sigma^n(\varphi_\rho(\alpha_0)+1) < \varphi_\rho(\alpha)$ by the induction hypothesis. Hence $\left|\frac{\varphi_\rho(\alpha)}{\beta}\right. \Delta$. $\square$

By iterated application of the Predicative Elimination Lemma we obtain

2.3.12 Theorem *(Elimination Theorem)* *Let* $\left|\frac{\alpha}{\rho}\right. \Delta$ *such that* $\rho =_{NF} \omega^{\rho_1} + \ldots + \omega^{\rho_n}$. *Then* $\left|\frac{\varphi_{\rho_1}(\varphi_{\rho_2}(\cdots\varphi_{\rho_n}(\alpha)\cdots))}{0}\right. \Delta$.

2.3.13 Theorem *(The upper bound for NT)* *If* $NT \vdash F$ *then* $tc(F) < \varepsilon_0$. *Hence*

$$\|NT\| = \|NT\|_{\Pi_1^1} \leq \varepsilon_0.$$

Proof If $NT \vdash F$ we get by (2.3) and (2.5)

$$\left|\frac{\omega+\omega}{r}\right. F \tag{i}$$

for $r := \max\{rnk(A_1), \ldots, rnk(A_n)\} < \omega$. By the Elimination Theorem (or just by iterated application of the Basis Elimination Lemma) this entails

$$\left|\frac{\varphi_0^r(\omega+\omega)}{0}\right. F. \tag{ii}$$

Hence $\left\|\frac{\varphi_0^r(\omega+\omega)}{}\right. F$ and we get $tc(F) < \varepsilon_0$ since $\varphi_0^r(\omega+\omega) < \varepsilon_0$ for all finite r. $\square$

2.4 The lower bound

We want to show that the bound given in Theorem 2.3.13 is the best possible one. By Theorem 1.8.7 it suffices to have Theorem 2.4.1 below.

2.4.1 Theorem *For every ordinal $\alpha < \varepsilon_0$ there is a primitive recursive well-order $\prec$ on the natural numbers of order type α such that $NT \vdash TI(\prec, X)$.*

The first step in proving Theorem 2.4.1 is to represent ordinals below ε_0 by primitive recursive well-orders. This is done by an arithmetization. We simultaneously define a set $\mathsf{On} \subseteq \mathbb{N}$ and a relation $a \prec b$ for $a, b \in \mathsf{On}$ together with an evaluation map $|\cdot|: \mathsf{On} \longrightarrow On$ such that On and $\prec$ become primitive recursive and $a \prec b \;\Leftrightarrow\; |a| < |b|$. We put

- $0 \in \mathsf{On}$ and $|0| = 0$

- If $z_1, \ldots, z_n \subseteq \mathsf{On}$ and $z_1 \succeq \ldots \succeq z_n$ then $\langle z_1, \ldots, z_n \rangle \in \mathsf{On}$ and $|\langle z_1, \ldots, z_n \rangle| = \omega^{|z_1|} + \ldots + \omega^{|z_n|}$

and

- $a \prec b :\Leftrightarrow \; a \in \mathsf{On} \wedge b \in \mathsf{On} \wedge [(a = 0 \wedge b \neq 0)$
  $\vee\, (lh(a) < lh(b) \wedge (\forall i < lh(a))((a)_i = (b)_i))$
  $\vee\, (\exists i < \min\{lh(a), lh(b)\})(\forall j < i)((a)_j = (b)_j \wedge (a)_i \prec (b)_i)]$.

Observe that On and $\prec$ are defined by simultaneous course of values recursion and thence are primitive recursive. It is also easy to check that $a \prec b \;\Leftrightarrow\; |a| < |b|$. The order $\langle \mathsf{On}, \prec \rangle$ is a well-order of order type ε_0. We may therefore represent every ordinal $\alpha < \varepsilon_0$ by an initial segment $\prec_\alpha$ of the well-order $\prec$. Thus we can talk about ordinals $< \varepsilon_0$ in $\mathcal{L}(NT)$. We will not distinguish between ordinals and their representations in $\mathcal{L}(NT)$ and regard formulas as $(\forall\alpha)[\ldots]$ as abbreviations for $(\forall x)[x \in \mathsf{On} \to \ldots]$ as well as $(\exists\alpha)[\ldots]$ as abbreviation for $(\exists x)[x \in \mathsf{On} \wedge \ldots]$. We also write $\alpha < \beta$ instead of $\alpha \prec \beta$. We introduce the following formulas:

- $\alpha \subseteq X \;:\Leftrightarrow\; (\forall\xi)[\xi < \alpha \to \xi \in X]$

- $\mathsf{Prog}(X) \;:\Leftrightarrow\; (\forall\alpha)[\alpha \subseteq X \to \alpha \in X]$

- $\mathsf{TI}(\alpha, X) \;:\Leftrightarrow\; \mathsf{Prog}(X) \to \alpha \subseteq X$

Our aim is to show $\mathsf{TI}(\alpha, X)$ for all $\alpha < \varepsilon_0$. Since $\varepsilon_0 = \sup\{exp^n(\omega, 0) \mid n \in \omega\}$ and $\mathsf{TI}(0, X)$ holds trivially we are done as soon as we succeed in proving

$$NT \vdash \mathsf{TI}(\alpha, X) \;\Rightarrow\; NT \vdash \mathsf{TI}(\omega^\alpha, X) \tag{2.8}$$

because $NT \vdash \mathsf{TI}(\alpha, X)$ and $\beta < \alpha$ obviously entails $NT \vdash \mathsf{TI}(\beta, X)$. The first observation is

$$NT \vdash F(X) \;\Rightarrow\; NT \vdash F(\{x \mid G(x)\}) \tag{2.9}$$

for all $\mathcal{L}(NT)$-formulas G. The formula $F(\{x \mid G(x)\})$ is obtained from $F(X)$ by replacing all occurrences of $t \in X$ by $G(t)$ and those of $t \notin X$ by $\neg G(t)$. To prove (2.9) assume

$$NT \vdash F(X) \tag{i}$$

and let $\mathfrak{S}$ be an arbitrary $\mathcal{L}(NT)$-structure and Φ an assignment of subsets of $\mathbb{N}$ to the set variables such that

$$\mathfrak{S} \models NT[\Phi]. \tag{ii}$$

We have to show

$$\mathfrak{S} \models F(\{x \mid G(x)\})[\Phi]. \tag{iii}$$

Define a new assignment

$$\Psi(Y) := \begin{cases} \Phi(Y) & \text{if } Y \neq X \\ \{n \in \mathfrak{S} \mid \mathfrak{S} \models G(x)[n, \Phi]\} & \text{otherwise.} \end{cases}$$

Then

$$\mathfrak{S} \models F(X)[\Psi] \text{ iff } \mathfrak{S} \models F(\{x \mid G(x)\})[\Phi]. \tag{iv}$$

We claim

$$\mathfrak{S} \models NT[\Psi]. \tag{v}$$

Then (v) together with (i) and (iv) prove (iii). To check (v) we have only to take care of formulas in NT which contain the set variable X. This can only happen in instances of the scheme of

Mathematical Induction or in identity axioms. Let

$$I(X) \quad :\Leftrightarrow \quad H(X,\underline{0}) \wedge (\forall x)[H(X,x) \to H(X,S(x))] \to (\forall x)H(X,x)$$

be an instance of Mathematical Induction. We have

$$\mathfrak{S} \models I(X)[\Psi] \text{ iff } \mathfrak{S} \models I(\{x \mid G(x)\})[\Phi]. \tag{vi}$$

The right formula in (vi), however, holds by (ii) since $H(\{x \mid G(x)\}, x)$ is also a formula in NT. Instances of identity axioms are treated analogously. ☐

The above proof shows the importance of formulating Mathematical Induction as a scheme. Let

$$\mathcal{J}(X) := \{\alpha \mid (\forall \xi)[\xi \subseteq X \to \xi + \omega^\alpha \subseteq X]\}$$

denote the *jump* of X. Then, if we assume

$$NT \vdash \mathsf{Prog}(X) \to \mathsf{Prog}(\mathcal{J}(X)), \tag{i}$$

we obtain

$$NT \vdash \mathsf{TI}(\alpha, \mathcal{J}(X)) \to \mathsf{TI}(\omega^\alpha, X). \tag{ii}$$

To prove (ii) assume (working informally in NT) $\mathsf{TI}(\alpha, \mathcal{J}(X))$, i.e.

$$\mathsf{Prog}(\mathcal{J}(X)) \to \alpha \subseteq \mathcal{J}(X) \tag{iii}$$

which entails

$$\mathsf{Prog}(\mathcal{J}(X)) \to \alpha \in \mathcal{J}(X). \tag{iv}$$

Choosing $\xi = 0$ in the definition of the jump turns (iv) into

$$\mathsf{Prog}(\mathcal{J}(X)) \to \omega^\alpha \subseteq X, \tag{v}$$

which, together with (i), gives

$$\mathsf{Prog}(X) \to \omega^\alpha \subseteq X, \tag{vi}$$

which is $\mathsf{TI}(\omega^\alpha, X)$. Once we have (ii) we also get (2.8) because $\mathsf{TI}(\alpha, X)$ implies $\mathsf{TI}(\alpha, \mathcal{J}(X))$ by (2.9).
It remains to prove (i). Again we work informally in NT. Assume

$$\mathsf{Prog}(X). \tag{vii}$$

We want to prove $\mathsf{Prog}(\mathcal{J}(X))$ i.e. $(\forall \alpha)[\alpha \subseteq \mathcal{J}(X) \to \alpha \in \mathcal{J}(X)]$. Thus, assuming also

$$\alpha \subseteq \mathcal{J}(X), \tag{viii}$$

we have to show $\alpha \in \mathcal{J}(X)$. i.e. $(\forall \xi)[\xi \subseteq X \to \xi + \omega^\alpha \subseteq X]$. That means that we have to prove $\eta \in X$ under the additional hypotheses

$$\xi \subseteq X \tag{ix}$$

and

$$\eta < \xi + \omega^\alpha. \tag{x}$$

If $\eta < \xi$ we obtain $\eta \in X$ by (ix). Let $\xi \leq \eta < \xi + \omega^\alpha$. If $\alpha = 0$ the $\eta = \xi$ and we obtain $\eta \in X$ by (ix) and (vii). If $\alpha > 0$ then there is a $\sigma < \alpha$ and a natural number (i.e. a numeral in NT), such that $\xi < \underbrace{\omega^\sigma + \ldots + \omega^\sigma}_{n-\text{fold}} =: \omega^\sigma \cdot n$ (c.f. the proof of the Predicative Elimination Lemma).

We show

$$\sigma < \alpha \to \xi + \omega^\sigma \cdot n \subseteq X \tag{xi}$$

by induction on n. For $n = 0$ this is (ix). For $n := m + 1$ we have

$$\xi + \omega^\sigma \cdot m \subseteq X \tag{xii}$$

by the induction hypothesis. From $\sigma < \alpha$ we obtain $\sigma \in \mathcal{J}(X)$ from (viii). This together with (xii) entails $\xi + \omega^\sigma \cdot n = \xi + \omega^\sigma \cdot m + \omega^\sigma \in X$. This finishes the proof of (i), hence also that of (2.8) which in turn implies Theorem 2.4.1. $\square$

Summing up we have shown

2.4.2 Theorem *(Ordinal Analysis of NT)* $\|NT\| = \|NT\|_{\Pi^1_1} = \varepsilon_0$.

As a consequence of Theorem 2.3.13 and Theorem 2.4.1 we get

2.4.3 Theorem *There is a Π^1_1–sentence $(\forall X)(\forall x)F(X,x)$ which is true in the standard structure $\mathbb{N}$ such that $NT \vdash F(X,\underline{n})$ for all $n \in \mathbb{N}$ but $NT \nvdash (\forall x)F(X,x)$.*

To prove the theorem choose $F(X,x) \;:\Leftrightarrow\; \mathsf{Prog}(X) \to x \in \mathsf{On} \to x \in X$. $\square$

Theorem 2.4.3 is a weakened form of GÖDEL's Theorem. GÖDEL's Theorem says that Theorem 2.4.3 holds already for a Π^0_1-sentence $(\forall x)F(x)$.

3. Ordinal analysis of non iterated inductive definitions

3.1 The theory ID_1

We want to axiomatize the theory for positively definable inductive definitions over the natural numbers. According to Theorem 1.7.7 we can express Π^1_1–relations by inductivley defined relations. Therefore we can dispend with set parameters in the theory and we will do so to save some case distinctions (and also to give examples for some of the phenomena which are characteristic for impredicative proof theory).

3.1.1 Definition The language $\mathcal{L}(ID_1)$ comprises the language of NT. For every X–positive formula $F(X, \vec{x})$ we introduce a new relation symbol I_F whose arity is the length of the tuple $\vec{x}$. The theory ID_1 comprises NT (but in the language without set parameters) together with the defining axioms for the set constants

$(ID_1{}^1)\,(\forall x)[F(\mathsf{I}_F, \vec{x}) \;\;\rightarrow\;\; x \in \mathsf{I}_F]$

and

$(ID_1{}^2)\,Cl_F(G) \;\;\rightarrow\;\; (\forall x)[x \in \mathsf{I}_F \rightarrow G(x)]$,

where

$$Cl_F(G) \equiv (\forall x)[F(G, \vec{x}) \rightarrow G(\vec{x})]$$

expresses that the "class" $\{\vec{x}|\ G(\vec{x})\}$ is closed under the operator Γ_F induced by $F(X, \vec{x})$. The notion $F(G, \vec{x})$ stands for the formula obtained from $F(X, \vec{x})$ replacing all occurrences $t \in X$ by $G(t)$ and $t \notin X$ by $\neg G(t)$.

The standard interpretation for I_F is of course the least fixed point I_F as introduced in Definition 1.6.1. The following theorem is left as an exercise.

