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Computability Theory of Hyperarithmetical Sets

Lecture by

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Preface

This text contains the somewhat extended material of a series of lectures given at the University of Münster. The aim of the course is to give an introduction to “higher” computability theory and to provide background material for the following courses in proof theory.
The prerequisites for the course are some basic facts about computable functions and mathematical logic. Some emphasis has been put on the notion of generalized inductive definitions. Whenever it seemed to be opportune we tried to obtain “classical” results by using generalized inductive definitions.
I am indebted to Dipl. Math. INGO LEPPER for the revising and supplementing the original text.

Münster, September 1999 WOLFRAM POHLERS

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1. Computable Functionals and Relations

1.1 Functionals and Relations

Let

$$^{\mathbb{N}}\mathbb{N} := \{\alpha \mid \alpha\colon \mathbb{N} \longrightarrow \mathbb{N}\}$$

be the space of all functions from the natural numbers into the natural numbers. In this lecture we will deal with the spaces

$$\mathbb{N}^{m,n} := \mathbb{N}^m \times (^{\mathbb{N}}\mathbb{N})^n.$$

The elements of this space will be denoted by lower case Gothic letters such as $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{a}_1$, $\mathfrak{a}_2 \dots$

1.1.1 Definition 1.) Let $D \subseteq \mathbb{N}^{m,n}$. An (m,n)*-ary partial functional* is a map $F\colon D \longrightarrow \mathbb{N}$. We denote this by

$$F\colon \mathbb{N}^{m,n} \longrightarrow_{\mathrm{p}} \mathbb{N}.$$

The set D is the *domain* of F – denoted by $\mathsf{dom}(F)$.
If $\mathsf{dom}(F) = \mathbb{N}^{m,n}$ we call F a *total* functional.
2.) An (m,n)-ary *relation* is a set $R \subseteq \mathbb{N}^{m,n}$. We use the notations $\mathfrak{a} \in R$ and $R(\mathfrak{a})$ synonymously to denote that $\mathfrak{a}$ belongs to R.

To distinguish notions from Ordinary Computation Theory (OCT) (or Classical Recursion Theory as it used to be called) from Hyperarithmetical Computation Theory (HCT) we refer to $(m,0)$-ary functionals as m-ary *functions* and to $(m,0)$-ary relations as m-ary *predicates*.
We use the common notations of OCT freely. E.g., $\langle x_1,\dots,x_n\rangle$ denotes the primitive–recursive coding function, $(x)_i$ its decoding and Seq the primitive–recursive set of *sequence codes*.
For $\mathfrak{a} = (x_1,\dots,x_m,\alpha_1,\dots,\alpha_n) \in \mathbb{N}^{m,n}$ and $k \in \mathbb{N}$ we put

$$\overline{\mathfrak{a}}(k) := (x_1,\dots,x_m,\overline{\alpha}_1(k),\dots,\overline{\alpha}_n(k)),$$

where

$$\overline{\alpha}(k) := \begin{cases} \langle\,\rangle & \text{if } k = 0 \\ \langle \alpha(0),\dots,\alpha(l)\rangle & \text{if } k = l+1 \end{cases}$$

denotes the course of values of α below k. We refer to $\overline{\mathfrak{a}}(k)$ as the *course of values* of the tuple $\mathfrak{a}$ below k.
If $\mathfrak{a}$ is as above, $\vec{y} = (y_1,\dots,y_k)$ and $\vec{\beta} = (\beta_1,\dots,\beta_l)$ we put

$$(\mathfrak{a},\vec{y},\vec{\beta}) := (x_1,\dots,x_m,y_1,\dots,y_k,\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_l).$$

1.1.2 Definition An (m,n)-ary relation R is *semi–decidable* (often also called *semi–recursive* or *recursively enumerable*) if there is a semi–recursive (which can be regarded as synonymous to recursively enumerable) $m+n$-ary predicate P_R such that

$$\mathfrak{a} \in R \;\;\Leftrightarrow\;\; (\exists x)P_R(\overline{\mathfrak{a}}(x)).$$

1.1.3 Discussion The definition of a semi-decidable relation meets the intuition of a "positively decidable" relation. We show that there is an algorithm which confirms $\mathfrak{a} \in R$. Since $P_R(\overline{\mathfrak{a}}(x))$ is semi–recursive in the sense of OCT there is a decidable predicate, say Q, such that

$$\begin{aligned}\mathfrak{a} \in R &\Leftrightarrow (\exists x)P_R(\overline{\mathfrak{a}}(x))\\ &\Leftrightarrow (\exists x)(\exists y)Q(\overline{\mathfrak{a}}(x), y).\end{aligned}$$

Now we decide $Q(\overline{\mathfrak{a}}((n)_0, (n)_1)$ for $n = 0, 1, \ldots$. This algorithm terminates if $\mathfrak{a} \in R$ but will give no information in case that $\mathfrak{a} \notin R$.

1.1.4 Definition Let $F, G\colon \mathbb{N}^{m,n} \longrightarrow_{\mathrm{p}} \mathbb{N}$. For $\mathfrak{a} \in \mathbb{N}^{m,n}$ we put

$$\begin{aligned}F(\mathfrak{a}) \simeq G(\mathfrak{a}) \;:\Leftrightarrow\; &(\mathfrak{a} \notin \mathsf{dom}(F) \wedge \mathfrak{a} \notin \mathsf{dom}(G))\\ &\vee (\mathfrak{a} \in \mathsf{dom}(F) \cap \mathsf{dom}(G) \wedge F(\mathfrak{a}) = G(\mathfrak{a})).\end{aligned}$$

Sometimes it is helpful to consider partial functionals as maps from $\mathbb{N}^{m,n}$ into $\mathbb{N} \cup \{\uparrow\}$. If we put

$$\tilde{F}(\mathfrak{a}) \colon\simeq \begin{cases} F(\mathfrak{a}) & \text{if } \mathfrak{a} \in \mathsf{dom}(F)\\ \uparrow & \text{otherwise}\end{cases}$$

then we get

$$F(\mathfrak{a}) \simeq G(\mathfrak{a}) \;\Leftrightarrow\; \tilde{F}(\mathfrak{a}) = \tilde{G}(\mathfrak{a}). \tag{1.1}$$

1.1.5 Definition Let $F\colon \mathbb{N}^{m,n} \longrightarrow_{\mathrm{p}} \mathbb{N}$. We call F *partial–computable* if its *graph*

$$G_F := \{(\mathfrak{a}, y) \mid\; F(\mathfrak{a}) \simeq y\}$$

is semi-decidable.
We call F *computable* if F is partial–computable and total.

1.1.6 Discussion The definition of a partial–computable functional meets the intuition of a positively computable functional. We indicate that there is an algorithm for F which terminates and yields $F(\mathfrak{a})$ in case that $\mathfrak{a} \in \mathsf{dom}(F)$. Since G_F is semi-decidable we get as in 1.1.3 a decidable predicate Q such that

$$F(\mathfrak{a}) \simeq x \;\Leftrightarrow\; (\exists z)(\exists y)Q(\overline{\mathfrak{a}}(z), y, x).$$

Again we decide $Q(\overline{\mathfrak{a}}((n)_0), (n)_1, (n)_2)$ for $n = 0, 1, \ldots$ and pick the first such n. Then $F(\mathfrak{a}) = (n)_2$. If F is computable, then it is total, and so this algorithm will always terminate.

We are now ready to study the closure properties of semi-decidable relations. It will turn out that most of the closure properties are just liftings of the closure properties of semi-decidable predicates.

1.1.7 Theorem *The semi-decidable relations are closed under*

- *the positive boolean operations* $\wedge$ *and* $\vee$*;*
- *bounded quantification on natural numbers;*
- *unbounded* $\exists$*–quantification over* $\mathbb{N}$ *and* ${}^{\mathbb{N}}\mathbb{N}$*;*
- *substitution with computable functionals.*

Proof: The only case which is new in comparison to OCT is the closure under second order quantification, i.e. quantifiers ranging over ${}^{\mathbb{N}}\mathbb{N}$. However, we will also give two examples for the more simple cases, e.g. closure under $\wedge$ and bounded $\forall$-quantification.
We have

$$\begin{aligned}R(\mathfrak{a}) \wedge Q(\mathfrak{a}) &\Leftrightarrow (\exists x)P_R(\overline{\mathfrak{a}}(x)) \wedge (\exists y)P_Q(\overline{\mathfrak{a}}(y))\\ &\Leftrightarrow (\exists u)[P_R(\overline{\mathfrak{a}}(u)\restriction(u)_0) \wedge P_Q(\overline{\mathfrak{a}}(u)\restriction(u)_1)]\end{aligned}$$

which shows that $R \wedge Q$ is semi-decidable.
For bounded $\forall$-quantification we have

$$(\forall x < y)R(\mathfrak{a}, x) \;\;\Leftrightarrow\;\; (\forall x < y)(\exists z)P_R(\overline{\mathfrak{a}}(z), x)$$

and the semi-computability of $(\forall x < y)R(\mathfrak{a}, x)$ follows immediately from the closure properties of semi-computable predicates.
For the new case we have

$$\begin{aligned}(\exists\alpha)R(\mathfrak{a},\alpha) \;&\Leftrightarrow\; (\exists\alpha)(\exists x)P_R(\overline{\mathfrak{a}}(x), \overline{\alpha}(x))\\ &\Leftrightarrow\; (\exists s)(\exists x)[\mathit{Seq}(s) \wedge \mathit{lh}(s) = x \wedge P_R(\overline{\mathfrak{a}}(x), s)].\end{aligned}$$

Hence $(\exists\alpha)R(\mathfrak{a},\alpha)$ is semi-decidable. □

We call the relation $(\exists x)R(\mathfrak{a}, x)$ the $\mathbb{N}$– or first order projection of $R(\mathfrak{a}, x)$ while $(\exists\alpha)R(\mathfrak{a},\alpha)$ is the ${}^{\mathbb{N}}\mathbb{N}$– or second order projection of $R(\mathfrak{a},\alpha)$. The motivation for this terminology becomes clear from Figure 1.1.1.

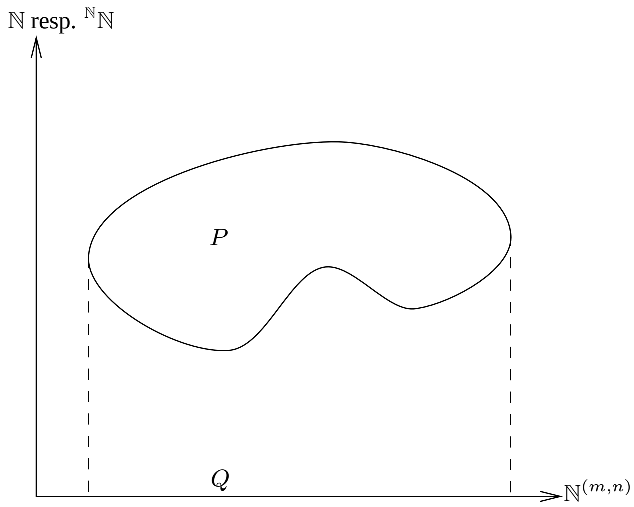

Figure 1.1.1: The $\mathbb{N}$– resp. ${}^{\mathbb{N}}\mathbb{N}$–projection of a relation

1.1.8 Definition The *characteristic functional* of an (m, n)–ary relation R is given by

$$\chi_R(\mathfrak{a}) := \begin{cases} 0 & \text{if } \mathfrak{a} \in R \\ 1 & \text{otherwise.} \end{cases}$$

Let us make some of the conventions explicit which we have been already using.
Quantifiers of the form $(\mathsf{Q}x)$, $(\mathsf{Q}y)$, …whose bound variables are indicated by lower case Roman letters are first order, i.e. quantifiers ranging over $\mathbb{N}$. To emphasize the first order of those quantifiers we sometimes (very rarely) will write $(\exists^0 x)$ or $(\forall^0 x)$.
Quantifiers of the form $(\mathsf{Q}\alpha)$, $(\mathsf{Q}\beta)$, …whose bound variables are indicated by lower case Greek letters are second order, i.e. quantifiers ranging over ${}^{\mathbb{N}}\mathbb{N}$. To emphasize the second order of those quantifiers we sometimes will write $(\exists^1\alpha)$ or $(\forall^1\alpha)$.
Sometimes we want to quantify over subsets of $\mathbb{N}$, i.e. over ${}^{\mathbb{N}}2$, the set of characteristic functions. This will be denoted by $(\mathsf{Q}\alpha^*)$, $(\mathsf{Q}\beta^*)$, $(\mathsf{Q}\alpha_1^*)$, …

1.1.9 Definition Let G be an $(m+1,n)$–ary functional. The *(unbounded) search operator* μ turns G into a (m,n)–ary functional (μG) which is defined by

$$(\mu G)(\mathfrak{a}) \simeq y \;\; :\Leftrightarrow \;\; G(\mathfrak{a},y) \simeq 0 \wedge (\forall u < y)(\exists z)[z \neq 0 \wedge G(\mathfrak{a},u) \simeq z]. \tag{1.2}$$

More sloppily we write $\mu x \,.\, G(\mathfrak{a},x)$ instead of $(\mu G)(\mathfrak{a})$ to emphasize the place at which μ searches for a zero of G.
The *bounded search operator* is defined by

$$\begin{aligned}\mu x < u \,.\, G(\mathfrak{a},x) \simeq y \;\; :\Leftrightarrow \;\; & (\forall x < y)(\exists z)[G(\mathfrak{a},x) \simeq z \wedge z \neq 0 \\ & \wedge ((G(\mathfrak{a},y) = 0 \wedge y < u) \vee y = u)].\end{aligned}$$

The bounded search operator searches for a zero below u and outputs u if no such zero exists.
As usual we define the substitution operator by

$$\mathsf{Sub}(G,H_1,\ldots,H_n)(\mathfrak{a}) \simeq G(H_1(\mathfrak{a}),\ldots,H_n(\mathfrak{a}))$$

1.1.10 Theorem *The partial–computable functionals are closed under unbounded search - and hence also under bounded search - and substitution.*

Proof: Having in mind the closure properties of semi-decidable relations the first claim follows by looking at (1.2). The second claim follows from

$$\begin{aligned}&\mathsf{Sub}(G,H_1,\ldots,H_n)(\mathfrak{a}) \simeq y \;\;\Leftrightarrow \\ &\qquad (\exists x_1)\ldots(\exists x_n)[H_1(\mathfrak{a}) \simeq x_1 \wedge \ldots \wedge H_n(\mathfrak{a}) \simeq x_n \wedge G(x_1,\ldots,x_n) \simeq y].\end{aligned}$$

$\square$

The possibilities for substitution, however, are not exhausted by the substitution operator. If H is an $(m+1,n)$–ary functional and G an $(m,n+1)$–ary functional then we may try to define

$$F(\mathfrak{a}) \simeq G(\mathfrak{a}, \lambda x \,.\, H(\mathfrak{a},x)). \tag{1.3}$$

The problem is that (1.3) is only defined if $\lambda x \,.\, H(\mathfrak{a},x)$ is total. The following lemma shows how this can be handled.

1.1.11 Lemma (Substitution Lemma) *Let G be an $(m,n+1)$–ary and H an $(m+1,n)$–ary partial–computable functional. Then there is a partial–computable functional F such that*

$$F(\mathfrak{a}) \simeq G(\mathfrak{a}, \lambda x \,.\, H(\mathfrak{a},x))$$

for all $\mathfrak{a}$ for which $\lambda x \,.\, H(\mathfrak{a},x)$ is total.

Proof: We have semi-decidable predicates P_G and P_H such that

$$G(\mathfrak{a},\alpha) \simeq u \;\;\Leftrightarrow\;\; (\exists z) P_G(\overline{\mathfrak{a}}(z), \overline{\alpha}(z), u) \tag{i}$$

and

$$H(\mathfrak{a},x) \simeq v \;\;\Leftrightarrow\;\; (\exists y) P_H(\overline{\mathfrak{a}}(y), x, v).$$

Using (i) we find a decidable predicate Q such that

$$G(\mathfrak{a},\alpha) \simeq u \;\;\Leftrightarrow\;\; (\exists z)(\exists x) Q(\overline{\mathfrak{a}}(z), \overline{\alpha}(z), u, x).$$

We put

$$F(\mathfrak{a}) :\simeq (\mu w \,.\, Q(\overline{\mathfrak{a}}((w)_0), \overline{\lambda x \,.\, H(\mathfrak{a},x)}((w)_0), (w)_1, (w)_2))_1.$$

Then $\mathfrak{a} \in \mathsf{dom}(F)$ if $\lambda x \,.\, H(\mathfrak{a},x)$ is total and $(\mathfrak{a}, \lambda x \,.\, H(\mathfrak{a},x)) \in \mathsf{dom}(G)$. Hence

$$\mathsf{dom}(\lambda \mathfrak{a} \,.\, G(\mathfrak{a}, \lambda x \,.\, H(\mathfrak{a},x))) \subseteq \mathsf{dom}(F)$$

but observe that the inclusion may well be proper. However, we have

$$G(\mathfrak{a}, \lambda x\,.\,H(\mathfrak{a}, x)) = F(\mathfrak{a})$$

for all $(\mathfrak{a}, \lambda x\,.\,H(\mathfrak{a}, x)) \in \mathsf{dom}(\lambda\mathfrak{a}\,.\,G(\mathfrak{a}, \lambda x\,.\,H(\mathfrak{a}, x)))$. We still have to show that F is partial–computable. Checking the graph of F we get

$$\begin{aligned} F(\mathfrak{a}) \simeq a \;\;\Leftrightarrow\;\; (\exists s)(\exists w)[&\mathit{Seq}(s) \wedge \mathit{Seq}(w) \wedge \mathit{lh}(w) = 3 \\ &\wedge \mathit{lh}(s) = w \\ &\wedge (\forall i < w) H(\mathfrak{a}, i) \simeq (s)_i \\ &\wedge Q(\overline{\mathfrak{a}}((w)_0), s\restriction(w)_0, (w)_1, (w)_2) \wedge (w)_1 = a \\ &\wedge (\forall j < w)\neg Q(\overline{\mathfrak{a}}((j)_0), s\restriction(j)_0, (j)_1, (j)_2)] \end{aligned}$$

where $s\restriction k$ stands for $\langle (s)_0, \ldots, (s)_{k\dot{-}1}\rangle$.
By the closure properties of semi-decidable and decidable predicates we get immediately that $F(\mathfrak{a}) \simeq a$ is a semi–decidable relation in $\mathfrak{a}$ and a. □

1.1.12 Lemma *The partial–computable functionals are closed under definition by cases: Let $G_1, \ldots, G_n$ be partial–computable and $R_1, \ldots, R_n$ pairwise disjoint semi–decidable relations and*

$$F(\mathfrak{a}) :\simeq \begin{cases} G_1(\mathfrak{a}) & \textit{if } R_1(\mathfrak{a}) \\ \vdots & \vdots \\ G_n(\mathfrak{a}) & \textit{if } R_n(\mathfrak{a}) \end{cases}$$

Then F is partial–computable.

Proof: We have

$$F(\mathfrak{a}) \simeq y \;\;\Leftrightarrow\;\; (R_1(\mathfrak{a}) \wedge G_1(\mathfrak{a}) \simeq y) \vee \ldots \vee (R_n(\mathfrak{a}) \wedge G_n(\mathfrak{a}) \simeq y)$$

which shows that F possesses a semi–decidable graph. □

The simplest example of a functional is the *application functional* which is defined by

$$\mathsf{App}(\alpha, n) :\simeq \alpha(n).$$

1.1.13 Theorem *The application functional is a $(1,1)$–ary computable functional.*

Proof: Since α is total App is total, too. For its graph we get

$$\mathsf{App}(\alpha, n) \simeq y \;\;\Leftrightarrow\;\; (\exists z)[n < z \wedge y = (\overline{\alpha}(z))_n].$$

□

To conclude this section we introduce the *decidable relations* which are often also called *recursive relations*.

1.1.14 Definition A relation $R \subseteq \mathbb{N}^{m,n}$ is *decidable* if its characteristic functional is computable.

All closure properties of decidable (i.e. recursive) predicates can be lifted to decidable relations. Therefore we state the following theorem without proof.

1.1.15 Theorem *The decidable relations are closed under:*

- *all boolean operations, i.e. $\neg, \wedge, \vee$;*
- *bounded quantification;*
- *substitution with computable functionals.*

However, as a consequence of Lemma 1.1.11, we get the following additional closure property.

1.1.16 Theorem *Let P be an $(m, n+1)$–ary decidable relation and H be an $(m+1, n)$–ary computable functional. Then the relation*

$$R := \{\mathfrak{a} \mid P(\mathfrak{a}, \lambda x \,.\, H(x, \mathfrak{a}))\}$$

is decidable.

Proof: We get

$$\chi_R(\mathfrak{a}) \simeq \chi_P(\mathfrak{a}, \lambda x \,.\, H(x, \mathfrak{a}))$$

and the right hand is a computable functional by Lemma 1.1.11 because $\lambda x \,.\, H(x, \mathfrak{a})$ is total. $\square$

In OCT we classify the semi–decidable predicates as $\mathbb{N}$–projections of decidable predicates. This too can be lifted to semi–decidable relations.

1.1.17 Theorem *An (m, n)–ary relation R is semi–decidable iff there is an $(m+1, n)$–ary decidable relation Q such that*

$$R(\mathfrak{a}) \;\Leftrightarrow\; (\exists z) Q(\mathfrak{a}, z),$$

i.e. the semi–decidable relations are exactly the $\mathbb{N}$–projections of the decidable relations.

Proof: Let R be semi–decidable. Then

$$\begin{aligned} R(\mathfrak{a}) \;&\Leftrightarrow\; (\exists z) P_R(\overline{\mathfrak{a}}(z)) \\ &\Leftrightarrow\; (\exists z)(\exists u) \tilde{Q}(\overline{\mathfrak{a}}(z), u) \end{aligned}$$

for some decidable predicate $\tilde{Q}$. Define

$$Q := \{(\mathfrak{a}, u) \mid \tilde{Q}(\overline{\mathfrak{a}}((u)_0), (u)_1)\}.$$

Then

$$R(\mathfrak{a}) \;\Leftrightarrow\; (\exists z) Q(\mathfrak{a}, z)$$

and Q is obviously decidable. $\square$

1.1.18 Theorem *Let R be an $(m+1, n)$–ary decidable relation and define*

$$F(\mathfrak{a}) :\simeq \mu w \,.\, R(\mathfrak{a}, w)\,.$$

Then F is an (m, n)–ary partial–computable functional.

Proof: We have

$$F(\mathfrak{a}) \simeq y \;\Leftrightarrow\; (\exists w)[R(\mathfrak{a}, y) \wedge (\forall u < y) \neg R(\mathfrak{a}, u)].$$

Thus F has a semi–decidable graph by Theorems 1.1.15 and 1.1.17. $\square$

1.2 The Normal–form Theorem

One of the most important theorems of OCT is KLEENE's Normal–form Theorem. The aim of this section is to lift this theorem to HCT. Recall that in OCT we defined W_e as the domain of a partial–computable function with index e. These domains are exactly the semi–decidable predicates. Thus $\{W_e \mid e \in \mathsf{Ind}(P)\}$ enumerates all semi–decidable predicates where $\mathsf{Ind}(P)$ is the set of indices of partial–computable functions. We use this enumeration to obtain an indexing of semi–computable functionals. Let R be an (m,n)–ary semi–decidable relation. Then there is an $e \in \mathsf{Ind}(P)$ such that

$$\begin{aligned} R(\mathfrak{a}) &\Leftrightarrow (\exists z)\mathsf{W}_e^{m+n}(\overline{\mathfrak{a}}(z)) \\ &\Leftrightarrow (\exists z)(\exists u)\mathsf{T}^{m+n}(e,\overline{\mathfrak{a}}(z),u) \end{aligned} \tag{1.4}$$

where T^{m+n} denotes the KLEENE predicate.
For a semi–computable (m,n)–ary functional we get from (1.4)

$$F(\mathfrak{a}) \simeq y \;\;\Leftrightarrow\;\; (\exists z)(\exists u)\mathsf{T}^{m+n+1}(e,\overline{\mathfrak{a}}(z),y,u).$$

Therefore we define

$$\mathsf{T}^{m,n} := \big\{(e,\mathfrak{a},w)\big|\; \mathsf{T}^{m+n+1}(e,\overline{\mathfrak{a}}((w)_0),(w)_1,(w)_2)\big\}.$$

Then $\mathsf{T}^{m,n}$ is an $(m+2,n)$–ary decidable relation for which we get

$$F(\mathfrak{a}) \simeq (\mu w\,.\,\mathsf{T}^{m,n}(e,\mathfrak{a},w))_1.$$

Therefore we have the following theorem.

1.2.1 Theorem (Normal–form Theorem) *There is an $(m+2,n)$–ary decidable relation $\mathsf{T}^{m,n}$ and a computable (even primitive–recursive) function U such that for all semi–computable (m,n)–ary functionals F there is an $e \in \mathbb{N}$ with*

$$F(\mathfrak{a}) \simeq U(\mu w\,.\,\mathsf{T}^{m,n}(e,\mathfrak{a},w)).$$

We agree about the notation

$$\{e\}^{m,n}(\mathfrak{a}) :\simeq U(\mu w\,.\,\mathsf{T}^{m,n}(e,\mathfrak{a},w))$$

and call e an *index* for F.

1.2.2 Theorem *The functional $\Phi^{m,n}(\mathfrak{a},e) \;:\simeq\; \{e\}^{m,n}(\mathfrak{a})$ is a partial–computable functional which is universal for the class of (m,n)–ary partial–computable functionals.*

Proof: The Normal–form Theorem entails the universality of the functional $\Phi^{m,n}$. To show its partial–computability we check its graph.

$$\begin{aligned} \Phi^{m,n}(\mathfrak{a},e) \simeq y &\Leftrightarrow \{e\}^{m,n}(\mathfrak{a}) \simeq y \\ &\Leftrightarrow (\exists w)[\mathsf{T}^{m,n}(e,\mathfrak{a},w) \land (\forall u < w)\neg\mathsf{T}^{m,n}(e,\mathfrak{a},u) \land y = U(w)]. \end{aligned}$$

Since $\mathsf{T}^{m,n}$ is a decidable relation the last line is semi–decidable by Theorems 1.1.15 and 1.1.17. □

We refer to Theorem 1.1.17 to obtain also a Normal–form Theorem for semi–decidable relations. In a first step we prove the following theorem.

1.2.3 Theorem *A relation is semi–decidable iff it is the domain of a partial–computable functional.*

Proof: Using the Normal–form Theorem we get

$$\mathfrak{a} \in \mathsf{dom}(F) \;\;\Leftrightarrow\;\; (\exists w)\mathsf{T}^{m,n}(e,\mathfrak{a},w)$$

showing that the domains of partial–computable functionals are semi–decidable. For the opposite direction let R be (m,n)–ary semi–decidable. By Theorem 1.1.17 we get a decidable relation Q such that

$$R(\mathfrak{a}) \quad \Leftrightarrow \quad (\exists z)Q(\mathfrak{a}, z).$$

Define

$$F(\mathfrak{a}) :\simeq \mu z\,.\,Q(\mathfrak{a}, z)\,.$$

Then F is partial–computable by Theorem 1.1.18 and we have

$$\mathsf{dom}(F) = \big\{\mathfrak{a} \,\big|\, (\exists z)Q(\mathfrak{a}, z)\big\} = R.$$

□

Now we define

$$\mathsf{W}_e^{m,n} := \mathsf{dom}(\{e\}^{m,n}) = \big\{\mathfrak{a} \,\big|\, (\exists w)\mathsf{T}^{m,n}(e, \mathfrak{a}, w)\big\}.$$

1.2.4 Theorem *The collection of (m,n)–ary semi–decidable relations is enumerated by $\mathsf{W}_e^{m,n}$, i.e.*

$$\big\{R \subseteq \mathbb{N}^{m,n} \,\big|\, R \textit{ is semi–decidable}\,\big\} = \big\{\mathsf{W}_e^{m,n} \,\big|\, e \in \mathbb{N}\big\}.$$

If $R = \mathsf{W}_e^{m,n}$ we call e an *index* for R.
The canonical next step is to lift the S_n^m–Theorem from OCT.

1.2.5 Theorem *($\mathsf{S}_k^{m,n}$–Theorem) There is a $k+1$–ary primitive–recursive function $\mathsf{S}_k^{m,n}$ such that*

$$\{e\}^{m+k,n}(\mathfrak{a}, y_1, \ldots, y_k) \simeq \{\mathsf{S}_k^{m,n}(e, y_1, \ldots, y_k)\}^{m,n}(\mathfrak{a}) \tag{1.5}$$

and

$$(\mathfrak{a}, y_1, \ldots, y_k) \in \mathsf{W}_e^{m+k,n} \quad \Leftrightarrow \quad \mathfrak{a} \in \mathsf{W}^{m,n}_{\mathsf{S}_k^{m,n}(e, y_1, \ldots, y_k)}. \tag{1.6}$$

Proof: We get

$$\begin{aligned}
\{e\}^{m+k,n}(\mathfrak{a}, \vec{y}) &\simeq U(\mu w\,.\,\mathsf{T}^{m+k,n}(e, \vec{y}, \mathfrak{a}, w))\\
&\simeq U(\mu w\,.\,\mathsf{T}^{m+k+n+1}(e, \vec{y}, \overline{\mathfrak{a}}((w)_0), (w)_1, (w)_2))\\
&\simeq U(\mu w\,.\,\mathsf{T}^{m+n+1}(\mathsf{S}_k^{m+n+1}(e, \vec{y}), \overline{\mathfrak{a}}((w)_0, (w)_1, (w)_2)))\\
&\simeq U(\mu w\,.\,\mathsf{T}^{m,n}(\mathsf{S}_k^{m+n+1}(e, \vec{y}), \mathfrak{a}, w))\\
&\simeq \{\mathsf{S}_k^{m+n+1}(e, \vec{y})\}^{m,n}(\mathfrak{a})
\end{aligned}$$

and we put

$$\mathsf{S}_k^{m,n}(e, \vec{y}) := \mathsf{S}_k^{m+n+1}(e, \vec{y})$$

where S_k^{m+n+1} is the function of OCT.
Since $\mathsf{W}_e^{m+k,n} = \mathsf{dom}(\{e\}^{m+k,n})$ we obtain (1.6) immediately from (1.5). □

The immediate consequence of the $\mathsf{S}_k^{m,n}$–Theorem is — as usual — the Recursion Theorem.