3.1.2 Theorem

$$\mathbb{N} \models \vec{n} \in \mathsf{I}_F \;\;\Leftrightarrow\;\; \mathbb{N} \models (\forall X)[Cl_F(X) \rightarrow \vec{n} \in X]$$

$$ID_1 \vdash (\forall x)[F(\mathsf{I}_F, \vec{x}) \leftrightarrow \vec{x} \in \mathsf{I}_F]$$

3.2 The language $\mathcal{L}_\infty(NT)$

We extend the language of $\mathcal{L}(NT)$ to an infinitary language containing infinitely long formulas.

3.2.1 Definition (The language $\mathcal{L}_{\infty,\omega}(NT)$) We define the language $\mathcal{L}_{\infty,\omega}(NT)$ as a TAIT–language parallel to Definition 1.3.1. It contains the same non logical symbols. The logical symbols are augmented by the infinite boolean operations $\bigvee$ and $\bigwedge$. The atomic formulas are unaltered. The language is closed under all first order operations and we have the additional clause

- If $\langle F_\xi \mid \xi \in I \rangle$ is a infinite sequence of $\mathcal{L}_{\infty,\omega}(NT)$–formulas containing at most finitely many free variables then $\bigwedge_{\xi \in I} F_\xi$ and $\bigvee_{\xi \in I} F_\xi$ are $\mathcal{L}_{\infty,\omega}(NT)$–formulas.

Again we are interested in the sentences of $\mathcal{L}_{\infty,\omega}(NT)$. The set of sentences is denoted by $\mathcal{L}_\infty(NT)$.

The semantics for $\mathcal{L}_\infty(NT)$ is defined in the obvious way. We get

$$\models \bigwedge_{\xi \in I} F_\xi \quad :\Leftrightarrow \quad \mathbb{N} \models F_\xi \ \text{ for all } \ \xi \in I$$

and

$$\models \bigvee_{\xi \in I} F_\xi \quad :\Leftrightarrow \quad \mathbb{N} \models F_\xi \ \text{ for some } \ \xi \in I.$$

Then it is obvious that we have

- $\bigwedge\limits_{\xi \in I} F_\xi \in \bigwedge$–type

and

- $\bigvee\limits_{\xi \in I} F_\xi \in \bigvee$–type

and

- $CS(\bigwedge\limits_{\xi \in I} F_\xi) = CS(\bigvee\limits_{\xi \in I} F_\xi) = \langle F_\xi \mid \xi \in I \rangle.$

The definition of the validity relation as given in Definition 1.3.10 now carries over to the language $\mathcal{L}_\infty(NT)$. Observe that we can dipsense with rule (Ax) because we don't have set parameters. By an easy induction on α we get

3.2.2 Lemma *For any finite set Δ of $\mathcal{L}_\infty(NT)$–sentences we have*

$$\models^{\alpha} \Delta \quad \Rightarrow \quad \mathbb{N} \models \bigvee \Delta.$$

Since we only deal with sentences the completeness of the validity relation is much easier to show.

3.2.3 Definition For every formula F in $\mathcal{L}_\infty(NT)$ we define its rank $rnk(F)$ by

$$rnk(F) := \sup\left\{ rnk(G) + 1 \,\middle|\, G \in CS(F) \right\}.$$

By an easy induction on $rnk(F)$ we obtain

3.2.4 Lemma *For $F \in \mathcal{L}_\infty(NT)$ we have*

$$\mathbb{N} \models F \quad \Rightarrow \quad \models^{rnk(F)} F\,.$$

3.3 Inductive definitions and $\mathcal{L}_\infty(NT)$

The stages of an inductive definition over $\mathbb{N}$ can be easily expressed in $\mathcal{L}_\infty(NT)$.

3.3.1 Definition Let $F(X,\vec{x})$ be a formula in $\mathcal{L}(NT)$. By recursion on $\alpha \leq \omega_1$ we define

$$\vec{t} \in \mathsf{I}_F^{<\alpha} :\equiv \bigvee_{\xi<\alpha} F(\mathsf{I}_F^{<\xi},\vec{t})$$

and dually

$$\vec{t} \notin \mathsf{I}_F^{<\alpha} :\equiv \bigwedge_{\xi<\alpha} \neg F(\mathsf{I}_F^{<\xi},\vec{t}).$$

As a shorthand we also use

$$\vec{t} \in \mathsf{I}_F^{\alpha} :\equiv F(\mathsf{I}_F^{<\alpha},\vec{t})$$

and

$$\vec{t} \notin \mathsf{I}_F^{\alpha} :\equiv \neg F(\mathsf{I}_F^{<\alpha},\vec{t}).$$

It it obvious that we have

$$\mathbb{N} \models \vec{t} \in \mathsf{I}_F^{\alpha} \;\;\Leftrightarrow\;\; \vec{t}^{\mathbb{N}} \in I_F^{\alpha} \tag{3.1}$$

where I_F^{α} denotes the stages of the inductive definiton induced by F in the sense of Definition 1.6.1.
For the rest of the lecture we will only regard the fragment of $\mathcal{L}_\infty(NT)$ which is obtained from the sentences defined in Definition 3.3.1 by closing them under first order operations.
If $F(X,\vec{x})$ is an X–positive $\mathcal{L}(NT)$–formula, we know by Theorem 1.7.13 $|F| \leq \omega_1^{CK}$. Hence $I_F = I_F^{<\omega_1^{CK}} = I_F^{<\omega_1}$. Let us use Ω as a symbol for either ω_1^{CK} or ω_1. There is an obvious embedding of the language $\mathcal{L}(ID_1)$ into our fragment of $\mathcal{L}_\infty(NT)$.

3.3.2 Lemma *If G is an $\mathcal{L}(ID_1)$–sentence we obtain G^* by replacing all occurrences of I_F in G by $\mathsf{I}_F^{<\Omega}$. Then*

$$\mathbb{N} \models G \;\;\Leftrightarrow\;\; N \models G^*.$$

3.4 The semi–formal system for $\mathcal{L}_\infty(NT)$

We introduced the theory ID_1 as a pure first order theory (i.e. a theory wich does not allow the formation of pseudo Π_1^1–sentences). Our observations in Section 1.1, however, based on the possibility of formation of pseudo Π_1^1–sentences. Therefore we have to start with a discussion in what sense a computation of the Π_1^1–ordinal for ID_1 is possible.
Our first observation is that extending the theory ID_1 to a theory ID_1^{ext} by adding free set parameters yields a conservative extension. This is obvious because any model for ID_1 can be extended to a model for ID_1^{ext} by assigning first order definable subsets of the domain of the model to the set variables. In the extended theory ID_1^{ext} is makes sents to talk about provable pseudo Π_1^1–sentences.
We are going to show that the computation of the ordinal

$$\kappa^{ID_1} := \sup\left\{|n|_F \,\middle|\, F(X,x) \text{ is } X\text{–positive } \wedge ID_1 \vdash \underline{n} \in \mathsf{I}_F\right\}$$

yields an ordinal analysis for ID_1^{ext}. First we remarke that $\left\{|n|_F \,\middle|\, ID_1 \vdash \underline{n} \in \mathsf{I}_F\right\}$ is a recursivley enumerable set which implies $\kappa^{ID_1} < \omega_1^{CK}$.
Next we observe

$$ID_1^{ext} \vdash TI(\prec, X) \;\;\Leftrightarrow\;\; ID_1 \vdash (\forall x \in \mathit{field}(\prec))\,[x \in Acc(\prec)]\,. \tag{3.2}$$

To check (3.2) let $F_{\prec}(X,x) :\equiv x \in \mathit{field}(\prec) \wedge (\forall y \prec x)[y \in X]$. Then $ID_1^{ext} \vdash TI(\prec, X)$ means $ID_1^{ext} \vdash Cl_{F_{\prec}}(X) \to (\forall x \in \mathit{field}(\prec))[x \in X]$. Hence $ID_1^{ext} \vdash Cl_{F_{\prec}}(Acc(\prec)) \to (\forall x \in \mathit{field}(\prec))[x \in Acc(\prec)]$. But since $Cl_{F_{\prec}}(Acc(\prec))$ is an axiom of ID_1^{ext} this entails $ID_1^{ext} \vdash (\forall x \in \mathit{field}(\prec))[x \in Acc(\prec)]$. For the opposite direction we observe that $Cl_{F_{\prec}}(X) \to (\forall x)[x \in Acc(\prec) \to x \in X]$ is an axiom of ID_1^{ext}. So from $ID_1^{ext} \vdash (\forall x \in \mathit{field}(\prec))[x \in Acc(\prec)]$ we immediatly get $Cl_{F_{\prec}}(X) \to (\forall x)[x \in \mathit{field}(\prec) \to x \in X]$, i.e. $ID_1^{ext} \vdash TI(\prec, X)$.
From (3.2) we get $otyp(\prec) = \sup\{otyp_{\prec}(x) \mid x \in \mathit{field}(\prec)\} \leq \sup\{|x|_{F_{\prec}} \mid x \in Acc(\prec)\} \leq \kappa^{ID_1}$ for every relation $\prec$ with $ID_1^{ext} \vdash TI(\prec, X)$.
But we also have

$$ID_1 \vdash \vec{t} \in \mathsf{I}_F \quad\Leftrightarrow\quad ID_1^{ext} \vdash Cl_F(X) \to \vec{t} \in X. \tag{3.3}$$

The direction from left to right holds since $(\forall x)Cl_F(X) \to \vec{t} \in \mathsf{I}_F \to \vec{t} \in X$ is an instance of the axiom ${ID_1}^2$ and the opposite direction follows because $ID_1 \vdash \vec{t} \in \mathsf{I}_F$ is obvious from the the instantiation $Cl_F(\mathsf{I}_F) \to t \in \mathsf{I}_F$ and the axioms $Cl_F(\mathsf{I}_F)$.
By the Stage Theorem 1.7.8 and (3.3) we then obtain $\kappa^{ID_1} \leq \|ID_1^{ext}\|_{\Pi_1^1}$. Hence

$$\|ID_1^{ext}\| \leq \kappa^{ID_1} \leq \|ID_1^{ext}\|_{\Pi_1^1} = \|ID_1^{ext}\|$$

which confirms our decision not to include set parameters.
First we observe

3.4.1 Lemma *We have*

$$\models^{\alpha} \Delta, \underline{\vec{t}} \in \mathsf{I}_F^{<\Omega} \quad\Rightarrow\quad \models^{\alpha} \Delta, \mathsf{I}_F^{<\alpha}$$

which means

$$|n|_F \leq tc(\underline{n} \in \mathsf{I}_F^{<\Omega}).$$

The proof is a straightforward induction on α which we omit since a similar property (Lemma 3.4.13) will be needed and proved for the semi–formal calculus below.
It becomes clear from Lemma 3.4.1 that the computation of an upper bound for κ^{ID_1} can be done analogously to that of an upper bound for $\|NT\|_{\Pi_1^1}$. Therfore the first step should be the computation of the truth complexities for the axioms of ID_1. Here we have even to be carful in transfering Theorem 2.1.6. The sentence $n \in \mathsf{I}_F$ is an atomic sentence of $\mathcal{L}(ID_1)$ but not an atomic sentence of $\mathcal{L}_\infty(NT)$. But observe that because of $rnk(\vec{t} \in \mathsf{I}_F^{<\Omega}) \leq \Omega$ we obtain by the Tautology Lemma (Lemma 2.3.1)

$$\models^{\Omega} \Delta, \vec{t} \notin \mathsf{I}_F^{<\Omega}, \vec{t} \in \mathsf{I}_F^{<\Omega}. \tag{3.4}$$

Thus Theorem 2.1.6 modifies to

3.4.2 Theorem *If* $\vdash^{m} \Delta(\vec{x})$ *holds for a finite set of* $\mathcal{L}(ID_1)$*–formulas then* $\models^{\Omega+m} \Delta\ (\underline{\vec{n}})$ *for all tuples* $\underline{\vec{n}}$ *of numerals.*

The truth complexities of the defining axioms for primitive recursive functions and relations are of course not altered. More caution is again needed for the identity axioms which of course include the scheme

$$(\forall \vec{x})(\forall \vec{y})[\vec{x} = \vec{y} \to \vec{x} \in \mathsf{I}_F \to \vec{y} \in \mathsf{I}_F].$$

But here we get

$$\models^{\Omega+n} (\forall \vec{x})(\forall \vec{y})[\vec{x} = \vec{y} \to \vec{x} \in \mathsf{I}_F^{<\Omega} \to \vec{y} \in \mathsf{I}_F^{<\Omega}]$$

for some $n < \omega$. By the Induction Lemma (Lemma 2.3.2) we obtain

$$\models^{\Omega+\omega+4} G^*$$

for all instances G of the scheme of Mathematical Induction in ID_1. It remains to check the truth complexities for the axioms $ID_1{}^1$ and $ID_1{}^2$. By Lemma 3.2.4 we obtain

$$\models^{\Omega+n} Cl_F(\mathsf{I}_F^{<\Omega})$$

since $rnk(Cl_F(\mathsf{I}_F^{<\Omega})) = \Omega + n$ for some $n < \omega$.
The same is of course also true for all instances of the axiom $ID_1{}^2$.
These observations show that the ordinal analysis for ID_1 needs something new. The truth complexities for the axioms of ID_1 are above Ω. The ordinal κ^{ID_1}, however, is an ordinal $< \Omega$ (regardless of the interpretation of Ω). Since a validation proof for a sentence $\vec{n} \in \mathsf{I}_F^{<\Omega}$ does not contain Ω–branchings it is also clear that $tc(\vec{n} \in \mathsf{I}_F^{<\Omega}) < \Omega$. So we need additional conditions which allow us to collapse the ordinal assigned to the infinitary derivations for sentences of the form $\vec{n} \in \mathsf{I}_F^{<\Omega}$ into ordinals below Ω.
But there is still another reason why cut–elimination alone cannot solve our problem. We define the semi–formal system for the language $\mathcal{L}_\infty(NT)$ as in Definition 2.3.3. Again we can dispense with the rule (Ax) because we do not have set parameters. But now we obtain the following theorem.