1.2.6 Theorem (Recursion Theorem) *Let G be an $(m+1,n)$–ary partial–computable functional. Then there is an e such that*

$$\{e\}^{m,n}(\mathfrak{a}) \simeq G(\mathfrak{a}, e).$$

Proof: We mimick the usual proof. Define

$$H(\mathfrak{a}, x) :\simeq G(\mathfrak{a}, \mathsf{S}_1^{m,n}(x, x)).$$

Then H is partial–computable by Theorem 1.1.10. Let e_0 be an index for H and define

$$e := \mathsf{S}_1^{m,n}(e_0, e_0).$$

Then

$$\begin{aligned}\{e\}^{m,n}(\mathfrak{a}) &\simeq \{\mathsf{S}_1^{m,n}(e_0, e_0)\}^{m,n}(\mathfrak{a}) \\ &\simeq \{e_0\}^{m+1,n}(\mathfrak{a}, e_0) \\ &\simeq H(\mathfrak{a}, e_0) \simeq G(\mathfrak{a}, \mathsf{S}_1^{m,n}(e_0, e_0)) \\ &\simeq G(\mathfrak{a}, e).\end{aligned}$$

□

As an application of the Recursion Theorem we show the closure of the partial–computable functionals under the Recursion Operator. The Recursion Operator turns an (m, n)–ary functional G and an $(m + 2, n)$–ary functional H into the $(m + 1, n)$–ary functional $\mathsf{Rec}(G, H)$ which is defined by

$$\mathsf{Rec}(G, H)(\mathfrak{a}, x) \simeq \begin{cases} G(\mathfrak{a}) & \text{if } x = 0 \\ H(\mathfrak{a}, y, z) & \text{if } x = y + 1 \text{ and } \mathsf{Rec}(G, H)(\mathfrak{a}, y) \simeq z. \end{cases}$$

1.2.7 Theorem *The partial–computable as well as the computable functionals are closed under the Recursion Operator.*

Proof: Let G and H be functionals of suitable arity. Define

$$F(\mathfrak{a}, x, e) \simeq \begin{cases} G(\mathfrak{a}) & \text{if } x = 0 \\ H(\mathfrak{a}, y, \{e\}^{m+1,n}(\mathfrak{a}, y)) & \text{if } x = y + 1. \end{cases}$$

Then F is partial–computable. Using the Recursion Theorem we obtain an index e such that

$$\{e\}^{m+1,n}(\mathfrak{a}, x) \simeq F(\mathfrak{a}, x, e).$$

Defining $E := \{e\}^{m+1,n}$ we obtain

$$E(\mathfrak{a}, x) \simeq \begin{cases} G(\mathfrak{a}) & \text{if } x = 0 \\ H(\mathfrak{a}, y, E(\mathfrak{a}, y)) & \text{if } x = y + 1 \end{cases}$$

by induction on x. Hence $E = \mathsf{Rec}(G, H)$. If moreover G and H are total, we get

$$(\forall \mathfrak{a})(\forall x)(\exists y)[E(\mathfrak{a}, x) \simeq y]$$

by induction on x. □

1.3 Computability relativized

If F is an $(1, 1)$–ary partial–computable functional and $\alpha \in {}^{\mathbb{N}}\mathbb{N}$ a given function then we may try to compute the function $\lambda x\,.\,F(\alpha, x)$. Since F is partial–computable we have

$$F(\alpha, x) \simeq y \;\;\Leftrightarrow\;\; (\exists w)Q(\overline{\alpha}((w)_0), (w)_1, x, y)$$

for some decidable predicate Q. Deciding $Q(\overline{\alpha}((w)_0), (w)_1, x, (w)_2)$ for $w = 0, 1, 2, \ldots$ and picking the least such w yields an algorithm for $\lambda x\,.\,F(\alpha, x)$ which asks for at most finitely many values of α. That means that a machine, e.g. a TURING–machine, could compute $\lambda x\,.\,F(\alpha, x)$ asking an oracle for the function α within finite time. In this situation we say that the function $\lambda x\,.\,F(\alpha, x)$ is computable relatively to α. Generalizing this to functionals leads to the following definition.

1.3.1 Definition A functional $F\colon \mathbb{N}^{m,n} \longrightarrow_{\mathrm{p}} \mathbb{N}$ is *partial–computable in* a given function α if there is an $(m, n + 1)$–ary partial–computable functional G such that

$$F(\mathfrak{a}) \simeq G(\mathfrak{a}, \alpha).$$

We call F *computable in* α if F is partial–computable in α and total. The functional F is *(partial–) computable in a set* $A \subseteq \mathbb{N}$ if F is (partial–)computable in its characteristic function χ_A.

1.3.2 Definition A relation $R \subseteq \mathbb{N}^{m,n}$ is *semi–decidable in* a function $\alpha \in {}^{\mathbb{N}}\mathbb{N}$ if R is the domain of a functional which is partial–computable in α.
We call R *decidable in* α if its characteristic functional χ_R is computable in α.
A relation R is *(semi–)decidable in a set* $A \subseteq \mathbb{N}$ if R is (semi–)decidable in its characteristic function χ_A.

The computability of functionals and relations carries over to the relativized case. We put

$$\begin{aligned}
&\mathsf{T}^{\alpha,m,n} := \big\{(e, \mathfrak{a}, w) \,\big|\; \mathsf{T}^{m,n+1}(e, \mathfrak{a}, \alpha, w)\big\}\\
&\mathsf{T}^{A,m,n} := \mathsf{T}^{\chi_A,m,n}\\
&\{e\}^{\alpha,m,n} := \lambda\mathfrak{a}\,.\, U(\mu w\,.\, \mathsf{T}^{\alpha,m,n}(e, \mathfrak{a}, w))\\
&\{e\}^{A,m,n} := \{e\}^{\chi_A,m,n}\\
&\Phi^{A,m,n} := \lambda e\mathfrak{a}\,.\, \{e\}^A(\mathfrak{a})\\
&\mathsf{W}_e^{\alpha,m,n} := \mathsf{dom}(\{e\}^{\alpha,m,n}) \text{ and } \mathsf{W}_e^{A,m,n} := \mathsf{W}_e^{\chi_A,m,n}.
\end{aligned}$$

To complete this section we reformulate the Normal–form Theorem, the $\mathsf{S}_k^{m,n}$–Theorem and the Recursion Theorem for the relativized case.

1.3.3 Theorem (Relativized Normal–form Theorem) *For any (m, n)–ary functional which is partial–computable in α there is an index e such that*

$$F(\mathfrak{a}) \simeq \{e\}^{\alpha,m,n}(\mathfrak{a}).$$

The functional $\Phi^\alpha(e, \mathfrak{a})$ is universal for the (m, n)–ary functionals which are partial–computable in α.
For any (m, n)–ary relation which is semi–decidable in α there is an index e such that

$$R = \mathsf{W}_e^{\alpha,m,n}.$$

To emphasize the relativized meaning we often talk about α–indices or A–indices, respectively.

1.3.4 Theorem (Relativized $\mathsf{S}_k^{m,n}$–Theorem) *There is an $k + 1$–ary primitive–recursive function $\mathsf{S}_k^{m,n}$ such that*

$$\{e\}^{\alpha,m+k,n}(\mathfrak{a}, y_1, \ldots, y_n) \simeq \{\mathsf{S}_k^{m,n}(e, y_1, \ldots, y_n)\}^\alpha(\mathfrak{a})$$

and

$$(\mathfrak{a}, y_1, \ldots, y_n) \in \mathsf{W}_e^{\alpha,m+k,n} \;\;\Leftrightarrow\;\; \mathfrak{a} \in \mathsf{W}^{\alpha,m,n}_{\mathsf{S}_k^{m,n}(e,y_1,\ldots,y_n)}.$$

1.3.5 Theorem (Relativized Recursion Theorem) *Let $G\colon \mathbb{N}^{m,n} \longrightarrow_p \mathbb{N}$ be partial–computable in α. Then there is an index e such that*

$$\{e\}^{\alpha,m,n}(\mathfrak{a}) \simeq G(\mathfrak{a}, e).$$

2. Degrees

This chapter will contain a brief introduction to *Degree Theory*. In Degree Theory we aim at classifying sets according to the difficulty of their decision problem. Two sets belong to the same degree if the solution of the decision problem for one set entails the solution of the decision problem for the other set and vice versa. There are different reducibility relations which are regarded in Computability Theory. Here we will only regard two of them. A quite narrow one — m–Reducibility — and the most general one — TURING–Reducibility.

2.1 m–Degrees

2.1.1 Definition Let $A, B \subseteq \mathbb{N}$. We say that A is *many–one reducible to* B, m*–reducible to* B for short, if there is a computable function, say f, such that

$$x \in A \quad \Leftrightarrow \quad f(x) \in B.$$

This will be denoted by $A \leq_m B$. In case that the reducing function f is one–one, we talk about *one–one Reducibility* or *1–Reducibility* and denote this by $A \leq_1 B$.

2.1.2 Discussion If $A \leq_m B$ or $A \leq_1 B$ we obviously can reduce the decision problem for A to that of B. To decide $x \in A$ we compute $f(x)$, which is possible because of the computability of f and then decide $f(x) \in B$.

There are some simple observations about m–Reducibility.

2.1.3 Lemma *The relation $\leq_m$ is reflexive and transitive. If $A \leq_m B$ then also $\neg A \leq_m \neg B$ where*

$$\neg A := \{x \in \mathbb{N} \mid x \notin A\}$$

denotes the complement of the set A.

Proof: We have $A \leq_m A$ via the identity. If $A \leq_m B$ via f and $B \leq_m C$ via g then $A \leq_m C$ via $g \circ f$.
If $A \leq_m B$ via f we get

$$x \in A \quad \Leftrightarrow \quad f(x) \in B$$

which implies also

$$x \notin A \quad \Leftrightarrow \quad f(x) \notin B.$$

Therefore we also have $\neg A \leq_m \neg B$ via f. $\square$

2.1.4 Definition We put

$$A \equiv_m B \quad :\Leftrightarrow \quad A \leq_m B \wedge B \leq_m A$$

and conclude from Lemma 2.1.3 that $\equiv_m$ is an equivalence relation. Its equivalence classes are called m*–degrees*. By

$$\deg_m(A) := \{B \subseteq \mathbb{N} \mid A \equiv_m B\}$$

we denote the m–degree of A.

We will not study the theory of m–degrees in this lecture. However, since we need m–degrees sometimes we decided to introduce them. Without proof we mention that $\mathsf{Pow}(\mathbb{N})$ together with $\leq_m$ is an upper semi–lattice. It is a known result of OCT that for all decidable sets $A \notin \{\emptyset, \mathbb{N}\}$ we have $K \not\leq_m A$ where $K := \big\{x \big| \ (\exists w)\mathsf{T}(x,x,w)\big\}$.
Just two simple facts about m–Reducibility.

2.1.5 Theorem *1) If B is decidable in α and $A \leq_m B$ then A is also decidable in α.*
2) If $A \leq_m B$ and B is semi–decidable in α then A is also semi–decidable in α.

Proof: 1) If $A \leq_m B$ via f then

$$\chi_A = \chi_B \circ f.$$

2) $A = \big\{x \big| \ f(x) \in B\big\}$ is semi–decidable in α since these sets are closed under substitution with computable functions. □

2.2 TURING–Reducibility

The most general reduction of the decision problem is given by TURING–Reducibility.

2.2.1 Definition We say that a set A is *decidable in* B if χ_A is computable in χ_B. This is denoted by

$$A \leq_T B$$

or briefly $A \leq B$ if there is no danger of confusion. Synonymously we say that A is TURING–*reducible to* B. We put

$$A \equiv_T B \ \ :\Leftrightarrow \ \ A \leq_T B \wedge B \leq_T A.$$

2.2.2 Discussion If $A \leq_T B$ and we want to decide $x \in A$ we compute $\chi_A(x)$. This is computable in B. If we assume that the decision problem for B is solved, we can use a decision procedure for B in the computation of $\chi_A(x)$. Therefore the decision problem for A is reduced to that of B.
This is, however, a reduction in a much weaker sense than m–Reducibility. So we have obviously

$$A \leq_m B \ \ \Rightarrow \ \ A \leq_T B$$

while the opposite direction is not true in general. In this sense $\leq_T$ is a coarser relation than $\leq_m$.

2.2.3 Theorem *The relation $\leq_T$ is reflexive and transitive. Therefore the relation $\equiv_T$ is an equivalence relation on* $\mathsf{Pow}(\mathbb{N})$.

Proof: Because of $\chi_A(x) = \mathsf{App}(\chi_A, x)$ we see that A is decidable in A. Hence $\leq_T$ is reflexive. If $A \leq_T B$ and $B \leq_T C$ we have computable functionals F and G such that

$$\chi_A(x) = F(\chi_B, x)$$

and

$$\chi_B(x) = G(\chi_C, x).$$

So we get

$$\chi_A(y) \simeq F(\lambda x \,.\, G(\chi_C, x), y)\,.$$

Since G is total $\lambda x\,.\,G(\alpha,x)$ is total for any α and we get by the Substitution Lemma (Lemma 1.1.11) that $\lambda\alpha y\,.\,F(\lambda x\,.\,G(\alpha,x),y)$ is a computable functional, say H. But then

$$\chi_A(y)=H(\chi_C,y)$$

which shows that $A\leq_T C$. ☐

2.2.4 Theorem (POST's Theorem) *A set $A\subseteq\mathbb{N}$ is decidable in $B\subseteq\mathbb{N}$ iff both A and $\neg A$ are semi–decidable in B.*

Proof: If A is decidable in B then both A and $\neg A$ are decidable in B. Hence also semi–decidable in B. This gives the easy direction. For the opposite direction assume that both A and $\neg A$ are semi–decidable in B. Then we get indices e_1 and e_2 such that

$$A=\{x\mid\ (\exists z)\mathsf{T}^{B,1,0}(e_1,x,z)\} \tag{i}$$

and

$$\neg A=\{x\mid\ (\exists z)\mathsf{T}^{B,1,0}(e_2,x,z)\}. \tag{ii}$$

Put

$$f(x):\simeq\mu z\,.\,[\mathsf{T}^{B,1,0}(e_1,x,z)\vee\mathsf{T}^{B,1,0}(e_2,x,z)].$$

Then f is partial–computable in B and we get from (i) and (ii) that f is also total. So f is computable in B and we have

$$A=\{x\in\mathbb{N}\mid\ \mathsf{T}^{B,1,0}(e_1,x,f(x))\}$$

which shows by Theorem 1.1.15 that A is decidable in B. ☐

2.2.5 Remark We formulated POST's theorem for sets in order to have it fit into this section. The proof, however, shows that it is true also for arbitrary relations.

2.2.6 Lemma *Let A be semi–decidable in B and $B\leq_T C$. Then A is semi–decidable in C.*

Proof: We have an index e such that

$$\begin{aligned}A&=\{x\in\mathbb{N}\mid\ (\exists z)\mathsf{T}^{B,1,0}(e,x,z)\}\\&=\{x\in\mathbb{N}\mid\ (\exists z)\mathsf{T}^{1,1}(e,\chi_B,x,z)\}.\end{aligned}$$

Because of $B\leq_T C$ there is an index e_0 such that

$$\begin{aligned}\chi_B&=\{e_0\}^{C,1,0}\\&=\lambda x\,.\,U(\mu w\,.\,\mathsf{T}^{C,1,0}(e_0,x,w))\\&=\lambda x\,.\,U(\mu w\,.\,\mathsf{T}^{1,1}(e_0,\chi_C,x,w)).\end{aligned} \tag{i}$$

Hence

$$\begin{aligned}A&=\{x\in\mathbb{N}\mid\ (\exists z)\mathsf{T}^{1,1}(e,\lambda x\,.\,U(\mu w\,.\,\mathsf{T}^{1,1}(e_0,\chi_C,x,w)),x,z\}\\&=\mathsf{dom}(\mu z\,.\,\mathsf{T}^{1,1}(e,\lambda x\,.\,U(\mu w\,.\,\mathsf{T}^{1,1}(e_0,\chi_C,x,w)))).\end{aligned} \tag{ii}$$

Since χ_B is total we get by (i) and the Substitution Lemma (Lemma 1.1.11) that the functional in the last line of (ii) is partial–computable in C. Hence A is semi–decidable in C. ☐

2.3 TURING–Degrees

We say that two sets $A, B \subseteq \mathbb{N}$ are TURING–*equivalent* iff $A \equiv_T B$. The class

$$\deg_T(A) = \{B \mid B \equiv_T A\}$$

forms the TURING–*degree* (or just *degree*) of A. We will denote degrees by lower case bold Roman letters, e.g., a, b, c, $\mathbf{a}_1, \ldots$. For degrees $\mathbf{a}, \mathbf{b}$ we define

$$\mathbf{a} \le \mathbf{b} \quad \Leftrightarrow \quad (\exists A \in \mathbf{a})(\exists B \in \mathbf{b})[A \le_T B]. \tag{2.1}$$

It follows from Theorem 2.2.3 that (2.1) is independent of the choice of A and B. We put

$$\mathbf{a} < \mathbf{b} \quad :\Leftrightarrow \quad \mathbf{a} \le \mathbf{b} \land \mathbf{a} \ne \mathbf{b}.$$

There is a minimal degree

$$\mathbf{0} := \deg_T(\emptyset)$$

which contains exactly the decidable sets. To show that for any degree a there is a strictly bigger degree $\mathbf{a}'$ we introduce the *jump operator* which is defined by

$$j(A) := \{x \mid (\exists w)\mathsf{T}^{A,1,0}(x, x, w)\} = \{x \mid x \in \mathsf{W}_x^{A,1,0}\}$$

for $A \subseteq \mathbb{N}$. We call $j(A)$ the *jump of* A.
For a degree a we introduce

$$\mathbf{a}' := \deg_T(j(A)) \text{ for some } A \in \mathbf{a}. \tag{2.2}$$

We will show later (cf. Theorem 3.1.1) that

$$A \le_T B \quad \Rightarrow \quad j(A) \le_T j(B).$$

Therefore $\mathbf{a}'$ in (2.2) is well–defined. We will moreover see that $A \le_m j(A)$. Hence also $A \le_T j(A)$ and $j(A)$ is obviously semi–decidable in A. We have, however, the following fact.

2.3.1 Theorem *The jump $j(A)$ is not decidable in A.*

Proof: Towards a proof by reductio ad absurdum assume $j(A) \le_T A$. Then $\neg j(A) \le_T A$ which entails that there is an index e such that

$$\neg j(A) = \mathsf{W}_e^{A,1,0}.$$

Hence

$$e \notin j(A) \quad \Leftrightarrow \quad e \in \mathsf{W}_e^{A,1,0} \quad \Leftrightarrow \quad e \in j(A).$$

A contradiction. □

As an immediate corollary of Theorem 2.3.1 we get

2.3.2 Theorem *For any degree* a *we have* $\mathbf{a} < \mathbf{a}'$.

Now the canonical questions arise

- Are the degrees linearly ordered by $\le$?
- Are there degrees between a and $\mathbf{a}'$?

These questions have already been asked by E. POST in 1944. It lasted until 1954 before they could be answered independently by R. FRIEDBERG and A. MUCHNIK. They proved the following theorem.

2.3.3 Theorem (FRIEDBERG,MUCHNIK**)** *There are semi–decidable sets A, B which are incomparable with respect to $\leq_T$, i.e. we have neither $A \leq_T B$ nor $B \leq_T A$.*

Proof: Before we start proving the theorem let us discuss it briefly. The proof will show that the theorem also holds in relativized form. It is just for simpler notations that we omitted the relativization.
We have $\emptyset \leq_T C$ for any set C and – as we will see soon – $A \leq_T j(\emptyset)$ for any semi–decidable set A. Thus if A and B are semi–decidable and incomparable we get the picture shown in Figure 2.3.1 where the arrows represent $\leq_T$. This shows that Theorem 2.3.3 in fact answers both questions.

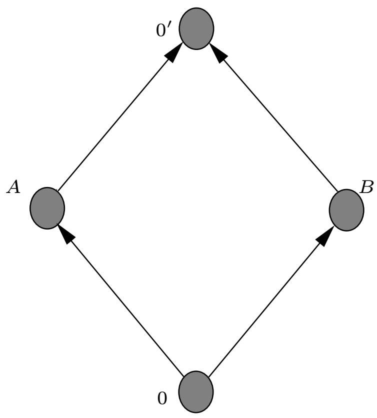

Figure 2.3.1: Two incomparable semi–decidable sets

The degrees are not linearly ordered and there are degrees between a and a′.
To prepare the technical part of the proof we start with a few heuristic remarks. Since we have $D \leq_T C$ for any decidable set D and any set C none of the sets A and B, which we are going to construct, must be decidable. Since we aim at $A \not\leq_T B$ as well as $B \not\leq_T A$ we have to ensure that $\chi_A \neq \{e\}^B$ for all e and also $\chi_B \neq \{e\}^A$ for all e, i.e.

$$(\forall e)(\exists y)[\chi_A(y) \not\simeq \{e\}^B(y)] \tag{i}$$

and

$$(\forall e)(\exists y)[\chi_B(y) \not\simeq \{e\}^A(y)]. \tag{ii}$$

To obtain (i) and (ii) it suffices to construct a function F which satisfies

$$(\forall e)[F(2e) \in A \;\Leftrightarrow\; \{e\}^{B,1,0}(F(2e)) \simeq 1] \tag{iii}$$

and

$$(\forall e)[F(2e+1) \in B \;\Leftrightarrow\; \{e\}^{A,1,0}(F(2e+1)) \simeq 1]. \tag{iv}$$

The function F, however, must not be computable. To see that assume that F is computable satisfying (iii) and (iv). We define

$$f(x,y) \simeq \begin{cases} 1 & \text{if } x \notin B \\ 0 & \text{if } x \in B. \end{cases}$$

Then f is computable in B and we obtain an index e such that

$$f = \{e\}^B.$$

Using the relativized S_1^m–Theorem this yields

$$\{S(e,x)\}^B(y) \simeq 1 \;\;\Leftrightarrow\;\; x \notin B$$

for all $x \in \mathbb{N}$. Hence

$$x \notin B \;\;\Leftrightarrow\;\; \{S(e,x)\}^B(F(2\cdot S(e,x)) \simeq 1 \;\;\Leftrightarrow\;\; F(2\cdot S(e,x)) \in A$$

by (iii). Since A is semi–decidable and F computable $\neg B$ is semi–decidable. Hence B is decidable by POST's Theorem. This, however, is impossible as we have seen above.
The problem is to *construct* F in such a way that F does not become computable. This cannot be simple because any *construction* of a non–computable function is close to conflict with CHURCH's Thesis. The basic idea is to approximate A, B and F stepwise by A_n, B_n and $\lambda x\,.\,F(n,x)$ such that $\overline{\chi}_{A_n}$, $\overline{\chi}_{B_n}$ and $\lambda x\,.\,F(n,x)$ are computable. In step n we compute either

$$y_n := \mu w < n\,.\left[\mathsf{T}^{1,1}(e, F(n,2e), \chi_{B_n}, w) \wedge U(w) = 1\right] \tag{v}$$

or

$$y_n := \mu w < n\,.\left[\mathsf{T}^{1,1}(e, F(n,2e+1), \chi_{A_n}, w) \wedge U(w) = 1\right] \tag{vi}$$

according to the shape of n which also determines e in an effective way. Whenever $y_n \neq n$ we put in the first case $F(n,2e)$ into A_{n+1} or — in the second case — $F(n,2e+1)$ in B_{n+1}. The obvious problem now is that at a later point $m > n$, where a larger portion A_m of A (or B_m of B) is known, the computation may change. Therefore we give $F(n+1,x)$ a value above y_n to ensure that the computations in (v) and (vi) will not be changed. The index n in $F(n,2e)$ is therefore the priority with which $F(n,2e)$ has to be put into A (or $F(n,2e+1)$ into B). Once we have reached the highest priority n we may put $F(x) := F(n,x)$. Of course we need to prove that such highest priorities exists. Though certainly still vague, we hope that these remarks will be helpful in the following technical part of the proof.
We put

$$A_0 := \emptyset,\;\; B_0 := \emptyset \;\;\text{and}\;\; F(0,x) := \begin{cases} 2^e & \text{if } x = 2e \\ 2^e & \text{if } x = 2e+1. \end{cases} \tag{vii}$$

Assume that A_n, B_n and $F(n,x)$ are defined for all x. We distinguish the following cases:

1) $(n)_0 = 2e$ for some $e \in \mathbb{N}$.

Then we compute

$$y_n := \mu w < n\,.\left[\mathsf{T}^{2,0}(e, F(n,2e), \overline{\chi}_{B_n}((w)_0), (w)_1) \wedge U((w)_1) = 1 \wedge F(n,2e) \notin A_n\right]. \tag{viii}$$

If $y_n = n$ we put

$$A_{n+1} := A_n,\;\; B_{n+1} := B_n \;\;\text{and}\;\; F(n+1,x) := F(n,x). \tag{ix}$$

Otherwise we define

$$A_{n+1} := A_n \cup \{F(n,2e)\},\;\; B_{n+1} := B_n \tag{x}$$

and

$$F(n+1,x) := \begin{cases} F(n,x) & \text{if } x \leq 2e \text{ or } x \equiv 0 \bmod 2 \\ F(n,x)\cdot 3^{y_n} & \text{if } 2e < x \text{ and } x \equiv 1 \bmod 2. \end{cases} \tag{xi}$$

2) $(n)_0 = 2e+1$ for some $e \in \mathbb{N}$.

Again we compute

$$y_n := \mu w < n \,.\, \big[\mathsf{T}^{2,0}(e, F(n, 2e+1), \overline{\chi}_{A_n}((w)_0), (w)_1) \wedge U((w)_1) = 1 \wedge F(n, 2e+1) \notin B_n\big]\,. \tag{xii}$$

If $y_n = n$ we put

$$A_{n+1} := A_n,\ \ B_{n+1} := B_n,\ \ F(n+1, x) := F(n, x) \tag{xiii}$$

and otherwise

$$A_{n+1} := A_n,\ \ B_{n+1} := B_n \cup \{F(n, 2e+1)\} \tag{xiv}$$

and

$$F(n+1, x) := \begin{cases} F(n, x) & \text{if } x \leq 2e+1 \text{ or } x \equiv 1 \bmod 2 \\ F(n, x) \cdot 3^{y_n} & \text{if } 2e+1 < x \text{ and } x \equiv 0 \bmod 2. \end{cases} \tag{xv}$$

One should observe that in (vii) through (xv) we define the functions $\lambda nx \,.\, \overline{\chi}_{A_n}(x)$, $\lambda nx \,.\, \overline{\chi}_{B_n}(x)$ and $\lambda nx \,.\, F(n, x)$ simultaneously by the Recursion Theorem. Hence all these functions are partial–computable. It follows by induction on n that all these functions are also total. By construction we have

$$F(n, x) \leq F(n+1, x)$$

which yields

$$m \leq n \ \ \Rightarrow \ \ F(m, x) \leq F(n, x)$$

by induction on n. Similarly we get

$$m \leq n \ \ \Rightarrow \ \ A_m \subseteq A_n \wedge B_m \subseteq B_n.$$

The essential step is to show:

$$V_x := \big\{n \big|\ F(n, x) \neq F(n+1, x)\big\} \text{ is finite.}$$

The proof is by induction on x. For $y < x$ the set V_y is finite by induction hypothesis. This entails the finiteness of the sets $\big\{F(n, y) \big|\ n \in \mathbb{N}\big\}$ for $y < x$. Hence

$$V := \bigcup_{y<x} \big\{F(n, y) \big|\ n \in \mathbb{N}\big\} \text{ is finite.}$$

We construct a one–one mapping from V_x into V. Let $n \in V_x$. If x is even then by (xi) and (xv) there is an $x_n < x$ such that $F(n, x_n) \notin B_n$ but $F(n, x_n) \in B_{n+1}$. For x odd we obtain by (xi) and (xv) an $x_n < x$ such that $F(n, x_n) \in A_{n+1} \setminus A_n$. Hence $F(n, x_n) \in V$ and for $m, n \in V_x$ with $m < n$ we get

$$F(m, x_m) \in B_{m+1} \subseteq B_n \not\ni F(n, x_n)$$

for x even or

$$F(m, x_m) \in A_{m+1} \subseteq A_n \not\ni F(n, x_n)$$

for x odd, respectively. Hence $F(m, x_m) \neq F(n, x_n)$ and

$$n \mapsto F(n, x_n)$$

is a one–one map from V_x into V. Therefore V_x is finite.
We define

$$A := \bigcup_{n\in\mathbb{N}} A_n;\ B := \bigcup_{n\in\mathbb{N}} B_n$$

and

$$F(x) := F(n,x) \text{ if } (\forall m)[m \geq n \;\Rightarrow\; F(m,x) = F(n,x)]. \tag{xvi}$$

Now we prove

$$\begin{aligned} &\{e\}^A(F(2e+1)) \simeq 1 \;\Rightarrow\; F(2e+1) \in B \\ &\{e\}^B(F(2e)) \simeq 1 \;\Rightarrow\; F(2e) \in A. \end{aligned} \tag{xvii}$$

Both lines of (xvii) are proved analogously. We show the first. From

$$\{e\}^A(F(2e+1)) \simeq 1$$

we get for some $w \in \mathbb{N}$

$$\mathsf{T}^{2,0}(e, F(2e+1), \overline{\chi}_A((w)_0), (w)_1) \wedge U((w)_1) = 1. \tag{xviii}$$

There are infinitely many $n \in \mathbb{N}$ such that

$$(n)_0 = 2e+1.$$

We choose n so big that

$$w < n,\;\; \overline{\chi}_{A_n}(w) = \overline{\chi}_A(w) \;\text{ and }\; F(2e+1) = F(n, 2e+1). \tag{xix}$$

Then (xix) and (xviii) yield

$$(\exists w < n)\left[\mathsf{T}^{2,0}(e, F(n,2e+1), \overline{\chi}_{A_n}((w)_0), (w)_1) \wedge U((w)_1) = 1\right]$$

and we either have $F(2e+1) = F(n,2e+1) \in B_n \subseteq B$ or obtain $F(2e+1) = F(n,2e+1) \in B_{n+1} \subseteq B$ by (xii) and (xiv). It remains to prove also the opposite directions in (xvii), i.e.

$$\begin{aligned} &F(2e+1) \in B \;\Rightarrow\; \{e\}^A(F(2e+1)) \simeq 1 \\ &F(2e) \in A \;\Rightarrow\; \{e\}^B(F(2e)) \simeq 1. \end{aligned} \tag{xx}$$

First we obtain

$$x = 2e \;\text{ or }\; x = 2e+1 \;\Rightarrow\; F(n,x) = 2^e \cdot 3^y \tag{xxi}$$

by an easy induction on n. As a consequence of (xxi) we get

$$\begin{aligned} &F(n, 2e_1) = F(m, 2e_2) \;\Rightarrow\; e_1 = e_2 \\ &F(n, 2e_1+1) = F(m, 2e_2+1) \;\Rightarrow\; e_1 = e_2. \end{aligned} \tag{xxii}$$

We prove the second line of (xx). The proof of the first runs analogously.
Let $F(2e) \in A$. Then there is an n such that

$$F(2e) \in A_{n+1} \setminus A_n$$

which implies

$$F(2e) = F(n, 2(n)_0).$$

According to (xvi) and the first line in (xxii) this yields $e = (n)_0$. Hence by (viii)

$$\mathsf{T}^{2,0}(e, F(n,2e), \overline{\chi}_{B_n}((y_n)_0), (y_n)_1) \wedge U((y_n)_1) = 1 \tag{xxiii}$$

and $y_n < n$. As soon as we can show

$$\overline{\chi}_B((y_n)_0) = \overline{\chi}_{B_n}((y_n)_0) \tag{xxiv}$$

we get $\{e\}^B(F(2e)) \simeq 1$ from (xxiii). Towards a contradiction assume

$$\overline{\chi}_B((y_n)_0) \neq \overline{\chi}_{B_n}((y_n)_0).$$

Then there is a $z < (y_n)_0 < y_n$ such that

$$\chi_B(z) \neq \chi_{B_n}(z).$$

But then $z \in B \setminus B_n$ which shows that there is an $m > n$ such that

$$z \in B_{m+1} \setminus B_m.$$

Hence $z = F(m, 2f+1)$ for some $f \in \mathbb{N}$. If $2e < 2f+1$ we get by (xi)

$$z = F(m, 2f+1) \geq F(n+1, 2f+1) = F(n, 2f+1) \cdot 3^{y_n}$$

which contradicts $z < y_n$. For $2e > 2f+1$, however, we get by (xv)

$$F(m+1, 2e) = F(m, 2e) \cdot 3^{y_n} > F(m, 2e) = F(n, 2e) = F(2e)$$

contradicting the definition of $F(2e)$ in (xvi). Hence (xxiv). □

3. The Arithmetical Hierarchy

3.1 The Jump operator revisited

The jump operator

$$j(A) := \{x \in \mathbb{N} \mid (\exists w)\mathsf{T}^{A,1,0}(x,x,w)\} = \{x \mid x \in \mathsf{W}_x^{A,1,0}\} \tag{3.1}$$

is introduced in Section 2.3. We are going to study its properties more profoundly in this section. It follows from (3.1) and Theorem 2.3.1 that $j(A)$ is semi–decidable but not decidable in A. The following theorem strengthens that.