3.4.3 Theorem *Let Γ be a finite sets of $\mathcal{L}_\infty(NT)$–sentences. Then*

$$\vdash^{\alpha}_{\rho} \Gamma \;\; \Rightarrow \;\; \models^{\alpha} \Gamma\,.$$

Proof We prove

$$\vdash^{\alpha}_{\rho} \Gamma, \Delta \;\text{ and }\; \mathbb{N} \not\models F \;\text{ for all }\; F \in \Delta \;\; \Rightarrow \;\; \models^{\alpha} \Gamma \tag{i}$$

by induction on α. The proof depends heavily on the fact that we only have sentences in $\mathcal{L}_\infty(NT)$. In the case of a cut we have the premises

$$\vdash^{\alpha_0}_{\rho} \Delta, \Gamma, F \tag{ii}$$

and

$$\vdash^{\alpha_0}_{\rho} \Delta, \Gamma, \neg F \tag{iii}$$

and either $\mathbb{N} \not\models F$ or $\mathbb{N} \not\models \neg F$. Using the induction hypothesis on the corresponding premise we get the claim. The remaining cases are obvious. $\Box$

It follows from Theorem 3.4.3 that cut–elimination cannot be the crucial point in the ordinal analysis of ID_1. The same is of course also true for stronger theories. *The hallmark for impredicative proof theory is not longer cut–elimination but collapsing.* Since ordinals above Ω are in general not collapsable into ordinal below Ω we have to control the ordinals assigned to the derivations. We follow the concept of *operator controlled derivations* which was introduced in [3] as a simplification of the method of local predicativity introduced in [9]. However, we will not copy BUCHHOLZ' proof but introduce a variant which even sharper pinpoints the role of collapsing.

3.4.4 Definition A *Skolem–hull operator* is a function $\mathcal{H}$ which maps sets of ordinals on sets of ordinals satisfying the conditions

- For all $X \subseteq On$ it is $X \subseteq \mathcal{H}(X)$
- If $Y \subseteq \mathcal{H}(X)$ then $\mathcal{H}(Y) \subseteq \mathcal{H}(X)$.

3.4.5 Definition For a sentence G in our fragment of $\mathcal{L}_\infty(NT)$ we define

$$par(G) := \{\alpha \mid \mathsf{I}_F^{<\alpha} \text{ occurs in } G\}.$$

For a finite set Δ of sentences of our fragment of $\mathcal{L}_\infty(NT)$ we define

$$par(\Delta) := \bigcup_{F \in \Delta} par(F).$$

3.4.6 Definition For a Skolem–hull operator we define the relation $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta$ by the clauses $(\bigwedge)$, $(\bigvee)$ and (cut) of Definition 2.3.3 with the additional conditions that we always have

- $\alpha \in \mathcal{H}(par(\Delta))$

and for an inference

$$\mathcal{H} \overset{\alpha_i}{\underset{\rho}{\vdash}} \Delta_\iota \text{ for } \iota \in I \quad \Rightarrow \quad \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta$$

with finite I also

- $par(\Delta_\iota) \subseteq \mathcal{H}(par(\Delta))$.

We define

$$\mathcal{H}_1 \subseteq \mathcal{H}_2 \quad :\Leftrightarrow \quad (\forall X) \subseteq On[\mathcal{H}_1(X) \subseteq \mathcal{H}_2(X)].$$

A Skolem–hull operator $\mathcal{H}$ is *Cantorian closed* iff

- $\{0, \Omega\} \subseteq \mathcal{H}(\emptyset)$,
- $\mathcal{H}(\emptyset) \cap \Omega$ is transitive

and it satisfies

- $\alpha \in \mathcal{H}(X) \quad \Leftrightarrow \quad SC(\alpha) \subseteq \mathcal{H}(X)$ for any set X of ordinals.

For a set $X \subseteq On$ and an operator $\mathcal{H}$ let

- $\mathcal{H}[X] := \lambda\Xi.\, \mathcal{H}(X \cup \Xi)$.

When writing $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta$ we tacitly assume that $\mathcal{H}$ is a Cantorian closed Skolem–hull operator. The Structural Lemma of Section 2.3 extents to

3.4.7 Lemma *If $\mathcal{H}_1 \subseteq \mathcal{H}_2$, $\alpha \leq \beta$, $\rho \leq \sigma$, $\Delta \subseteq \Gamma$, $\beta \in \mathcal{H}_2(par(\Gamma))$ and $\mathcal{H}_1 \overset{\alpha}{\underset{\rho}{\vdash}} \Delta$ then $\mathcal{H}_2 \overset{\beta}{\underset{\sigma}{\vdash}} \Gamma$.*

The remaining facts of Section 2.3 carry over to controlled semi–formal derivations.

3.4.8 Lemma *(Inversion Lemma) If $F \in \bigwedge$–type and $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, F$ then $\mathcal{H}[par(F)] \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, G$ for all $G \in CS(F)$.*

3.4.9 Lemma *($\vee$–Exportation) If $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, F_1 \vee \cdots \vee F_n$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, F_1, \ldots, F_n$.*

3.4.10 Lemma *If $F \in \mathsf{D}(\mathbb{N})$ and $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, \neg F$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta$.*

3.4.11 Lemma *(Reduction Lemma) Let $F \in \bigwedge$–type, $\rho = rnk(F)$ and $par(F) \subseteq \mathcal{H}(par(\Delta))$. If $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, F$ and $\mathcal{H} \overset{\beta}{\underset{\rho}{\vdash}} \Gamma, \neg F$ then $\mathcal{H} \overset{\alpha+\beta}{\underset{\rho}{\vdash}} \Delta, \Gamma$.*

Proof The proof is of course very similar to that of Lemma 2.3.9 but we need to put extra care on the controlling operator. We induct on β. Let us first assume that the critical formula of the last inference " $\mathcal{H}\,\frac{\beta_\iota}{\rho}\,\Gamma_i, \neg F$ for $\iota \in I \;\Rightarrow\; \mathcal{H}\,\frac{\beta}{\rho}\,\Gamma, \neg F$ " is different from $\neg F$. Then we still have $par(F) \subseteq \mathcal{H}(par(\Delta))$ and obtain $\mathcal{H}\,\frac{\alpha+\beta_\iota}{\rho}\,\Delta, \Gamma_\iota$ by the induction hypothesis. Since $\alpha + \beta_\iota < \alpha + \beta$ – and in the case of finite I also $par(\Delta, \Gamma_\iota) \subseteq \mathcal{H}(par(\Delta, \Gamma))$ – we obtain $\mathcal{H}\,\frac{\alpha+\beta}{\rho}\,\Delta, \Gamma$ by the same inference.

Now assume that $\neg F$ is the main formula of the last inference in $\mathcal{H}\,\frac{\beta}{\rho}\,\Gamma, \neg F$. Then we have the premise $\mathcal{H}\,\frac{\beta_0}{\rho}\,\Delta, \neg F, \neg G$ for some $G \in CS(F)$ with

$$par(\Delta, F, G) \subseteq \mathcal{H}(par(\Delta, F)) \tag{i}$$

and obtain $\mathcal{H}\,\frac{\alpha+\beta_0}{\rho}\,\Delta, \Gamma, \neg G$ by the induction hypothesis. By inversion, the Structural Lemma and the hypothesis $par(F) \subseteq \mathcal{H}(par(\Delta)) \subseteq \mathcal{H}(par(\Delta, \Gamma))$ we also have $\mathcal{H}\,\frac{\alpha+\beta_0}{\rho}\,\Delta, \Gamma, G$. It is $rnk(G) < \rho$ but to apply a cut we also have to check

$$par(\Delta, \Gamma, G) \subseteq \mathcal{H}(par(\Delta, \Gamma)). \tag{ii}$$

But this is secured by (i) and the hypothesis $par(F) \subseteq \mathcal{H}(par(\Delta)) \subseteq \mathcal{H}(par(\Delta, \Gamma))$. $\square$

3.4.12 Theorem *(Cut elimination for controlled derivations) Let $\mathcal{H}$ be a Cantorian closed Skolem–hull operator. Then*

(i) $\quad \mathcal{H}\,\frac{\alpha}{\rho+1}\,\Delta \;\Rightarrow\; \mathcal{H}\,\frac{2^\alpha}{\rho}\,\Delta$

and

(ii) $\quad \mathcal{H}\,\frac{\alpha}{\beta+\omega^\rho}\,\Delta$ *and* $\rho \in \mathcal{H}(par(\Delta)) \;\Rightarrow\; \mathcal{H}\,\frac{\varphi_\rho(\alpha)}{\beta}\,\Delta.$

Proof We show (i) by induction of α. If the last inference $\mathcal{H}\,\frac{\alpha_\iota}{\rho+1}\,\Delta_\iota$ for $\iota \in I \;\Rightarrow\; \mathcal{H}\,\frac{\alpha}{\rho+1}\,\Delta$ is not a cut of rank ρ we have $\mathcal{H}\,\frac{2^{\alpha_\iota}}{\rho}\,\Delta_\iota$ by induction hypothesis and $par(\Delta_\iota) \subseteq \mathcal{H}(par(\Delta))$ in the case of finite I. So we get $\mathcal{H}\,\frac{2^\alpha}{\rho}\,\Delta$ by the same inference.
In case that the last inference is a cut $\mathcal{H}\,\frac{\alpha_0}{\rho+1}\,\Delta, F \quad \mathcal{H}\,\frac{\alpha_0}{\rho+1}\,\Delta, \neg F \;\Rightarrow\; \mathcal{H}\,\frac{\alpha}{\rho+1}\,\Delta$ of rank ρ we obtain $\mathcal{H}\,\frac{2^{\alpha_0}}{\rho}\,\Delta, F$ and $\mathcal{H}\,\frac{2^{\alpha_0}}{\rho}\,\Delta, \neg F$ by the induction hypothesis. But either $F \in \bigwedge$–type or $\neg F \in \bigwedge$–type and $par(F) = par(\neg F) \subseteq \mathcal{H}(par(\Delta))$. Therefore we may apply the Reduction Lemma (Lemma 3.4.11) and the fact that $2^{\alpha_0} + 2^{\alpha_0} \leq 2^\alpha$ to obtain $\mathcal{H}\,\frac{2^\alpha}{\rho}\,\Delta$. $\square$

We close this section by showing a extension of Lemma 3.4.1 to operator controlled derivations. This will be one of the key properties of the collapsing procedure in the following section.

3.4.13 Lemma *(Boundedness) If $\mathcal{H}\,\frac{\alpha}{\rho}\,\Delta(\vec{t} \in \mathsf{I}_F^{<\beta})$ then $\mathcal{H}[\{\beta\}]\,\frac{\alpha}{\rho}\,\Delta(\vec{t} \in \mathsf{I}_F^{<\gamma})$ holds for all γ such that $\alpha \leq \gamma \leq \beta$.*

Proof We induct on α. In the cases that $\vec{t} \in \mathsf{I}_F^{<\beta}$ is not the main formula of the last inference

$$\mathcal{H}\,\frac{\alpha_\iota}{\rho}\,\Delta_\iota(\vec{t} \in \mathsf{I}_F^{<\beta}) \text{ for } \iota \in I \;\Rightarrow\; \mathcal{H}\,\frac{\alpha}{\rho}\,\Delta(\vec{t} \in \mathsf{I}_F^{<\beta}) \tag{i}$$

we get

$$\mathcal{H}[\{\beta\}]\,\frac{\alpha_\iota}{\rho}\,\Delta_\iota(\vec{t} \in \mathsf{I}_F^{<\gamma}) \tag{ii}$$

by induction hypothesis. If I is finite we have $par(\Delta_\iota(\vec{t} \in \mathsf{I}_F^{<\beta})) \subseteq \mathcal{H}(par(\Delta(\vec{t} \in \mathsf{I}_F^{<\beta})))$ which entails $par(\Delta_\iota(\vec{t} \in \mathsf{I}_F^{<\gamma})) \subseteq \mathcal{H}[\{\beta\}](par(\Delta(\vec{t} \in \mathsf{I}_F^{<\gamma})))$ and we obtain

$$\mathcal{H}[\{\beta\}] \vert\!\frac{\alpha}{\rho}\, \Delta(\vec{t} \in \mathsf{I}_F^{<\gamma}) \tag{iii}$$

from (ii) by the same inference.
If $\vec{t} \in \mathsf{I}_F^{<\beta}$ is the main formula we are in the case of an $(\bigvee)$ inference with the premise

$$\mathcal{H} \vert\!\frac{\alpha_0}{\rho}\, \Delta_0, \vec{t} \in \mathsf{I}_F^{<\beta}, \vec{t} \in \mathsf{I}_F^{\xi} \tag{iv}$$

for some $\xi < \beta$. Applying the induction hypothesis twice we obtain

$$\mathcal{H}[\{\beta,\xi\}] \vert\!\frac{\alpha_0}{\rho}\, \Delta_0, \vec{t} \in \mathsf{I}_F^{<\gamma}, \vec{t} \in \mathsf{I}_F^{\alpha_0}. \tag{v}$$

From $\alpha_0 \in \mathcal{H}(par(\Delta_0, \vec{t} \in \mathsf{I}_F^{<\beta}, \vec{t} \in \mathsf{I}_F^{\xi}))$ and $\xi \in \mathcal{H}(par(\Delta_0, \vec{t} \in \mathsf{I}_F^{<\beta}))$ we obtain $\alpha_o \in \mathcal{H}(par(\Delta_0, \mathsf{I}_F^{<\beta})) \subseteq \mathcal{H}[\{\beta\}](par(\Delta_0, \mathsf{I}_F^{<\gamma}))$ and $\mathcal{H}[\{\beta,\xi\}](par(=))\mathcal{H}[\{\beta\}](par())$. Since $\alpha_0 < \alpha \le \gamma$ we can apply an inference $(\bigvee)$ to obtain

$$\mathcal{H}[\{\beta\}] \vert\!\frac{\alpha}{\rho}\, \Delta_0, \vec{t} \in \mathsf{I}_F^{<\gamma}. \qquad \square$$

3.5 The collapsing theorem for ID_1

Let $\mathcal{H}$ by an Cantorian close operator. We define its iterations $\mathcal{H}_\alpha$.

3.5.1 Definition For $X \subseteq On$ let $\mathcal{H}_\alpha(X)$ be the least set of ordinals containing $X \cup \{0, \Omega\}$ which is closed under $\mathcal{H}$ and the collapsing function $\psi_\mathcal{H} \restriction \alpha$ where

$$\psi_\mathcal{H}(\alpha) := \min\left\{\xi \mid \ \xi \notin \mathcal{H}_\alpha(\emptyset)\right\}.$$

We need a few facts about the operators $\mathcal{H}_\alpha$. Here it is comfortable to think on Ω as the first uncountable cardinal. Interpreting Ω as ω_1^{CK} makes the following considerations much harder.
First we observe

$$|\mathcal{H}_\alpha(X)| = \max\{|X|, \omega\} \tag{3.5}$$

which implies

$$\psi_\mathcal{H}(\alpha) < \Omega \tag{3.6}$$

showing that $\psi_\mathcal{H}$ is in fact collapsing. Clearly the operators $\mathcal{H}_\alpha$ are Cantorian closed and cumulative, i.e.

$$\alpha \le \beta \ \Rightarrow\ \mathcal{H}_\alpha \subseteq \mathcal{H}_\beta \text{ and } \psi_\mathcal{H}(\alpha) \le \psi_\mathcal{H}(\beta). \tag{3.7}$$

Since for $\alpha \in \mathcal{H}_\beta(\emptyset) \cap \beta$ we get $\psi_\mathcal{H}(\alpha) \in \mathcal{H}_\beta(\emptyset)$ we have

$$\alpha \in \mathcal{H}_\beta(\emptyset) \cap \beta \ \Rightarrow\ \psi_\mathcal{H}(\alpha) < \psi_\mathcal{H}(\beta). \tag{3.8}$$

From (3.8) we get

$$\mathcal{H}_\alpha(\emptyset) \cap \Omega = \psi_\mathcal{H}(\alpha). \tag{3.9}$$

The "$\supseteq$"–direction follows from the definition of $\psi_\mathcal{H}(\alpha)$ and (3.6). For the opposite inclusion observe that $\psi_\mathcal{H}(\alpha)$ is strongly critial and show

$$\xi \in \mathcal{H}_\alpha(\emptyset) \cap \Omega \ \Rightarrow\ \xi < \psi_\mathcal{H}(\alpha)$$

by induction on the definition of $\xi \in \mathcal{H}_\alpha(\emptyset)$. In case that $\xi = \psi_{\mathcal{H}}(\eta)$ we have $\eta \in \mathcal{H}_\alpha(\emptyset) \cap \alpha$ which by (3.8) implies $\xi = \psi_{\mathcal{H}}(\eta) < \psi_{\mathcal{H}}(\alpha)$.
From (3.9) we see that all the iterations $\mathcal{H}_\alpha$ are again Cantorian closed operators.

3.5.2 Lemma *Let $\mathcal{H}$ be an Cantorian closed operator. Then $(\mathcal{H}_\alpha)_\beta(X) = \mathcal{H}_{\alpha+\beta}(X)$ for all X and $\psi_{\mathcal{H}_\alpha}(\beta) = \psi_{\mathcal{H}}(\alpha+\beta)$.*

Proof This is a straight forward induction on β. □

The following observation will be crucial for the ordinal analysis of ID_1.

3.5.3 Definition We say that a sentence in our fragment of $\mathcal{L}_\infty(NT)$ is in $\bigvee^\Omega$–type if it does not contain subformulas of the shape $\vec{t} \notin \mathsf{I}_F^{<\Omega}$.

3.5.4 Lemma *(Collapsing Lemma) Let $\Delta \subseteq \bigvee^\Omega$–type such that $par(\Delta) \subseteq \mathcal{H}(\emptyset)$ and $\mathcal{H} \vert\!\frac{\beta}{\Omega}\, \neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}), \dots, \neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega}), \Delta$. Then $\mathcal{H}_{\omega^\beta+1} \vert\!\frac{\psi_{\mathcal{H}}(\omega^\beta)}{\Omega}\, \Delta$.*

The proof is by induction on β. The key property is

$$\beta \in \mathcal{H}(\emptyset) \text{ and } \omega^\beta < \gamma \;\Rightarrow\; \psi_{\mathcal{H}}(\omega^\beta) < \psi_{\mathcal{H}}(\gamma) \tag{i}$$

which is obvious by (3.8) since we have $\omega^\beta \in \mathcal{H}(\emptyset) \cap \gamma \subseteq \mathcal{H}_\gamma(\emptyset) \cap \gamma$. Other observations are

$$\mathcal{H}(par(\Delta)) = \mathcal{H}(\emptyset) \tag{ii}$$

because $par(\Delta) \subseteq \mathcal{H}(\emptyset)$ and

$$\beta \in \mathcal{H}(\emptyset) \;\Rightarrow\; \omega^\beta \in \mathcal{H}_{\omega^\beta+1}(\emptyset) \text{ and } \psi_{\mathcal{H}}(\omega^\beta) \in \mathcal{H}_{\omega^\beta+1}(\emptyset) \tag{iii}$$

which is clear by (3.7) and definition.
Let us first assume that the main part of the last inference

$$\left.\begin{array}{r}\mathcal{H} \vert\!\frac{\beta_\iota}{\Omega}\, \neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}), \dots, \neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega}), \Delta_\iota \\ \text{for } \iota \in I\end{array}\right\} \;\Rightarrow\; \mathcal{H} \vert\!\frac{\beta}{\Omega}\, \neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}), \dots, \neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega}), \Delta \tag{iv}$$

belongs to a sentence in Δ. Observe that $par(\neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}), \dots, \neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega})) = \{\Omega\}$. So we only have to bother about the parameters of Δ. We claim

$$par(\Delta_\iota) \subseteq \mathcal{H}(\emptyset). \tag{v}$$

If I is finite then we have $par(\Delta_\iota) \subseteq \mathcal{H}_{\alpha+1}(par(\Delta)) = \mathcal{H}(\emptyset)$ because $par(\Delta) \subseteq \mathcal{H}(\emptyset)$.
If I is infinite the main formula of the inference is $\vec{t} \notin \mathsf{I}_F^{<\xi}$ for $\xi < \Omega$ because $\Delta \subseteq \bigvee^\Omega$–type. Then $\Delta_\iota = \Delta, G$ for $G \in CS(\vec{t} \notin \mathsf{I}_F^{<\xi})$ which means that $par(\Delta_\iota) \subseteq par(\Delta) \cup \{\eta\}$ for some $\eta < \xi$. But $\xi \in \mathcal{H}(\emptyset) \cap \Omega$ entails $\xi \subseteq \mathcal{H}(\emptyset)$ by transitivity. Hence $par(\Delta_\iota) \subseteq \mathcal{H}(\emptyset)$ for all $\iota \in I$ and the proof of (v) is completed.
Next we claim

$$\Delta_\iota \subseteq \bigvee\nolimits^\Omega\text{–type}. \tag{vi}$$

This follows from $\Delta \subseteq \bigvee^\Omega$–type for inferences which are no cuts. In case that the inference in (iv) is a cut its cut–sentence is of rank $< \Omega$ which ensures that it belongs to $\bigvee^\Omega$–type, too. Because of (v) and (vi) the induction hypothesis applies to the premises of (iv) and we obtain

$$\mathcal{H}_{\omega^{\beta_\iota}+1} \vert\!\frac{\psi_{\mathcal{H}}(\omega^{\beta_\iota})}{\Omega}\, \Delta_\iota. \tag{vii}$$

From $\beta_\iota \in \mathcal{H}(\emptyset)$ we obtain $\psi_{\mathcal{H}}(\omega^{\beta_\iota}) < \psi_{\mathcal{H}}(\beta)$ by (i) and from $\beta \in \mathcal{H}(\emptyset)$ also $\psi_{\mathcal{H}}(\omega^{\beta}) \in \mathcal{H}_{\omega^\beta+1}(\emptyset)$. Since also $par(\Delta_\iota) \subseteq \mathcal{H}(\emptyset) \subseteq \mathcal{H}_{\omega^\beta+1}(\emptyset)$ we obtain

$$\mathcal{H}_{\omega^\beta+1} \left|\frac{\psi_{\mathcal{H}}(\omega^\beta)}{\Omega}\right. \Delta \tag{viii}$$

from (vii) by the same inference.
Now assume that the main formula ot the last inference is

$$\neg Cl_{F_i}(\mathsf{I}_{F_i}^{<\Omega}) \equiv (\exists x)[F_i(\mathsf{I}_{F_i}^{<\Omega}, x) \wedge x \notin \mathsf{I}_{F_i}^{<\Omega}]. \tag{ix}$$

Then we have the premise

$$\mathcal{H} \left|\frac{\beta_0}{\Omega}\right. \neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}), \ldots, \neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega}), F_i(\mathsf{I}_{F_i}^{<\Omega}, t) \wedge t \notin \mathsf{I}_{F_i}^{<\Omega}, \Delta \tag{x}$$

with $\beta_0 \in \mathcal{H}(par(\Delta) \cup \{\Omega\}) = \mathcal{H}(\emptyset)$. By inversion we obtain from (x)

$$\mathcal{H} \left|\frac{\beta_0}{\Omega}\right. \neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}), \ldots, \neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega}), F_i(\mathsf{I}_{F_i}^{<\Omega}, t), \Delta \tag{xi}$$

and

$$\mathcal{H} \left|\frac{\beta_0}{\Omega}\right. \neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}), \ldots, \neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega}), t \notin \mathsf{I}_{F_i}^{<\Omega}, \Delta. \tag{xii}$$

Applying the induction hypothesis to (xi) and then using boundedness gives

$$\mathcal{H}_{\omega^{\beta_0}+1} \left|\frac{\psi_{\mathcal{H}}(\omega^{\beta_0})}{\Omega}\right. F_i(\mathsf{I}_{F_i}^{<\psi_{\mathcal{H}}(\omega^{\beta_0})}, t), \Delta, \tag{xiii}$$

i.e.

$$\mathcal{H}_{\omega^{\beta_0}+1} \left|\frac{\psi_{\mathcal{H}}(\omega^{\beta_0})}{\Omega}\right. t \in \mathsf{I}_{F_i}^{\psi_{\mathcal{H}}(\omega^{\beta_0})}, \Delta. \tag{xiv}$$

From (xii) we obtain by inversion

$$\mathcal{H} \left|\frac{\beta_0}{\Omega}\right. \neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}), \ldots, \neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega}), t \notin \mathsf{I}_{F_i}^{\psi_{\mathcal{H}}(\omega^{\beta_0})}, \Delta \tag{xv}$$

which entails

$$\mathcal{H}_{\omega^{\beta_0}+1} \left|\frac{\beta_0}{\Omega}\right. \neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}), \ldots, \neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega}), t \notin \mathsf{I}_{F_i}^{\psi_{\mathcal{H}}(\omega^{\beta_0})}, \Delta. \tag{xvi}$$

Since $\psi_{\mathcal{H}}(\omega^{\beta_0}) \in \mathcal{H}_{\omega^{\beta_0}+1}(\emptyset)$ the induction hypothesis applies to (xvi) and we obtain

$$(\mathcal{H}_{\omega^{\beta_0}+1})_{\omega^{\beta_0}+1} \left|\frac{\psi_{\mathcal{H}}(\omega^{\beta_0}+\omega^{\beta_0})}{\Omega}\right. t \notin \mathsf{I}_{F_i}^{\psi_{\mathcal{H}}(\omega^{\beta_0})}, \Delta. \tag{xvii}$$

By Lemma 3.5.2 this entails

$$\mathcal{H}_{\omega^{\beta_0}+1+\omega^{\beta_0}+1} \left|\frac{\psi_{\mathcal{H}}(\omega^{\beta_0}+\omega^{\beta_0})}{\Omega}\right. t \notin \mathsf{I}_{F_i}^{\psi_{\mathcal{H}}(\omega^{\beta_0})}, \Delta. \tag{xviii}$$

Now we obtain

$$\mathcal{H}_{\omega^\beta+1} \left|\frac{\psi_{\mathcal{H}}(\omega^\beta)}{\Omega}\right. \Delta$$

from (xiv) and (xvii) by the Structural Lemma and (cut). $\square$

3.5.5 Remark Although we will not need it for the ordinal analysis of ID_1 we want to remark that the Collapsing Lemma may be strengthened to

$$\mathcal{H} \left|\frac{\beta}{\Omega+1}\right. \neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}), \ldots, \neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega}), \Delta \;\Rightarrow\; \mathcal{H}_{\omega^\beta+1} \left|\frac{\psi_{\mathcal{H}}(\omega^\beta)}{\psi_{\mathcal{H}}(\omega^\beta)}\right. \Delta.$$

For $k = 0$ it can be modified to

$$\mathcal{H}\,\frac{\beta}{\Omega}\;\Delta \;\;\Rightarrow\;\; \mathcal{H}_{\beta+1}\,\frac{\psi_{\mathcal{H}}(\beta)}{\psi_{\mathcal{H}}(\beta)}\;\Delta$$

Proof We have to do three things. First we observe that in the case of a cut of rank $< \Omega$ we have $par(F) \subseteq \mathcal{H}(\emptyset) \cap \Omega \subseteq \mathcal{H}_{\omega^\beta} \cap \Omega = \psi_{\mathcal{H}}(\omega^\beta)$. Since $rnk(F) < \max par(F) + \omega$ we obtain $rnk(F) < \psi_{\mathcal{H}}(\omega^\beta)$. If the cut rank is $\Omega + 1$ we have the additional case of a cut of rank Ω. Then the cut sentence is $t \in \mathsf{I}_F^{<\Omega}$ and we have the premises

$$\mathcal{H}\,\frac{\beta_0}{\Omega+1}\;\neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}), \ldots, \neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega}), \Delta, t \in \mathsf{I}_F^{<\Omega} \tag{i}$$

and

$$\mathcal{H}\,\frac{\beta_0}{\Omega+1}\;\neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}), \ldots, \neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega}), \Delta, t \notin \mathsf{I}_F^{<\Omega}. \tag{ii}$$