3.1.1 Theorem *1) A set $A \subseteq \mathbb{N}$ is semi-decidable in B iff $A \leq_m j(B)$.*
2) We have $A \leq_T B$ iff $j(A) \leq_m j(B)$.
Both claims hold uniformly in A, i.e. an index of the m–reducing computable function f can be computed from a B–index for the set A.

Proof: 1) Define

$$\begin{aligned} K_0^B &:= \{(x,y) \mid (y)_1 \in \mathsf{W}_{(y)_0}^{B,1,0}\} \\ &= \{(x,y) \mid (\exists w)\mathsf{T}^{B,1,0}((y)_0,(y)_1,w)\}. \end{aligned}$$

The predicate K_0^B is semi–decidable in B. Let e_0 be an index for K_0^B. Then we get

$$\begin{aligned} (y)_1 \in \mathsf{W}_{(y)_0}^{B,1,0} &\Leftrightarrow (x,y) \in K_0^B \\ &\Leftrightarrow (x,y) \in \mathsf{W}_{e_0}^{B,2,0} \\ &\Leftrightarrow x \in \mathsf{W}_{\mathsf{S}_1^{2,0}(e_0,y)}^{B,1,0}. \end{aligned} \tag{i}$$

If A is semi–decidable in B we have an index e for A, i.e. $A = \mathsf{W}_e^{B,1,0}$, and define a function f by

$$f(x) := \mathsf{S}_1^{2,0}(e_0,\langle e,x\rangle).$$

Then f is computable and an index for f can be computed from e. According to (i) we get

$$\begin{aligned} f(x) \in j(B) &\Leftrightarrow f(x) \in \mathsf{W}_{f(x)}^{B,1,0} \\ &\Leftrightarrow f(x) \in \mathsf{W}_{\mathsf{S}_1^{2,0}(e_0,\langle e,x\rangle)}^{B,1,0} \\ &\Leftrightarrow x \in \mathsf{W}_e^{B,1,0} \\ &\Leftrightarrow x \in A \end{aligned}$$

which shows that $A \leq_m j(B)$ via f.
For the opposite direction we assume $A \leq_m j(B)$ via f. But then

$$x \in A \;\Leftrightarrow\; (\exists w)\mathsf{T}^{B,1,0}(f(x),f(x),w)$$

which shows immediately that A is semi–decidable in B.
2) We start with the "if"–direction. Since $j(A)$ is semi–decidable in A we get from $A \leq_T B$ by Lemma 2.2.6 that $j(A)$ is also semi–decidable in B. Hence $j(A) \leq_m j(B)$ by 1). To obtain also the uniformity we need to know that a B–index of $j(A)$ can be computed from a B–index of A, i.e. we need a computable function, say h, with

$$\chi_A = \{e\}^{B,1,0} \Rightarrow j(A) = \mathsf{W}^{B,1,0}_{h(e)}.$$

Though simple a rigid proof is quite tedious and depends heavily on our special definition of indices. Therefore we restrict ourselves to a rough sketch. We have

$$\begin{aligned} x \in j(A) &\Leftrightarrow (\exists w)\mathsf{T}^{1,1}(x, \chi_A, x, w) \\ &\Leftrightarrow (\exists w)\mathsf{T}^{1,1}(x, \lambda y\,.\,\{e\}^B(y), x, w)\,. \end{aligned} \tag{ii}$$

The function $g := \mu w\,.\,\mathsf{T}^{1,1}(x, \lambda y\,.\,\{e\}^B(y), x, w)$ is obvious partial–computable in B and its index depends only on e. This dependence is effective which means that there is a computable (even primitive–recursive) function, say h, such that $h(e)$ is a B–index for g. Hence by (ii)

$$j(A) = \mathsf{dom}(\{h(e)\}^B) = \mathsf{W}^{B,1,0}_{h(e)}.$$

For the "only–if"–direction assume $j(A) \leq_m j(B)$. Since A as well as $\neg A$ are semi–decidable in A we get

$$A \leq_m j(A) \leq_m j(B)$$

and also

$$\neg A \leq_m j(A) \leq_m j(B)$$

by part 1). By the transitivity of $\leq_m$ and part 1) this implies that A and $\neg A$ are both semi–decidable in B. Using POST's Theorem (Theorem 2.2.6) we obtain $A \leq_T B$. $\square$

3.1.2 Definition The n–th jump of a set $A \subseteq \mathbb{N}$ is defined by

$$\begin{aligned} &A^{(0)} := A \\ &A^{(n+1)} := j(A^{(n)}). \end{aligned}$$

3.1.3 Lemma *We have*

$$n \leq k \Rightarrow A^{(n)} \leq_m A^{(k)} \tag{3.2}$$

and

$$A \leq_m B \Rightarrow A^{(n)} \leq_m B^{(n)}. \tag{3.3}$$

Claim (3.2) holds uniformly in n and k, i.e. an index for the reducing function can be computed from n and k, while claim (3.3) holds uniformly in n and the index of the function which reduces A to B.

Proof: We show (3.2) by induction on k. The claim is obvious for $k = n$. For $k = l + 1 > n$ we have

$$A^{(n)} \leq_m A^{(l)}$$

via $\{f(n, l)\}$ by the induction hypothesis. By part 1) of Theorem 3.1.1 we have $A^{(l)} \leq_m A^{(l+1)}$ via some function g. Hence $A^{(n)} \leq_m A^{(l+1)}$. To show also the uniformity we observe that g does not depend on A. Since

$$j(B) = \{x \mid (x, \chi_B) \in \mathsf{W}^{1,1}_x\}$$

we see that there is an $e \in \mathbb{N}$ such that

$$j(B) = \mathsf{W}^{B,1,0}_e$$

holds for any $B \subseteq \mathbb{N}$. So, according to Theorem 3.1.1, g depends only on the constant e and the index of the reducing function $g \circ \{f(n, l)\}$ can be computed from n and e.

We prove (3.3) by induction on n. For $n = 0$ we have $A \leq_m B$ by hypothesis. In the successor case we have $A^{(n)} \leq_m B^{(n)}$ by the induction hypothesis and obtain $A^{(n+1)} \leq_m B^{(n+1)}$ by Theorem 3.1.1 2). The index of the reducing function is computed from a $B^{(n)}$–index of $A^{(n)}$ which in turn depends on the reducing function for $A^{(n)} \leq_m B^{(n)}$. This function, however, can by induction hypothesis be computed from an index of the reducing function for $A \leq_m B$. $\square$

3.2 The Arithmetical Hierarchy

The Arithmetical Hierarchy classifies the subsets of $\mathbb{N}$ which can be defined arithmetically. The most obvious classification is according to the complexity of the defining formula. Therefore we introduce first a classification of the arithmetical formulas.

3.2.1 Definition Let φ be a formula in the language of arithmetic, i.e. the only non–logical symbols occurring in φ are constants for natural numbers, for primitive–recursive functions and of predicates which can be decided primitive–recursively. In an arithmetical formula all quantifiers are supposed to range over individuals, i.e. we are in first order, however, we allow free function variables ξ,η, $\xi_1, \ldots$.
We say that φ is a Δ^0_0–formula, if φ contains at most bounded quantifiers.
We say that φ is Σ^0_1, if there is a Δ^0_0–formula $\psi(x)$ such that $\varphi \equiv (\exists x)\psi(x)$.
Dually φ is Π^0_1 if $\neg\varphi$ is Σ^0_1.
A formula φ is in Σ^0_{n+1} if there is a formula $\psi(x)$ in Π^0_n such that $\varphi \equiv (\exists x)\psi(x)$.
Dually φ is Π^0_{n+1} if $\neg\varphi$ is $\exists^0_{n+1}$.

3.2.2 Remark In the above definition we assume that the language of arithmetic is given as a TAIT–language (cf. [4]), i.e. a language containing $\neq$ as basic symbol in which $\neg\varphi$ is defined by

$$\begin{aligned}
&\neg(s = t) :\equiv s \neq t\\
&\neg(s \neq t) :\equiv s = t\\
&\neg(Rt_1, \ldots, t_n) :\equiv (\neg R)t_1, \ldots, t_n\\
&\neg((\neg R)t_1, \ldots, t_n) :\equiv Rt_1, \ldots, t_n\\
&\neg(\varphi \wedge \psi) :\equiv \neg\varphi \vee \neg\psi\\
&\neg(\varphi \vee \psi) :\equiv \neg\varphi \wedge \neg\psi\\
&\neg(\forall x)\varphi(x) :\equiv (\exists x)\neg\varphi(x)\\
&\neg(\exists x)\varphi(x) :\equiv (\forall x)\neg\varphi(x)
\end{aligned}$$

where $\neg R$ is a relation constant whose interpretation is the complement of the interpretation of R.
We obviously have

$$\varphi \in \Sigma^0_n \;\;\Leftrightarrow\;\; \varphi \equiv (\exists x_1)(\forall x_2)\ldots(\mathsf{Q}x_n)\psi(\vec{x})$$

and

$$\varphi \in \Pi^0_n \;\;\Leftrightarrow\;\; \varphi \equiv (\forall x_1)(\exists x_2)\ldots(\mathsf{Q}x_n)\psi(\vec{x})$$

where $\psi(\vec{x})$ is a Δ^0_0–formula and $(\exists x_1)(\forall x_2)\ldots(\mathsf{Q}x_n)$ as well as $(\forall x_1)(\exists x_2)\ldots(\mathsf{Q}x_n)$ are alternating strings of $\mathbb{N}$–quantifiers.

3.2.3 Definition A relation $R \subseteq \mathbb{N}^{m,n}$ is definable with parameters $\beta_1, \ldots, \beta_l$ by a formula

$$\varphi(\xi_1, \ldots, \xi_l, x_1, \ldots, x_n, \eta_1, \ldots, \eta_m)$$

if φ possesses only the indicated free variables and

$$R = \{\mathfrak{a} \in \mathbb{N}^{m,n} |\ \mathbb{N} \models \varphi[\beta_1, \ldots, \beta_l, \mathfrak{a}]\}.$$

3.2.4 Definition 1) A relation is $\Sigma_n^0[A]$ if it is definable with parameter χ_A by a Σ_n^0–formula. $\Pi_n^0[A]$-relations are defined analogously.
2) A relation is $\Delta_n^0[A]$ if it is both, $\Sigma_n^0[A]$ and $\Pi_n^0[A]$.
Instead of $\Sigma_n^0[\emptyset]$, $\Pi_n^0[\emptyset]$, and $\Delta_n^0[\emptyset]$ we write Σ_n^0, Π_n^0 and Δ_n^0.
3) A relation is called *arithmetical (in A)* if it is in $\Delta_n^0[A]$ for some $n \in \mathbb{N}$. To unify notations we put

$$\Delta_0^1[A] := \Pi_0^1[A] := \Sigma_0^1[A] := \{R|\ R \text{ is arithmetical in } A\}.$$

3.2.5 Theorem *1) The Δ_0^0–predicates are exactly the primitive–recursively decidable predicates.*
2) The Σ_1^0–relations are exactly the semi–decidable relations.

3.2.6 Theorem (Post**)** *1) A relation R is semi–decidable in a set $A \subseteq \mathbb{N}$ iff R is $\Sigma_1^0[A]$.*
2) A relation is decidable in a set $A \subseteq \mathbb{N}$ iff it is $\Delta_1^0[A]$.

The proofs of Theorems 3.2.5 and 3.2.6 are obvious from our previous knowledge.

3.2.7 Definition Let $\mathcal{F}$ denote one of the complexity classes introduced in Definition 3.2.4. We say that a partial functional is an $\mathcal{F}$–functional iff its graph belongs to $\mathcal{F}$.

3.2.8 Lemma *Any total $\Sigma_n^0[A]$–functional is already in $\Delta_n^0[A]$.*

Proof: The proof needs already the closure of Σ_n^0 under $\wedge$ and $\exists^0$–quantification. Let F be a total $\Sigma_n^0[A]$–functional. Then

$$\begin{aligned} \neg G_F(\mathfrak{a}, y) &\Leftrightarrow F(\mathfrak{a}) \not\simeq y \\ &\Leftrightarrow (\exists z)[F(\mathfrak{a}) \simeq z \wedge z \neq y]. \end{aligned}$$

Which shows that both, the graph of F and its complement, are in $\Sigma_n^0[A]$. Hence $G_F \in \Delta_n^0[A]$. □

We list the closure properties of these newly introduced relation–classes in the table shown in Figure 3.2.1. The positive closure properties, i.e. those which carry a "yes", are shown by induction on k. We already proved them for the case $k = 1$ with the exception of the closure of Σ_1^0 under $\exists^1$–quantification. However, we want to postpone this property because it does not carry over to $k > 1$. So assume that we have the positive closure properties for k. Let R_1 and R_2 be Σ_{k+1}^0–relations. Then we have

$$R_1(\mathfrak{a}) \ \Leftrightarrow\ (\exists x) Q_1(\mathfrak{a}, x)$$

and

$$R_2(\mathfrak{a}) \ \Leftrightarrow\ (\exists y) Q_2(\mathfrak{a}, y)$$

for $Q_i \in \Pi_k^0$. Hence

$$\begin{aligned} R_1(\mathfrak{a}) \genfrac{}{}{0pt}{}{\wedge}{\vee} R_2(\mathfrak{a}) &\Leftrightarrow \\ &\Leftrightarrow \end{aligned}$$

and the expression in square–brackets is Π_k^0 by the induction hypothesis.
The closure of Σ_{k+1}^0–relations under $\exists^0$–quantification follows by contraction of quantifiers, i.e. by

$$(\mathsf{Q}x)(\mathsf{Q}y) R(\mathfrak{a}, x, y) \ \Leftrightarrow\ (\mathsf{Q}u) R(\mathfrak{a}, (u)_0, (u)_1). \tag{3.4}$$

Relation–class	$\neg$	$\vee$	$\wedge$	$\exists^<$	$\forall^<$	$\exists^0$	$\forall^0$	$\exists^1$	$\forall^1$	Substitution with
primitive–recursive	yes	yes	yes	yes	yes	no	no	no	no	primitive–recursive functionals
Δ^0_1–relations	yes	yes	yes	yes	yes	no	no	no	no	computable functionals
Σ^0_1–relations	no	yes	yes	yes	yes	yes	no	yes	no	computable functionals
Π^0_1–relations	no	yes	yes	yes	yes	no	yes	no	yes	computable functionals
Δ^0_{k+1}–relations	yes	yes	yes	yes	yes	no	no	no	no	total Σ^0_{k+1}-functionals
Σ^0_{k+1}–relations	no	yes	yes	yes	yes	yes	no	no	no	total Σ^0_{k+1}-functionals
Π^0_{k+1}–relations	no	yes	yes	yes	yes	no	yes	no	no	total Σ^0_{k+1}-functionals
arithmetical	yes	yes	yes	yes	yes	yes	yes	no	no	total arithmetical functionals

Figure 3.2.1: Closure Properties of Relation–Classes

So we are left with bounded $\forall$–quantification and substitution with total Σ^0_{k+1}–functionals. Assume

$$\begin{aligned} P(\mathfrak{a}) &\Leftrightarrow (\forall x < n)R(\mathfrak{a}, x) \\ &\Leftrightarrow (\forall x < n)(\exists y)Q(\mathfrak{a}, x, y) \end{aligned}$$

for R a Σ^0_{k+1}– and Q a Π^0_k–relation. Then

$$\begin{aligned} P(\mathfrak{a}) &\Leftrightarrow (\forall x < n)(\exists y)Q(\mathfrak{a}, x, y) \\ &\Leftrightarrow (\exists s)[\mathit{Seq}(s) \wedge \mathit{lh}(s) = n \wedge (\forall x < n)Q(\mathfrak{a}, x, (s)_x)] \end{aligned}$$

and the expression in square–brackets is Π^0_k by induction hypothesis. Hence P is Σ^0_{k+1}.
By duality we get the dual closure properties for Π^0_{k+1}–relations.
If F is a total Σ^0_{k+1}–functional and

$$P(\mathfrak{a}) \Leftrightarrow R(\mathfrak{a}, F(\mathfrak{a}))$$

for a Σ^0_{k+1}– (Π^0_{k+1}–)relation R we get

$$\begin{aligned} P(\mathfrak{a}) &\Leftrightarrow (\exists z)[F(\mathfrak{a}) \simeq z \wedge R(\mathfrak{a}, z)] \\ &\Leftrightarrow (\forall z)[F(\mathfrak{a}) \simeq z \Rightarrow R(\mathfrak{a}, z)]. \end{aligned}$$

Applying Lemma 3.2.8 – which is possible since we know that Σ^0_{k+1} is closed under $\exists^0$–quantification and $\wedge$, we get that P is in Σ^0_{k+1} or Π^0_{k+1}, respectively.
The closure properties for Δ^0_k–relations follow by combining those of Π^0_k– and Σ^0_k–relations (we still regard only the positive closure properties) and the shown (positive) closure properties for arithmetical relations follow from those of Δ^0_n–relations.
It remains to show that Σ^0_1 is closed under $\exists^1$–quantification. For a Σ^0_1–relation R we get

$$\begin{aligned} (\exists\alpha)R(\mathfrak{a}, \alpha) &\Leftrightarrow (\exists\alpha)(\exists y)Q(\overline{\mathfrak{a}}(y), \overline{\alpha}(y)) \\ &\Leftrightarrow (\exists s)[\mathit{Seq}(s) \wedge \mathit{lh}(s) = y \wedge Q(\overline{\mathfrak{a}}(y), s)] \end{aligned}$$

where Q is a semi–decidable predicate. But then the expression in square–brackets is also semi–decidable which implies that $\{\mathfrak{a} \mid (\exists\alpha)R(\mathfrak{a}, \alpha)\}$ is a semi–decidable relation. By duality we obtain that Π^0_1 is closed under $\forall^1$–quantification. Observe that the closure under second order

quantifiers cannot be lifted to the higher levels of the hierarchy. As soon as we have a quantifier string of the form $(\exists\alpha)(\forall x)q(\alpha,x)$ it is obvious that we cannot replace α by a finite sequence. A rigid proof will be in the Analytical Hierarchy Theorem in the next chapter.
Let $\varphi(x_1,\dots,x_m,\xi_1,\dots,\xi_n)$ be any first order formula in the language of arithmetic. Then $\varphi(\vec{x},\vec{\xi})$ is logically equivalent to a formula in prenex form and we may use the quantifier contraction (3.4) to see that

$$\mathbb{N} \models (\forall\vec{x})(\forall\vec{\xi})[\varphi(\vec{x},\vec{\xi}) \iff \psi(\vec{x},\vec{\xi})]$$

for a formula ψ which is either in Σ^0_k or Π^0_k for some $k \in \mathbb{N}$. Then

$$R := \{\mathfrak{a} \mid \mathbb{N} \models \varphi[\mathfrak{a}]\}$$

is Δ^0_{k+1}, i.e. arithmetical. We put this into a lemma.

3.2.9 Lemma *Let $\varphi(x_1,\dots,x_m,\xi_1,\dots,\xi_n)$ be a first order formula in the language of arithmetic. Then the relation*

$$R := \{\mathfrak{a} \in \mathbb{N}^{m,n} \mid \mathbb{N} \models \varphi[\mathfrak{a}]\}$$

is arithmetical.

We are now going to investigate the connection of the arithmetical hierarchy to the jump hierarchy introduced in Definition 3.1.2.

3.2.10 Theorem *A relation R is $\Sigma^0_{k+1}[A]$ iff it is semi–decidable in $A^{(k)}$.*

Proof: We prove the theorem by induction on k. We begin with the "if"–direction. For $k=0$ this is Theorem 3.2.6 1). For the induction step assume that $R \in \Sigma^0_{k+2}[A]$. Then

$$\mathfrak{a} \in R \;\Leftrightarrow\; (\exists x)P(\mathfrak{a},x)$$

for a $\Pi^0_{k+1}[A]$–relation P. Then $\neg P \in \Sigma^0_{k+1}[A]$ and $\neg P$ is semi–decidable in $A^{(k)}$ by induction hypothesis. By Theorem 3.1.1 1) this implies $\neg P \leq_T A^{(k+1)}$ and, since $P \leq_T \neg P$, $P \leq_T A^{(k+1)}$. Since R is the $\mathbb{N}$–projection of the relation P which is decidable in $A^{(k+1)}$ we get by the relativization of Theorem 1.1.17 that R is semi–decidable in $A^{(k+1)}$. For the "only if" direction let R be semi–decidable in $A^{(k+1)}$. Since $A^{(k+1)}$ is semi–decidable in $A^{(k)}$ we get $A^{(k+1)} \in \Sigma^0_{k+1}[A]$ by induction hypothesis. Let e be an $A^{(k+1)}$–index for R. Then we obtain

$$\begin{aligned} R(\mathfrak{a}) \;&\Leftrightarrow\; \mathfrak{a} \in \mathsf{W}_e^{A^{(k+1)},m,n}\\ &\Leftrightarrow\; (\exists w)\mathsf{T}^{m,n+1}(e,\mathfrak{a},\chi_{A^{(k+1)}},w)\\ &\Leftrightarrow\; (\exists w)\mathsf{T}^{m+n+2}(e,\overline{\mathfrak{a}}((w)_0),\overline{\chi}_{A^{(k+1)}}((w)_0),(w)_1,(w)_2)\\ &\Leftrightarrow\; (\exists s)(\exists w)[\mathit{Seq}(s) \wedge \mathit{lh}(s)=(w)_0\\ &\qquad\qquad \wedge (\forall i<(w)_0)(\chi_{A^{(k+1)}}(i)=(s)_i) \wedge \mathsf{T}^{m+n+2}(e,\overline{\mathfrak{a}}((w)_0),s,(w)_1,(w)_2)]. \end{aligned} \tag{i}$$

The predicates Seq, T, $=$ etc. are all in Δ^0_0. So we only have to check the complexity of $\chi_{A^{(k+1)}}(i)=y$. Because of

$$\chi_{A^{(k+1)}}(i)=y \;\Leftrightarrow\; (y=0 \wedge x \in A^{(k+1)}) \vee (y=1 \wedge x \notin A^{(k+1)})$$

and the fact that $A^{(k+1)} \in \Sigma^0_{k+1}[A]$ we get

$$\{(i,y) \mid \chi_{A^{(k+1)}}(i)=y\} \in \Delta^0_{k+2}[A]. \tag{ii}$$

But (i) together with (ii) show $R \in \Sigma^0_{k+2}[A]$. $\square$

As a consequence of Theorem 3.2.10 we get the following generalization of POST's theorem.

3.2.11 Theorem *A relation R is in $\Delta^0_{k+1}[A]$ iff R is decidable in $A^{(k)}$.*

Proof: We have $R \in \Delta^0_{k+1}[A]$ iff R and $\neg R$ are semi–decidable in $A^{(k)}$ by Theorem 3.2.10. By POST's theorem this holds if and only if R is decidable in $A^{(k)}$. □

The next theorem will help us in confirming also the negative closure properties listed in Figure 3.2.1.

3.2.12 Lemma *We have*

$$A^{(k+1)} \notin \Delta^0_{k+1}[A]$$

for all $k \in \mathbb{N}$.

Proof: We have shown in Theorem 2.3.1 that $j(M) \not\leq_T M$ for any set M. By Theorem 3.2.11, however, this means $A^{(k+1)} \notin \Delta^0_{k+1}[A]$. □

3.2.13 Theorem (Arithmetical Hierarchy Theorem) *We have*

1) $\Delta^0_{k+1}[A] \subsetneqq \Sigma^0_{k+1}[A]$

2) $\Delta^0_{k+1}[A] \subsetneqq \Pi^0_{k+1}[A]$

3) $\Sigma^0_{k+1}[A] \cup \Pi^0_{k+1}[A] \subsetneqq \Delta^0_{k+2}[A]$.

Proof: All inclusions are obvious by definition. It remains to show that these inclusions are proper. According to Lemma 3.2.12 we have

$$A^{(k+1)} \notin \Delta^0_{k+1}[A] \text{ but } A^{(k+1)} \in \Sigma^0_{k+1}[A]$$

by Theorem 3.2.10. This proves 1) and 2) is an immediate consequence of 1).
To prove 3) regard the "effective union" of $A^{(k+1)}$ and $\neg A^{(k+1)}$ which is given by

$$B := \{2x \mid x \in A^{(k+1)}\} \cup \{2x+1 \mid x \notin A^{(k+1)}\}.$$

Then $B \in \Delta^0_{k+2}[A]$ and $A^{(k+1)} \leq_m B$ via $\lambda x\,.\,2x$ as well as $\neg A^{(k+1)} \leq_m B$ via $\lambda x\,.\,2x+1$. Hence neither $B \in \Sigma^0_{k+1}[A]$ nor $B \in \Pi^0_{k+1}[A]$ because any of both assumptions would lead to $A^{(k+1)} \in \Delta^0_{k+1}[A]$ which contradicts Lemma 3.2.12. □

It follows from the Arithmetical Hierarchy Theorem that $\Sigma^0_k[A]$ cannot be closed under negation and $\forall^0$–quantification. Dually $\Pi^0_k[A]$ cannot be closed under negation and $\exists^0$–quantification. Since any first order quantifier

$$(\mathsf{Q}x)[\ldots x \ldots]$$

can be replaced by a second order quantifier

$$(\mathsf{Q}\alpha)[\ldots \alpha(0) \ldots]$$

we see that $\Sigma^0_k[A]$ cannot be closed under $\forall^1$–quantifiers and dually that $\Pi^0_k[A]$ cannot be closed under $\exists^1$–quantifiers. So the only open items in Figure 3.2.1 are closure of $\Sigma^0_k[A]$ and $\exists^1$–quantifiers and $\Pi^0_k[A]$ and $\forall^1$–quantifier for $k > 1$. We have to postpone that until the next chapter.
Up to now we get a picture of the Arithmetical Hierarchy as shown in Figure 3.2.2.
Let us recall the notion of an universal relation.

3.2.14 Definition Let $\mathfrak{R}$ be a collection of (m,n)–ary relations. An $(m+1,n)$–ary relation U is *universal for* $\mathfrak{R}$ if for any $R \in \mathfrak{R}$ there is an $e \in \mathbb{N}$ such that

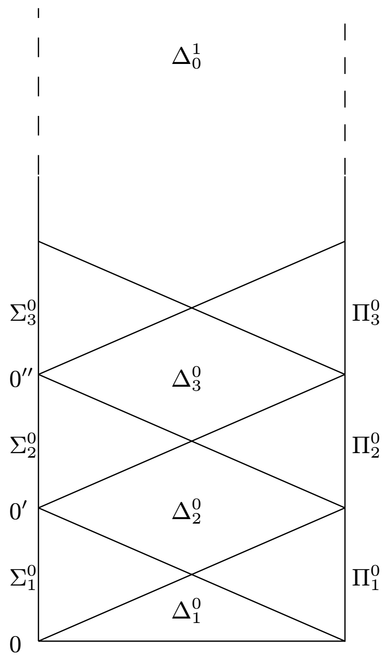

Figure 3.2.2: The Arithmetical Hierarchy

$$R(\mathfrak{a}) \;\Leftrightarrow\; U(\mathfrak{a}, e).$$

A collection $\mathfrak{K}$ of relations is a *universal class* if for any (m, n) there is an $(m+1, n)$–ary relation $U^{m,n} \in \mathfrak{K}$ which is universal for the (m, n)–ary relations in $\mathfrak{K}$.
$\mathfrak{K}$ is an *acceptable universal class* if $\mathfrak{K}$ is a universal class and there are $k + 1$–ary computable functions $\mathsf{S}_k^{m,n}$ such that

$$U^{m+k,n}(\mathfrak{a}, y_1, \ldots, y_k, e) \;\Leftrightarrow\; U^{m,n}(\mathfrak{a}, \mathsf{S}_k^{m,n}(e, y_1, \ldots, y_k)).$$

If $\mathfrak{K}$ is a universal class and

$$R(\mathfrak{a}) \;\Leftrightarrow\; U^{m,n}(\mathfrak{a}, e)$$

we call e an *$\mathfrak{K}$–index* for R.

We have already seen that the class of $\Sigma_1^0[A]$–relations, i.e. the class of relations which are semi–decidable in A, is an acceptable universal class. This can be lifted to all levels of the Arithmetical Hierarchy.

3.2.15 Theorem *The classes of $\Sigma_k^0[A]$ and $\Pi_k^0[A]$ are acceptable universal.*

Proof: By Theorem 3.2.10 we get for an (m, n)–ary $\Sigma_{k+1}^0[A]$–relation R

$$R(\mathfrak{a}) \;\Leftrightarrow\; \mathfrak{a} \in \mathsf{W}_e^{A^{(k)},m,n}.$$

Putting

$$\mathsf{U}^{\Sigma_{k+1}^0[A],m,n} := \left\{(\mathfrak{a}, e) \,\middle|\, \mathfrak{a} \in \mathsf{W}_e^{A^{(k)},m,n}\right\}$$

defines a universal relation for the (m, n)–ary relations in $\Sigma_{k+1}^0[A]$. The acceptability, however, is an immediate consequence of the relativized $\mathsf{S}_k^{m,n}$–Theorem.

By dualization we get universal relations $\mathsf{U}^{\Pi^0_{k+1}[A],m,n}$ for the (m,n)–ary $\Pi^0_{k+1}[A]$–relations.
$\square$

3.3 The Limits of the Arithmetical Hierarchy

The collection of all arithmetically definable sets forms a countable set. This shows that we are far from having characterized all subsets of $\mathbb{N}$. We will indicate that we are even still far from having characterized all definable subsets of $\mathbb{N}$.
Put

$$A^{(\omega)} := \{x \mid (x)_0 \in A^{((x)_1)}\}.$$

We may regard $A^{(\omega)}$ as "effective" union of all $A^{(n)}$. Effective because for any $x \in A^{(\omega)}$ we can by computing $(x)_1$ effectively determine to which member of $\bigcup A^{(n)}$ the element $(x)_0$ belongs. Clearly any effective union has to be pairwise disjunct.
Because of the effectiveness of $A^{(\omega)}$ we get

$$x \in A^{(n)} \;\;\Leftrightarrow\;\; \langle x, n\rangle \in A^{(\omega)}$$

which shows

$$A^{(n)} \leq_m A^{(\omega)} \tag{3.5}$$

for any n.

3.3.1 Theorem *The set $A^{(\omega)}$ is not arithmetical in A.*

Proof: Towards an indirect proof assume that $A^{(\omega)} \in \Delta^0_k[A]$ for some k. Hence $A^{(\omega)} \leq_T A^{(k)} \leq_T A^{(k+1)} \leq_T A^{(\omega)}$ which implies $A^{(k)} \equiv_T A^{(k+1)}$. But then $A^{(k+1)} \in \Delta^0_{k+1}[A]$ by Theorem 3.2.11 which contradicts Lemma 3.2.12. $\square$

Building $A^{(\omega)}$ means to iterate the jump operator arbitrarily finitely often. But when we have $A^{(\omega)}$ we can go on building $j(A^{(\omega)}), j(j(A^{(\omega)}))\ldots$ Such an infinite iteration of jumps, however, needs a theory of ordinals, which we postpone until Chapter 5. First we want to extend the hierarchy by allowing second order formulas in the defining formulas of relations.

4. The Analytical Hierarchy

4.1 Second order arithmetic

In order to extend the arithmetical hierarchy we extend the hierarchy of arithmetical formulas. We are going to allow quantifiers on functions which are supposed to range over ${}^{\mathbb{N}}\mathbb{N}$. We briefly denote by

$$\mathbb{N} \models \varphi$$

that the sentence φ is valid in the standard interpretation. Let us start with a classification of the second order arithmetical formulas according to their second order quantifier–complexity.