But then we may apply the induction hypothesis to (i) and then proceed as in the last case in the proof of the Collapsing Lemma. The resulting cut sentence is $t \in \mathsf{I}_F^{<\psi_{\mathcal{H}}(\omega^{\beta_0})}$ which shows that the cut sentence has rank $< \psi_{\mathcal{H}}(\omega^\beta)$.
Finally we observe that only in this case we needed the fact that ω^β is additively indecomposable. This case is not needed if $k = 0$ and we may replace ω^β by β. $\square$

3.6 The upper bound

In order to get an upper bound for κ^{ID_1} Theorem 3.4.2 is not longer sufficient. What we need is

3.6.1 Theorem *If* $\,\frac{m}{}\,\Delta(\vec{x})$ *holds for a finite set of* $\mathcal{L}(ID_1)$*–formulas then* $\mathcal{H}\,\frac{\Omega+m}{0}\,\Delta(\underline{\vec{n}})$ *for all tuples* $\underline{\vec{n}}$ *of numerals and all Cantorian closed Skolem–hull operators* $\mathcal{H}$*.*

The key here is

3.6.2 Lemma *(Controlled Tautology)* *For every* $\mathcal{L}_\infty(NT)$*-sentence and Cantorian closed Skolem–hull operator* $\mathcal{H}$ *we have* $\mathcal{H}\,\frac{2\cdot rnk(F)}{0}\,\Delta, \neg F, F$*.*

The proof by induction on $rnk(F)$ is easy. First observe that $2 \cdot rnk(F) \in \mathcal{H}(par(F))$ for every Cantorian closed Skolem–hull operator because $rnk(F) = \max par(F) + n$ for some $n < \omega$. Assume without loss of generality that $F \in \bigwedge$–type. By induction hypothesis we have

$$\mathcal{H}\,\frac{2\cdot rnk(G)}{0}\;\Delta, \neg F, F, G, \neg G \tag{i}$$

for all $G \in CS(F)$. Since $par(\Delta, \neg F, F, G, \neg G) \subseteq \mathcal{H}(par(\Delta, \neg F, F, G))$ we obtain from (i)

$$\mathcal{H}\,\frac{2\cdot rnk(G)+1}{0}\;\Delta, \neg F, F, G, \tag{ii}$$

for all $G \in CS(F)$ by an inference $(\bigvee)$. From (ii) and $2 \cdot rnk(G) + 1 < 2 \cdot (rnk(G) + 1) \leq 2 \cdot rnk(F)$, however, we immediately get

$$\mathcal{H}\,\frac{2\cdot rnk(F)}{0}\;\Delta, \neg F, F$$

by an inference $(\bigwedge)$. $\square$

Now it is an easy exercise to prove Theorem 3.6.1 by induction on m using Lemma 3.6.2 in case that $\,\frac{m}{}\,\Delta, \vec{t} \in \mathsf{I}_F, \neg\vec{t} \in \mathsf{I}_F$ holds by (AxL).

It is obvious that all defining axioms and also all identity axioms are controlled derivable with a derivation depth below ω. Ruminating the proof of the Induction Lemma (Lemma 2.3.2) shows that this proof is controlled by any Cantorian closed Skolem–hull operator. Summing up we get

$$\mathcal{H} \big|\!\frac{\Omega+\omega+4}{0}\; F^* \tag{3.10}$$

for every axiom G of NT in the language $\mathcal{L}(ID_1)$ where $\mathcal{H}$ may be an aribtrary Cantorian closed Skolem–hull operator.
So it remains to check the schemes ${ID_1}^1$ and ${ID_1}^2$. By the Collapsing Lemma (Lemma 3.5.4) we have only to deal with ${ID_1}^2$.

3.6.3 Lemma *(Generalized Induction) Let $F(X,\vec{x})$ be an X–positive NT formula. Then*

$$\mathcal{H} \big|\!\frac{2\cdot rnk(G)+\omega\cdot(\alpha+1)}{0}\; \neg Cl_F(G), \vec{n} \notin \mathsf{I}_F^{\alpha}, G(\vec{n})$$

holds for any sentence $G(\vec{n})$ in our fragment of $\mathcal{L}_\infty(NT)$ and for any Cantorian closed Skolem–hull operator $\mathcal{H}$.

From the Generalized Induction Lemma we obtain

$$\mathcal{H} \big|\!\frac{\Omega\cdot 2+3}{0}\; \neg Cl_F(G), (\forall \vec{x})[\vec{x} \in \mathsf{I}_F^{<\Omega} \to G(\vec{x})] \tag{3.11}$$

which is the translation of the scheme ${ID_1}^2$.
The proof of Lemma 3.6.3 still needs a preparing lemma.

3.6.4 Lemma *(Monotonicity Lemma) Let $F(X,\vec{x})$ be an X–positive $\mathcal{L}(NT)$–formula. Then*

$$\mathcal{H} \big|\!\frac{\alpha}{\rho}\; \Delta, \neg G(\vec{n}), H(\vec{n}) \text{ for all } \vec{n} \;\Rightarrow\; \mathcal{H} \big|\!\frac{\alpha+2\cdot rnk(F)}{\rho}\; \Delta, \neg F(G,\vec{n}), F(H,\vec{n})$$

for all $\vec{n}$.

Proof Induction on $rnk(F)$. In the case that X does not occur in $F(X,\vec{x})$ we have

$$\mathcal{H} \big|\!\frac{2\cdot rnk(F)}{0}\; \Delta, \neg F, F$$

by the Tautology Lemma (Lemma 3.6.2). In the case that $F \equiv (\vec{x} \in X)$ we obtain the claim from the hypothesis $\mathcal{H} \big|\!\frac{\alpha}{\rho} \Delta, \neg G(\vec{n}), H(\vec{n})$. The remaining cases are as in the proof of the controlled Tautology Lemma. $\square$

Proof of the Generalized Induction Lemma. We have

$$\mathcal{H} \big|\!\frac{2\cdot rnk(G)+\omega\cdot\alpha}{0}\; \neg Cl_F(G), \vec{n} \notin \mathsf{I}_F^{<\alpha}, G(\vec{n}) \tag{i}$$

by an inference $(\bigwedge)$ with empty premises if $\alpha = 0$ or by induction hypothesis. From (i) we obtain

$$\mathcal{H} \big|\!\frac{2\cdot rnk(G)+\omega\cdot\alpha+2\cdot rnk(F)}{0}\; \neg Cl_F(G), \vec{n} \notin \mathsf{I}_F^{\alpha}, F(G,\vec{n}) \tag{ii}$$

by the Monotonicity Lemma. By controlled Tautology we have

$$\mathcal{H} \big|\!\frac{2\cdot rnk(G)}{0}\; \neg Cl_F(G), \vec{n} \notin \mathsf{I}_F^{\alpha}, \neg G(\vec{n}), G(\vec{n}). \tag{iii}$$

From (ii) and (iii) we get

$$\mathcal{H} \big|\!\frac{2\cdot rnk(G)+\omega\cdot\alpha+(2\cdot rnk(F))+1}{0}\; \neg Cl_F(G), \vec{n} \notin \mathsf{I}_F^{\alpha}, F(G,\vec{n}) \wedge \neg G(\vec{n}), G(\vec{n}) \tag{iv}$$

by an inference $(\bigwedge)$. From (iv) we finally obtain

$$\mathcal{H}\,\Big|\frac{2\cdot rnk(G)+\omega\cdot\alpha+2\cdot rnk(F)+2}{0}\;\neg Cl_F(G), \vec{n}\notin \mathsf{I}_F^{\alpha}, G(\vec{n})$$

by an inference $(\bigvee)$. Since $2\cdot rnk(G)+\omega\cdot\alpha+2\cdot rnk(F)+2 < 2\cdot rnk(G)+\omega\cdot(\alpha+1)$ we are done. $\square$

3.6.5 Theorem *If $ID_1 \vdash F(\vec{x})$ then there are finitely many axioms $Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}),\dots,Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega})$ and an $n<\omega$ such that*

$$\mathcal{H}\,\Big|\frac{\Omega\cdot 2+\omega}{\Omega+n}\;\neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}),\dots,\neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega}), F(\vec{m})$$

holds for any tuple $\vec{m}$ of the length of $\vec{x}$ and for any Cantorian closed Skolem–hull operator.

Proof If $ID_1 \vdash F(\vec{x})$ then there are finitely many axioms $A_1,\dots,A_r$ and a natural number p such that $\big|\!\frac{p}{}\, \neg A_1,\dots,\neg A_r, F(\vec{x})$. By Theorem 3.6.1 this implies

$$\mathcal{H}\,\Big|\frac{\Omega+p}{0}\;\neg A_1^*,\dots,\neg A_r^*, F^*(\vec{m}) \tag{i}$$

for any Cantorian closed Skolem–hull operator $\mathcal{H}$. From (i), (3.10) and (3.11) we obtain the claim by some cuts. $\square$

Let $\mathcal{B}_0(X)$ the least set $Y \supseteq X$ such that $\{0,\Omega\}\subseteq Y$ and Y is closed under $+$ and φ. Then $\mathcal{B}_0$ is Cantorian closed and we obtain a hierarchy $\mathcal{B}_\alpha$ of Cantorian closed operators. We put $\psi(\alpha) := \psi_{\mathcal{B}_0}(\alpha)$. The ordinal $\psi(\varepsilon_{\Omega+1})$ is then the BACHMANN–HOWARD ordinal.

3.6.6 Theorem *(The Upper Bound for ID_1) It is $\kappa^{ID_1}\le\psi(\varepsilon_{\Omega+1})$.*

Proof If $ID_1 \vdash \underline{m}\in \mathsf{I}_F$ we obtain by Theorem 3.6.5

$$\mathcal{B}_0\,\Big|\frac{\Omega\cdot 2+\omega}{\Omega+n}\;\neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}),\dots,\neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega}), \underline{m}\in\mathsf{I}_F^{<\Omega}. \tag{i}$$

By Theorem 3.4.12 we obtain an $\alpha<\varepsilon_{\Omega+1}$ such that

$$\mathcal{B}_0\,\Big|\frac{\alpha}{\Omega}\;\neg Cl_{F_1}(\mathsf{I}_{F_1}^{<\Omega}),\dots,\neg Cl_{F_k}(\mathsf{I}_{F_k}^{<\Omega}), \underline{m}\in\mathsf{I}_F^{<\Omega}. \tag{ii}$$

From (ii) and the Collapsing Lemma (Lemma 3.5.4) it follows

$$\mathcal{B}_{\omega^\alpha+1}\,\Big|\frac{\psi(\omega^\alpha)}{\Omega}\;\underline{m}\in\mathsf{I}_F^{<\Omega}$$

which by Theorem 3.4.3 implies $tc(\underline{m}\in\mathsf{I}_F^{<\Omega})\le\psi(\omega^\alpha)<\psi(\varepsilon_{\Omega+1})$. By Lemma 3.4.1 the claim follows. $\square$

3.7 The lower bound

3.7.1 Coding ordinals in $\mathcal{L}(NT)$

It follows from the previous sections that $\mathcal{B}_{\varepsilon_{\Omega+1}}(\emptyset)$ is the set of ordinals which turned out to be relevant in the computation of an upper bound for κ^{ID_1}. To prove that $\psi(\varepsilon_{\Omega+1})$ is the exact bound it suffices to prove $n\in Acc^\alpha(\prec)$ for some arithmetical definable relation $\prec$ and all $\alpha<\psi(\varepsilon_{\Omega+1})$. If we succeed in showing that for a primitive recursive relation $\prec$ we have by Observation 1.7.10 $\|ID_1\|=\psi(\varepsilon_{\Omega+1})$.
Since we cannot talk about ordinals in $\mathcal{L}(ID_1)$ we need codes for the ordinals in $\mathcal{B}_{\varepsilon_{\Omega+1}}(\emptyset)$. The only parameters occurring on $\mathcal{B}_{\varepsilon_{\Omega+1}}(\emptyset)$ are 0 and Ω. Therefore every ordinal in $\mathcal{B}_{\varepsilon_{\Omega+1}}(\emptyset)$ possesses a term notation which is built up from $0,\Omega$ by the functions $+$, φ and ψ. This term notation, however, is not unique. In order to show that the set of term notations together with the induced

$<$–relation on the terms is primitive recursive we need a unique term notation. This forces us to inspect the set $\mathcal{B}_\alpha(\emptyset)$ more closely.
We define

$$\alpha =_{NF} \psi(\beta) \quad :\Leftrightarrow \quad \alpha = \psi(\beta) \wedge \beta \in \mathcal{B}_\beta(\emptyset).$$

Then

$$\alpha =_{NF} \psi(\beta_1) \wedge \alpha =_{NF} \psi(\beta_2) \quad \Rightarrow \quad \beta_1 = \beta_2 \tag{3.12}$$

since $\beta_1 < \beta_2$ would imply $\psi(\beta_1) < \psi(\beta_2)$ by (3.8) because $\beta_1 \in \mathcal{B}_{\beta_1}(\emptyset) \subseteq \mathcal{B}_{\beta_2}(\emptyset)$. Now we define a set of *ordinal terms* T by the clauses

(T_0) $\quad \{0, \Omega\} \subseteq T$

(T_1) $\quad \alpha \notin SC \wedge SC(\alpha) \subseteq T \quad \Rightarrow \quad \alpha \in T$

(T_2) $\quad \beta \in T \wedge \alpha =_{NF} \psi(\beta) \quad \Rightarrow \quad \alpha \in T.$

We want to prove

$$T = \mathcal{B}_{\Omega^\Gamma}(\emptyset) \tag{3.13}$$

for $\Omega^\Gamma := \min\{\alpha \in SC \mid \Omega < \alpha\}$.
The inclusion $\subseteq$ in (3.13) is obvious. Troublesome is the converse inclusion. The idea is of course to prove

$$\xi \in \mathcal{B}_{\Omega^\Gamma}(\emptyset) \quad \Rightarrow \quad \xi \in T \tag{3.14}$$

by induction on the definition of $\xi \in \mathcal{B}_{\Omega^\Gamma}(\emptyset)$. We will therefore redefine the sets $\mathcal{B}_\alpha(\emptyset)$ more carefully by the following clauses.

(B_0) $\quad \{0, \Omega\} \subseteq B_\alpha^n$

(B_1) $\quad \xi \notin SC \wedge SC(\xi) \subseteq B_\alpha^{n-1} \quad \Rightarrow \quad \xi \in B_\alpha^n$

(B_2) $\quad \eta \in B_\alpha^{n-1} \cap \alpha \quad \Rightarrow \quad \psi(\eta) \in B_\alpha^n$

(B_3) $\quad B_\alpha := \bigcup_{n\in\omega} B_\alpha^n \wedge \psi(\alpha) := \min\{\xi \mid \xi \notin B_\alpha\}.$

It is easy to check that $B_\alpha = \mathcal{B}_\alpha(\emptyset)$ for all $\alpha \leq \Omega^\Gamma$ which justifies the use of the same symbol ψ for the functions $\min\{\xi \mid \xi \notin \mathcal{B}_\alpha(\emptyset)\}$ and $\min\{\xi \mid \xi \notin B_\alpha\}$. So (3.14) can be shown by proving

$$\xi \in B_\alpha^n \quad \Rightarrow \quad \xi \in T \tag{3.15}$$

for all $\alpha < \Omega^\Gamma$ by induction on n. What is still troublesome in pursuing this strategy is case (B_2). In this case we don't know if $\psi(\eta)$ is in normal–form, i.e. if $\eta \in B_\eta$. Therefore we show first

3.7.1 Lemma *For every ordinal $\alpha < \Omega^\Gamma$ the ordinal $\alpha_{\mathsf{nf}} := \min\{\xi \mid \alpha \leq \xi \in B_\alpha\}$ exists and it is $\psi(\alpha) =_{NF} \psi(\alpha_{\mathsf{nf}})$.*

Proof Since $\Omega^\Gamma = \sup_{n\in\omega} \varphi_\Omega^n(0)$ and $\varphi_\Omega^n(0) \in B_\alpha$ for any α it follows that α_{nf} exists. By definition $[\alpha, \alpha_{\mathsf{nf}}) \cap B_\alpha = \emptyset$ which implies $B_\alpha = B_{\alpha_{\mathsf{nf}}}$ and thus also $\psi(\alpha) = \psi(\alpha_{\mathsf{nf}})$. Since $\alpha_{\mathsf{nf}} \in B_\alpha = B_{\alpha_{\mathsf{nf}}}$ we have $\psi(\alpha) =_{NF} \psi(\alpha_{\mathsf{nf}})$. □

Our troubles are solved as soon as we can show

$$\eta \in B_\alpha^n \quad \Rightarrow \quad \eta_{\mathsf{nf}} \in B_\alpha^n. \tag{3.16}$$

Then we may argue in case (B_2) that for $\eta \in B_\alpha^{n-1}$ we also have $\eta_{\mathsf{nf}} \in B_\alpha^{n-1}$ and thus $\eta_{\mathsf{nf}} \in T$ which entails $\psi(\eta) =_{NF} \psi(\eta_{\mathsf{nf}}) \in T$.

We obtain (3.16) as a special case of the following lemma whose proof is admittedly tedious. Also we cannot learn much from its proof. Therefore one commonly includes the normal–form condition into clause (B_2) which then becomes

$(B_2)'$ $\eta \in B_\alpha^{n-1} \cap \alpha \wedge \eta \in B_\eta \;\;\Rightarrow\;\; \psi(\eta) \in B_\alpha^n$.

The proof of (3.15) then becomes trivial.

3.7.2 Lemma *Let* $\delta(\alpha) := \min\{\xi \mid \alpha \leq \xi \in B_\delta\}$. *Then* $\alpha \in B_\beta^n$ *implies* $\delta(\alpha) \in B_\beta^n$ *for all* $\alpha < \Omega^\Gamma$.

Proof We show the lemma by induction on n. First observe that by the miminality of $\delta(\alpha)$ we get

$$\alpha \in \mathbb{H} \;\;\Rightarrow\;\; \delta(\alpha) \in \mathbb{H} \text{ and } \alpha \in SC \;\;\Rightarrow\;\; \delta(\alpha) \in SC. \tag{i}$$

The lemma is trivial if $\alpha \in B_\delta$. Then $\delta(\alpha) = \alpha$. Therefore we assume

$$\alpha \notin B_\delta. \tag{ii}$$

Then $\alpha < \delta(\alpha)$ and for $\alpha < \Omega$ we get by (3.9) $\delta(\alpha) = \Omega \in B_\delta^n$ for any n. Therefore we may also assume

$$\Omega \leq \alpha. \tag{iii}$$

We have

$$\xi \notin SC \wedge \xi \in B_\beta^n \;\;\Rightarrow\;\; SC(\xi) \subseteq B_\beta^{n-1}. \tag{iv}$$

Since $(\Omega, \Omega^\Gamma) \cap SC = \emptyset$ we obtain by induction hypothesis

$$\delta(SC(\alpha)) := \{\delta(\xi) \mid \xi \in SC(\alpha)\} \subseteq B_\beta^{n-1} \cap B_\delta. \tag{v}$$

We are done if we can prove

$$SC(\delta(\alpha)) \subseteq B_\beta^{n-1} \cap B_\delta. \tag{vi}$$

We prove (vi) by induction on the number of ordinals in $SC(\alpha)$. First assume $\alpha =_{NF} \alpha_1 + \cdots + \alpha_k$. Since $\alpha_j \leq \delta(\alpha_j) \in \mathbb{H}$ we obtain $\alpha \leq \delta(\alpha_1) + \cdots + \delta(\alpha_k)$ and because $\delta(\alpha_1) + \cdots \delta(\alpha_n) \in B_\delta$ even $\alpha < \delta(\alpha_1) + \cdots \delta(\alpha_k)$. Let $i := \min\{j \leq k \mid \alpha_j < \delta(\alpha_j)\}$. We claim

$$\delta(\alpha) = \alpha_1 + \cdots + \alpha_{i-1} + \delta(\alpha_i) = \delta(\alpha_1) + \cdots + \delta(\alpha_{i-1}) + \delta(\alpha_i). \tag{vii}$$

From (vii) we obtain (vi) by induction hypothesis. Let $\eta := \alpha_1 + \cdots + \alpha_{i-1}$. We have $\alpha < \eta + \delta(\alpha_i)$. Hence $\delta(\alpha) \leq \eta + \delta(\alpha_i)$. If we assume $\delta(\alpha) < \eta + \delta(\alpha_i)$ there is an $\varepsilon \in B_\delta$ such that $\eta + \alpha_i \leq \alpha < \varepsilon < \eta + \delta(\alpha_i)$. But then we obtain an ε_1 such that $\varepsilon = \eta + \varepsilon_1$ and $\alpha_i < \varepsilon_1 < \delta(\alpha_i)$. But $\varepsilon \in B_\delta$ entails $\varepsilon_1 \in B_\delta$ which contradicts the definition of $\delta(\alpha_1)$.
Next assume $\alpha =_{NF} \varphi_{\alpha_1}(\alpha_2)$. If $\delta(\alpha_1) = \alpha_1$ we immediately obtain $\delta(\varphi_{\alpha_1}(\alpha_2)) = \varphi_{\alpha_1}(\delta(\alpha_2)) = \varphi_{\delta(\alpha_1)}(\delta(\alpha_2))$ which entails (vi) byinduction hypothesis. If $\alpha_1 < \delta(\alpha_1)$ and $\alpha \leq \delta(\alpha_2)$ we obtain $\delta(\alpha) \leq \delta(\alpha_2) \leq \delta(\alpha)$ and (vi) follows by induction hypothesis. So assume $\alpha_1 < \delta(\alpha_1)$ and $\delta(\alpha_2) < \alpha$. Let

$$\alpha_3 := \min\{\xi \mid \alpha \leq \varphi_{\delta(\alpha_1)}(\xi)\}. \tag{viii}$$

We claim

$$\alpha_3 \in B_\beta^{n-1} \cap B_\delta. \tag{ix}$$

From (ix) we get $\delta(\alpha) \leq \varphi_{\delta(\alpha_1)}(\alpha_3)$. If we assume $\delta(\alpha) < \varphi_{\delta(\alpha_1)}(\alpha_3)$ we have $\alpha = \varphi_{\alpha_1}(\alpha_2) < \varphi_{\delta(\alpha_1)}(\alpha_3)$. Since $\delta(\alpha) \in \mathbb{H}$ we obtain $\delta(\alpha) =_{NF} \varphi_{\xi_1}(\xi_2)$. The assumption $\xi_1 = \delta(\alpha_1)$ yields $\alpha < \delta(\alpha) = \varphi_{\delta(\alpha_1)}(\xi_2) < \varphi_{\delta(\alpha_1)}(\alpha_3)$ and thus $\xi_2 < \alpha_3$ conctradicting the minimality of α_3.

Assuming $\delta(\alpha_1) < \xi_1$ yields $\delta(\alpha) < \alpha_3$ and $\alpha < \delta(\alpha) = \varphi_{\delta(\alpha_1)}(\delta(\alpha))$ again contradicting the minimality of α_3. So it remains $\xi_1 < \delta(\alpha_1)$. But since $\xi_1 \in B_\alpha$ this implies $\xi_1 < \alpha_1$ which in turn entails $\alpha < \xi_2 \in B_\delta \cap \delta(\alpha)$ contradiction the definition of $\delta(\alpha)$. Therefore we have

$$\delta(\alpha) = \varphi_{\delta(\alpha_1)}(\alpha_3) \tag{x}$$

and obtain (vi) from (x) by induction hypothesis and (ix).
It remains to prove (ix). We are done if $\alpha_3 = 0$. If we assume $\alpha_3 \in Lim$ we get $\alpha =_{NF} \varphi_{\alpha_1}(\alpha_2) = \varphi_{\delta(\alpha_1)}(\alpha_3)$ by the continuity of $\varphi_{\delta(\alpha_1)}$. Because $\alpha_1 < \delta(\alpha_1)$ we then obtain $\alpha_2 = \alpha$ contradicting $\alpha_2 \leq \delta(\alpha_2) < \alpha$. It remains the case that $\alpha_3 = \eta + 1$. Then $\varphi_{\delta(\alpha_1)}(\eta) < \alpha =_{NF} \varphi_{\alpha_1}(\alpha_2) < \varphi_{\delta(\alpha_1)}(\eta + 1)$. Because of $\alpha_1 < \delta(\alpha_1)$ this implies $\varphi_{\delta(\alpha_1)}(\eta) \leq \alpha_2 \leq \delta(\alpha_2) < \alpha = \varphi_{\delta(\alpha_1)}(\eta + 1)$. But $\alpha_2 = \varphi_{\delta(\alpha_1)}(\eta)$ is excluded because otherwise we get $\varphi_{\alpha_1}(\alpha_2) = \alpha_2 < \varphi_{\alpha_1}(\alpha_2)$. Since $\delta(\alpha_2) \in B_\beta^{n-1} \cap B_\delta$ we have shown

$$B_\delta \cap B_\beta^{n-1} \cap (\varphi_{\delta(\alpha_1)}(\eta), \varphi_{\delta(\alpha_1)}(\eta + 1)) \neq \emptyset. \tag{xi}$$

To finish the proof we show that in general we have

$$B_\beta^n \cap (\varphi_\xi(\eta), \varphi_\xi(\eta + 1)) \neq \emptyset \;\;\Rightarrow\;\; \eta + 1 \in B_\beta^n. \tag{3.17}$$

From (xi) and (3.17) we then obtain $\alpha_3 \in B_\delta \cap B_\beta^{n-1}$, i.e. (ix).
To prove (3.17) we first show

$$\gamma \in [\varphi_\xi(\eta), \varphi_\xi(\eta + 1)) \;\;\Rightarrow\;\; SC(\eta) \subseteq SC(\gamma) \tag{xii}$$

by induction on the number of elements in $SC(\gamma)$.
If $\gamma =_{NF} \gamma_1 + \cdots + \gamma_k$ we have $\gamma_1 \in [\varphi_\xi(\eta), \varphi_\xi(\eta+1))$ and obtain $SC(\eta) \subseteq SC(\gamma_1) \subseteq SC(\gamma)$. If $\gamma =_{NF} \varphi_{\gamma_1}(\gamma_2)$ then $\gamma_1 \leq \xi$ because $\xi < \gamma_1$ entails $\gamma \leq \eta + 1 < \varphi_\xi(\eta)$. If $\xi = \gamma_1$ then $\eta = \gamma_2$ and $SC(\eta) = SC(\gamma_2) \subseteq SC(\gamma)$. If $\gamma_1 < \xi$ then $\varphi_\xi(\eta) \leq \gamma_2 < \gamma < \varphi_\xi(\eta + 1)$ and we obtain $SC(\eta) \subseteq SC(\gamma_2) \subseteq SC(\gamma)$ by induction hypothesis. If finally $\gamma \in SC$ then $\gamma = \varphi_\xi(\eta) = \eta$ and the claim is obvious.
We prove (3.17) by induction on n. Let $\sigma \in B_\beta^n \cap (\varphi_\xi(\eta), \varphi_\xi(\eta + 1))$. Then $\sigma \notin SC$ and we have $SC(\sigma) \subseteq B_\beta^{n-1}$. By (xii) we get $SC(\eta) \subseteq SC(\sigma) \subseteq B_\beta^{n-1}$. Since $0 \in B_\beta^{n-1}$ we also have $SC(\eta + 1) \subseteq B_\beta^{n-1}$ and thus obtain $\eta + 1 \in B_\beta^n$. □

Having established $\mathcal{B}_{\Omega^\Gamma}(\emptyset) = B_{\Omega^\Gamma} = T$ we want to develop a primitive recursive notation system for the ordinals in T. What is still annoying is the normal–form condition in clause (T_2). In order to define a set On of notions for ordinals in T together with a $<$–relation in On by simultaneous course–of–values recursion we should try to replace the condition $\beta \in B_\beta$ in $\alpha =_{NF} \psi(\beta)$ by a condition which refers only to proper subterms of β. We observe that we have

$$\begin{aligned} \xi \in B_\beta \;\;\Leftrightarrow\;\; & \xi = 0 \vee \xi = \Omega \vee \\ & (\xi \notin SC \wedge SC(\xi) \subseteq B_\beta) \vee \\ & (\xi = \psi(\eta) \wedge \eta \in B_\beta \cap \beta). \end{aligned} \tag{3.18}$$

From (3.18) we read off the following definition.