4.1.1 Definition A formula φ is a Π^1_1–formula if $\varphi \equiv (\forall\alpha)\psi(\alpha)$ and $\psi(\alpha)$ is Σ^0_1.
Dually a formula φ is Σ^1_1 iff $\neg\varphi$ is Π^1_1.
A formula φ is Π^1_{k+1} iff $\varphi \equiv (\forall\alpha)\psi(\alpha)$ and $\psi(\alpha)$ is Σ^1_k.
Dually φ is Σ^1_{k+1} iff $\neg\varphi$ is Π^1_{k+1}.

Again we get

$$\varphi \in \Pi^1_k \quad\Leftrightarrow\quad \varphi \equiv (\forall\alpha_1)(\exists\alpha_2)\ldots(\mathsf{Q}\alpha_k)(\breve{\mathsf{Q}}x)\psi(\vec{\alpha}, x)$$

and

$$\varphi \in \Sigma^1_k \quad\Leftrightarrow\quad \varphi \equiv (\exists\alpha_1)(\forall\alpha_2)\ldots(\mathsf{Q}\alpha_k)(\breve{\mathsf{Q}}x)\psi(\vec{\alpha}, x)$$

where $\psi(\vec{\alpha}, x)$ is quantifier free.
A formula is *analytic* if it is Σ^1_n or Π^1_n for some n. We introduce the abbreviation

$$(\alpha)_x := \lambda u\,.\,\alpha(\langle x, u\rangle)\,. \tag{4.1}$$

4.1.2 Lemma *For any formula φ in the language of second order arithmetic and $\mathsf{Q} \in \{\forall, \exists\}$ we have*

$$(\mathsf{Q}x)\varphi(x) \quad\Leftrightarrow\quad (\mathsf{Q}\alpha)\varphi(\alpha(0)) \tag{4.2}$$

$$(\mathsf{Q}\alpha)(\mathsf{Q}\beta)\varphi(\alpha, \beta) \quad\Leftrightarrow\quad (\mathsf{Q}\gamma)\varphi((\gamma)_0, (\gamma)_1) \tag{4.3}$$

$$(\forall x)(\exists\alpha)\varphi(x, \alpha) \quad\Leftrightarrow\quad (\exists\beta)(\forall y)\varphi(y, (\beta)_y) \tag{4.4}$$

$$(\exists x)(\forall\alpha)\varphi(x, \alpha) \quad\Leftrightarrow\quad (\forall\beta)(\exists y)\varphi(y, (\beta)_y) \tag{4.5}$$

Proof: Claim (4.2) holds obviously and (4.3) becomes clear by putting

$$\gamma(x) := \begin{cases} \alpha((x)_1) & \text{if } (x)_0 = 0 \\ \beta((x)_1) & \text{if } (x)_0 = 1. \end{cases}$$

The direction from right to left in (4.4) holds for logical reasons. For the opposite direction assume

$$(\forall x)(\exists\alpha)\varphi(x, \alpha)$$

and choose α_x for every $x \in \mathbb{N}$. Defining

$$\beta(y) := \alpha_{(y)_0}((y)_1)$$

we get

$$(\forall y)\varphi(y,(\beta)_y).$$

Hence

$$(\exists\beta)(\forall y)\varphi(y,(\beta)_y).$$

Equation (4.5) follows from (4.4) by taking negations. $\square$

We observe that every formula in the language of Second Order Arithmetic is logically equivalent to some formula in prenex form. Using Lemma 4.1.2 it becomes equivalent to some formula of the form

$$(\forall\alpha_1)(\exists\alpha_2)\dots(\mathsf{Q}\alpha_n)(\breve{\mathsf{Q}}x)\varphi(\alpha_1,\dots,\alpha_n,x)$$

or

$$(\exists\alpha_1)(\forall\alpha_2)\dots(\breve{\mathsf{Q}}\alpha_n)(\mathsf{Q}x)\varphi(\alpha_1,\dots,\alpha_n,x)$$

where $\varphi(\alpha_1,\dots,\alpha_n,x)$ is quantifier free and $\breve{\mathsf{Q}}$ denotes the quantifier which is dual to Q. Hence every formula in the language of Second Order Arithmetic is equivalent to some analytical formula.

4.2 Analytical relations

4.2.1 Definition 1) A relation is $\Pi^1_n[A](\Sigma^1_n[A])$ iff it is definable with parameter χ_A by some Π^1_n– (Σ^1_n–)formula. Again we write Π^1_k and Σ^1_k instead of $\Pi^1_k[\emptyset]$ and $\Sigma^1_k[\emptyset]$.
2) A relation is $\Delta^1_k[A]$ iff it is both $\Sigma^1_k[A]$ and $\Pi^1_k[A]$.
3) A relation is analytical (in A) if it is in Δ^1_k ($\Delta^1_k[A]$) for some k.

A picture of the analytical hierarchy is given in Figure 4.2.1.

4.2.2 Remark By the considerations in the end of the previous section we get that a relation is definable in second order arithmetic iff it is analytical.

To obtain the closure properties of analytical relations we begin with the lowest level.

4.2.3 Lemma *The relations in* $\Pi^1_1[A]$ *are closed under*

- *the positive boolean operations* $\wedge$, $\vee$
- *all* $\mathbb{N}$*–quantifications*
- $\forall^1$*–quantifications*
- *substitution with total functionals having* $\Pi^1_1[A]$*–graphs*

Proof: Let

$$P_1(\mathfrak{a}) \;\Leftrightarrow\; (\forall\alpha)(\exists y)Q_1(\alpha,y,\mathfrak{a})$$

and

$$P_2(\mathfrak{a}) \;\Leftrightarrow\; (\forall\beta)(\exists z)Q_2(\beta,z,\mathfrak{a})$$

be Π^1_1–relations. Then

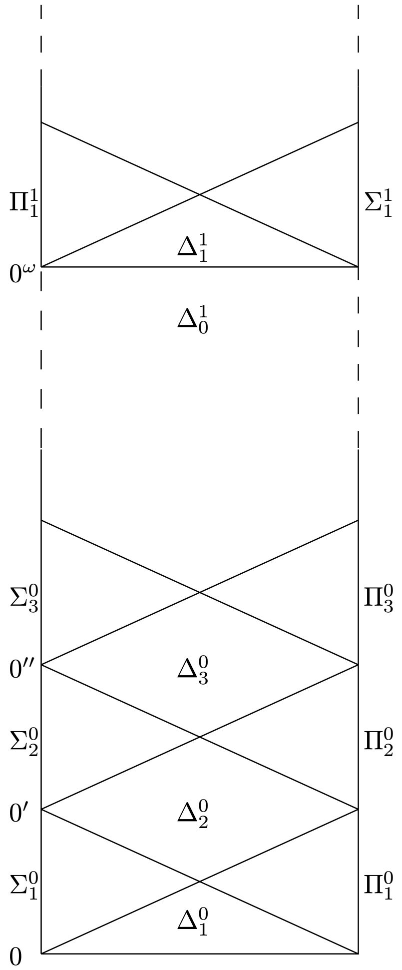

Figure 4.2.1: The Analytical Hierarchy

$$
\begin{aligned}
P_1(\mathfrak{a}) \substack{\wedge\\ \vee} P_2(\mathfrak{a}) \;&\Leftrightarrow\; (\forall\alpha)(\exists y)Q_1(\alpha, y, \mathfrak{a}) \substack{\wedge\\ \vee} (\forall\beta)(\exists z)Q_2(\beta, z, \mathfrak{a}) \\
&\Leftrightarrow\; (\forall\alpha)(\forall\beta)(\exists y)(\exists z)\,[Q_1(\alpha, y, \mathfrak{a}) \substack{\wedge\\ \vee} Q_2(\beta, z, \mathfrak{a})] \\
&\Leftrightarrow\; (\forall\gamma)(\exists u)\,[Q_1((\gamma)_0, (u)_0, \mathfrak{a}) \substack{\wedge\\ \vee} Q_2((\gamma)_1, (u)_1, \mathfrak{a})]\,.
\end{aligned}
$$

This gives the closure under positive boolean operations. Closure under $\forall^1$–quantification follows from the quantifier contractions (4.3); closure under $\forall^0$–quantification is obtained by converting it into a $\forall^1$–quantifier according to (4.2) and then using quantifier contraction; closure under $\exists^0$–quantification follows from the choice–principle (4.5).
Let's turn to closure under substitution. For a total functional F we get

$$
\begin{aligned}
F(\mathfrak{a}) \neq y \;&\Leftrightarrow\; (\exists x)\,[F(\mathfrak{a}) \simeq x \wedge x \neq y] \\
&\Leftrightarrow\; (\forall x)\,[F(\mathfrak{a}) \simeq x \Rightarrow x \neq y]
\end{aligned}
$$

which shows that $G_F \in \Delta^1_1[A]$ for $\Pi^1_1[A]$– and for $\Sigma^1_1[A]$–functionals F (here we use that $\Sigma^1_1[A]$ has the dual closure properties of $\Pi^1_1[A]$). For a $\Pi^1_1[A]$–relation P we obtain

$$
\begin{aligned}
P(\mathfrak{a}, F(\mathfrak{a})) \;&\Leftrightarrow\; (\exists y)\,[F(\mathfrak{a}) \simeq y \wedge P(\mathfrak{a}, y)] \\
&\Leftrightarrow\; (\forall y)\,[F(\mathfrak{a}) \simeq y \Rightarrow P(\mathfrak{a}, y)]\,.
\end{aligned} \tag{4.6}
$$

This shows that $\Pi^1_1[A]$ (as well as $\Sigma^1_1[A]$) is closed under substitution with total $\Pi^1_1[A]$–functionals. □

By dualization we obtain from Lemma 4.2.3.

4.2.4 Lemma *The $\Sigma^1_1[A]$–relations are closed under*

- *the positive boolean operations $\wedge$, $\vee$*
- *all $\mathbb{N}$–quantifications*
- *$\exists^1$–quantifications*
- *substitution with total $\Pi^1_1[A]$–functionals*

The $\Delta^1_1[A]$–relations are closed under

- *all boolean operations*
- *all $\mathbb{N}$–quantifications*
- *substitution with total $\Pi^1_1[A]$–functionals.*

Using induction on k in the same way as we did it in the case of the arithmetical hierarchy gives the positive closure properties of the levels of the analytical hierarchy as displayed in Figure 4.2.2. To answer the obvious question whether the $\Pi^1_k[A]$ and $\Sigma^1_k[A]$ form a proper hierarchy we check

Relation–class	$\neg$	$\vee$	$\wedge$	$\exists^<$	$\forall^<$	$\exists^0$	$\forall^0$	$\exists^1$	$\forall^1$	acceptable universal	Substitution with
primitive–recursive	yes	yes	yes	yes	yes	no	no	no	no	no	primitive–recursive functions
Δ^0_1	yes	yes	yes	yes	yes	no	no	no	no	no	computable functionals
Σ^0_1	no	yes	yes	yes	yes	yes	no	yes	no	yes	computable functionals
Π^0_1	no	yes	yes	yes	yes	no	yes	no	yes	yes	computable functionals
Δ^0_{k+1}	yes	yes	yes	yes	yes	no	no	no	no	no	total Σ^0_{k+1}-functionals
Σ^0_{k+1}	no	yes	yes	yes	yes	yes	no	no	no	yes	total Σ^0_{k+1}-functionals
Π^0_{k+1}	no	yes	yes	yes	yes	no	yes	no	no	yes	total Σ^0_{k+1}-functionals
Δ^1_0	yes	yes	yes	yes	yes	yes	yes	no	no	no	total arithmetical–functionals
Π^1_{k+1}	no	yes	yes	yes	yes	yes	yes	no	yes	yes	total Π^1_{k+1}–functionals
Σ^1_{k+1}	no	yes	yes	yes	yes	yes	yes	yes	no	yes	total Π^1_{k+1}–functionals
Δ^1_{k+1}	no	yes	yes	yes	yes	yes	yes	no	no	no	total Π^1_{k+1}–functionals
analytical	yes	yes	yes	yes	yes	yes	yes	yes	yes	no	total analytical–functionals

Figure 4.2.2: Closure Properties of Relation–Classes

the universality of these classes.

4.2.5 Theorem *The classes $\Pi^1_{k+1}[A]$ and $\Sigma^1_{k+1}[A]$ are acceptable universal.*

Proof: For $P \in \Pi^1_1[A]$ we have

$$P(\mathfrak{a}) \;\Leftrightarrow\; (\forall\alpha)(\exists x)R(\mathfrak{a},\alpha,x)$$

such that $\{(\mathfrak{a},\alpha)|\ (\exists x)R(\mathfrak{a},\alpha,x)\}$ is $\Sigma^0_1[A]$. By the universality of $\Sigma^0_1[A]$ we therefore get an index e such that

$$(\exists x)R(\mathfrak{a},\alpha,x) \;\Leftrightarrow\; (\mathfrak{a},\alpha) \in \mathsf{W}^{A,m,n+1}_e.$$

We define

$$\mathsf{U}^{\Pi^1_1[A],m,n} := \{(e,\mathfrak{a})|\ (\forall\alpha)\left[(\mathfrak{a},\alpha) \in \mathsf{W}^{A,m,n+1}_e\right]\}. \tag{4.7}$$

Then $\mathsf{U}^{\Pi^1_1[A],m,n}$ is by construction universal for $\Pi^1_1[A]$. We usually write $\mathfrak{a} \in \mathsf{U}^{\Pi^1_1[A],m,n}_e$ instead of $(e,\mathfrak{a}) \in \mathsf{U}^{\Pi^1_1[A],m,n}$. To see that it is also acceptable we do the following computation

$$\begin{aligned}(\mathfrak{a},y_1,\dots,y_k) \in \mathsf{U}^{\Pi^1_1[A],m+k,n}_e &\Leftrightarrow (\forall\alpha)\left[(\mathfrak{a},\alpha,\vec{y}) \in \mathsf{W}^{A,m+k,n+1}_e\right]\\ &\Leftrightarrow (\forall\alpha)\left[(\mathfrak{a},\alpha) \in \mathsf{W}^{A,m,n}_{\mathsf{S}^{m,n+1}_k(e,\vec{y})}\right]\\ &\Leftrightarrow \mathfrak{a} \in \mathsf{U}^{\Pi^1_1[A],m,n}_{\mathsf{S}^{m,n+1}_k(e,\vec{y})}.\end{aligned}$$

Since $R \in \Sigma^1_1[A] \;\Leftrightarrow\; \neg R \in \Pi^1_1[A]$ we may put

$$\mathsf{U}^{\Sigma^1_1[A],m,n} := \{(e,\mathfrak{a})|\ (\exists\alpha)\left[(\mathfrak{a},\alpha) \notin \mathsf{W}^{A,m,n+1}_e\right]\} \tag{4.8}$$

and obtain by duality that $\mathsf{U}^{\Sigma^1_1[A]}$ is acceptable universal for $\Sigma^1_1[A]$. $\square$

Using induction on k we can lift Theorem 4.2.5 to all levels of the analytical hierarchy. We just put

$$\mathsf{U}^{\Pi^1_{k+1}[A],m,n} := \{(e,\mathfrak{a})|\ (\forall\alpha)\left[(\mathfrak{a},\alpha) \in \mathsf{U}^{\Sigma^1_k[A],m,n+1}\right]\} \tag{4.9}$$

and

$$\mathsf{U}^{\Sigma^1_{k+1}[A],m,n} := \{(e,\mathfrak{a})|\ (\exists\alpha)\left[(\mathfrak{a},\alpha) \in \mathsf{U}^{\Pi^1_k[A],m,n+1}\right]\}. \tag{4.10}$$

Turning this into a theorem we get:

4.2.6 Theorem *The classes $\Pi^1_{k+1}[A]$ as well as the classes $\Sigma^1_{k+1}[A]$ are acceptable universal for all k. The universal predicates $\mathsf{U}^{\Pi^1_{k+1}[A]}$ and $\mathsf{U}^{\Sigma^1_{k+1}[A]}$ are defined in (4.6) through (4.10). Putting*

$$\begin{aligned}\mathsf{U}^{\Delta^1_{k+1}[A],m,n} := \{(e,\mathfrak{a})|\ &\mathit{Seq}(e) \wedge \mathit{lh}(e) = 2 \wedge ((e)_0,\mathfrak{a}) \in \mathsf{U}^{\Sigma^1_{k+1}[A],m,n}\\ &\wedge (\forall\mathfrak{b})\left[((e)_0,\mathfrak{b}) \in \mathsf{U}^{\Sigma^1_{k+1}[A],m,n} \Leftrightarrow ((e)_1,\mathfrak{b}) \in \mathsf{U}^{\Pi^1_{k+1}[A],m,n}\right]\}\end{aligned}$$

we get also indices for $\Delta^1_{k+1}[A]$–relations. Note, however, that $\mathsf{U}^{\Delta^1_{k+1}[A],m,n}$ is not a $\Delta^1_{k+1}[A]$–relation.

To show that the analytical hierarchy does not collapse we need the following lemma.

4.2.7 Lemma *For all $k > 0$ there is a relation K^A_k such that $K^A_k \in \Pi^1_k[A] \setminus \Sigma^1_k[A]$.*

Proof: We use a diagonalization argument. Towards an indirect proof we assume that $\Pi^1_k[A] \subseteq \Sigma^1_k[A]$.
Define

$$K_k^A := \{x \mid x \notin \mathsf{U}_x^{\Sigma_k^1[A],1,0}\}$$

then $K_k^A \in \Pi_k^1[A] \subseteq \Sigma_k^1[A]$. Let e be a $\Sigma_k^1[A]$–index for K_k^A. Then

$$e \in K_k^A \;\Leftrightarrow\; e \in \mathsf{U}_e^{\Sigma_k^1[A],1,0} \;\Leftrightarrow\; e \notin K_k^A,$$

which is absurd. □

4.2.8 Theorem *For all k we have*

- $\Delta_{k+1}^1[A] \subsetneqq \Pi_{k+1}^1[A]$
- $\Delta_{k+1}^1[A] \subsetneqq \Sigma_{k+1}^1[A]$
- $\Pi_k^1[A] \cup \Sigma_k^1[A] \subsetneqq \Delta_{k+1}^1[A]$.

Proof: By Lemma 4.2.7 we have $K_{k+1}^A \in \Pi_{k+1}^1[A] \setminus \Delta_{k+1}^1[A]$ or $\neg K_{k+1}^A \in \Sigma_{k+1}^1[A] \setminus \Delta_{k+1}^1[A]$, respectively. For $k > 0$ we put $B := \{2e \mid e \in K_k^A\} \cup \{2e+1 \mid e \notin K_k^A\}$ and get $B \in \Delta_{k+1}^1[A]$, but since $K_k^A \leq_m B$ as well as $\neg K_k^A \leq_m B$ neither $B \in \Pi_k^1[A]$ nor $B \in \Sigma_k^1[A]$ is possible. It remains the case of $k = 0$. We have already seen that

$$A^{(\omega)} := \{x \mid (x)_0 \in A^{((x)_1)}\}$$

is not arithmetical. We give $\Delta_1^1[A]$–definition of $A^{(\omega)}$. First we describe the jump–hierarchy. We define

$$\begin{aligned}
\mathsf{JH}_A(\alpha) \;:\Leftrightarrow\; & (\forall n)(\forall x)[(\alpha(x) \neq 0 \;\Rightarrow\; \textsf{\textit{Seq}}(x) \wedge \textsf{\textit{lh}}(x) = 2) \\
& \wedge\, \alpha(\langle n, x\rangle) \leq 1 \\
& \wedge\, (\alpha(\langle 0, x\rangle) = 0 \;\Leftrightarrow\; x \in A) \\
& \wedge\, (\alpha(\langle n+1, x\rangle) = 0 \;\Leftrightarrow\; (\exists x)(\exists u)[\textsf{\textit{Seq}}(s) \wedge \textsf{\textit{lh}}(s) = (u)_0 \\
& \quad \wedge\, (\forall j < (u)_0)((s)_j = \alpha(\langle n, j\rangle) \wedge \mathsf{T}^{2,0}(x, x, s, (u)_1))])]
\end{aligned}$$

Then we prove

$$\mathsf{JH}(\alpha) \wedge \mathsf{JH}(\beta) \;\Rightarrow\; (\forall n)(\forall x)\,[\alpha(\langle n, x\rangle) = \beta(\langle n, x\rangle)]$$

by induction on n. Hence

$$\mathsf{JH}(\alpha) \wedge \mathsf{JH}(\beta) \;\Rightarrow\; \alpha = \beta$$

and we see that

$$\begin{aligned}
\{x \mid \alpha(\langle n+1, x\rangle) = 0\} &= \{x \mid (\exists u)\mathsf{T}^{1,1}(x, x, \{x \mid \alpha(\langle n, x\rangle) = 0\}, u)\} \\
&= j(\{x \mid \alpha(\langle n, x\rangle) = 0\}).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
n \in A^{(\omega)} \;&\Leftrightarrow\; (\exists\alpha)\,[\mathsf{JH}_A(\alpha) \wedge \alpha(n) = 0] \\
&\Leftrightarrow\; (\forall\alpha)\,[\mathsf{JH}_A(\alpha) \Rightarrow \alpha(n) = 0]\,.
\end{aligned}$$

Since $\mathsf{JH}_A(\alpha)$ is arithmetical in A we see that $A^{(\omega)}$ is $\Delta_1^1[A]$. By Theorem 3.3.1, however, $A^{(\omega)} \notin \Pi_0^1[A] \cup \Sigma_0^1[A]$. □

5. The Theory of Countable Ordinals

In Chapter 3 we succeeded in characterizing the Arithmetical Hierarchy by iteration of the jump operator. We indicated that iterating the jump infinitely often leads outside the Arithmetical Hierarchy. We even proved that already its ω–fold iteration $\emptyset^{(\omega)}$ is outside the arithmetically definable sets. We are going to study the effects of transfinite iterations of various operators. To prepare that we need an introduction to the theory of transfinite numbers, i.e. ordinals. It has become common to regard ordinals set–theoretical, i.e. as sets which are well–ordered by the membership relation $\in$. In presence of the axiom of foundation any hereditarily transitive set is already an ordinal. This is probably the easiest way to introduce ordinals. However, since we can restrict ourselves to countable ordinals and don't want to require too much pre–knowledge in Set Theory, we are going to develop the theory of countable ordinals in a more old fashioned way. This should be profitable even for somebody who already knows set–theoretical ordinals.

5.1 Ordinals as order–types

5.1.1 Definition Let N be some set.
1) Let $R \subseteq N \times N$ be a binary predicate. For binary predicates we sometimes prefer the infix notation, i.e. we write $x\ R\ y$ instead of $(x, y) \in R$ or $R(x, y)$. We define

$$\mathit{field}(R) = \{x|\ (\exists y)\,[x\ R\ y \lor y\ R\ x]\} \tag{5.1}$$

and call $\mathit{field}(R)$ the *field* of the predicate R. We put

$$x\ R_{\neq}\ y \quad :\Leftrightarrow \quad x\ R\ y \land x \neq y \tag{5.2}$$

and call $R_{\neq}$ the *strict predicate* associated with R. In case that R is *irreflexive*, i.e. if

- $(\forall x \in \mathit{field}(R))\,[\neg(x\ R\ x)]$,

R and $R_{\neq}$ are the same predicates.
2) A predicate $\preceq\ \subseteq N \times N$ is a *partial ordering* if $\preceq$ is reflexive, anti–symmetrical and transitive, i.e. if

- $(\forall x \in \mathit{field}(\preceq))\,[x \preceq x]$
- $(\forall x \in \mathit{field}(\preceq))(\forall y \in \mathit{field}(\preceq))\,[x \preceq y \land y \preceq x \Rightarrow x = y]$

and

- $(\forall x \in \mathit{field}(\preceq))(\forall y \in \mathit{field}(\preceq))(\forall z \in \mathit{field}(\preceq))\,[x \preceq y \land y \preceq z \Rightarrow x \preceq z]$.

If $\mathit{field}(\preceq) \subseteq \mathbb{N}$ we talk about a *countable partial ordering*.
3) A predicate $\preceq\ \subseteq N \times N$ is an *ordering* if $\preceq$ is a partial ordering which is linear, i.e. if

- $(\forall x \in \mathit{field}(\preceq))(\forall y \in \mathit{field}(\preceq))\,[x \preceq y \lor y \preceq x]$.

For a partial–ordering $\preceq$ we denote its associated strict predicate by $\prec$. Hence

$$x \prec y \quad \Leftrightarrow \quad x \preceq y \land x \neq y.$$

We call $\prec$ a *strict partial ordering*. Vice versa we can associate a partial ordering

$$x \preceq y \quad :\Leftrightarrow \quad x \prec y \lor (x \in \mathit{field}(\prec) \land x = y) \tag{5.3}$$

to every irreflexive and transitive predicate $\prec \subseteq N \times N$.
4) A predicate $R \subseteq N \times N$ is *well–founded* if every nonempty subset of *field*(R) possesses a R–least element, i.e. if

$$(\forall M)\,[M \subseteq \mathit{field}(R) \wedge M \neq \emptyset \Rightarrow (\exists x \in M)(\forall y)(y\ R\ x \Rightarrow y = x \vee y \notin M)]\,.$$

5) A *well–ordering* is an ordering which is well–founded.
6) Two orderings $\preceq_1$ and $\preceq_2$ are *equivalent* if there is a strictly order–preserving map from *field*$(\preceq_1)$ onto *field*$(\preceq_2)$, i.e. if we have

$$f\colon \mathit{field}(\preceq_1) \xrightarrow{\mathit{onto}} \mathit{field}(\preceq_2)$$

such that

$$(\forall x \in \mathit{field}(\preceq_1))(\forall y \in \mathit{field}(\preceq_1))\,[x \prec_1 y \Rightarrow f(x) \prec_2 f(y)]$$

where $\prec_1$ and $\prec_2$ are the corresponding strict orderings as defined in (5.3).
By $\preceq_1 \equiv \preceq_2$ we denote the equivalence of $\preceq_1$ and $\preceq_2$.

For well–founded predicates R we have the *principle of transfinite induction* along R which says

$$(\forall x)\,[(\forall y)(y\ R_{\neq}\ x \Rightarrow \varphi(y)) \Rightarrow \varphi(x)] \Rightarrow (\forall x)\varphi(x). \tag{5.4}$$

To realize (5.4) observe that its premise entails $x \notin \mathit{field}(R) \Rightarrow \varphi(x)$. Thus assume

$$\{x \mid \neg\varphi(x)\} \cap \mathit{field}(R) \neq \emptyset.$$

Since R is well–founded we find a $z \in \{x \mid \neg\varphi(x)\} \cap \mathit{field}(R)$ such that $(\forall y)\,[y\ R_{\neq}\ z \;\Rightarrow\; \varphi(y)]$. This, however, implies $\varphi(z)$ by the premise of (5.4). A contradiction. Another important principle is that of *transfinite recursion* along a well–ordering $\preceq$. Let G be a total functional. Then there is a functional F satisfying the equation

$$F(\mathfrak{a}, x) = G(\mathfrak{a}, \lambda z \prec x\,.\,F(\mathfrak{a}, z)) \tag{5.5}$$

where

$$(\lambda z \prec x\,.\,F(\mathfrak{a}, z))(n) := \begin{cases} F(\mathfrak{a}, n) & \text{if } n \prec x \\ 0 & \text{otherwise.} \end{cases}$$

The principle of transfinite recursion is provable within a framework of Set Theory. We will, however, regard (5.5) as an axiom. But for computable G and decidable $\preceq$ we can prove that there is a computable functional F satisfying (5.5). We use the Recursion Theorem to obtain an index e such that

$$\{e\}^{m+1,n}(\mathfrak{a}, x) \simeq G(\mathfrak{a}, \lambda z \prec x\,.\,\{e\}^{m+1,n}(\mathfrak{a}, z)).$$

Now we show by transfinite induction along $\preceq$ that

$$(\forall \mathfrak{a})(\exists y)[\{e\}^{m+1,n}(\mathfrak{a}, x) \simeq y].$$

Putting $F := \{e\}^{m+1,n}$ we have a computable solution of (5.5).
The following lemma is an immediate consequence of the definition of the equivalence of orderings.

5.1.2 Lemma *The equivalence of orderings is a reflexive, transitive and symmetric relation.*

5.1.3 Definition A *countable ordinal* is the equivalence class of a countable well–ordering.

We are going to denote ordinals by lower case Greek letters in the end of the alphabet, e.g. $\sigma, \tau, \xi, \mu, \ldots$ The *order–type* of a well–ordering $\prec$ is the ordinal which is represented by $\prec$. The

order–type of $\prec$ is often denoted by $\mathit{otyp}(\prec)$. The class of countable ordinals is denoted by On. We want to show that there is a strict well-ordering $<$ on On. To define $<$ we introduce some notations.
A *segment* of an ordering $\preceq$ is a set $M \subseteq \mathit{field}(\preceq)$ such that

$$(\forall x \in M)(\forall z \in \mathit{field}(\preceq))\,[z \preceq x \Rightarrow z \in M]\,.$$

We talk about a *proper segment* if M is a segment but $M \neq \mathit{field}(\preceq)$.
For an element $z \in \mathit{field}(\preceq)$ we introduce the segment induced by z

$$\preceq\restriction z = \bigl\{(x,y) \bigm| y \prec z \wedge x \preceq y\bigr\}.$$

The segment $\preceq\restriction z$ of $\preceq$ is obviously always proper. Moreover we have

$$\mathit{field}(\preceq\restriction z) = \bigl\{x \in \mathit{field}(\preceq) \bigm| x \prec z\bigr\}.$$

5.1.4 Definition Let σ, τ be ordinals. We say that σ is less than τ, written as $\sigma < \tau$, if there are well–orderings $\preceq_\sigma$ and $\preceq_\tau$ representing σ and τ, respectively, and a $z \in \mathit{field}(\preceq_\tau)$ such that $\preceq_\sigma \equiv \preceq_\tau\restriction z$, i.e.

$$\sigma < \tau \quad :\Leftrightarrow \quad (\exists \preceq_\sigma \in \sigma)(\exists \preceq_\tau \in \tau)(\exists z \in \mathit{field}(\preceq_\tau))\,[\preceq_\sigma \equiv \preceq_\tau\restriction z]\,. \tag{5.6}$$

5.1.5 Theorem *The relation $\sigma < \tau$ defined in (5.6) is an irreflexive well–ordering on the ordinals.*

Proof: The proof is easy but a bit lengthy. Therefore we concentrate on the more tricky parts. If $\preceq_1 \equiv \preceq_2$, $\preceq_3 \equiv \preceq_4$ and $\preceq_2 \equiv \preceq_3\restriction z$ for some $z \in \mathit{field}(\preceq_3)$, we get $\preceq_1 \equiv \preceq_4\restriction f(z)$ if f is an order–isomorphism between $\preceq_3$ and $\preceq_4$. Hence (5.6) is well–defined. Irreflexivity and transitivity are equally easy to check.
The most difficult part is to check linearity. Let $\preceq_1$ and $\preceq_2$ be two well–orderings such that $\preceq_1 \not\equiv \preceq_2$. We have to show that there is either a $z \in \mathit{field}(\preceq_1)$ such that $\preceq_1\restriction z \equiv \preceq_2$ or a $z \in \mathit{field}(z_2)$ such that $\preceq_1 \equiv \preceq_2\restriction z$. Putting

$$f(x) := \min\bigl\{z \in \mathit{field}(\preceq_2) \bigm| (\forall y \prec_1 x)\,[f(y) \prec_2 z]\bigr\}$$

we get a partial function

$$f\colon \mathit{field}(\preceq_1) \longrightarrow_{\mathrm{p}} \mathit{field}(\preceq_2)$$

which is order–preserving by definition. By construction $\mathsf{dom}(f)$ and $\mathsf{rng}(f)$ are segments of $\preceq_1$ and $\preceq_2$, respectively. More precisely $\preceq_1\restriction\mathsf{dom}(f)$ and $\preceq_2\restriction\mathsf{rng}(f)$ are segments. But we will often use the more sloppy way of talking as above. Since $\preceq_1 \not\equiv \preceq_2$ either $\mathsf{dom}(f)$ or $\mathsf{rng}(f)$ has to be proper. In the first case we get for $z := \min_{\prec_1}\bigl\{x \in \mathit{field}(\preceq_1) \bigm| x \notin \mathsf{dom}(f)\bigr\}$ that $\preceq_1\restriction z \equiv \preceq_2$ and in the second $\preceq_1 \equiv \preceq_2\restriction z$ for $z := \min_{\prec_2}\bigl\{x \in \mathit{field}(\preceq_2) \bigm| x \notin \mathsf{rng}(f)\bigr\}$.
To see that $<$ is well–founded on On take some $M \subseteq \mathit{On}$ such that $M \neq \emptyset$. Assume that M does not possess a $<$–least element. Pick any $\sigma \in M$ and let $\preceq$ be a well–ordering representing σ. Then there is a $\tau \in M$ such that $\tau < \sigma$. Therefore we find a $z_0 \in \mathit{field}(\preceq)$ such that $\preceq\restriction z_0$ represents τ. Assuming we already defined the sequence

$$z_0 \succ z_1 \succ \ldots \succ z_n$$

such that $\tau_i := \mathit{otyp}(\preceq\restriction z_i) \in M$ for $i = 0, \ldots, n$ we find some $\tau_{n+1} < \tau_n$ in M and therefore also some $z_{n+1} \prec z_n$ such that $\tau_{n+1} = \mathit{otyp}(\preceq\restriction z_{n+1})$. This gives an infinite strictly descending sequence $z_0 \succ \ldots \succ z_n \succ \ldots$ in $\mathit{field}(\preceq)$ which is impossible because $\bigl\{z_i \bigm| i \in \mathbb{N}\bigr\} \subseteq \mathit{field}(\preceq)$ would not have a $\preceq$–least element. $\square$

We just used the fact that in a well–ordering there are no infinite strictly descending sequences. This is in fact equivalent to being a well–ordering.

5.1.6 Theorem *A binary predicate R is well–founded iff there are no infinite $R_{\neq}$–descending sequences.*

Proof: We have just seen that a well–founded predicate does not allow infinite strictly descending sequences. For the opposite direction assume that every R–descending sequence is finite. Towards an indirect proof let M be a nonempty subset of $\mathit{field}(R)$ without R–least element. Then we may choose some $z_0 \in M$. Suppose that we already have chosen $z_0, \ldots, z_n \in M$ such that

$$z_0 \succ z_1 \succ \ldots \succ z_n.$$

Since M has no R–least element there is an $z_{n+1}\ R_{\neq}\ z_n$ and we may thus construct an infinite $R_{\neq}$–descending sequence. □

If $M \subseteq \mathit{On}$ is bounded, i.e. if there is some $\alpha \in \mathit{On}$ such that $(\forall \xi \in M)\,[\xi \leq \alpha]$ then we define

$$\sup M := \min\bigl\{\eta \in \mathit{On} \bigm| \ (\forall \xi \in M)\,[\xi \leq \eta]\bigr\}. \tag{5.7}$$

5.1.7 Theorem *The class On of countable ordinals is unbounded in the countable ordinals, i.e. for every countable ordinal σ there is a countable ordinal τ such that $\sigma < \tau$.*

Proof: Let $\sigma \in \mathit{On}$ and $\preceq$ a well–ordering representing σ. Put

$$\begin{aligned} x \prec' y \quad :\Leftrightarrow \quad & \mathit{Seq}(x) \wedge \mathit{Seq}(y) \wedge \mathit{lh}(x) = \mathit{lh}(y) = 2 \\ & \wedge [(x)_0 = 0 \wedge (y)_0 = 0 \wedge (x)_1 \prec (y)_1) \\ & \ \ \vee ((x)_0 = 0 \wedge (x)_1 \in \mathit{field}(\prec) \wedge (y)_0 = 1 \wedge (y)_1 = 1)], \end{aligned}$$

i.e. we add a single point $\langle 1, 1\rangle$ on top of the well–ordering $\prec$. Then we get

$$\preceq \equiv \preceq' \restriction \langle 1, 1\rangle.$$

Hence

$$\sigma = \mathit{otyp}(\preceq) < \mathit{otyp}(\preceq') =: \tau.$$ □

Using Theorem 5.1.7 we define the *successor*

$$\sigma + 1 := \min\bigl\{\xi \in \mathit{On} \bigm| \ \sigma < \xi\bigr\}. \tag{5.8}$$

We put

$$0 := \min \mathit{On} \tag{5.9}$$

and get

$$0 = \mathit{otyp}(\emptyset).$$

5.1.8 Definition An ordinal σ is a *successor–ordinal* if there is an ordinal τ such that $\sigma = \tau + 1$. An ordinal σ is a *limit–ordinal* if $\sigma \neq 0$ and σ is not a successor ordinal. We denote the class of limit ordinals by Lim.

We obtain

$$\lambda \in \mathsf{Lim} \ \Leftrightarrow \ \lambda \neq 0 \wedge (\forall \xi < \lambda)\,[\xi + 1 < \lambda] \tag{5.10}$$

because $\xi < \lambda$ implies $\xi + 1 \leq \lambda$ and $\xi + 1 = \lambda$ is excluded by the definition of Lim. An equivalent formulation of (5.10) is

$$\lambda \in \mathsf{Lim} \ \Leftrightarrow \ \lambda \neq 0 \wedge (\forall \xi < \lambda)(\exists \eta < \lambda)\,[\xi < \eta]\,. \tag{5.11}$$

Equation (5.11) follows immediately from (5.10) with $\xi+1$ as witness for η and the opposite direction follows because $\xi<\eta<\lambda$ implies $\xi+1\leq\eta<\lambda$ and $<$ is transitive on the countable ordinals.
We use (5.11) to prove

$$\omega := \mathit{otyp}(<) \in \mathsf{Lim} \tag{5.12}$$

where $<$ stands for the standard ordering of the natural numbers. It is obvious that $\omega \neq 0$ and for $\sigma<\omega$ we obtain an $n\in\mathbb{N}$ such that $\sigma=\mathit{otyp}(<\restriction n) < \mathit{otyp}(<\restriction n+1) < \omega$. Hence $\omega\in\mathsf{Lim}$ by (5.11)
Ordinals $<\omega$ are *finite*. Finite ordinals are represented by $<\restriction n$ for $n\in\mathbb{N}$, i.e. by orderings of the form $0<1<\ldots<n-1$. Therefore we often identify finite ordinals and natural numbers.
For a well–ordering $\preceq$ and $x\in\mathit{field}(\preceq)$ we define

$$\mathit{otyp}_{\preceq}(x) := \mathit{otyp}(\preceq\restriction x). \tag{5.13}$$

Then we obtain

$$\mathit{otyp}_{\preceq}(x) = \sup\bigl\{\mathit{otyp}_{\preceq}(y)+1 \bigm|\ y\prec x\bigr\}. \tag{5.14}$$

To prove (5.13) we observe that $\sigma := \sup\bigl\{\mathit{otyp}_{\preceq}(y)+1 \bigm|\ y\prec x\bigr\} \leq \mathit{otyp}_{\preceq}(x)$. If we assume $\sigma<\mathit{otyp}_{\preceq}(x)=\mathit{otyp}(\preceq\restriction x)$ we get a $y_0\prec x$ such that $\sigma=\mathit{otyp}_{\preceq}(y_0)$ and this leads to

$$\sigma=\mathit{otyp}_{\preceq}(y_0)<\mathit{otyp}_{\preceq}(y_0)+1\leq\sigma$$

contradicting that $<$ is irreflexive on *On*.
In a similar way we show

$$\mathit{otyp}(\preceq) = \sup\bigl\{\mathit{otyp}_{\preceq}(y)+1 \bigm|\ y\in\mathit{field}(\preceq)\bigr\}. \tag{5.15}$$

Putting $\sigma := \sup\bigl\{\mathit{otyp}_{\preceq}(y)+1 \bigm|\ y\in\mathit{field}(\preceq)\bigr\}$ we obviously have $\sigma\leq\mathit{otyp}(\preceq)$. The assumption $\sigma<\mathit{otyp}(\preceq)$ leads again to the contradiction that then there is a $y\in\mathit{field}\preceq$ such that

$$\sigma=\mathit{otyp}_{\preceq}(y)<\mathit{otyp}_{\preceq}(y)+1\leq\sigma. \qquad \square$$

Generalizing (5.14) and (5.15) we define

$$\mathit{otyp}_R(x) := \sup\bigl\{\mathit{otyp}_R(y)+1 \bigm|\ y\ R_{\neq}\ x\bigr\} \tag{5.16}$$

and

$$\mathit{otyp}(R) := \sup\bigl\{\mathit{otyp}_R(y)+1 \bigm|\ y\in\mathit{field}(R)\bigr\} \tag{5.17}$$

for arbitrary well-founded orderings $\preceq$ by transfinite recursion along R.
We close this section by examining the complexity of the notions of partial–ordering, ordering and well–ordering in the Analytical Hierarchy. We express a binary predicate R by the characteristic function of its contractions

$$\langle R\rangle := \bigl\{\langle x,y\rangle \bigm|\ (x,y)\in R\bigr\}.$$

We define

$$\begin{aligned}
\mathbb{CF}(\alpha) \quad &:\Leftrightarrow \quad (\forall x)\,[\alpha(x)\leq 1 \land (\alpha(x)=0 \Rightarrow \mathit{Seq}(x)\land \mathit{lh}(x)=2)]\\
\mathbb{PO}(\alpha) \quad &:\Leftrightarrow \quad \mathbb{CF}(\alpha)\\
&\qquad\quad \land (\forall x)(\forall y)\,[\alpha(\langle x,y\rangle)=0 \Rightarrow \alpha(\langle x,x\rangle)=0 \land \alpha(\langle y,y\rangle)=0]\\
&\qquad\quad \land (\forall x)(\forall y)\,[\alpha(\langle x,y\rangle)=0 \land \alpha(\langle y,x\rangle)=0 \Rightarrow x=y]\\
&\qquad\quad \land (\forall x)(\forall y)(\forall z)\,[\alpha(\langle x,y\rangle)=0 \land \alpha(\langle y,z\rangle)=0 \Rightarrow \alpha(\langle x,z\rangle)=0]\,,
\end{aligned}$$

$$\begin{aligned}\mathbb{LO}(\alpha) \;\;:\Leftrightarrow\;\; & \mathbb{PO}(\alpha) \wedge (\forall x)(\forall y)[\alpha(\langle x,x\rangle)=0 \wedge \alpha(\langle y,y\rangle)=0 \\ & \qquad\qquad\qquad\qquad \Rightarrow\;\; \alpha(\langle x,y\rangle)=0 \vee \alpha(\langle y,x\rangle)=0]\end{aligned}$$

$$\begin{aligned}\mathbb{WF}(\alpha) \;\;:\Leftrightarrow\;\; & (\forall\beta^*)[(\forall x)(\beta^*(x)=0 \Rightarrow \alpha(\langle x,x\rangle)=0) \wedge (\exists x)\,[\beta^*(x)=0] \\ & \Rightarrow (\exists z)(\beta^*(z)=0 \wedge (\forall y)[\alpha(\langle y,z\rangle)=0 \Rightarrow y=z \vee \beta^*(y)=1])]\end{aligned}$$

and finally

$$\mathbb{WO}(\alpha) \;\;:\Leftrightarrow\;\; \mathbb{LO}(\alpha) \wedge \mathbb{WF}(\alpha).$$

Then $\mathbb{PO}(\alpha)$ expresses that α is the characteristic function of the contraction of a partial–ordering, $\mathbb{LO}(\alpha)$ that α is the characteristic function of the contraction of an ordering, $\mathbb{WF}(\alpha)$ expresses that α is the characteristic function of the contraction of a well-founded binary predicate and $\mathbb{WO}(\alpha)$ denotes that α is the characteristic function of the contraction of a well–ordering. We moreover have

5.1.9 Theorem *The relations $\mathbb{PO}(\alpha)$, $\mathbb{LO}(\alpha)$ are Π^0_1 and the relations $\mathbb{WF}(\alpha)$ and $\mathbb{WO}(\alpha)$ are Π^1_1.*

5.2 Trees

An extremely important tool in the investigation of hyperarithmetical set are trees. We are going to introduce trees as sets (of codes of) finite sequences which are closed under initial segments.

5.2.1 Definition Let $s, t \in \mathbb{N}$. We put

$$s \subseteq t \;\;:\Leftrightarrow\;\; \mathit{Seq}(s) \wedge \mathit{Seq}(t) \wedge \mathit{lh}(s) \leq \mathit{lh}(t) \wedge (\forall i < \mathit{lh}(s))\,[(s)_i = (t)_i] \tag{5.18}$$

and say that s is an *initial segment* of t.
A *tree* is a nonempty set $B \subseteq \mathit{Seq}$ which is closed under initial segments, i.e. we put

$$\mathbb{T}(B) \;\;:\Leftrightarrow\;\; (\forall s)\,[s \in B \Rightarrow \mathit{Seq}(s)] \;\wedge B \neq \emptyset \wedge (\forall s)(\forall t)\,[t \in B \wedge s \subseteq t \Rightarrow s \in B]\,.$$

For any tree B we have $\langle\rangle \in B$ by (5.19). We call $\langle\rangle$ the *root* of the tree B. Trees should be visualized as shown in Figure 5.2.1.
Notice that writing $\mathbb{T}(B)$ as an analytical formula, i.e.

$$\begin{aligned}\mathbb{T}(\alpha) \;\;:\Leftrightarrow\;\; & (\forall x)\,[\alpha(x) \leq 1 \wedge (\alpha(x)=0 \rightarrow \mathit{Seq}(x))] \;\wedge \alpha(\langle\rangle)=0 \\ & \wedge (\forall x)(\forall y)\,[\alpha(x)=0 \wedge y \subseteq x \rightarrow \alpha(y)=0]\,,\end{aligned} \tag{5.19}$$

shows that it is a $(0,1)$–ary Π^0_1–predicate.
If $s^\frown\langle x\rangle \in B$ then we also have $s \in B$. We call $s^\frown\langle x\rangle$ an *immediate B–predecessor* of s and s the *immediate B–successor* of $s^\frown\langle x\rangle$.
A *path in* a tree B is a subset $P \subseteq B$ which is a linearly ordered by $\subseteq$ and closed under immediate successors. A *path through* a tree B is a path in B which also satisfies

$$s \in P \wedge (\exists x)\,[s^\frown\langle x\rangle \in B] \;\;\Rightarrow\;\; (\exists x)\,[s^\frown\langle x\rangle \in P]\,.$$

A tree is *well–founded* if it does not contain infinite paths, i.e. if

$$\mathbb{T}(B) \wedge (\forall\beta)(\exists z)\left[\overline{\beta}(z) \notin B\right].$$

Expressing that by an analytical formula we put

$$\mathbb{WT}(\alpha) \;\;:\Leftrightarrow\;\; \mathbb{T}(\alpha) \wedge (\forall\beta)(\exists z)\left[\alpha(\overline{\beta}(z))=1\right]. \tag{5.20}$$

From (5.19) and (5.20) we have the following lemma.

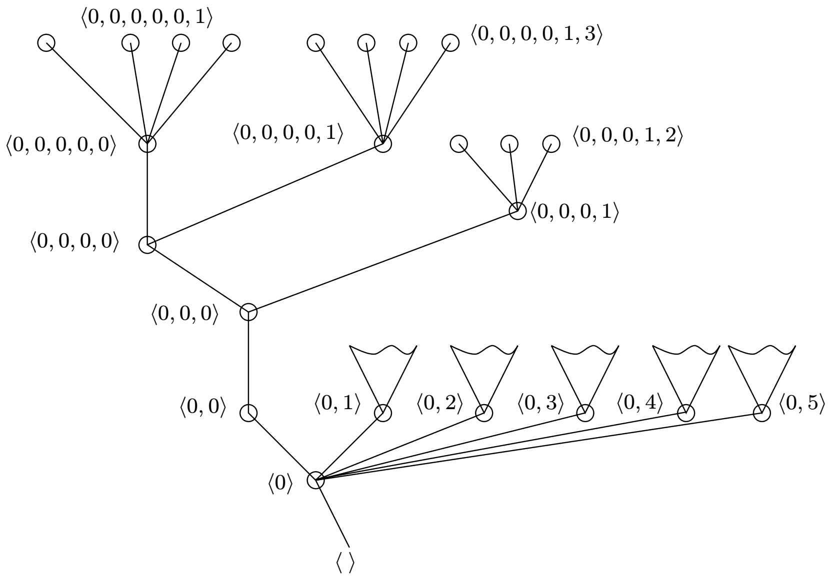

Figure 5.2.1: Visualization of a tree

5.2.2 Lemma *The* $(0,1)$*–ary relations* $\mathbb{T}$ *and* $\mathbb{WT}$ *are* Π^0_1 *and* Π^1_1*, respectively.*

5.2.3 Theorem (Bar induction) *For well–founded trees we have the* principle of bar induction, *i.e. if* B *is a well-founded tree we have*

(BI) $\qquad (\forall s)\,[(\forall x)(s^\frown\langle x\rangle \in B \Rightarrow \varphi(s^\frown\langle x\rangle)) \Rightarrow \varphi(s)] \;\;\Rightarrow\;\; \varphi(\langle\rangle).$

Proof: We prove

$$(\forall s)\,[(\forall x)(s^\frown\langle x\rangle \in B \Rightarrow \varphi(s^\frown\langle x\rangle)) \Rightarrow \varphi(s)] \;\;\Rightarrow\;\; (\forall s \in B)\varphi(s). \tag{5.21}$$

Towards an indirect proof assume

$$(\forall s)\,[(\forall x)(s^\frown\langle x\rangle \in B \Rightarrow \varphi(s^\frown\langle x\rangle)) \Rightarrow \varphi(s)] \tag{i}$$

and

$$s \in B \wedge \neg\varphi(s)$$

for some s. We are going to construct an infinite path $s_0, s_1, \ldots$ in B. Put

$$s_0 := s$$

and assume that $s_0, \ldots, s_n$ are already defined such that

$$s_i \in B \wedge \neg\varphi(s_i) \wedge \;\; \text{“}s_i \text{ immediately succeeds } s_{i+1}\text{”}$$

holds for $i = 0, \ldots, n$ or $i = 0, \ldots, n-1$, respectively. But then there is an x such that

$$s_n^\frown\langle x\rangle \in B \wedge \neg\varphi(s_n^\frown\langle x\rangle)$$

because otherwise we get $\varphi(s_n)$ by (i). Putting

$$s_{n+1} := s_n^\frown\langle x\rangle$$

we obtain an infinite path $s_0, s_1, \ldots$ in B which contradicts the well-foundedness of B. $\square$

There should be a connection between bar induction and transfinite induction along well–founded predicates. To make this explicit we introduce the predicate

$$s \leq^*_B t \;\;:\Leftrightarrow\;\; s \in B \wedge t \in B \wedge t \subseteq s. \tag{5.22}$$

We denote the strict version of $\leq^*_B$ by $<^*_B$. The predicate $\leq^*_B$ is obviously the reflexive and transitive hull of the immediate B–successor predicate. Therefore any infinite path in B induces an infinite $<^*_B$–descending sequence. Conversely, every $<^*_B$–descending sequence is an infinite path in B. Together with Theorem 5.1.6 we get

5.2.4 Theorem *A tree B is well–founded iff the predicate $\leq^*_B$ is well–founded.*

According to Theorem 5.2.4 we may regard bar induction as a special case of transfinite induction. For a tree B and a node $s \in B$ we may regard the *subtree* of B above s which is defined by

$$B{\restriction}s := \big\{t \in \mathit{Seq} \big|\; s^\frown t \in B\big\}. \tag{5.23}$$

Then we have

$$\mathbb{T}(B) \wedge s \in B \;\;\Rightarrow\;\; \mathbb{T}(B{\restriction}s)$$

and obviously also

$$\mathbb{WT}(B) \wedge s \in B \;\;\Rightarrow\;\; \mathbb{WT}(B{\restriction}s).$$

We call a tree B *finitely branching* if every node in B has only finitely many predecessors, i.e. if

$$(\forall s \in B)\,\big[\big|\big\{x \big|\; s^\frown\langle x\rangle \in B\big\}\big| < \aleph_0\big]$$

where $|M|$ denotes the cardinality of a set M and $\aleph_0$ the first infinite cardinal. An important property of finitely branching trees is KÖNIG's lemma.

5.2.5 Lemma (KÖNIG's Lemma) *Any tree which is finitely branching but infinite possesses an infinite path.*

Proof: We assume that B is finitely branching but infinite. We construct an infinite path $s_0, s_1, \ldots$ in B. Put

$$s_0 = \langle\rangle$$

and assume that $s_0, \ldots, s_n$ are defined such that

$$s_i \in B \wedge |B{\restriction}s_i| \geq \aleph_0$$

holds for $i = 0, \ldots, n$. Since

$$\aleph_0 \leq |B{\restriction}s_n| = \Big|\{\langle\rangle\} \cup \bigcup_{x \in \mathbb{N}} \big\{B{\restriction}s_n^\frown\langle x\rangle \big|\; s_n^\frown\langle x\rangle \in B\big\}\Big|$$

and

$$\Big|\big\{x \big|\; s_n^\frown\langle x\rangle \in B\big\}\Big| < \aleph_0$$

there is an x such that

$$s_n^\frown\langle x\rangle \in B \wedge |B{\restriction}s_n^\frown\langle x\rangle| \geq \aleph_0.$$

Defining

$$s_{n+1} := s_n^\frown\langle x\rangle$$

we obtain an infinite path. $\square$

We call a tree *boundedly branching* if there is a $k \in \mathbb{N}$ such that

$$(\forall s \in B)(\forall x)\,[s^\frown\langle x\rangle \in B \Rightarrow x \leq k]\,. \tag{5.24}$$

We call k a *branching bound*. If B is boundedly branching with branching bound k we obviously have

$$(\forall s \in B)(\forall i < \mathit{lh}(s))\,[(s)_i \leq k]\,.$$

Every boundedly branching tree is finitely branching. The important fact about boundedly branching trees is that their finiteness can be expressed by an arithmetical formula. For a boundedly branching tree B we get

$$B \text{ is finite} \quad \Leftrightarrow \quad (\exists n)(\forall s)\,[\mathit{Seq}(s) \wedge \mathit{lh}(s) = n \Rightarrow s \notin B]\,. \tag{5.25}$$

Combining (5.25) with KÖNIG's Lemma we get

5.2.6 Theorem (Finiteness Theorem) *Let B be a boundedly branching tree. Then*

$$(\forall\beta)(\exists z)\,\left[\overline{\beta}(z) \notin B\right] \quad \Leftrightarrow \quad (\exists n)(\forall s)[\mathit{Seq}(s) \wedge \mathit{lh}(s) = n \Rightarrow s \notin B]. \tag{5.26}$$

The importance of the Finiteness Theorem is that it shows that for boundedly branching trees the Π^1_1–property of being well-founded can be expressed arithmetically.
A binary tree is a boundedly branching tree with branching bound 1. The Finiteness Theorem for binary trees is also known as Weak KÖNIG's Lemma.
We also want to establish a connection between well–founded tress and ordinals. The key here is Theorem 5.2.4 and the definitions in (5.16) and (5.17), respectively.

5.2.7 Definition Let B be a well-founded tree. For $s \in B$ we define

$$\mathit{otyp}_B(s) := \mathit{otyp}_{\leq^*_B}(s)$$

and

$$\mathit{otyp}(B) = \mathit{otyp}_B(\langle\rangle).$$

By (5.16) we have

$$\mathit{otyp}_B(s) = \sup\left\{\mathit{otyp}_B(t) + 1 \,\middle|\, t <^*_B s\right\}. \tag{5.27}$$

For $t <^*_B s$, however, we find an x such that $t \leq^*_B s^\frown\langle x\rangle <^*_B s$. Because of $\mathit{otyp}_B(t) \leq \mathit{otyp}_B(s^\frown\langle x\rangle)$ we obtain from (5.27)

$$\mathit{otyp}_B(s) = \sup\left\{\mathit{otyp}_B(s^\frown\langle x\rangle) + 1 \,\middle|\, s^\frown\langle x\rangle \in B\right\}. \tag{5.28}$$

Since $\mathit{otyp}(B) = \mathit{otyp}_B(\langle\rangle) = \sup\left\{\mathit{otyp}_{\leq^*_B}(s) + 1 \,\middle|\, \langle\rangle <^*_B s\right\}$ we get

$$\mathit{otyp}(B) = \mathit{otyp}(\leq^*_B \restriction \langle\rangle). \tag{5.29}$$

The tree predicate $\leq^*_B$ is a partial ordering. However, sometimes it is desirable to have an ordering on a tree. We are going to linearize the order $\leq^*_B$ using an idea which goes back to KLEENE and BROUWER. To their honor this ordering is called KLEENE–BROUWER–ordering.

5.2.8 Definition (KLEENE–BROUWER–ordering) For a sequence number s and $x < \mathit{lh}(s)$ put

$$s\restriction x := \langle (s)_0, \ldots, (s)_{x \dot{-} 1}\rangle.$$

Let B be a tree. For s, t in B we define

$$s <^{KB}_B t \;\; :\Leftrightarrow \;\; t \subsetneq s \vee (\exists x < \mathit{lh}(s))\,[s\restriction x = t\restriction x \wedge (s)_x < (t)_x]\,.$$

The predicate $<_B^{KB}$ is irreflexive. The associated partial order is

$$s \leq_B^{KB} t \quad :\Leftrightarrow \quad s <_B^{KB} t \vee (s \in B \wedge s = t).$$

A visualization of the KLEENE–BROUWER–ordering is given in Figure 5.2.2.

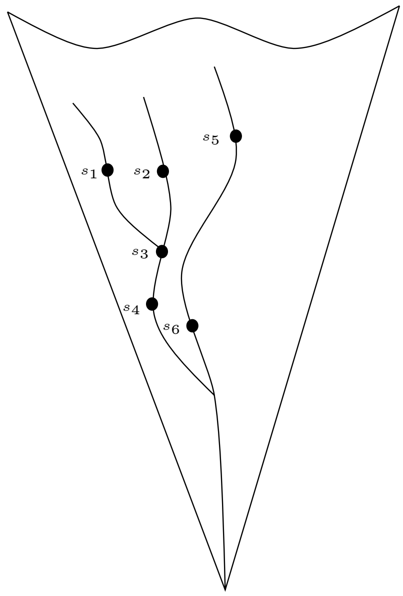

Figure 5.2.2: Visualization of the KLEENE–BROUWER–ordering
The nodes $s_1,\ldots,s_6$ are in increasing order

5.2.9 Lemma *For any tree B the predicate $\leq_B^{KB}$ is an ordering on B, $<_B^{KB}$ is a strict ordering on B.*

Proof: It suffices to show that $\leq_B^{KB}$ is irreflexive, transitive and linear. Irreflexivity follows by definition. Transitivity is easy but a bit cumbersome because of the many cases one has to consider. A proof is sketched in Figure 5.2.3. To check linearity notice that for any $s \neq t \in B$ there is a maximal x such that $s{\restriction}x = t{\restriction}x$. If $t{\restriction}x = s$ then $s \subseteq t$, hence $t \leq_B^{KB} s$ and if $s{\restriction}x = t$ then $t \subseteq s$, hence $s \leq_B^{KB} t$. Otherwise we either have $(s)_x < (t)_x$ and obtain $s <_B^{KB} t$ or $(s)_x > (t)_x$ and obtain $t <_B^{KB} s$. $\square$

5.2.10 Theorem *A tree B is well–founded iff its* KLEENE–BROUWER*–ordering $\leq_B^{KB}$ is well–founded.*

Proof: We start with the easy direction. Assume that B is not well-founded. Then there exists an infinite path $s_0, s_1, \ldots$ in B. According to Definition 5.2.8 this implies $s_0 >_B^{KB} s_1 >_B^{KB} \ldots$ and we obtain an infinite $<_B^{KB}$–descending sequence which, according to Theorem 5.1.6 contradicts the well–foundedness of $\leq_B^{KB}$.
For the opposite direction we use KÖNIG's Lemma. Let

$$s_0 >_B^{KB} s_1 >_B^{KB} \ldots >_B^{KB} s_i >_B^{KB} s_{i+1} \ldots$$

be an infinite $<_B^{KB}$–descending sequence and put $S := \left\{ s_i \mid i \in \mathbb{N} \right\}$. Define

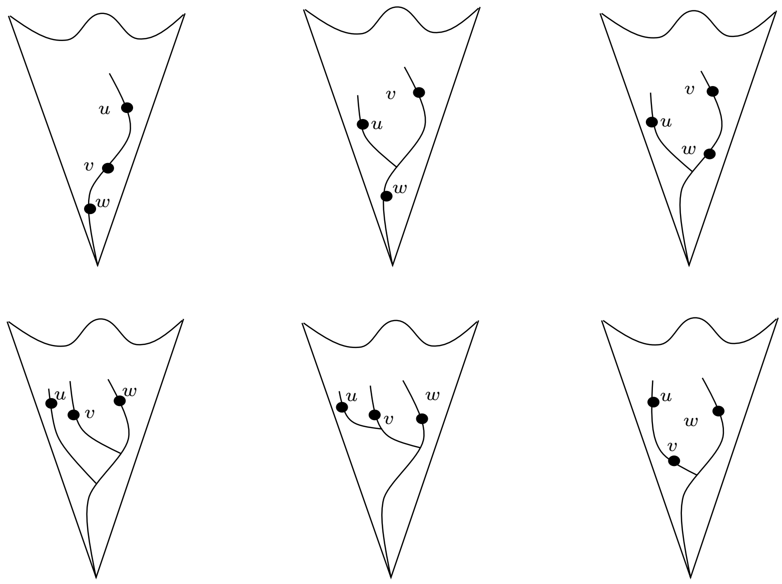

Figure 5.2.3: How to prove $u <_B^{KB} v <_B^{KB} w \Rightarrow u <_B^{KB} w$ in the KLEENE–BROUWER–ordering

$$B' := \{t \mid \mathit{Seq}(t) \wedge (\exists s \in S)\,[t \subseteq s]\}.$$

Then $B' \subseteq B$ is obviously an infinite tree. We claim that B' is finitely branching. Chose any $t \in B'$ and regard

$$M_t := \{x \mid t^\frown\langle x\rangle \in B'\}.$$

For any $x \in M_t$ there is an $s^x \in S$ such that $t^\frown\langle x\rangle \subseteq s^x$ and for x, y in M_t we get $s^x <_B^{KB} s^y$ if $x < y$. The set $\{r \in S \mid s <_B^{KB} r\}$, however, is finite by construction of S. Therefore M_t is finite for any $t \in B'$. It follows from KÖNIG's Lemma that B' contains an infinite path P. But P is also a path in B. Hence B is not well–founded. □

5.3 Recursive Ordinals

This lecture is only concerned with countable ordinals. However, we don't want to hide that there are also bigger – uncountable – ordinals. Usually one puts

$$\omega_1 := \sup\{\sigma \mid \sigma \text{ is a countable ordinal }\}.$$

We have seen in Theorem 5.1.7 that the countable ordinals are unbounded in the countable ordinals. Therefore ω_1 can't be a countable ordinal itself. In this section we will introduce an even smaller class of ordinals.