3.7.3 Definition Let

$$K(\xi) := \begin{cases} \emptyset & \text{if } \xi = 0 \text{ or } \xi = \Omega \\ \bigcup \{K(\eta) \mid \eta \in SC(\xi)\} & \text{if } \xi \notin SC \\ \{\eta\} \cup K(\eta) & \text{if } \xi = \psi(\eta). \end{cases}$$

From (3.18) and Definition 3.7.3 we immediately get

3.7.4 Lemma *It is $\xi \in B_\beta$ iff $K(\xi) \subseteq \beta$.*

3.7.5 Corollary *We have $\alpha =_{NF} \psi(\beta)$ iff $\alpha = \psi(\beta)$ and $K(\beta) \subseteq \beta$.*

3.7.6 Definition We use the facts about ordinals in T to define sets $\mathsf{SC} \subseteq \mathsf{H} \subseteq \mathsf{On} \subseteq \mathbb{N}$ of ordinal notations together with a finite set $\mathsf{K}(a) \subseteq \mathsf{On}$ of subterms of $a \in \mathsf{On}$, relations $\prec \subseteq \mathsf{On} \times \mathsf{On}$ and $\equiv \subseteq \mathsf{On} \times \mathsf{On}$ and an evaluation function $|\ |_{\mathcal{O}} \colon \mathsf{On} \longrightarrow T$ by the following clauses.

Definition of SC, H and On.

- $\langle 0 \rangle \in \mathsf{On}$, $\langle 1 \rangle \in \mathsf{SC}$, $|\langle 0 \rangle|_{\mathcal{O}} := 0$ and $|\langle 1 \rangle|_{\mathcal{O}} := \omega_1$
- If $a_1, \dots, a_n \in \mathcal{H}$ and $a_1 \succeq \cdots \succeq a_n$ then $\langle 1, a_1, \dots, a_n \rangle \in \mathsf{On}$ and $|\langle 1, a_1, \dots, a_n \rangle|_{\mathcal{O}} := |a_1|_{\mathcal{O}} + \cdots + |a_n|_{\mathcal{O}}$
- If $a, b \in \mathsf{On}$ then $\langle 2, a, b \rangle \in \mathsf{H}$ and $|\langle 2, a, b \rangle|_{\mathcal{O}} = \varphi_{|a|_{\mathcal{O}}}(|b|_{\mathcal{O}})$
- If $a \in \mathsf{On}$ and $b \prec a$ for all $b \in \mathsf{K}(a)$ then $\langle 3, a \rangle \in \mathsf{SC}$ and $|\langle 3, a \rangle|_{\mathcal{O}} := \psi(|a|_{\mathcal{O}})$

Definition of $\mathsf{K}(a)$.

- $\mathsf{K}(\langle 0 \rangle) = \mathsf{K}(\langle 1 \rangle) = \emptyset$
- $\mathsf{K}(\langle 1, a_1, \dots, a_n \rangle) = \mathsf{K}(a_1) \cup \cdots \cup \mathsf{K}(a_n)$
- If $b \prec \langle 2, a, b \rangle$ then $\mathsf{K}(\langle 2, a, b \rangle) = \mathsf{K}(a) \cup \mathsf{K}(b)$
- $\mathsf{K}(\langle 3, a \rangle) = \{a\} \cup \mathsf{K}(a)$

Let $a, b \in \mathsf{On}$. Then $a \prec b$ iff one of the following conditions is satisfied.

- $a = \langle 0 \rangle$ and $b \neq \langle 0 \rangle$
- $a = \langle 1, a_1, \dots, a_m \rangle$, $b = \langle 1, b_1, \dots, b_n \rangle$ and $(\exists i < m)(\forall j \leq i)[a_j \equiv b_j \wedge a_{i+1} \prec b_{i+1}]$ or $(\forall j \leq m)[a_j \equiv b_j] \wedge m < n$
- $a = \langle 1, a_1, \dots, a_n \rangle$, $b \in \mathsf{H}$ and $a_1 \prec b$
- $a \in \mathsf{H}$, $b = \langle 1, b_1, \dots, b_n \rangle$ and $a \preceq b_1$
- $a = \langle 2, a_1, a_2 \rangle$, $b = \langle 2, b_1, b_2 \rangle$ and one of the following conditions is satisfied

  $a_1 \prec b_1$ and $a_2 \prec b$

  $a_1 = b_1$ and $a_2 \prec b_2$

  $b_1 \prec a_1$ and $a \prec b_2$

- $a = \langle 2, a_1, a_2 \rangle$, $a_2 \prec a$, $b \in \mathsf{SC}$ and $a_1, a_2 \prec b$
- $a \in \mathsf{SC}$, $b = \langle 2, b_1, b_2 \rangle$, $b_2 \prec b$ and $a \preceq b_1$ or $a \preceq b_2$
- $a = \langle 3, a_1 \rangle$, $b = \langle 3, b_1 \rangle$ and $a_1 \prec b_1$
- $a = \langle 3, a_1 \rangle$ and $b = \langle 1 \rangle$

For $a, b \in \mathsf{On}$ we define $a \equiv b$ if one of the following conditions is satisfied

- $(a)_0 \neq 2$ and $(b)_0 \neq 2$ and $a = b$
- $a \in \mathsf{SC}$, $b_1 \prec a$ and $b = \langle 2, b_1, a \rangle$
- $b \in \mathsf{SC}$, $a_1 \prec b$ and $a = \langle 2, a_1, b \rangle$

- $a = \langle 2, a_1, a_2 \rangle$, $b = \langle 2, b_1, b_2 \rangle$ and one of the following conditions is satisfied

  $a_1 \prec b_1$ and $a_2 \equiv b$

  $a_1 = b_2$ and $a_2 \equiv b_2$

  $b_1 \prec a_1$ and $a \equiv b_2$.

- The relation $\equiv$ is transitive, reflexive and symmetrical.

Collecting all the known facts about T and observing that On, SC, H, $K(a)$, $\prec$ and $\equiv$ are defined by simultaneous course–of–values recursion we obtain the following theorem.

3.7.7 Theorem *The sets* On, H *and* SC *as well as the relations* $\prec$ *and* $\equiv$ *are primitive recursive. The map* $|\ |_{\mathcal{O}} \colon \mathsf{On} \longrightarrow T$ *is onto such that* $a \prec b$ *iff* $|a|_{\mathcal{O}} < |b|_{\mathcal{O}}$ *and* $a \equiv b$ *iff* $|a|_{\mathcal{O}} = |b|_{\mathcal{O}}$.

3.7.8 Corollary $\psi(\varepsilon_{\Omega+1}) < \omega_1^{CK}$.

3.7.2 The well–ordering proof

In view of Theorem 3.7.7 we may talk about the ordinals in $\mathcal{B}_{\Omega^\Gamma}(\emptyset)$ in $\mathcal{L}(NT)$ and thus also in $\mathcal{L}(ID_1)$. For the sake of better readability we will, however, not use the codes but identify ordinals in $\mathcal{B}_{\Omega^\Gamma}(\emptyset)$ and their codes. We will denote (codes of) ordinals by lower case greek letters and write $\alpha < \beta$ instead of $\alpha \prec \beta$. We use the abbreviations introduced in Section 2.4.
The aim of this section is to show that there is a primitive recursiv relation $<_0$ such that for every $\alpha < \psi(\varepsilon_{\Omega+1})$ we get $ID_1 \vdash \alpha \in Acc(<_0)$. The strategy of the proof will be the following.

- We first define a relation $<_1$ which is not longer arithmetical definable but needs a fixed point in its definition such that $\mathsf{TI}_1(\Omega, X)$ holds trivially and then use the well–ordering proof of Section 2.4 to obtain $\mathsf{TI}_1(\alpha, X)$ porvable in ID_1^{ext} for all $\alpha <_1 \varepsilon_{\Omega+1}$.
- Then we use a *condensing argument* to show that $\mathsf{TI}_1(\alpha, X)$ implies $\psi(\alpha) \in Acc(<_0)$.

3.7.9 Definition For ordinals α, β we define

- $\alpha <_0 \beta \ :\Leftrightarrow\ \alpha < \beta < \Omega$.

By $\xi \subseteq_0 X$ we denote the formula $(\forall \eta <_0 \xi)[\eta \in X]$.
Let Acc be the fixed point of the operator induced by $\xi \subseteq_0 X$, i.e. $\mathsf{Acc} \equiv Acc(<_0)$.
For $\alpha, \beta \in \mathsf{On}$ we define

- $\alpha <_1 \beta \ :\Leftrightarrow\ \alpha < \beta \land SC(\alpha) \cap \Omega \subseteq \mathsf{Acc}$.

$\xi \subseteq_1 X$ stands for $(\forall \eta <_1 \xi)[\eta \in X]$.
Let

- $\mathsf{Prog}_i(F) :\equiv (\forall \xi \in \mathit{field}(<_i))[(\forall \eta <_i \xi) F(\eta) \ \rightarrow\ F(\xi)]$
- $\mathsf{TI}_i(\alpha, F) :\equiv \alpha \in \mathit{field}(<_i) \land \mathsf{Prog}_i(F) \ \rightarrow\ (\forall \xi <_i \alpha) F(\xi)$.

Observe that by the axioms of ID_1 and Theorem 3.1.2 we have

$$ID_1 \vdash \alpha \in \mathsf{Acc} \ \leftrightarrow\ \alpha < \Omega \land \alpha \subseteq \mathsf{Acc} \tag{3.19}$$

$$ID_1 \vdash \mathsf{Prog}_0(\mathsf{Acc}) \tag{3.20}$$

$$ID_1 \vdash \mathsf{Prog}_0(F) \ \rightarrow\ (\forall \xi)[\xi \in \mathsf{Acc} \rightarrow F(\xi)] \tag{3.21}$$

3.7.10 Lemma $ID_1 \vdash \mathsf{Acc} \subseteq \Omega$.

Proof Since $\mathsf{Prog}_0(\mathit{field}(<_0))$ holds trivially we get $\mathsf{Acc} \subseteq \mathit{field}(<_0) = \{\alpha \mid \alpha < \Omega\}$ by (3.21). $\square$

3.7.11 Lemma *Let* $\mathsf{Prog}(F) :\equiv (\forall\alpha)[(\forall\xi < \alpha)F(\xi) \rightarrow F(\alpha)]$. *Then* $ID_1 \vdash \mathsf{Prog}(F) \rightarrow \mathsf{Prog}_0(F)$ *and thus also* $ID_1 \vdash \mathsf{Prog}(F) \rightarrow (\forall\xi \in \mathsf{Acc})F(\xi)$.

Proof $(\forall\xi <_0 \alpha)F(\xi)$ implies $(\forall\xi < \alpha)F(\xi)$ for $\alpha < \Omega$. Together with $\mathsf{Prog}(F)$ we therefore get $F(\alpha)$, i.e. we have $\mathsf{Prog}_0(F)$. Together with (3.21) we obtain the second claim, too, $\square$

3.7.12 Lemma (ID_1) *The class* Acc *is closed under ordinal addition.*

Proof Let $\mathsf{Acc}_+ := \{\xi \mid (\forall\eta \in \mathsf{Acc})[\eta + \xi \in \mathsf{Acc}]\}$. We claim

$$\mathsf{Prog}_0(\mathsf{Acc}_+). \tag{i}$$

To prove (i) we have the hypothesis

$$\alpha < \Omega \wedge (\forall\xi < \alpha)[\xi \in \mathsf{Acc}_+] \tag{ii}$$

and have to show $\alpha \in \mathsf{Acc}_+$ i.e.

$$(\forall\eta \in \mathsf{Acc})[\eta + \alpha \in \mathsf{Acc}]. \tag{iii}$$

By (3.19) it suffices to have

$$\eta + \alpha \subseteq \mathsf{Acc} \tag{iv}$$

to get (iii). Let $\xi < \eta+\alpha$. If $\xi < \eta$ then we get $\xi \in \mathsf{Acc}$ from $\eta \in \mathsf{Acc}$ by (3.19). If $\eta \leq \xi < \eta+\alpha$ there is a $\rho < \alpha$ such that $\xi = \eta + \rho$. Then we obtain $\eta + \rho \in \mathsf{Acc}$ by (ii).
From (i) we obtain

$$(\forall\xi \in \mathsf{Acc})[\xi \in \mathsf{Acc}_+] \tag{v}$$

by (3.21) which means

$$(\forall\xi \in \mathsf{Acc})(\forall\eta \in \mathsf{Acc})[\xi + \eta \in \mathsf{Acc}].$$ $\square$

3.7.13 Lemma $ID_1 \vdash \mathsf{Prog}_1(F) \rightarrow \mathsf{Prog}_0(F)$.

Proof We have the premises $\mathsf{Prog}_1(F)$, $\alpha < \Omega$ and $(\forall\xi <_0 \alpha)F(\xi)$ and have to show $F(\alpha)$. If $\xi <_1 \alpha$ we get $\xi <_0 \alpha$ by $\alpha < \Omega$ and thus $F(\xi)$ by $(\forall\xi <_0 \alpha)F(\xi)$. Hence $(\forall\xi <_1 \alpha)F(\xi)$ which entails $F(\alpha)$ by $\mathsf{Prog}_1(F)$. $\square$

3.7.14 Lemma (ID_1) *The class* Acc *is closed under* $\lambda\xi,\eta\,.\,\varphi_\xi(\eta)$.

Let $\mathsf{M} := \{\alpha \mid SC(\alpha) \cap \Omega \subseteq \mathsf{Acc}\}$ and define

$$\mathsf{Acc}_\varphi := \{\alpha \mid (\forall\xi \in \mathsf{Acc})[\xi < \varphi_\alpha(\xi) \rightarrow \varphi_\alpha(\xi) \in \mathsf{Acc}] \vee \alpha \notin \mathsf{M} \vee \Omega \leq \alpha\}. \tag{i}$$