5.3.1 Definition An ordinal is called recursive (in A) if it is represented by some (in A) decidable countable well-ordering. We define

$$\omega_1^{CK} := \sup\{\sigma \in \mathit{On} \mid \sigma \text{ is recursive}\}$$

and

$$\omega_1^{CK}[A] := \sup\{\sigma \in \mathit{On} \mid \sigma \text{ is recursive in } A\}.$$

It is obvious that we have $\omega_1^{CK} \leq \omega_1$ and $\omega_1^{CK}[A] \leq \omega_1$ for any set $A \subseteq \mathbb{N}$. Observe that the (in A) recursive ordinals form a segment of the ordinals. To see that let σ be a (in A) recursive ordinal and $\preceq$ a (in A) decidable well–ordering representing σ. For $\tau < \sigma$ there is a $z \in \mathit{field}(\preceq)$ such that $\preceq\restriction z$ represents τ. Since $\preceq\restriction z$ is again decidable (in A) the ordinal τ is recursive (in A), too. Notice that we did **not** claim that the relation $\tau < \sigma$ is decidable.

5.3.2 Lemma *The ordinal ω_1^{CK} is a limit ordinal which is not recursive.*

Proof: Let σ be a recursive ordinal and $\preceq$ a decidable well-ordering representing σ. We construct a well–ordering $\preceq'$ as in the proof of Theorem 5.1.7. Obviously $\preceq'$ is again decidable. Therefore ω_1^{CK} cannot be recursive and also not a successor ordinal. $\square$

As a consequence of Lemma 5.3.2 and the fact that the (in A) recursive ordinals form a segment of the countable ordinals we get

$$\omega_1^{CK} = \min\big\{\xi \in \mathit{On} \big|\ \xi \text{ is not recursive}\big\} \tag{5.30}$$

and

$$\omega_1^{CK}[A] = \min\big\{\xi \in \mathit{On} \big|\ \xi \text{ is not recursive in } A\big\} \tag{5.31}$$

which entails

5.3.3 Lemma *An ordinal σ is recursive iff $\sigma < \omega_1^{CK}$. An ordinal σ is recursive in A iff $\sigma < \omega_1^{CK}[A]$.*

The ordinal ω_1^{CK} is therefore the least ordinal which cannot be represented by a decidable well–ordering. In that sense ω_1^{CK} is the "effective" counterpart of the ordinal ω_1 which is the least ordinal which cannot be represented by a countable well–ordering.
We are going to introduce the *light face* versions of the relations $\mathbb{CF}$, $\mathbb{PO}$, $\mathbb{LO}$, $\mathbb{WF}$, $\mathbb{T}$ and $\mathbb{WT}$. We put

$\mathit{CF}(e)$ $:\Leftrightarrow$ "e is index of a characteristic function"

$\mathit{PO}(e)$ $:\Leftrightarrow$ "e is index of a partial ordering"

$\mathit{PO}^A(e)$ $:\Leftrightarrow$ "e is A–index of a partial ordering"

$\mathit{LO}(e)$ $:\Leftrightarrow$ "e is index of an ordering"

$\mathit{LO}^A(e)$ $:\Leftrightarrow$ "e is A–index of an ordering"

$\mathit{WF}(e)$ $:\Leftrightarrow$ "e is index of a well–founded binary predicate"

$\mathit{WF}^A(e)$ $:\Leftrightarrow$ "e is A–index of a well–founded binary predicate"

$\mathit{WO}(e)$ $:\Leftrightarrow$ "e is index of a well–ordering"

$\mathit{WO}^A(e)$ $:\Leftrightarrow$ "e is A–index of a well–ordering"

$\mathit{Tree}(e)$ $:\Leftrightarrow$ "e is index of a tree"

$\mathit{Tree}^A(e)$ $:\Leftrightarrow$ "e is A–index of a tree"

$$\mathit{WT}(e) \quad :\Leftrightarrow \quad \text{“}e \text{ is index of a well–founded tree”}$$

$$\mathit{WT}^A(e) \quad :\Leftrightarrow \quad \text{“}e \text{ is } A\text{–index of a well–founded tree”}$$

All these predicates are arithmetical or analytical. To check their complexity recall that

$$\{e\}^{n,0}(\vec{x}) \simeq y \quad \Leftrightarrow \quad (\exists u)\left[\mathsf{T}^{n,0}(e,\vec{x},u) \wedge U(u) = y\right].$$

Hence $\{(e,\vec{x},y) \mid \{e\}^{n,0}(\vec{x}) \simeq y\} \in \Sigma^0_1$. Therefore we get

$$\mathit{CF}(e) \quad \Leftrightarrow \quad (\forall x)(\forall y)(\exists z)\left[\{e\}^{2,0}(x,y) \simeq z \wedge z \leq 1\right]$$

and

$$\begin{aligned}\mathit{PO}(e) \quad \Leftrightarrow \quad & \mathit{CF}(e)\\ & \wedge (\forall x)(\forall y)\left[\{e\}^{2,0}(x,y) = 0 \;\; \rightarrow \;\; \{e\}^{2,0}(x,x) = 0 \wedge \{e\}^{2,0}(y,y) = 0\right]\\ & \wedge (\forall x)(\forall y)\left[\{e\}^{2,0}(x,y) = 0 \wedge \{e\}^{2,0}(y,x) = 0 \;\; \rightarrow \;\; x = y\right]\\ & \wedge (\forall x)(\forall y)(\forall z)\left[\{e\}^{2,0}(x,y) = 0 \wedge \{e\}^{2,0}(y,z) = 0 \;\; \rightarrow \;\; \{e\}^{2,0}(x,z) = 0\right].\end{aligned}$$

as well as

$$\begin{aligned}\mathit{LO}(e) \quad \Leftrightarrow \quad & \mathit{PO}(e)\\ & \wedge (\forall x)(\forall y)[(\{e\}^{2,0}(x,x) = 0 \wedge \{e\}^{2,0}(y,y) = 0)\\ & \qquad\qquad \rightarrow \;\; (\{e\}^{2,0}(x,y) = 0 \vee \{e\}^{2,0}(y,x) = 0)].\end{aligned}$$

similarly we get

$$\begin{aligned}\mathit{Tree}(e) \quad \Leftrightarrow \quad & (\forall x)(\exists z)\left[\{e\}^{1,0}(x) \simeq z \wedge z \leq 1\right]\\ & \wedge \{e\}^{1,0}(\langle\rangle) = 0\\ & \wedge (\forall x)\left[\{e\}^{1,0}(x) = 0 \rightarrow \mathit{Seq}(x)\right]\\ & \wedge (\forall x)(\forall y)\left[\{e\}^{1,0}(x) = 0 \wedge y \subseteq x \rightarrow \{e\}^{1,0}(y) = 0\right].\end{aligned}$$

These predicates are arithmetical. As examples for analyitcal predicates we take

$$\begin{aligned}\mathit{WF}(e) \quad \Leftrightarrow \quad & \mathit{CF}(e)\\ & \wedge (\forall\alpha)\{(\exists x)(\alpha(x) = 0) \wedge (\forall x)\left[\alpha(x) = 0 \rightarrow \{e\}^{2,0}(x,x) = 0\right]\\ & \qquad \rightarrow \;\; (\exists z)\left[\alpha(z) = 0 \wedge (\forall u)(\{e\}^{2,0}(u,z) = 0 \rightarrow u = z \vee \alpha(u) = 1)\right]\}.\end{aligned}$$

Therefore we have

$$\mathit{WO}(e) \quad \Leftrightarrow \quad \mathit{LO}(e) \wedge \mathit{WF}(e)$$

and

$$\mathit{WT}(e) \quad \Leftrightarrow \quad \mathit{Tree}(e) \wedge (\forall\alpha)(\exists z)\left[\{e\}^{1,0}(\overline{\alpha}(z)) = 1\right].$$

Summing up we get the following theorem.

5.3.4 Theorem *The predicates $\mathit{PO}(e)$, $\mathit{LO}(e)$ and $\mathit{Tree}(e)$ are all Π^0_2. The predicates $\mathit{WF}(e)$, $\mathit{WO}(e)$ and $\mathit{WT}(e)$ are Π^1_1.*

5.3.5 Definition If $\mathit{WO}(e)$ we put

$$\mathit{otyp}^{\mathit{WO}}(e) := \mathit{otyp}(\{(x,y) \mid \{e\}^{2,0}(x,y) = 0\}).$$

For $\mathit{WO}(e)^A$ let

$$\mathit{otyp}^{\mathit{WO}^A}(e) := \mathit{otyp}(\{(x,y) \mid \{e\}^{A,2,0}(x,y) = 0\}).$$

For $\mathit{WT}(e)$ we put

$$\mathit{otyp}^{\mathit{Tree}}(e) := \mathit{otyp}(\{x \mid \{e\}^{1,0}(x) = 0\}).$$

And for $\mathit{WT}^A(e)$ we let

$$\mathit{otyp}^{\mathit{Tree}^A}(e) := \mathit{otyp}(\{x \mid \{e\}^{A,1,0}(x) = 0\}).$$

5.4 KLEENE's Ordinal Notations

Before we look closer at the connections between recursive ordinals and the ordinals which are given by well–founded trees we introduce another form of ordinals via effective abstract notations. This approach is due to S. C. KLEENE. The idea is to introduce simultaneously a set $\mathcal{O}$ of ordinal notations together with an evaluation function $|\cdot|_{\mathcal{O}} \colon \mathcal{O} \longrightarrow \mathit{On}$ and an order relation $<_{\mathcal{O}}$ such that $a <_{\mathcal{O}} b \;\Rightarrow\; |a|_{\mathcal{O}} < |b|_{\mathcal{O}}$.

5.4.1 Definition We define the set $\mathcal{O}$ of *ordinal notations*, the $\mathcal{O}$–evaluation $|\ |_{\mathcal{O}}$ and the order–predicate $<_{\mathcal{O}}$ on $\mathcal{O}$ simultaneously by the following clauses.

1) $1 \in \mathcal{O}$, $|1|_{\mathcal{O}} := 0$ and $1 \le_{\mathcal{O}} a$ for all $a \in \mathcal{O}$.
2) If $a \in \mathcal{O}$ then $2^a \in \mathcal{O}$, $|2^a|_{\mathcal{O}} := |a|_{\mathcal{O}} + 1$ and $c <_{\mathcal{O}} 2^a$ for all $c \le_{\mathcal{O}} a$.
3) Let e be the index of a computable function such that

$$(\forall x)\left[\{e\}^{1,0}(x) \in \mathcal{O} \wedge \{e\}^{1,0}(x) <_{\mathcal{O}} \{e\}^{1,0}(x+1)\right]$$

then $3 \cdot 5^e \in \mathcal{O}$, $|3 \cdot 5^e|_{\mathcal{O}} = \sup\left\{|\{e\}^{1,0}(n)|_{\mathcal{O}} \mid n \in \mathbb{N}\right\}$ and $c <_{\mathcal{O}} 3 \cdot 5^e$ iff there is an $n \in \mathbb{N}$ such that $c \le_{\mathcal{O}} \{e\}^{1,0}(n)$.

An ordinal σ is KLEENE–*recursive* iff there is an $a \in \mathcal{O}$ such that $\sigma = |a|_{\mathcal{O}}$.

As a first consequence of Definition 5.4.1 we obtain

5.4.2 Lemma *The predicate $<_{\mathcal{O}}$ is transitive on $\mathcal{O}$ and we have*

$$a <_{\mathcal{O}} b \;\Rightarrow\; |a|_{\mathcal{O}} < |b|_{\mathcal{O}}.$$

Proof: We show

$$a <_{\mathcal{O}} b \wedge b <_{\mathcal{O}} c \;\Rightarrow\; a <_{\mathcal{O}} c$$

by induction on $|c|_{\mathcal{O}}$.
If $c = 1$ we have nothing to show.
If $c = 2^{c_0}$ then $a <_{\mathcal{O}} b \le_{\mathcal{O}} c_0$ and $|c_0|_{\mathcal{O}} < |c|_{\mathcal{O}}$. By the induction hypothesis we get $a \le_{\mathcal{O}} c_0$ which entails $a <_{\mathcal{O}} c$.
If $c = 3 \cdot 5^e$ we get $a <_{\mathcal{O}} b \le_{\mathcal{O}} \{e\}^{1,0}(n)$ for some $n \in \mathbb{N}$. Then $|\{e\}^{1,0}(n)|_{\mathcal{O}} < |c|_{\mathcal{O}}$. Hence $a \le_{\mathcal{O}} \{e\}^{1,0}(n)$ by induction hypothesis which implies $a <_{\mathcal{O}} c$.
The second claim is an easy consequence of the definition which we leave as an exercise. □

As a consequence of the second claim in Lemma 5.4.2 we get

5.4.3 Corollary *The predicate $<_{\mathcal{O}}$ on $\mathcal{O}$ is well–founded.*

Proof: Any infinite $<_{\mathcal{O}}$–descending sequence induces by Lemma 5.4.2 an infinite descending sequence in the ordinals. □

5.4.4 Theorem *The* KLEENE*–recursive ordinals form a segment of the countable ordinals, i.e. if* $a \in \mathcal{O}$ *and* $\sigma < |a|_{\mathcal{O}}$ *then there is a* $b \in \mathcal{O}$ *such that* $\sigma = |b|_{\mathcal{O}}$.

Proof: We induct on $|a|_{\mathcal{O}}$. For $a = 1$ we have nothing to show. For $a = 2^{a_0}$ and $\sigma < |a|_{\mathcal{O}}$ we get $\sigma \leq |a_0|_{\mathcal{O}}$. Therefore we either have $\sigma = |a_0|_{\mathcal{O}}$ or obtain a $b \in \mathcal{O}$ such that $\sigma = |b|_{\mathcal{O}}$ by the induction hypothesis.
For $a = 3 \cdot 5^e$ and $\sigma < |a|_{\mathcal{O}}$ we get $\sigma < |\{e\}^{1,0}(n)|$ for some $n \in \mathbb{N}$. Then there is a $b \in \mathcal{O}$ such that $\sigma = |b|_{\mathcal{O}}$ by induction hypothesis. $\square$

5.4.5 Lemma *There is a binary computable function* $+_{\mathcal{O}}$ *such that for all* $a, b, c \in \mathbb{N}$ *the following hold*

1) $(a \in \mathcal{O} \wedge b \in \mathcal{O}) \;\Leftrightarrow\; a +_{\mathcal{O}} b \in \mathcal{O}$

2) $(a \in \mathcal{O} \wedge b \in \mathcal{O}) \;\Rightarrow\; |a +_{\mathcal{O}} b|_{\mathcal{O}} = |a|_{\mathcal{O}} + |b|_{\mathcal{O}}$

3) $(a \in \mathcal{O} \wedge b \in \mathcal{O} \wedge b \neq 1) \;\Rightarrow\; a <_{\mathcal{O}} a +_{\mathcal{O}} b$

4) $(a \in \mathcal{O} \wedge c <_{\mathcal{O}} b) \;\Leftrightarrow\; a +_{\mathcal{O}} c <_{\mathcal{O}} a +_{\mathcal{O}} b$

5) $(a \in \mathcal{O} \wedge b = c \in \mathcal{O}) \;\Leftrightarrow\; a +_{\mathcal{O}} b = a +_{\mathcal{O}} c$

Proof: Let h be a recursive function such that for all $e, a, d, n \in \mathbb{N}$

$$\{h(e,a,d)\}(n) \simeq \{e\}(a, \{d\}(n))$$

holds. By using different indices for the same function we are able to make h one–one. Define

$$g(e,a,b) = \begin{cases} a & \text{if } b = 1 \\ 2^{\{e\}(a,y)} & \text{if } b = 2^y \neq 1 \\ 3 \cdot 5^{h(e,a,y)} & \text{if } b = 3 \cdot 5^y \\ 7 & \text{otherwise,} \end{cases}$$

and use the Recursion Theorem to obtain an index e such that

$$\{e\}(a,b) \simeq g(e,a,b). \tag{i}$$

Putting

$$a +_{\mathcal{O}} b := \{e\}(a,b)$$

we get a partial–computable function for which one easily sees by induction that $a +_{\mathcal{O}} b$ is defined for all $a, b \in \mathcal{O}$. Surprisingly, the new function is total. Suppose $a +_{\mathcal{O}} b$ is not defined. Then, as h is total, we have $b = 2^y \neq 1$ for some $y < b$. By induction on $\mathbb{N}$ we can convince ourselves that $+_{\mathcal{O}}$ is total.
The rest of the proof, being an interesting but lengthy exercise in induction, is left to the reader. There is only one step of the proof where h is required to be one–one. $\square$

We postpone the study of the complexities of the set $\mathcal{O}$ and the predicate $<_{\mathcal{O}}$ until chapter 7 and devote the rest of this section to the study of the connections between the different notions of recursive ordinals we just introduced. The easiest connection to establish is the one between decidable well–founded trees and recursive ordinals.

5.4.6 Lemma *There is a computable function* f *such that for alle* $e \in \mathbb{N}$

$$\mathit{WT}(e) \;\Leftrightarrow\; \mathit{WO}(f(e))$$

and

$$\mathit{WT}(e) \;\Rightarrow\; \mathit{WO}(f(e)) \wedge \mathit{otyp}^{\mathit{Tree}}(e) \leq \mathit{otyp}^{\mathit{WO}}(f(e)).$$

The ordertype of a decidable tree is therefore a recursive ordinal.

Proof: If B is any decidable tree then the associated KLEENE–BROUWER–ordering $\leq_B^{KB}$ is also decidable. Moreover, an index for $\leq_B^{KB}$ is effectively computable from an index of B. Since $s <_B^* t$ entails $s <_B^{KB} t$ we get by induction on $\mathit{otyp}_B(t)$

$$\mathit{otyp}_B(t) = \sup\{\mathit{otyp}_B(s)+1 \mid s <_B^* t\} \leq \sup\{\mathit{otyp}_{\leq_B^{KB}}(s)+1 \mid s <_B^{KB} t\} = \mathit{otyp}_{\leq_B^{KB}}(t).$$

Therefore we obtain together with Lemma 5.3.3 that the ordertypes of decidable trees are recursive ordinals.
Unfortunately it is not sufficient to take f as the function that takes each $e \in \mathbb{N}$ to an index of the KLEENE–BROUWER–ordering induced by $\{e\}$: If e is the characteristic function of a finite set of sequences that is not closed under initial segments then $<_B^{KB}$ may still be a well-ordering (take for example $B := \{\langle\rangle, \langle 0,0\rangle\}$). Fortunately we can overcome this obstacle. For $B \subseteq \mathbb{N}$ we put

$$s <_B^{MM} t \;\;:\Leftrightarrow\;\; s <_B^{KB} t \vee \langle\rangle \notin B \vee [t \in B \wedge (\exists t_0 \subseteq t)(t_0 \notin B)].$$

Obviously, if B is a tree, then $<_B^{MM} = <_B^{KB}$ holds. Furthermore, it is not hard to see that $<_B^{MM}$ is not well–founded if B is not a tree. So, we just have to let f be the (computable) function that takes each $e \in \mathbb{N}$ to an index of the induced $<^{MM}$–ordering. Note that $\{e\}$ is total iff $\{f(e)\}$ is. $\square$

The other connections are a bit more complicated. As an auxiliary lemma we need the following *Recursion Lemma* which sometimes is also called *Definition by bar recursion*. Observe that, for a relation R and a partial–computable functional H, the validity of $R(\mathfrak{b}, H(\mathfrak{a}))$ implies $H(\mathfrak{a})\downarrow$.

5.4.7 Lemma (Recursion Lemma) *Let R be an $(m+2,n)$-ary relation and $\prec$ be an irreflexive well–founded predicate. For any $(m+2,n)$–ary in A partial–computable functional H such that*

$$(\forall \mathfrak{a})(\forall e)(\forall x \in \mathit{field}(\prec))\left[(\forall y \prec x) R(\mathfrak{a}, y, \{e\}^{A,m+1,n}(\mathfrak{a}, y)) \;\;\Rightarrow\;\; R(\mathfrak{a}, x, H(\mathfrak{a}, x, e))\right] \tag{5.32}$$

there is an in A partial–computable functional F such that

$$(\forall \mathfrak{a})(\forall x \in \mathit{field}(\prec))\,[R(\mathfrak{a}, x, F(\mathfrak{a}, x))]. \tag{5.33}$$

If H is total, then so is F.

Proof: We use the Recursion Theorem to obtain an A–index f with

$$\{f\}^A(\mathfrak{a}, x) \simeq H(\mathfrak{a}, x, f). \tag{i}$$

We show

$$(\forall \mathfrak{a})(\forall x \in \mathit{field}(\prec)) R(\mathfrak{a}, x, \{f\}^A(\mathfrak{a}, x))$$

by transfinite induction along $\prec$. We have

$$(\forall \mathfrak{a})(\forall y \prec x) R(\mathfrak{a}, y, \{f\}^A(\mathfrak{a}, y)) \tag{ii}$$

by the induction hypothesis. Then by (i) and (5.32) we obtain from (ii)

$$R(\mathfrak{a}, x, H(\mathfrak{a}, x, f))$$

which is

$$R(\mathfrak{a}, x, \{f\}^A(\mathfrak{a}, x)).$$

Putting $F := \{f\}^A$ finishes this proof. $\square$

The Recursion Lemma is the main tool in the proof of the following theorem which establishes the connections between recursive ordinals, order–types of decidable trees and KLEENE–recursive ordinals.

5.4.8 Theorem *There are computable functions f and g such that*

$$a \in \mathit{WT} \;\Rightarrow\; f(a) \in \mathit{WO} \wedge \mathit{otyp}^{\mathit{Tree}}(a) \leq \mathit{otyp}^{\mathit{WO}}(f(a)) \tag{5.34}$$

and

$$a \in \mathcal{O} \;\Rightarrow\; g(a) \in \mathit{WT} \wedge |a|_{\mathcal{O}} = \mathit{otyp}^{\mathit{Tree}}(g(a)). \tag{5.35}$$

For $a \in \mathit{WO}$ let $\preceq$ be the induced well–ordering. There exists a partial–computable function h with

$$(\forall x \in \mathit{field}(\preceq))\, [h(x) \in \mathcal{O} \wedge \mathit{otyp}_{\preceq}(x) \leq |h(x)|_{\mathcal{O}}]\,. \tag{5.36}$$

Additionally we get

$$a \in \mathit{WO} \;\Rightarrow\; (\exists b \in \mathcal{O})\, \big[\mathit{otyp}^{\mathit{WO}}(a) \leq |b|_{\mathcal{O}}\big]\,. \tag{5.37}$$

Proof: Equation (5.34) is Lemma 5.4.6.
To show (5.35) we use the Recursion Lemma along the well–founded predicate $<_{\mathcal{O}}$. We assume $a \in \mathcal{O}$ and the recursion hypothesis

$$(\forall x <_{\mathcal{O}} a)\, \big[\{e\}^{1,0}(x) \in \mathit{WT} \wedge |x|_{\mathcal{O}} = \mathit{otyp}^{\mathit{Tree}}(\{e\}^{1,0}(x))\big]$$

and define a computable function G such that

$$G(e,a) \in \mathit{WT} \wedge |a|_{\mathcal{O}} = \mathit{otyp}^{\mathit{Tree}}(G(e,a)). \tag{i}$$

We put

$$G(e,a) := \begin{cases} \text{index of } \{\langle\rangle\} & \text{if } a = 1 \\ \text{index of } \{\langle\rangle\} \cup \big\{\langle 0\rangle^\frown s \big|\ \{\{e\}^{1,0}(y)\}^{1,0}(s) = 0\big\} & \text{if } a = 2^y \neq 1 \\ \text{index of } \{\langle\rangle\} \cup \big\{\langle n\rangle^\frown s \big|\ \{\{e\}^{1,0}(\{y\}^{1,0}(n))\}^{1,0}(s) = 0\big\} & \text{if } a = 3 \cdot 5^y \\ 0 & \text{otherwise} \end{cases}$$

The function G satifies (i) by construction. We may therefore apply the Recursion Lemma to obtain a computable function g such that (5.35) holds.
We want to use the Recursion Lemma to define a partial–computable function h such that (5.36) holds. We assume $x \in \mathit{field}(\prec)$ and the recursion hypothesis

$$(\forall z \prec x)\, \big[\{e\}^{1,0}(z) \in \mathcal{O} \wedge \mathit{otyp}_{\preceq}(z) \leq |\{e\}^{1,0}(z)|_{\mathcal{O}}\big] \tag{ii}$$

and have to define a partial–computable function H such that

$$H(e,x) \in \mathcal{O} \wedge \mathit{otyp}_{\preceq}(x) \leq |H(e,x)|_{\mathcal{O}}. \tag{iii}$$

Here, however, we encounter the difficulty that we cannot in general decide whether $\mathit{otyp}_{\preceq}(x) \in \mathsf{Lim}$. As a remedy we use a trick. We introduce a new well–ordering $\preceq'$ which is the reflexive hull of the predicate defined by

$$\begin{aligned} a \prec' b \;:\Leftrightarrow\;& \mathit{Seq}(a) \wedge \mathit{Seq}(b) \wedge \mathit{lh}(a) = \mathit{lh}(b) = 2 \\ & \wedge\; [(a)_0 \prec (b)_0 \vee ((a)_0 = (b)_0 \wedge (a)_0 \preceq (a)_0 \wedge (a)_1 < (b)_1)]\,. \end{aligned}$$

The Ordering $\preceq'$ is again decidable and a well-ordering such that $\mathit{otyp}(\preceq) \leq \mathit{otyp}(\preceq')$. (It is $\mathit{otyp}(\preceq') = \omega \cdot \mathit{otyp}(\preceq)$ for those who know ordinal arithmetic.) The ordering $\preceq'$ has the advantage that we can decide whether $x \in \mathit{field}(\preceq')$ is a limit point. We have

$$\mathit{otyp}_{\preceq'}(x) \in \mathsf{Lim} \;\Leftrightarrow\; (x)_0 \neq 0 \wedge (x)_1 = 0$$

where we assume without loss of generality that 0 is the $\preceq$–least element. Moreover we can also compute a fundamental sequence for $\mathit{otyp}_{\preceq'}(\langle x, 0\rangle)$. We put

$$F(x,0) := \langle 0, 0\rangle \tag{iv}$$

and

$$F(x,n+1) := \begin{cases} \langle n,0\rangle & \text{if } (F(x,n))_0 \prec n \prec x \\ \langle (F(x,n))_0, (F(x,n))_1+1\rangle & \text{otherwise.} \end{cases}$$

Then F is a computable function. We have

$$(\forall n)\,[F(x,n) \prec' F(x,n+1)] \tag{v}$$

and prove

$$x \neq 0 \quad\Rightarrow\quad (\forall n)\,[F(x,n) \prec' \langle x,0\rangle]\,.$$

by induction on n. For $n = 0$ this follows from $x \neq 0$ and (iv). From the induction hypothesis $F(x,n) \prec' \langle x,0\rangle$ we get $(F(x,n))_0 \prec x$ and obtain $F(x,n+1) = \langle n,0\rangle \prec' \langle x,0\rangle$ if $(F(x,n))_0 \prec n \prec x$ or $F(x,n+1) = \langle (F(x,n))_0, (F(x,n))_1+1\rangle \prec' \langle x,0\rangle$ otherwise.
Hence

$$\sup\big\{\mathit{otyp}_{\preceq'}(F(x,n))\,\big|\; n\in\mathbb{N}\big\} \leq \mathit{otyp}_{\preceq'}(\langle x,0\rangle). \tag{vi}$$

To obtain equalitity in (vi) we assume $\langle y,n\rangle \prec' \langle x,0\rangle$ and show that there is a $k\in\mathbb{N}$ such that $\langle y,n\rangle \prec' F(x,k)$. From $\langle y,n\rangle \prec' \langle x,0\rangle$ we get $y \prec x$. If $y \preceq (F(x,y))_0$ then

$$F(x,y+1) = \langle (F(x,y))_0, (F(x,y))_1+1\rangle$$

and we find a $k\in\mathbb{N}$ such that $\langle y,n\rangle \prec' F(x,k)$. If $(F(x,y))_0 \prec y$ then

$$F(x,y+1) = \langle y,0\rangle$$

and again we find a $k\in\mathbb{N}$ such that $\langle y,n\rangle \prec' F(x,k)$. Hence

$$\sup\big\{\mathit{otyp}_{\preceq'}(F(x,n))\,\big|\; n\in\mathbb{N}\big\} = \mathit{otyp}_{\preceq'}(\langle x,0\rangle).$$

Together with (v) this shows that $(\mathit{otyp}_{\preceq'}(F(x,n)))_{n\in\mathbb{N}}$ is a fundamental sequence for $\mathit{otyp}_{\preceq'}(\langle x,0\rangle)$.
We use the Recursion Lemma to obtain (5.36) for $\preceq'$ instead of $\preceq$ and assume the recursion hypothesis (ii) for $\preceq'$ instead of $\preceq$. We define

$$H(e,x) := \begin{cases} 1 & \text{if } x = \langle 0,0\rangle \\ 2^{\{e\}^{1,0}(\langle u,v\rangle)} & \text{if } x = \langle u,v+1\rangle \\ 3\cdot 5^z & \text{if } x = \langle u,0\rangle \text{ and } u\neq 0 \end{cases}$$

where z is such that $\{z\}^{1,0}(0) = 1$ and

$$\{z\}^{1,0}(n+1) = \{z\}^{1,0}(n) +_{\mathcal{O}} \{e\}^{1,0}(F(u,n)) +_{\mathcal{O}} 2^1$$

hold. Then, according to Lemma 5.4.5, $H(e,x)$ satisfies (iii) with $\preceq$ replaced by $\preceq'$ and we have (5.36).
If $a \in \mathit{WO}$ we find a decidable well-ordering $\preceq'$ and a $z \in \mathit{field}(\preceq')$ such that $\mathit{otyp}^{\mathit{WO}}(a) = \mathit{otyp}_{\preceq'}(z)$. Without loss of generality we may assume that $\preceq'$ is an ordering of the kind we just have considered. Hence

$$\mathit{otyp}^{\mathit{WO}}(a) = \mathit{otyp}_{\preceq'}(z) \leq |h(z)|_{\mathcal{O}}$$

by (5.36) which finishes the proof. □

It follows from Theorem 5.4.8 that the different approaches to obtain representations for “effective” ordinals all lead to the same class. This proves the following theorem.

5.4.9 Theorem *We have*

$$\begin{aligned}
\omega_1^{CK} &= \sup\big\{\sigma \in \mathit{On} \,\big|\; (\exists a \in \mathit{WO})\,\big[\sigma = \mathit{otyp}^{WO}(a)\big]\big\} \\
&= \sup\big\{\sigma \in \mathit{On} \,\big|\; (\exists a \in \mathit{WT})\,\big[\sigma = \mathit{otyp}^{Tree}(a)\big]\big\} \\
&= \sup\big\{\sigma \in \mathit{On} \,\big|\; (\exists a \in \mathcal{O})\,[\sigma = |a|_{\mathcal{O}}]\big\}.
\end{aligned}$$

6. Generalized Inductive Definitions

In the previous chapter we left the question for the complexity of KLEENE's O unanswered. This chapter will show that the principles used in the definition of O can be systematically studied. This will lead to the fundamental notion of generalized inductive definitions.

6.1 Clauses and operators

Inductive definitions are ubiquitous in Mathematics and especially in Mathematical Logic. Usually we use clauses in inductive definitions. The simplest example of an inductive definition is that of the set of natural numbers. We might say that the natural numbers are inductively defined by the following clauses:

- 0 is a natural number
- If 0 is a natural number, its successor $S(n)$ is also a natural number.

We develop an abstract notion for clauses. Let N be a nonempty set.

6.1.1 Definition A clause over N has the form

(C) $\quad R \to r$

where $R \subseteq N^n$ and $r \in N^n$. We call R the set of premises and r the conclusion of the clause (C).
A set $S \subseteq N^n$ satisfies clause (C) iff $R \subseteq S$ implies $r \in S$.
A *system of clauses* is a set $\Phi = \big\{R_i \to r_i \big|\ i \in I\big\}$ of clauses $R_i \to r_i$.
A set $S \subseteq N^n$ is closed under Φ if S satisfies $R_i \to r_i$ for all $i \in I$.
The least subset of N^n which is closed under a system Φ of clauses is the set which is *inductively defined* by Φ.

Examples for systems of clauses are:

- $\emptyset \to 0$
- $\{n\} \to n+1$

which defines the natural numbers inductively on $\mathbb{N}$.

- $\big\{\emptyset \to s \big|\ s \in S\big\}$
- $\big\{\{x_1, \ldots, x_n\} \to \sum\limits_{i=1}^{n} \alpha_i x_i \big|\ n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in K\big\}$

which defines the subspace of a vector space V over K spanned by some $S \subseteq V$. More examples are easy to find.
The important feature of an inductively defined set $S \subseteq N^n$ is that we have a *"principle of induction on the definition"* of S, which is:
"If a set $S \subseteq N^n$ is inductively defined by some system $\Phi = \big\{R_i \to r_i \big|\ i \in I\big\}$ of clauses and φ is some 'property' which is preserved by all clauses in Φ, i.e. if

$$(\forall x \in R_i)\varphi(x) \Rightarrow \varphi(r_i) \text{ for all } i \in I,$$

then $\varphi(s)$ holds for all $s \in S$."
This principle is obvious from the definition of the set inductively defined by Φ as the least set which is closed under φ. Properly φ being preserved by all clauses in Φ means that $\{x \mid \varphi(x)\}$ is closed under Φ. Since S is the least Φ–closed set, we have $(\forall s \in S)\varphi(s)$.
Observe that the principle of induction on the definition of the natural numbers is exactly the familiar principle of Mathematical Induction. Most induction principles are instances of the principle of induction on some inductive definitions. We are going to study this on the example of transfinite induction along a well–founded predicate. Let $\prec \subseteq N \times N$ be a binary predicate. We introduce the system of clauses

$$\text{(A)} \quad \big\{\{y \mid y \prec x\} \to x \mid x \in N\big\}$$

and call the set $\mathit{Acc}(\prec) \subseteq N$ which is inductively defined by (A) the *accessible part of* $\prec$. The principle of induction on the inductive definition of $\mathit{Acc}(\prec)$ takes the form

$$(\forall x)\,[(\forall y)(y \prec x \to \varphi(y)) \to \varphi(x)] \Rightarrow (\forall x \in \mathit{Acc}(\prec))\varphi(x). \tag{6.1}$$

If we assume that $\prec$ is well–founded we get

$$\mathit{Acc}(\prec) = N. \tag{6.2}$$

$\mathit{Acc}(\prec) \subseteq N$ holds by definition. Let $x \in N$. If $x \notin \mathit{field}(\prec)$ we have trivially $(\forall y)(y \prec x \to y \in \mathit{Acc}(\prec))$. This, however, implies $x \in \mathit{Acc}(\prec)$ by (A). If we assume that there is an $x \in \mathit{field}(\prec)$ which does not belong to $\mathit{Acc}(\prec)$ then we get a least such x by the well–foundedness of $\prec$. But then $y \in \mathit{Acc}(\prec)$ for all $y \prec x$ which again entails $x \in \mathit{Acc}(\prec)$ by (A). Hence $\mathit{field}(\prec) \subseteq \mathit{Acc}(\prec)$. By (6.1) and (6.2) we obtain

$$(\forall x)\,[(\forall y)(y \prec x \to \varphi(y)) \to \varphi(x)] \Rightarrow (\forall x)\varphi(x)$$

and also

$$(\forall x \in \mathit{field}(\prec))\,[(\forall y)(y \prec x \to \varphi(y)) \to \varphi(x)] \Rightarrow (\forall x \in \mathit{field}(\prec))\varphi(x)$$

which is the principle of transfinite induction.
Towards a theory of inductively defined sets we generalize the notion of an inductive definition. A system of clauses $\mathcal{C} = \{R_i \to r_i \mid i \in I\}$ on an infinite set N induces an operator

$$\Gamma_{\mathcal{C}} \colon \mathsf{Pow}(N^n) \longrightarrow \mathsf{Pow}(N^n)$$

which is defined by

$$\Gamma_{\mathcal{C}}(S) = \{r \in N^n \mid (\exists R)\,[R \subseteq S \land R \to r \in \mathcal{C}]\}.$$

If $S \subseteq T$ we obviously have $\Gamma_{\mathcal{C}}(S) \subseteq \Gamma_{\mathcal{C}}(T)$. An operator

$$\Gamma \colon \mathsf{Pow}(N^n) \longrightarrow \mathsf{Pow}(N^n)$$

having the property

$$S \subseteq T \to \Gamma(S) \subseteq \Gamma(T)$$

is called monotone.
Generalizing the situation of systems of clauses we introduce the following definition.

6.1.2 Definition Let N be an infinite set. A monotone operator

$$\Gamma \colon \mathsf{Pow}(N^n) \longrightarrow \mathsf{Pow}(N^n)$$

is a *generalized inductive definition* on N
A set $S \subseteq N^n$ is Γ*–closed* if $\Gamma(S) \subseteq S$. A set $S \subseteq N^n$ is a *fixed–point* of Γ if

$$\Gamma(S) = S.$$

We denote the – with respect to set inclusion – least fixed–point of an operator Γ by I_Γ. We call I_Γ *the* fixed–point of Γ.
A set $S \subseteq N^n$ is *inductively definable* if there is an inductive definition Γ and a tuple $\vec{k} \in N^m$ such that

$$S = \{\vec{x} \in N^n \mid (\vec{x}, \vec{k}) \in I_\Gamma\}.$$

6.1.3 Lemma *Let Γ be a generalized inductive definition on N. The fixed–point of Γ is the least Γ–closed set, i.e.*

$$I_\Gamma = \bigcap \{S \subseteq N^n \mid \Gamma(S) \subseteq S\}.$$

Proof: Put

$$\mathcal{D} := \{S \subseteq N^n \mid \Gamma(S) \subseteq S\}$$

and

$$D = \bigcap \mathcal{D}.$$

For any $S \in \mathcal{D}$ we have $D \subseteq S$ and therefore also $\Gamma(D) \subseteq \Gamma(S) \subseteq S$ by the monotonicity of Γ. Hence

$$\Gamma(D) \subseteq \bigcap \mathcal{D} = D. \tag{i}$$

From (i) we get again by the monotonicity of Γ

$$\Gamma(\Gamma(D)) \subseteq \Gamma(D) \tag{ii}$$

which proves $\Gamma(D) \in \mathcal{D}$. Hence

$$D \subseteq \Gamma(D). \tag{iii}$$

Thus D is a fixed–point by (ii) and (iii). Since $D \subseteq F$ for any fixed–point F holds by definition of D, it is the least fixed–point. $\square$

6.2 The stages of an inductive definition

Describing inductively defined sets by fixed–points of monotone operators means to define them explicitly. This does not really meet the meaning we associate with the phrase “inductive”. An inductive definition should come step by step. Given a generalized inductive definition $\Gamma\colon \mathsf{Pow}(N^n) \longrightarrow \mathsf{Pow}(N^n)$ we may try to construct the fixed–point stepwise by forming

$$\Gamma(\emptyset), \Gamma(\Gamma(\emptyset)), \Gamma^3(\emptyset), \ldots$$

But in general we cannot expect to obtain the fixed–point after finitely many steps. Therefore we will have to iterate Γ transfinitely often.

6.2.1 Definition Let N be a countable infinte set and let $\Gamma\colon \mathsf{Pow}(N^n) \longrightarrow \mathsf{Pow}(N^n)$ be an inductive definition. We define by transfinite recursion

$$I_\Gamma^\sigma := \Gamma(\bigcup_{\tau<\sigma} I_\Gamma^\tau)$$

and call I_Γ^σ the σ–th stage of the fixed–point I_Γ.

The countability of N is needed since we have only introduced countable ordinals.
It follows easily from Definition 6.2.1 that for finite ordinals $n < \omega$ we have

$$I_\Gamma^n = \Gamma^{1+n}(\emptyset).$$

To simplify notations we put

$$I_\Gamma^{<\sigma} := \bigcup_{\xi<\sigma} I_\Gamma^\xi. \tag{6.3}$$

Then $\sigma < \tau \Rightarrow I_\Gamma^{<\sigma} \subseteq I_\Gamma^{<\tau}$ and by the monotonicity of the operator Γ we obtain

$$\sigma < \tau \Rightarrow I_\Gamma^\sigma = \Gamma(I_\Gamma^{<\sigma}) \subseteq \Gamma(I_\Gamma^{<\tau}) = I_\Gamma^\tau. \tag{6.4}$$

We have $I_\Gamma^\sigma \subseteq N^n$ by definition. Hence all I_Γ^τ are countable. By (6.4) it follows by a cardinality argument that there is a countable ordinal $\sigma < \omega_1$, such that $I_\Gamma^{<\sigma} = I_\Gamma^\sigma$. We define

$$||\Gamma|| := \min\{\sigma \mid\ I_\Gamma^{<\sigma} = I_\Gamma^\sigma\} \tag{6.5}$$

and call $||\Gamma||$ the *closure ordinal* of the inductive definition Γ.

6.2.2 Theorem *The fixed–point I_Γ of an inductive definition is the union of its stages I_Γ^σ. We have especially*

$$I_\Gamma = I_\Gamma^{||\Gamma||}.$$

Proof: First we show

$$I_\Gamma^\xi \subseteq I_\Gamma \tag{i}$$

by induction on ξ. The induction hypothesis yields $I_\Gamma^{<\xi} \subseteq I_\Gamma$. By the monotonicity of Γ this entails $I_\Gamma^\xi = \Gamma(I_\Gamma^{<\xi}) \subseteq \Gamma(I_\Gamma) = I_\Gamma$. By definition of $||\Gamma||$ we have $\Gamma(I_\Gamma^{<||\Gamma||}) = I_\Gamma^{||\Gamma||} = I_\Gamma^{<||\Gamma||}$ which shows that $I_\Gamma^{<||\Gamma||}$ is Γ–closed. Hence

$$I_\Gamma \subseteq I_\Gamma^{<||\Gamma||} \tag{ii}$$

and the claim follows by (i) and (ii). ☐

Observe that by (6.4) and the definition of $||\Gamma||$ we have $I_\Gamma^\sigma = I_\Gamma^{||\Gamma||}$ for all $\sigma \geq ||\Gamma||$.

6.2.3 Definition Let Γ be an inductive definition on N. For $n \in N$ we put

$$|n|_\Gamma := \begin{cases} \min\{\sigma \mid\ n \in I_\Gamma^\sigma\} & \text{if } n \in I_\Gamma \\ \omega_1 & \text{otherwise} \end{cases}$$

and call $|n|_\Gamma$ the Γ–*inductive norm of n.*

6.2.4 Theorem *Let Γ be an inductive definition. Then*

$$||\Gamma|| = \sup\{|x|_\Gamma + 1 \mid\ x \in I_\Gamma\}.$$

Proof: We have $|x|_\Gamma < ||\Gamma||$ for all $x \in I_\Gamma$ by definition. Hence $\sigma := \sup\{|x|_\Gamma + 1 \mid\ x \in I_\Gamma\} \leq ||\Gamma||$. Assuming $\sigma < ||\Gamma||$ we get $I_\Gamma^{<\sigma} \subsetneqq I_\Gamma^\sigma$ and find some $x \in I_\Gamma$ such that $\sigma \leq |x|_\Gamma < |x|_\Gamma + 1 \leq \sigma$. A contradiction. ☐

Determining the closure ordinal of an inductive definition is — as we will see — an interesting problem. In the general case, however, all we can say is that it is some countable ordinal. Yet, in special situations we may know more.

6.2.5 Theorem *Let Φ be a finite system of finite clauses, i.e. a finite set Φ such that for all $R \to r \in \Phi$ the set R is finite. Let Γ_Φ be the induced operator. Then $||\Gamma_\Phi|| \leq \omega$.*

Proof: Let I_Φ be the fixed–point of Γ_Φ. We show

$$r \in I_\Phi \Rightarrow |r|_{\Gamma_\Phi} < \omega$$

by induction on the inductive definition of I_Φ. For $r \in I_\Phi$ and $R \to r$ we get $|s|_{\Gamma_\Phi} < \omega$ for all $s \in R$. Since R is finite and there are only finitely many $R \to r \in \Phi$ we obtain

$$\sigma := \sup\big\{|s|_{\Gamma_\Phi} \,\big|\, (\exists R)\,[s \in R \wedge R \to r \in \Phi]\big\} < \omega.$$

Hence $|r|_{\Gamma_\Phi} \leq \sigma + 1 < \omega$. By Theorem 6.2.4 we get $||\Gamma_\Phi|| \leq \omega$.

6.3 Arithmetically definable inductive definitions

We will now concentrate on inductive definitions on the space $\mathbb{N}^{m,n}$. To introduce definable operators we extend the language of arithmetic by n–ary predicate variables which we are going to denote by capital Roman letters in the end of the alphabet, e.g. X, Y, Z, $X_1, \dots$ We will moreover enrich the language by variables for functionals for which we are going to use F, G, $F_1, \dots$ as syntactical variables. Observe that we then obtain additional terms $t(\mathfrak{a})$ which may contain occurences of functional variables and new atomic formulas of the shape $(\vec{x} \in X)$.

6.3.1 Definition An operator $\Gamma\colon \mathsf{Pow}(\mathbb{N}^n) \longrightarrow \mathsf{Pow}(\mathbb{N}^n)$ is definable if there is a formula $\varphi(X, \vec{x})$ in the language of arithmetic whose only free variables are those shown such that

$$\Gamma(S) = \big\{\vec{x} \in \mathbb{N}^n \,\big|\, \mathbb{N} \models \varphi\,[S, \vec{x}]\big\}.$$

We call Γ *arithmetically* or *elementary definable* if its defining formula $\varphi(X, \vec{x})$ does not contain second order quantifiers, i.e. quantifiers ranging over functions. If there are additional function parameters in $\varphi(X, \vec{x}, \vec{\alpha})$ we say that Γ is *definable with parameters*.

Observe that in the case that an operator is definable with parameters, say

$$\Gamma = \big\{\vec{x} \,\big|\, \mathbb{N} \models \varphi[\vec{x}, \vec{\alpha}]\big\},$$

we may denote the dependence on the parameters by $\Gamma(\vec{\alpha})$, i.e. we obtain a relation

$$Q_\Gamma(\vec{x}, \vec{\alpha}) \quad :\Leftrightarrow \quad \vec{x} \in \Gamma(\vec{\alpha}).$$

In this sense we will also talk about relations which are definable by operators.
In order to obtain inductive definitions we need monotone operators. To ensure that definable operators are monotone we have to restrict the class of defining formulas.

6.3.2 Definition We inductively define the class of X–positive formulas by the following clauses:

1) If X does not occur in $\varphi(X)$ then $\varphi(X)$ is X–positive
2) The formula $t \in X$ is X–positive
3) The X–positive formulas are closed under
   - the positive boolean operations $\vee$, $\wedge$
   - quantification over numbers and functions.

6.3.3 Lemma *Let $\varphi(X, x_1, \dots, x_m, \alpha_1, \dots, \alpha_n)$ be an X–positive formula without further free variables. The operator*

$$\Gamma_\varphi(S) := \big\{(x_1,\dots,x_m)\in\mathbb{N}^m \big|\ \mathbb{N}\models[S,x_1,\dots,x_m,f_1,\dots,f_n]\big\},$$

where $f_1,\dots,f_n$ is a fixed n–tuple of functions from $\mathbb{N}$ to $\mathbb{N}$, is a monotone operator.

Proof: Let $S\subseteq T\subseteq\mathbb{N}$. We have to show

$$\varphi\,[S,x_1,\dots,x_m,f_1,\dots,f_n] \Rightarrow \varphi\,[T,x_1,\dots,x_m,f_1,\dots,f_n] \tag{i}$$

and prove (i) by induction on the definition of "$\varphi(X,\vec{x})$ is an X–positive formula". If X does not occur in $\varphi(X,\vec{x},\vec{\alpha})$ then both formulas in (i) are identical. If $\varphi(X,\vec{x},\vec{\alpha})\equiv(\vec{t}\in X)$ then $(\vec{t}^{\,\mathbb{N}}\in S)\Rightarrow(\vec{t}^{\,\mathbb{N}}\in T)$ holds by the hypothesis $S\subseteq T$. The remaining cases follow immediately from the induction hypothesis. □

A monotone operator which is definable by an X–positive formula is called *positively definable*. It is of course unlikely that all definable monotone operators are positively definable. However, it follows from the CRAIG–LYNDON interpolation theorem that at least those definable operators whose monotonicity is logically provable are positively definable. This is because if

$$\models(\forall x)(x\in X\to x\in Y)\to(\forall\vec{y})\,[\varphi(X,\vec{y})\to\varphi(Y,\vec{y})]$$

then there is an interpolation formula, say $\psi(Y,\vec{y})$, in which Y occurs at most positively such that

$$\models(\forall x)(x\in X\to x\in Y)\to(\forall\vec{y})\,[\varphi(X,\vec{y})\to\psi(Y,\vec{y})] \tag{i}$$

and

$$\models(\forall\vec{y})\,[\psi(Y,\vec{y})\to\varphi(Y,\vec{y})]\,. \tag{ii}$$

Choosing $X=Y$ in (i) yields

$$\models(\forall\vec{y})\,[\varphi(Y,\vec{y})\to\psi(Y,\vec{y})] \tag{iii}$$

and (ii) and (iii) show that $\varphi(Y,\vec{y})$ is logically equivalent to a Y–positive formula. □

If Γ is an operator which is definable by some formula φ we write shortly I_φ for I_{Γ_φ}, $||\varphi||$ for $||\Gamma_\varphi||$ and $|n|_\varphi$ for $|n|_{\Gamma_\varphi}$.

6.3.4 Definition Let Γ_φ be the operator which is definable by the X–positive formula $\varphi(X,\vec{x},\vec{\alpha})$ with parameters. For any choice of a tuple of functions $\vec{\alpha}$ we obtain its fixed–point $I_{\varphi(\vec{\alpha})}$ which we denote by $I_\varphi(\vec{\alpha})$. This defines an (n,m)–ary relation. Observe that we may write

$$\vec{x}\in I_\varphi(\vec{\alpha})\ \ \Leftrightarrow\ \ \varphi(I_\varphi,\vec{x},\vec{\alpha})$$

since $\vec{\alpha}$ is not really an argument of I_φ.
A relation $P\subseteq\mathbb{N}^{m,n}$ is *positively arithmetical inductive* over $\mathbb{N}$ if there is an X–positive arithmetical formula $\varphi(X,\vec{x},\vec{y},\vec{\alpha})$ with no other free variables and a tuple $\vec{m}$ such that

$$P=\big\{(\vec{x},\vec{\alpha})\big|\ (\vec{x},\vec{m})\in I_\varphi(\vec{\alpha})\big\}.$$

6.3.5 Remark This is not the strongest way to obtain relations by fixed–points. Another way would be to augment the language by (m,n)–ary relation variables $\mathfrak{X}$, $\mathfrak{Y}$, …and then define operators from a formula $\varphi(\mathfrak{X},\mathfrak{a})$ by

$$\Gamma_\varphi(\mathfrak{S}) := \big\{\mathfrak{a}\big|\ \mathbb{N}\models\varphi[\mathfrak{S},\mathfrak{a}]\big\}.$$

Then one may regard fixed points of such operators. However, in this lecture we will only regard relations whose "function part" comes from the parameters in the defining formula.

We usually omit the the phrase "positively arithmetical" and talk just about inductive relations or relations which are inductive with parameters.
The rest of the section is devoted to the study of the closure properties of inductive relations.

6.3.6 Lemma (Simultaneous inductive definitions) *For any X, Y–positive formulas $\varphi(X, Y, \vec{x}, \vec{\alpha})$ and $\psi(X, Y, \vec{y}, \vec{\alpha})$ we define*

$$I_\varphi^\xi(\vec{\alpha}) := \{\vec{x} \in \mathbb{N}^m \mid \varphi(I_\varphi^{<\xi}, I_\psi^{<\xi}, \vec{x}, \vec{\alpha})\}$$

and

$$I_\psi^\xi(\vec{\alpha}) := \{\vec{y} \in \mathbb{N}^m \mid \psi(I_\varphi^{<\xi}, I_\psi^{<\xi}, \vec{y}, \vec{\alpha})\}.$$

Then we find a Z–positive formula $\chi(Z, z, \vec{x}, \vec{y}, \vec{\alpha})$ and tuples $\vec{m}, \vec{n}$ of the adequate length such that

$$\vec{x} \in I_\varphi(\vec{\alpha}) \;\Leftrightarrow\; (0, \vec{x}, \vec{m}) \in I_\chi(\vec{\alpha})$$

and

$$\vec{y} \in I_\psi(\vec{\alpha}) \;\Leftrightarrow\; (1, \vec{y}, \vec{n}) \in I_\chi(\vec{\alpha})$$

where $I_\varphi(\vec{\alpha}) := \bigcup_{\xi \in On} I_\varphi^\xi(\vec{\alpha})$ and $I_\psi(\vec{\alpha}) := \bigcup_{\xi \in On} I_\psi^\xi(\vec{\alpha})$.

Proof: Choose $\vec{m}$ and $\vec{n}$ of the appropriate arity and put

$$\begin{aligned} \chi(Z, z, \vec{x}, \vec{y}, \vec{\alpha}) :\equiv \quad & [z = 0 \land \varphi(\{\vec{u} \mid (0, \vec{u}, \vec{m}) \in Z\}, \{\vec{v} \mid (1, \vec{n}, \vec{v}) \in Z\}, \vec{x}, \vec{\alpha})] \\ \lor \; & [z = 1 \land \psi(\{\vec{u} \mid (0, \vec{u}, \vec{m}) \in Z\}, \{\vec{v} \mid (1, \vec{n}, \vec{v}) \in Z\}, \vec{y}, \vec{\alpha})]. \end{aligned}$$

Then $\chi(Z, z, \vec{x}, \vec{y}, \vec{\alpha})$ is Z–positive and we show by transfinite induction on ξ

$$\vec{x} \in I_\varphi^\xi(\vec{\alpha}) \;\Leftrightarrow\; (0, \vec{x}, \vec{m}) \in I_\chi^\xi(\vec{\alpha})$$

as well as

$$\vec{y} \in I_\psi^\xi(\vec{\alpha}) \;\Leftrightarrow\; (1, \vec{n}, \vec{y}) \in I_\chi^\xi(\vec{\alpha}).$$

From the induction hypothesis we get

$$\begin{aligned} \vec{x} \in I_\varphi^\xi(\vec{\alpha}) \;&\Leftrightarrow\; \varphi(I_\varphi^{<\xi}, I_\varphi^{<\xi}, \vec{x}, \vec{\alpha}) \\ &\Leftrightarrow\; \varphi(\{\vec{u} \mid (0, \vec{u}, \vec{m}) \in I_\chi^{<\xi}\}, \{\vec{v} \mid (1, \vec{n}, \vec{v}) \in I_\chi^{<\xi}\}, \vec{x}, \vec{\alpha}) \\ &\Leftrightarrow\; \chi(I_\chi^{<\xi}, 0, \vec{x}, \vec{m}, \vec{\alpha}) \\ &\Leftrightarrow\; (0, \vec{x}, \vec{m}) \in I_\chi^\xi(\vec{\alpha}). \end{aligned}$$

Completely analogously we get

$$\begin{aligned} \vec{y} \in I_\psi^\xi(\vec{\alpha}) \;&\Leftrightarrow\; \chi(I_\chi^{<\xi}, 1, \vec{n}, \vec{y}, \vec{\alpha}) \\ &\Leftrightarrow\; (1, \vec{n}, \vec{y}) \in I_\chi^\xi(\vec{\alpha}). \end{aligned}$$

□

In a next step we want to show that the inductive predicates are closed under "positively inductive in".

6.3.7 Lemma *Let $\varphi(X, \vec{x}, \vec{\alpha})$ be an X–positive arithmetical formula and let $\psi(X, Y, \vec{y}, \vec{\alpha})$ be an X, Y–positive arithmetical formula. Put $\tilde{\psi}(X, \vec{y}, \vec{\alpha}) :\equiv \psi(X, I_\varphi(\vec{\alpha}), \vec{y}, \vec{\alpha})$. Then there is an X–positive arithmetical formula $\chi(X, z, \vec{x}, \vec{y}, \vec{\alpha})$ without additional function parameters and a tuple $\vec{m} \in \mathbb{N}^k$ such that*

$$\vec{y} \in I_{\tilde{\psi}}(\vec{\alpha}) \;\Leftrightarrow\; (1, \vec{m}, \vec{y}) \in I_\chi(\vec{\alpha}).$$

Proof: The difficulty is the fact, that $\tilde{\psi}$ is not longer an arithmetical formula. The idea of the proof is to construct I_φ and $I_{\tilde{\psi}}$ simultaneously instead of first finishing I_φ and then start constructing $I_{\tilde{\psi}}$. To improve readability we suppress the parameters $\vec{\alpha}$. We choose tuples $\vec{m}$ and $\vec{n}$ of adequate lengths and put

$$\begin{aligned} \chi(Z, z, \vec{x}, \vec{y}) :\equiv \quad & \big[z = 0 \wedge \varphi(\{\vec{u} |\ (0, \vec{u}, \vec{n}) \in Z\}, \vec{x})\big] \\ \vee \ & \big[z = 1 \wedge \psi(\{\vec{v} |\ (1, \vec{m}, \vec{v}) \in Z\}, \{\vec{u} |\ (0, \vec{u}, \vec{n}) \in Z\}, \vec{y})\big] . \end{aligned}$$

We introduce the abbreviations

$$J_0^\xi := \{\vec{x} |\ (0, \vec{x}, \vec{n}) \in I_\chi^\xi\}$$

and

$$J_1^\xi := \{\vec{y} |\ (1, \vec{m}, \vec{y}) \in I_\chi^\xi\}.$$

We first prove

$$J_0^\xi = I_\varphi^\xi \tag{i}$$

by induction on ξ. From the induction hypothesis $J_0^{<\xi} = I_\varphi^{<\xi}$ we get

$$\begin{aligned} \vec{x} \in J_0^\xi \ & \Leftrightarrow \ (0, \vec{x}, \vec{n}) \in I_\chi^\xi \\ & \Leftrightarrow \ \chi(I_\chi^{<\xi}, 0, \vec{x}, \vec{n}) \\ & \Leftrightarrow \ \varphi(J_0^{<\xi}, \vec{x}) \\ & \Leftrightarrow \ \varphi(I_\varphi^{<\xi}, \vec{x}) \\ & \Leftrightarrow \ \vec{x} \in I_\varphi^\xi. \end{aligned}$$

Obviously $||\chi|| \geq ||\varphi||$ holds. Now we get

$$\begin{aligned} \vec{y} \in J_1^\xi \ & \Leftrightarrow \ (1, \vec{m}, \vec{y}) \in I_\chi^\xi \\ & \Leftrightarrow \ \chi(I_\chi^{<\xi}, 1, \vec{m}, \vec{y}) \\ & \Leftrightarrow \ \psi(J_1^{<\xi}, I_\varphi^{<\xi}, \vec{y}). \end{aligned} \tag{ii}$$

It remains to show that this inductive definition which only uses the initial part $I_\varphi^{<\xi}$ instead of I_φ will eventually catch up with that of $I_{\tilde{\psi}}$. We first show

$$J_1^\xi \subseteq I_{\tilde{\psi}}^\xi \tag{iii}$$

by induction on ξ. This, however, is immediate from $I_\varphi^{<\xi} \subseteq I_\varphi$, the induction hypothesis $J_1^{<\xi} \subseteq I_{\tilde{\psi}}^{<\xi}$, (ii) and the X, Y–positivity of $\psi(X, Y, \vec{y})$. To obtain also the converse inclusion we show

$$\vec{y} \in I_{\tilde{\psi}}^\xi \Rightarrow (1, \vec{m}, \vec{y}) \in I_\chi \tag{iv}$$

by induction on ξ. Using the induction hypothesis and (i) we see

$$\begin{aligned} \vec{y} \in I_{\tilde{\psi}}^\xi \ & \Leftrightarrow \ \psi(I_{\tilde{\psi}}^{<\xi}, I_\varphi, \vec{y}) \\ & \Leftrightarrow \ \psi(I_{\tilde{\psi}}^{<\xi}, J_0, \vec{y}) \\ & \Rightarrow \ \psi(\{\vec{v} |\ (1, \vec{m}, \vec{v}) \in I_\chi\}, \{\vec{u} |\ (0, \vec{u}, \vec{n}) \in I_\chi\}, \vec{y}) \\ & \Leftrightarrow \ \chi(I_\chi, 1, \vec{m}, \vec{y}) \\ & \Leftrightarrow \ (1, \vec{m}, \vec{y}) \in I_\chi. \end{aligned}$$

From (iii) and (iv) we finally get

$$I_{\tilde{\psi}} = \{\vec{y} |\ (1, \vec{m}, \vec{y}) \in I_\chi\}.$$

□

Lemma 6.3.7 generalizes of course to inductive relations. That means we have the following theorem.

6.3.8 Theorem (Substitution Theorem) *Assume that $S_1, \ldots, S_n$ are positively inductive relations and $\varphi(X, Y_1, \ldots, Y_n, \vec{x}, \vec{\alpha})$ is an $X, Y_1, \ldots, Y_n$–positive arithmetical formula. Then the fixed–point of the operator defined by $\varphi(X, S_1, \ldots, S_n, \vec{x}, \vec{\alpha})$ is positively inductive.*

As an easy observation we get

6.3.9 Theorem *Every arithmetical definable relation is positively inductive.*

Proof: Let $P = \big\{(\vec{x}, \vec{\alpha}) \in \mathbb{N}^{m,n} \big|\ \varphi(\vec{x}, \vec{\alpha})\big\}$ for an arithmetical formula $\varphi(\vec{x}, \vec{\alpha})$. Then $\varphi(\vec{x}, \vec{\alpha})$ is X–positive. For its fixed point we get

$$\vec{x} \in I_\varphi(\vec{\alpha}) \ \Leftrightarrow\ \mathbb{N} \models \varphi\,[\vec{x}, \vec{\alpha}]\,,$$

i.e. $I_\varphi = P$. □

6.3.10 Definition A relation $S \subseteq \mathbb{N}^{m,n}$ is *coinductive* if its complement $\mathbb{N}^{m,n} \setminus S$ is inductive. A relation is *hyperelementary* if it is both, inductive and coinductive.

From Theorem 6.3.9 and the Substitution Theorem (Theorem 6.3.8) we already get the basic closure properties of inductive, coinductive and hyperelementary predicates.