We claim

$$\mathsf{Prog}_1(\mathsf{Acc}_\varphi). \tag{ii}$$

To prove (ii) we have the hypothesis

$$(\forall\xi <_1 \alpha)[\xi \in \mathsf{Acc}_\varphi] \tag{iii}$$

and have to show

$$\alpha \in \mathsf{Acc}_\varphi. \tag{iv}$$

For $\alpha \notin \mathsf{M}$ or $\Omega \leq \alpha$ (iii) is obvious. Therefore assume

$$\alpha \in \mathsf{M} \cap \Omega. \tag{v}$$

we have to show

$$(\forall \xi \in \mathsf{Acc})[\xi < \varphi_\alpha(\xi) \to \varphi_\alpha(\xi) \in \mathsf{Acc}]. \tag{vi}$$

According to Lemma 3.7.11 we may assume that we have

$$(\forall \eta < \xi)[\eta < \varphi_\alpha(\eta) \to \varphi_\alpha(\eta) \in \mathsf{Acc}] \tag{vii}$$

and have to show

$$\varphi_\alpha(\xi) \in \mathsf{Acc} \tag{viii}$$

for which by (3.19) ist suffices to prove

$$\rho < \varphi_\alpha(\xi) \quad \to \quad \rho \in \mathsf{Acc}. \tag{ix}$$

We show (ix) by Mathematical Induction on the length of the term notation of ρ. If $\rho =_{NF} \rho_1 + \cdots + \rho_n$ we have $\rho_i \in \mathsf{Acc}$ by induction hypothesis and obtain $\rho \in \mathsf{Acc}$ by Lemma 3.7.12. If $\rho \in \mathsf{SC}$ then we have $\rho \leq \alpha$ or $\rho \leq \xi$. If $\rho \leq \xi$ we get $\rho \in \mathsf{Acc}$ from $\xi \in \mathsf{Acc}$. If $\rho \leq \alpha$ we have $\rho \leq \mu$ for some $\mu \in SC(\alpha)$. Since $\alpha \in \mathsf{M}$ we have $\mu \in \mathsf{Acc}$ and thence also $\rho \in \mathsf{Acc}$.
Now assume $\rho \in \mathsf{H} \setminus \mathsf{SC}$. Then $\rho =_{NF} \varphi_{\rho_1}(\rho_2)$. There are the following cases.

1. $\rho_1 = \alpha$ and $\rho_2 < \xi$. Then we obtain $\varphi_{\rho_1}(\rho_2) \in \mathsf{Acc}$ by (vii).
2. $\alpha < \rho_1$ and $\rho < \xi$. Then $\rho \in \mathsf{Acc}$ follows from $\xi \in \mathsf{Acc}$.
3. $\rho_1 < \alpha$ and $\rho_2 < \varphi_\alpha(\xi)$. Then $SC(\rho_1) \cap \Omega$ is majorized by some $\mu \in SC(\alpha) \cap \Omega \subseteq \mathsf{Acc}$ which means $SC(\rho_1) \cap \Omega \subseteq \mathsf{Acc}$ and therefore $\rho_1 <_1 \alpha$. By (ii) we obtain $\rho_1 \in \mathsf{Acc}_\varphi$. By induction hypothesis we have $\rho_2 \in \mathsf{Acc}$ and which entails $\varphi_{\rho_1}(\rho_2) \in \mathsf{Acc}$. This finishes the proof of (ii). We have to show

$$\alpha, \beta \in \mathsf{Acc} \quad \Rightarrow \quad \varphi_\alpha(\beta) \in \mathsf{Acc}. \tag{x}$$

From $\alpha, \beta \in \mathsf{Acc}$ we get $\alpha, \beta < \Omega$. Therefore $SC(\alpha) \subseteq \alpha$ which implies $SC(\alpha) \cap \Omega \subseteq \mathsf{Acc}$. Hence $\alpha \in \mathsf{M} \cap \Omega$. From (ii) and Lemma 3.7.13 we obtain $\mathsf{Prog}_0(\mathsf{Acc}_\varphi)$ and thence $\mathsf{Acc}_\varphi \subseteq \mathsf{Acc}$ by (3.21). Together with $\beta \in \mathsf{Acc}$ this implies $\varphi_\alpha(\beta) \in \mathsf{Acc}$. ☐

3.7.15 Lemma (ID_1) *Define* $\mathsf{Acc}_\Omega := \{\alpha \mid \alpha \notin \mathsf{M} \vee (\exists \xi \in K(\alpha))[\alpha \leq \xi] \vee \psi(\alpha) \in \mathsf{Acc}\}$. *Then we obtain* $\mathsf{Prog}_1(\mathsf{Acc}_\Omega)$.

Proof Assume

$$\alpha \in \mathit{field}(<_1) \ \text{ and } \ (\forall \eta <_1 \alpha)[\eta \in \mathsf{Acc}_\Omega]. \tag{i}$$

We have to show

$$\alpha \in \mathsf{Acc}_\Omega. \tag{ii}$$

For $\alpha \notin \mathsf{M}$ or $(\exists \xi \in K(\alpha))[\alpha \leq \xi]$ (ii) is obvious. Therefore assume $\alpha \in \mathsf{M}$ and $K(\alpha) \subseteq \alpha$. To prove (ii) it remains to show

$$\psi(\alpha) \in \mathsf{Acc}. \tag{iii}$$

For (iii) in turn it suffices to have

$$\rho < \psi(\alpha) \quad \to \quad \rho \in \mathsf{Acc}. \tag{iv}$$

We prove (iv) by Mathematical Induction on the length of the term notation of ρ. If $\rho \notin SC$ we get $SC(\rho) \subseteq \mathsf{Acc}$ by induction hypothesis and thence $\rho \in \mathsf{Acc}$ by Lemma 3.7.12 and Lemma 3.7.14. If $\rho \in SC$ then there is a ρ_0 such that $\mathsf{K}(\rho_0) \subseteq \rho_0 < \alpha$ and $\rho = \psi(\rho_0)$. For $\xi \in SC(\rho_0) \cap \Omega$ we either have $\xi = 0$ or $\xi =_{NF} \psi(\eta)$ for some η. In the second case we get $\eta \in \mathsf{K}(\xi) \subseteq \mathsf{K}(\rho_0) \subseteq \alpha$ which implies $\xi = \psi(\eta) < \psi(\alpha)$. Hence $SC(\rho_0) \cap \Omega \subseteq \psi(\alpha)$. By induction hypothesis we therefore obtain $SC(\rho_0) \cap \Omega \subseteq \mathsf{Acc}$. Hence $\rho_0 <_1 \alpha$ and therefore $\rho_0 \in \mathsf{Acc}_\Omega$ by (i). Since we have $\mathsf{K}(\rho_0) \subseteq \rho_0$ and just showed $\rho_0 \in \mathsf{M}$ this implies $\rho = \psi(\rho_0) \in \mathsf{Acc}$. $\square$

3.7.16 Lemma *(Condensation Lemma) If $\mathsf{K}(\alpha) \subseteq \alpha$ and $\alpha \in \mathsf{M}$ then $ID_1 \vdash \mathsf{TI}_1(\alpha, F)$, implies $ID_1 \vdash \psi(\alpha) \in \mathsf{Acc}$.*

Proof We especially have

$$ID_1 \vdash \mathsf{TI}_1(\alpha, \mathsf{Acc}_\Omega). \tag{i}$$

From (i) and Lemma 3.7.15 we obtain

$$(\forall \xi <_1 \alpha)[\xi \in \mathsf{Acc}_\Omega] \tag{ii}$$

and from (ii) and Lemma 3.7.15

$$\alpha \in \mathsf{Acc}_\Omega. \tag{iii}$$

But (iii) together with the other hypotheses yield $\psi(\alpha) \in \mathsf{Acc}$. $\square$

3.7.17 Lemma $ID_1 \vdash \mathsf{TI}_1(\Omega + 1, F) \wedge \mathsf{K}(\Omega + 1) \subseteq \Omega + 1 \wedge \Omega + 1 \in \mathsf{M}$.

Proof Since $SC(\Omega{+}1) = \{0\}$ and $\mathsf{K}(\Omega{+}1) = \emptyset$ we obviously have $\mathsf{K}(\Omega{+}1) \subseteq \Omega{+}1 \wedge \Omega{+}1 \in \mathsf{M}$. Assuming $\mathsf{Prog}_1(F)$ we have to show $(\forall \xi <_1 \Omega{+}1)[F(\xi)]$. If $\xi <_1 \Omega$ we obtain $SC(\xi) \subseteq \mathsf{Acc}$ and thus $\xi \in \mathsf{Acc}$ by Lemma 3.7.12 and Lemma 3.7.14. By Lemma 3.7.13 we get $\mathsf{Prog}_0(F)$ which then by (3.21) entails $F(\xi)$. So we have $(\forall \xi <_1 \Omega)[F(\xi)]$ which by $\mathsf{Prog}_1(F)$ also implies $F(\Omega)$. $\square$

3.7.18 Lemma

$$ID_1 \vdash \mathsf{TI}_1(\alpha, F) \wedge \mathsf{K}(\alpha) \subseteq \alpha \wedge \alpha \in \mathsf{M} \;\Rightarrow\; ID_1 \vdash \mathsf{TI}_1(\omega^\alpha, F) \wedge \mathsf{K}(\omega^\alpha) \subseteq \omega^\alpha \wedge \omega^\alpha \in \mathsf{M}.$$

Proof We show $ID_1 \vdash \mathsf{TI}_1(\alpha, F) \Rightarrow ID_1 \vdash \mathsf{TI}_1(\omega^\alpha, F)$ literally as (2.8). Because of $SC(\omega^\alpha) \cap \Omega = SC(\alpha) \cap \Omega \cup \{0\}$ and $\mathsf{K}(\omega^\alpha) = \mathsf{K}(\alpha)$ the remaining claims follow trivially. $\square$

3.7.19 Theorem *(The lower bound for ID_1) For every ordinal $\alpha < \psi(\varepsilon_{\Omega+1})$ there is a primitiv recursive ordering $\prec$ such that $ID_1 \vdash \underline{n} \in Acc(\prec)$ and $\alpha \le |n|_{Acc(\prec)}$.*

Proof We have outlined in Theorem 3.7.7 that $<_0$ is primitive recursive. Defining a sequence $\zeta_0 = \Omega + 1$ and $\zeta_{n+1} = \omega^{\zeta_n}$ we obtain by Lemma 3.7.17 and Lemma 3.7.18

$$ID_1 \vdash \mathsf{TI}_1(\zeta_n, F) \wedge \mathsf{K}(\zeta_n) \subseteq \zeta_n \wedge \zeta_n \in \mathsf{M}$$

for all n. Hence $\psi(\zeta_n) \in \mathsf{Acc} = Acc(<_0)$ by the Condensation Lemma (Lemma 3.7.16). By Observation 1.7.10 we have $|n|_{Acc(<_0)} = otyp_{<_0}(n) = |n|_{\mathcal{O}}$. Hence $|\psi(\zeta_n)|_{Acc(<_0)} = \psi(\zeta_n)$ and the claim follows because $\sup_n \zeta_n = \psi(\varepsilon_{\Omega+1})$. $\square$

3.7.20 Corollary *We have $\|ID_1\| = \kappa^{ID_1} = \psi(\varepsilon_{\Omega+1})$ and $\|ID_1^{ext}\| = \|ID_1^{ext}\|_{\Pi^1_1} = \kappa^{ID_1^{ext}} = \psi(\varepsilon_{\Omega+1})$.*

Index

Notations

Key–words

Bibliography

[1] P. Aczel, H. Simmons and S. S. Wainer (editors), **Proof theory** (Leeds 1990), Cambridge University Press, Cambridge, 1992.

[2] A. Beckmann and W. Pohlers, *Application of cut–free infinitary derivations to generalized recursion theory*, **Annals of Pure and Applied Logic**, vol. 94 (1998), pp. 1–19.

[3] W. Buchholz, *A simplified version of local predicativity*, **Proof theory** (P. Aczel et al., editors), Cambridge University Press, Cambridge, 1992, pp. 115–147.

[4] S. R. Buss (editor), **Handbook of Proof Theory**, Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Company, 1998.

[5] G. Gentzen, *Die Widerspruchsfreiheit der reinen Zahlentheorie*, **Mathematische Annalen**, vol. 112 (1936), pp. 493–565.

[6] ——, *Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie*, **Forschungen zur Logik und Grundlegung der exakten Wissenschaften**, vol. 4 (1938), pp. 19–44.

[7] ——, *Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie*, **Mathematische Annalen**, vol. 119 (1943), pp. 140–161.

[8] J.-Y. Girard, **Proof theory and logical complexity**, vol. 1, Bibliopolis, Naples, 1987.

[9] W. Pohlers, *Cut-elimination for impredicative infinitary systems I. Ordinal analysis for ID_1*, **Archiv für Mathematische Logik und Grundlagenforschung**, vol. 21 (1981), pp. 113–129.

[10] ——, **Proof theory. An introduction**, Lecture Notes in Mathematics, vol. 1407, Springer-Verlag, Berlin/Heidelberg/New York, 1989.

[11] ——, *A short course in ordinal analysis*, **Proof theory** (P. Aczel et al., editors), Cambridge University Press, Cambridge, 1992, pp. 27–78.

[12] ——, *Subsystems of set theory and second order number theory*, **Handbook of Proof Theory** (S. R. Buss, editor), Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Company, 1998, pp. 209–335.

[13] K. Schütte, **Proof theory**, Grundlehren der mathematischen Wissenschaften, vol. 225, Springer-Verlag, Heidelberg/New York, 1977.

[14] G. Takeuti, **Proof theory**, 2. ed., Studies in Logic and the Foundations of Mathematics, vol. 81, North-Holland Publishing Company, Amsterdam, 1987.