6.3.11 Theorem *The inductive and coinductive relations are closed under*

- *positive boolean operations*
- *quantification over numbers*
- *substitution with hyperelementary relations and functions.*

The hyperelementary relations are closed under

- *all boolean operations*
- *quantification over numbers*
- *substitution with hyperelementary relations and functions.*

Proof: Assume that a relation Q is obtained from inductive relations by positive boolean operations and quantification over numbers from inductive relations $S_1, \ldots, S_n$. Then there is a $Y_1, \ldots, Y_n$–positive formula $\varphi(Y_1, \ldots, Y_n, \mathfrak{a})$ such that

$$Q(\mathfrak{a}) \ \Leftrightarrow\ \varphi(S_1, \ldots, S_n, \mathfrak{a}).$$

Regarding $\varphi(Y_1, \ldots, Y_n, \mathfrak{a})$ as X–positive for a dummy variable X we obtain

$$\begin{aligned} Q(\mathfrak{a}) \ &\Leftrightarrow\ \varphi(S_1, \ldots, S_n, \mathfrak{a}) \\ &\Leftrightarrow\ \mathfrak{a} \in I_{\varphi(S_1, \ldots, S_n)} \end{aligned}$$

and Q is inductive by the Substitution Theorem 6.3.8.
There are different possibilities to substitute functions or relations into inductive relations. The most simple one is to define a predicate

$$Q := \big\{\mathfrak{a} \big|\ (f(\mathfrak{a}), \mathfrak{a}) \in S\big\} \tag{i}$$

for an inductive set S and a hyperelementary function f. A function is hyperelementary iff its graph is hyperelementary. Recall that for total functions it suffices to have an inductive (or coinductive) graph in order to be hyperelementary. We get from (i)

$$\mathfrak{a} \in Q \quad \Leftrightarrow \quad (\exists z)\,[f(\mathfrak{a}) = z \wedge (z, \mathfrak{a}) \in S]\,.$$

By the already known closure properites of inductive sets and the fact that $f(\mathfrak{a}) = z$ is inductive we conclude that the predicate Q is inductive. The other possibility to substitute hyperelelementary predicates into relations is to form a relation

$$Q := \big\{(\vec{x}, \vec{\alpha})\big|\ (\vec{y}, \vec{\alpha}, \big\{x\big|\ H(\vec{y}, \vec{\alpha}, x)\big\}) \in R\big\}$$

for a hyperelementary relation H. Let

$$H = \big\{(\vec{y}, \vec{\alpha}, x)\big|\ (\vec{y}, x, \vec{m}_+) \in I_{\psi_+}(\vec{\alpha})\big\} = \big\{(\vec{y}, \vec{\alpha}, x)\big|\ (\vec{y}, x, \vec{m}_-) \notin I_{\psi_-}(\vec{\alpha})\big\}$$

and

$$R = \big\{(\vec{y}, \vec{\alpha}, \alpha^*)\big|\ (\vec{y}, \vec{m}_0) \in I_\varphi(\vec{\alpha}, \alpha^*)\big\}.$$

Then

$$\begin{aligned} Q(\vec{y}, \vec{\alpha}) \;&\Leftrightarrow\; (\vec{y}, \vec{m}_0) \in I_\varphi(\vec{\alpha}, \big\{x\big|\ H(\vec{y}, \vec{\alpha}, x)\big\}) \\ &\Leftrightarrow\; \varphi(I_\varphi, \vec{y}, \vec{\alpha}, \big\{x\big|\ H(\vec{y}, \vec{\alpha}, x)\big\}, \vec{m}_0) \\ &\Leftrightarrow\; \varphi(I_\varphi, \vec{y}, \vec{\alpha}, \big\{x\big|\ (\vec{y}, x, \vec{m}_+) \in I_{\psi_+}(\vec{\alpha})\big\}, \vec{m}_0) \\ &\Leftrightarrow\; \varphi(I_\varphi, \vec{y}, \vec{\alpha}, \big\{x\big|\ (\vec{y}, x, \vec{m}_-) \in I_{\psi_-}(\vec{\alpha})\big\}, \vec{m}_0). \end{aligned}$$

Choosing the positive version I_{ψ_+} or the negative version I_{ψ_-} according to the occurence of the predicate variable Y in $\varphi(X, (\vec{y}, \vec{\alpha}), Y, \vec{y})$ we obtain the claim from Lemma 6.3.7 $\square$

By Lemma 6.1.3 we obtain an upper bound for the complexitiy of arithmetical positive inductive definitions in the analytical hierarchy. We get

6.3.12 Theorem *Every inductive relation is Π^1_1. The coinductive relations are therefore Σ^1_1 and the hyperelementary relations Δ^1_1.*

Proof: Let

$$S = \big\{(\vec{x}, \vec{\alpha})\big|\ (\vec{x}, \vec{k}) \in I_\varphi(\vec{\alpha})\big\} \tag{i}$$

for some X–positive formula $\varphi(X, \vec{x}, \vec{y}, \vec{\alpha})$. By Lemma 6.1.3 I_φ is the least $\Gamma_{\varphi(\vec{\alpha})}$–closed set which implies

$$(\vec{x}, \vec{\alpha}) \in S \quad \Leftrightarrow \quad (\forall X)[(\forall \vec{u})(\forall \vec{v})(\varphi(X, \vec{u}, \vec{v}, \vec{\alpha}) \Rightarrow (\vec{u}, \vec{v}) \in X) \quad \Rightarrow \quad (\vec{x}, \vec{k}) \in X]$$

This is a Π^1_1–definition of S. The remaining claims follow immediately. $\square$

6.4 The stage comparison theorem

Defining the inductive norm $|n|_\varphi$ for objects in I_φ opens the possibility to use elements $n \in I_\varphi$ as ordinal notations. Ordinal notations, however, are of little use as long as we don't know how to compare them. The aim of the present section is to show that the stage comparison predicate is also an inductive predicate.

6.4.1 Definition Let $\varphi(X, \vec{x})$ and $\psi(X, \vec{y})$ be X–positive elementary formulas. We introduce the *stage comparison predicates*

$$\vec{x} \leq^*_{\varphi,\psi} \vec{y} \quad :\Leftrightarrow \quad \vec{x} \in I_\varphi \wedge (\vec{y} \in I_\psi \Rightarrow |\vec{x}|_\varphi \leq |\vec{y}|_\psi) \tag{6.6}$$

and

$$\vec{x} <^*_{\varphi,\psi} \vec{y} \quad :\Leftrightarrow \quad \vec{x} \in I_\varphi \wedge (\vec{y} \in I_\psi \Rightarrow |\vec{x}|_\varphi < |\vec{y}|_\psi). \tag{6.7}$$

Recall that we defined $|\vec{n}|_\varphi = \omega_1$ for $n \notin I_\varphi$. That means that we have

$$\vec{n} \in I_\varphi \quad \Leftrightarrow \quad |\vec{n}|_\varphi < \omega_1.$$

The definitions in (6.6) therefore simplify to

$$\vec{x} \leq^*_{\varphi,\psi} \vec{y} \;\; \Leftrightarrow \;\; \vec{x} \in I_\varphi \wedge |\vec{x}|_\varphi \leq |\vec{y}|_\psi \tag{6.8}$$

and

$$\vec{x} <^*_{\varphi,\psi} \vec{y} \;\; \Leftrightarrow \;\; \vec{x} \in I_\varphi \wedge |\vec{x}|_\varphi < |\vec{y}|_\psi \tag{6.9}$$

respectively.

6.4.2 Theorem (Stage Comparison Theorem) *The stage comparison predicates $\leq^*_{\varphi,\psi}$ and $<^*_{\varphi,\psi}$ as defined in (6.6) and (6.7) are positively inductive.*

Proof: To find the defining formula for the stage comparison predicate we just rewrite its definition in modified form. We have

$$\begin{aligned} \vec{x} \leq^*_{\varphi,\psi} \vec{y} \;\; &\Leftrightarrow \;\; \vec{x} \in I_\varphi^{|\vec{y}|_\psi} \\ &\Leftrightarrow \;\; \varphi(I_\varphi^{<|\vec{y}|_\psi}, \vec{x}) \\ &\Leftrightarrow \;\; \varphi(\{\vec{u} \mid\; |\vec{u}|_\varphi < |\vec{y}|_\psi\}, \vec{x}) \\ &\Leftrightarrow \;\; \varphi(\{\vec{u} \mid\; \neg|\vec{y}|_\psi \leq |\vec{u}|_\varphi\}, \vec{x}). \end{aligned} \tag{i}$$

But for $\vec{u} \in I_\varphi$ we get

$$\begin{aligned} |\vec{y}|_\psi \leq |\vec{u}|_\varphi \;\; &\Leftrightarrow \;\; \vec{y} \in I_\psi^{|\vec{u}|_\varphi} \\ &\Leftrightarrow \;\; \psi(\{\vec{v} \mid\; |\vec{v}|_\psi < |\vec{u}|_\varphi\}, \vec{y}) \\ &\Leftrightarrow \;\; \psi(\{\vec{v} \mid\; \neg(\vec{u} \leq^*_{\varphi,\psi} \vec{v})\}, \vec{y}). \end{aligned} \tag{ii}$$

For the last equivalence observe that

$$\neg(\vec{u} \leq^*_{\varphi,\psi} \vec{v}) \;\; \Leftrightarrow \;\; \vec{u} \notin I_\varphi \vee (\vec{v} \in I_\psi \wedge |\vec{v}|_\psi < |\vec{u}|_\varphi).$$

Therefore assuming $\vec{u} \in I_\varphi$ we have

$$\{\vec{v} \mid\; |\vec{v}|_\psi < |\vec{u}|_\varphi\} = \{\vec{v} \mid\; \neg(\vec{u} \leq_{\varphi,\psi} \vec{v})\}.$$

For $\vec{u} \notin I_\varphi$, however, we have

$$\{\vec{v} \mid\; \neg(\vec{u} \leq_{\varphi,\psi} \vec{v})\} = \mathbb{N}^n.$$

Thus assuming

$$\psi(\mathbb{N}^n, \vec{y}) \tag{iii}$$

we can dispense with the premise $\vec{u} \in I_\varphi$. However, assuming (iii) means no loss of generality. If $\neg\psi(\mathbb{N}^n, \vec{y})$ we modify the formula to

$$\tilde{\psi}(X, \vec{y}) :\equiv \psi(X, \vec{y}) \vee (\forall \vec{z})(\vec{z} \in X)$$

and observe that

$$I_{\tilde{\psi}}^\xi = I_\psi^\xi$$

holds for all $\xi \in \mathit{On}$. Now, plugging (ii) into (i) we get

$$\vec{x} \leq^*_{\varphi,\psi} \vec{y} \;\; \Leftrightarrow \;\; \varphi(\{\vec{u} \mid\; \neg\psi(\{\vec{v} \mid\; \neg(\vec{u} \leq^*_{\varphi,\psi} \vec{v})\}, \vec{y})\}, \vec{x}). \tag{iv}$$

Defining

$$\chi(Z,\vec{x},\vec{y}) :\equiv \varphi(\{\vec{u}|\ \neg\psi(\{\vec{v}|\ \neg(\vec{u},\vec{v}) \in Z\},\vec{y})\},\vec{x})$$

we obtain a Z–positive formula. By (iv) we have

$$\vec{x} \leq^*_{\varphi,\psi} \vec{y} \ \Leftrightarrow\ \chi(\leq^*_{\varphi,\psi},\vec{x},\vec{y}).$$

Hence

$$I_\chi \subseteq \leq^*_{\varphi,\psi}$$

and it remains to show that $\leq^*_{\varphi,\psi}$ is indeed the least fixed–point. We prove

$$\vec{x} \leq^*_{\varphi,\psi} \vec{y} \Rightarrow (\vec{x},\vec{y}) \in I_\chi$$

by induction on $|\vec{x}|_\varphi$. Towards an indirect proof assume

$$\vec{x} \leq^*_{\varphi,\psi} \vec{y} \wedge (\vec{x},\vec{y}) \notin I_\chi.$$

Then we have

$$\neg\varphi(\{\vec{u}|\ \neg\psi(\{\vec{v}|\ \neg(\vec{u},\vec{v}) \in I_\chi\},\vec{y})\},\vec{x})$$

and

$$\varphi(I_\varphi^{<|x|_\varphi},\vec{x})$$

which implies

$$I_\varphi^{<|\vec{x}|_\varphi} \nsubseteq \{\vec{u}|\ \neg\psi(\{\vec{v}|\ \neg(\vec{u},\vec{v}) \in I_\chi\},\vec{y})\}. \tag{v}$$

By (v) there is a $\vec{x}_0 \in I_\varphi^{<|\vec{x}|_\varphi}$ such that

$$\psi(\{\vec{v}|\ \neg(\vec{x}_0,\vec{v}) \in I_\chi\},\vec{y}). \tag{vi}$$

By induction hypothesis, however, we have

$$\{\vec{v}|\ \neg(\vec{x}_0,\vec{v}) \in I_\chi\} \subseteq \{\vec{v}|\ \neg(\vec{x}_0 \leq_{\varphi,\psi} \vec{v})\}. \tag{vii}$$

From (vi) and (vii) we obtain

$$\psi(\{\vec{v}|\ \neg(\vec{x}_0 \leq_{\varphi,\psi} \vec{v})\},\vec{y})$$

which is

$$\psi(I_\psi^{<|\vec{x}_0|_\varphi},\vec{y}).$$

Hence $\vec{y} \in I_\psi^{|\vec{x}_0|_\varphi}$ which means

$$|\vec{y}|_\psi \leq |\vec{x}_0|_\varphi < |\vec{x}|_\varphi$$

in contradiction to $\vec{x} \leq^*_{\varphi,\psi} \vec{y}$.
The proof of the fact that $<^*_{\varphi,\psi}$ is a fixed–point is completely dual and left as an exercise. $\square$

If there is no danger of confusion we write $\vec{x} \leq^* \vec{y}$ and $\vec{x} <^* \vec{y}$ instead of $\vec{x} \leq^*_{\varphi,\psi} \vec{y}$ and $\vec{x} <^*_{\varphi,\psi} \vec{y}$.

We want to extend the norm definition $|x|_\varphi$ to elements of inductive sets. If $S = \{\vec{x}|\ (\vec{x},\vec{k}) \in I_\varphi\}$ it makes no sense to define $|\vec{x}|_S = |(\vec{x},\vec{k})|_\varphi$ since this would leave gaps. However, if we define a predicate

$$\begin{aligned}\vec{x} <_S \vec{y} \ &\Leftrightarrow\ \vec{x} \in S \wedge \vec{y} \in S \wedge |(\vec{x},k)|_\varphi < |(\vec{y},\vec{k})|_\varphi\\ &\Leftrightarrow\ \vec{y} \in S \wedge (\vec{x},\vec{k}) <^*_{\varphi,\varphi} (\vec{y},\vec{k})\end{aligned}$$

this defines a well–founded predicate and we may define

$$|\vec{x}|_S := \sup\{|\vec{y}|_S + 1 \mid \vec{y} <_S \vec{x}\}. \tag{6.10}$$

Observe that we can do the same constuction for inductive relations. Assume that

$$S = \{(\vec{\alpha}, \vec{x}) \mid (\vec{x}, \vec{k}) \in I_\varphi(\vec{\alpha})\}.$$

Define

$$(\vec{\alpha}, \vec{x}) <_S (\vec{\beta}, \vec{y}) \;\Leftrightarrow\; (\vec{y}, \vec{k}) \in I_\varphi(\vec{\beta}) \wedge (\vec{x}, \vec{k}) <^*_{\varphi(\vec{\alpha}),\varphi(\vec{\beta})} (\vec{y}, \vec{k}) \tag{6.11}$$

and then $|(\vec{\alpha}, \vec{x})|_S$ as in (6.10). We will, however, for the sake of simpler notations, mostly talk of predicates or even rather sets. But you should always tacitly check how far the results relativize. This will be the case nearly everywhere. We try to mention the cases where this becomes wrong. To enter a more general framework we introduce the following notations.

6.4.3 Definition Let $S \subseteq \mathbb{N}^{m,n}$ and

$$\mu \colon S \xrightarrow{onto} \lambda \in On$$

be a mapping. We call μ an *inductive norm* if there are an inductive relation J and a coinductive relation $\breve{J}$ such that for all $\mathfrak{b} \in S$ we have

$$\begin{aligned} \mathfrak{a} \in S \wedge \mu(\mathfrak{a}) \le \mu(\mathfrak{b}) \;&\Leftrightarrow\; J(\mathfrak{a}, \mathfrak{b}) \\ &\Leftrightarrow\; \breve{J}(\mathfrak{a}, \mathfrak{b}). \end{aligned} \tag{6.12}$$

There is a uniform way of expressing J and $\breve{J}$. We prove

6.4.4 Lemma *Let λ be an ordinal and $\mu \colon S \xrightarrow{onto} \lambda$ be a mapping onto λ. The norm given by μ is inductive iff the relations*

$$\mathfrak{a} \preceq^*_S \mathfrak{b} \;:\Leftrightarrow\; \mathfrak{a} \in S \wedge [\mathfrak{b} \in S \Rightarrow \mu(\mathfrak{a}) \le \mu(\mathfrak{b})] \tag{6.13}$$

and

$$\mathfrak{a} \prec^*_S \mathfrak{b} \;:\Leftrightarrow\; \mathfrak{a} \in S \wedge [\mathfrak{b} \in S \Rightarrow \mu(\mathfrak{a}) < \mu(\mathfrak{b})] \tag{6.14}$$

are inductive.

Proof: If $\preceq^*_S$ and $\prec^*_S$ are both inductive we put

$$J(\mathfrak{a}, \mathfrak{b}) \;:\Leftrightarrow\; \mathfrak{a} \preceq^*_S \mathfrak{b}$$

and

$$\breve{J}(\mathfrak{a}, \mathfrak{b}) \;:\Leftrightarrow\; \neg(\mathfrak{b} \prec^*_S \mathfrak{a})$$

and check easily that J and $\breve{J}$ satisfy (6.12).
Thus assume that μ is an inductive norm whose accompanying predicates are J and $\breve{J}$. Then we obtain

$$\mathfrak{a} \preceq^*_S \mathfrak{b} \;\Leftrightarrow\; \mathfrak{a} \in S \wedge \Big[J(\mathfrak{a}, \mathfrak{b}) \vee \neg\breve{J}(\mathfrak{b}, \mathfrak{a})\Big]$$

and

$$\mathfrak{a} \prec^*_S \mathfrak{b} \;\Leftrightarrow\; \mathfrak{a} \in S \wedge \breve{J}(\mathfrak{b}, \mathfrak{a}). \qquad \square$$

It will follow that the norm defined in (6.10) is inductive. We prove

6.4.5 Theorem *Let S be an inductive relation. Say $S = \{(\vec{\alpha}, \vec{x}) \mid (\vec{x}, \vec{k}) \in I_\varphi(\vec{\alpha})\}$ for some X–positive arithmetical formula $\varphi(X, \vec{x}, \vec{y}, \vec{\alpha})$. Then the norm defined in (6.10)*

$$\begin{aligned} |\ |_S \colon S &\longrightarrow ||S|| := \sup\{|\mathfrak{a}|_S + 1 |\ \mathfrak{a} \in S\} \\ \mathfrak{a} &\longmapsto |\mathfrak{a}|_S := \sup\{|\mathfrak{b}|_S + 1 |\ \mathfrak{b} <_S \mathfrak{a}\} \end{aligned}$$

is an inductive norm. This shows that every inductive set possesses an inductive norm.

Proof: By definition $|\ |_S$ is a map from S onto $||S||$. Because of

$$(\vec{\alpha}, \vec{x}) \preceq^*_S (\vec{\beta}, \vec{y}) \quad \Leftrightarrow \quad (\vec{x}, \vec{k}) \leq^*_{\varphi(\vec{\alpha}),\varphi(\vec{\beta})} (\vec{y}, \vec{k})$$

and

$$(\vec{\alpha}, \vec{x}) \prec^*_S (\vec{\beta}, \vec{y}) \quad \Leftrightarrow \quad (\vec{x}, \vec{k}) <^*_{\varphi(\vec{\alpha}),\varphi(\vec{\beta})} (\vec{y}, \vec{k})$$

we obtain $\preceq^*_S$ and $\prec^*_S$ as inductive. Hence $|\ |_S$ is inductive by Lemma 6.4.4. □

7. Inductive Definitions, Π^1_1–sets and the ordinal ω_1^{CK}

7.1 Π^1_1–sets vs. inductive sets

In Theorem 6.3.12 we have shown that all elementary positive inductive sets are Π^1_1–definable. Our next aim is to show that conversely every Π^1_1–set is inductively definable. The first step is to define a normal–form for Π^1_1–relations. Let P be some (m,n)–ary Π^1_1–relation. Then

$$P(\mathfrak{a}) \;\Leftrightarrow\; (\forall\alpha)(\exists y)R(\alpha,y,\mathfrak{a})$$

and the relation $(\exists y)R(\alpha,y,\mathfrak{a})$ is semi–decidable. But then there is some decidable predicate R' such that

$$(\exists y)R(\alpha,y,\mathfrak{a}) \;\Leftrightarrow\; (\exists y)R'(\overline{\alpha}(y),y,\overline{\mathfrak{a}}(y))$$

and we define

$$R_P(s,\mathfrak{a}) \;:\Leftrightarrow\; R'(s,\mathit{lh}(s),\overline{\mathfrak{a}}(\mathit{lh}(s))).$$

Then we get

7.1.1 Lemma (Π^1_1–normal form) *For every Π^1_1–relation P there is a decidable relation R_P such that*

$$P(\mathfrak{a}) \;\Leftrightarrow\; (\forall\alpha)(\exists y)R_P(\overline{\alpha}(y),\mathfrak{a}).$$

We use Lemma 7.1.1 in the following definition

7.1.2 Definition Let

$$P(\mathfrak{a}) \;\Leftrightarrow\; (\forall\alpha)(\exists y)R_P(\overline{\alpha}(y),\mathfrak{a})$$

be a Π^1_1–relation in normal form. We define

$$T_P(\mathfrak{a}) := \big\{s\in \mathit{Seq}\,|\;(\forall s_0)(s_0\subsetneq s\Rightarrow \neg R_P(s_0,\mathfrak{a}))\big\} \tag{7.1}$$

and call T_P the *tree of unsecured sequences* for P.

It is an immediate consequence of (7.1) that $T_P(\mathfrak{a})$ is a tree. We have

$$\begin{aligned} P(\mathfrak{a}) \;&\Leftrightarrow\; (\forall\alpha)(\exists y)R_P(\overline{\alpha}(y),\mathfrak{a})\\ &\Leftrightarrow\; (\forall\alpha)(\exists y)\,[\overline{\alpha}(y)\notin T_P(\mathfrak{a})]\\ &\Leftrightarrow\; T_P(\mathfrak{a}) \text{ is well-founded.}\end{aligned}$$

Observe that the quantifier in (7.1) is bounded. Hence $T_P(\mathfrak{a})$ is decidable in $\mathfrak{a}$ which means that its characteristic function has the form $\lambda x\,.\,F_P(\mathfrak{a},x)$ for some computable functional F_P and we have shown

7.1.3 Theorem *For every Π^1_1–relation P there is a computable functional F_P such that*

$$P(\mathfrak{a}) \;\Leftrightarrow\; \lambda x\,.\,F_P(\mathfrak{a},x)\in\mathbb{WT}.$$

If P is an n–ary predicate then $T_P(\vec{x})$ is decidable and an index for $T_P(\vec{x})$ can be computed from $\vec{x}$. Thus Theorem 7.1.3 modifies to

7.1.4 Theorem *For every Π^1_1–predicate P there is a computable function T_P such that*

$$P(\vec{x}) \;\Leftrightarrow\; T_P(\vec{x}) \in \mathit{WT}.$$

By Theorem 7.1.4 we have $P \leq_m \mathit{WT}$ for every Π^1_1–predicate P. We say that WT is Π^1_1–*complete*.
To establish the connection between Π^1_1–predicates and inductive sets we study well–founded trees in terms of fixed–points. Let T be a tree and put

$$\varphi_T(X,x) :\equiv (\forall y)\,[x^\frown\langle y\rangle \in T \;\Rightarrow\; x^\frown\langle y\rangle \in X]\,. \tag{7.2}$$

Then $\varphi_T(X,x)$ is an X–positive formula. Denote its fixed–point by I_T. We prove

$$T \in \mathbb{WT} \wedge s \in T \;\Rightarrow\; s \in I_T^{\mathit{otyp}_T(s)} \tag{7.3}$$

by induction on $\mathit{otyp}_T(s)$. If

$$\mathit{otyp}_T(s) = \sup\bigl\{\mathit{otyp}_T(t)+1 \bigm|\, t <_T^* s\bigr\} = 0$$

then $\bigl\{t \bigm|\, t <_T^* s\bigr\} = \emptyset$ which implies that $s^\frown\langle y\rangle \notin T$ for all y. But then $\varphi_T(\emptyset,s)$ which shows $s \in I_T^0$. If $\mathit{otyp}_T(s) =: \sigma > 0$ then $\mathit{otyp}_T(s^\frown\langle y\rangle) < \sigma$ for all y such that $s^\frown\langle y\rangle \in T$. By the induction hypothesis we get

$$(\forall y)\,\bigl[s^\frown\langle y\rangle \in T \Rightarrow s^\frown\langle y\rangle \in I_T^{<\sigma}\bigr]$$

which is $\varphi_T(I_T^{<\sigma},s)$. Hence $s \in I_T^\sigma$.
From (7.3) we get

$$T \in \mathbb{WT} \wedge s \in T \Rightarrow |s|_T \leq \mathit{otyp}_T(s) \tag{7.4}$$

where $|s|_T := |s|_{\varphi_T}$ denotes the φ_T–norm of s. To obtain also the converse inequality we prove

$$s \in T \wedge s \in I_T \Rightarrow T{\upharpoonright}s \in \mathbb{WT} \wedge \mathit{otyp}(T{\upharpoonright}s) \leq |s|_T \tag{7.5}$$

by induction on $|s|_T$. If $|s|_T = 0$ we have $(\forall y)\,[s^\frown\langle y\rangle \notin T]$ which shows $T{\upharpoonright}s = \langle\rangle$ and $\mathit{otyp}(T{\upharpoonright}s) = 0$. So assume $|s|_T =: \sigma > 0$. Since $s \in I_T^\sigma$ we get $\varphi_T(I_T^{<\sigma},s)$ which is

$$(\forall y)\,\bigl[s^\frown\langle y\rangle \in T \Rightarrow s^\frown\langle y\rangle \in I_T^{<\sigma}\bigr]\,. \tag{i}$$

By induction hypothesis we get

$$(\forall y)\,[s^\frown\langle y\rangle \in T \Rightarrow T{\upharpoonright}s^\frown\langle y\rangle \in \mathbb{WT} \wedge \mathit{otyp}(T{\upharpoonright}s^\frown\langle y\rangle) < \sigma]\,. \tag{ii}$$

An infinite path in $T{\upharpoonright}s$ would induce an infinite path in one of the trees $T{\upharpoonright}s^\frown\langle y\rangle$ which is impossible by (ii). So $T{\upharpoonright}s$ is well–founded and by (5.28) we get

$$\mathit{otyp}(T{\upharpoonright}s) = \mathit{otyp}_{T{\upharpoonright}s}(\langle\rangle) = \sup\bigl\{\mathit{otyp}(T{\upharpoonright}s^\frown\langle y\rangle)+1 \bigm|\, s^\frown\langle y\rangle \in T\bigr\} \leq \sigma. \qquad \Box$$

It follows from (7.5) that a tree T is well–founded if $\langle\rangle \in I_T$. Conversely we have $\langle\rangle \in I_T$ for well–founded trees T by (7.4). Therefore we have shown

7.1.5 Theorem *A tree T is well–founded iff $\langle\rangle \in I_T$. For well–founded trees we get*

$$s \in T \Rightarrow \mathit{otyp}_T(s) = |s|_T \tag{7.6}$$

and

$$\mathit{otyp}(T) + 1 = ||\varphi_T||. \tag{7.7}$$

Proof: We proved everything but (7.7). But this is simple because $\mathit{otyp}(T) = \mathit{otyp}_T(\langle\rangle) = |\langle\rangle|_T > |s|_T$ for all $s \in T$ such that $s \neq \langle\rangle$. By Theorem 6.2.4, however, it then follows

$$||\varphi_T|| = |\langle\rangle|_T + 1 = \mathit{otyp}(T) + 1. \qquad \square$$

The link between Π^1_1–relations and inductive relations is given by Theorems 7.1.3 and 7.1.5. We get

7.1.6 Theorem *The Π^1_1–relations are exactly the positively inductive relations on $\mathbb{N}$.*

Proof: We have by Theorem 6.3.12 that positively inductive relations are Π^1_1. Conversely if P is a Π^1_1–relation then we get by Theorem 7.1.3

$$P(\mathfrak{a}) \;\Leftrightarrow\; \lambda x \,.\, F_P(\mathfrak{a}, x) \in \mathbb{WT}.$$

Putting $\varphi_P(X, s, \mathfrak{a}) \;:\Leftrightarrow\; (\forall y)\,[F_P(\mathfrak{a}, s^\frown\langle y\rangle) = 0 \Rightarrow s^\frown\langle y\rangle \in X]$ we get by Theorem 7.1.5

$$P(\mathfrak{a}) \;\Leftrightarrow\; (\langle\rangle, \mathfrak{a}) \in I_{\varphi_P}.$$

Hence P is inductive. $\square$

Dealing with predicates we can sharpen Theorem 7.1.6 as follows

7.1.7 Theorem *There is an X–positive elementary formula $\varphi_P(X, s, \vec{x})$ such that for any Π^1_1–predicate P we have*

$$P(\vec{x}) \;\Leftrightarrow\; (\langle\rangle, \vec{x}) \in I_{\varphi_P}.$$

Proof: By Theorem 7.1.4 we have

$$P(\vec{x}) \;\Leftrightarrow\; T_P(\vec{x}) \in \mathit{WT}$$

for a computable function T_P. We define

$$\varphi_P(X, s, \vec{x}) \;\Leftrightarrow\; (\forall y)\left[\{T_P(\vec{x})\}^{1,0}(s^\frown\langle y\rangle) = 0 \;\Rightarrow\; (s^\frown\langle y\rangle, \vec{x}) \in X\right]. \tag{7.8}$$

Then $\varphi_P(X, s, \vec{x})$ is X–positive and elementary and by Theorem 7.1.5 we get

$$T_P(\vec{x}) \in \mathit{WT} \;\Leftrightarrow\; (\langle\rangle, \vec{x}) \in I_{\varphi_P}. \qquad \square$$

7.1.8 Remark Although we proved in Theorem 6.3.12 that fixed–points of arithmetically definable monotone operators are Π^1_1–definable we did *not* prove the converse proposition in Theorem 7.1.6 (or 7.1.7). All we showed is that Π^1_1–relations are inductive but not necessarily fixed–points. The additional parameter — which is $\langle\rangle$ in our setting — is indispensable, even for certain Δ^1_1–relations. A proof of this fact, however, is outside the scope of this lecture. It can be found in [1].

As a consequence of Theorem 7.1.6 and 7.1.7 we get the following corollaries.

7.1.9 Corollary *The Π^1_1–relations are exactly the positively inductive relations on $\mathbb{N}$. The Σ^1_1–relations are exactly the positively coinductive relations on $\mathbb{N}$. The Δ^1_1–relations are exactly the hyperelementary relations on $\mathbb{N}$.*

7.1.10 Corollary *The Π^1_1–predicates are exactly the positively elementary inductive predicates on $\mathbb{N}$. The Σ^1_1–predicates are exactly the positively elementary coinductive predicates on $\mathbb{N}$ and the Δ^1_1–predicates exactly the hyperelementary predicates.*

7.2 The inductive closure ordinal of $\mathbb{N}$

Let us return to the general situation. Developing the theory of inductive definitions in Chapter 6 we did not make use of special features of the structure $\mathbb{N}$ of natural numbers. The fact that we restricted ourselves to unary predicate variables was sheer lazyness. Without the possibility of contracting n–ary predicate variables to unary ones we could have developed the same theory using n–ary predicate variables. [But observe that we did make use of special features of $\mathbb{N}$ in Section 7.1.] Let $\mathcal{A}$ by any structure and call a first order formula in the language of $\mathcal{A}$ $\mathcal{L}_{\mathcal{A}}$–elementary if it contains no function or set parameters. We define

$$\kappa^{\mathcal{A}} := \sup\big\{||\varphi||\,|\ \varphi(X,\vec{x}) \text{ is an } X\text{–positive } \mathcal{L}_{\mathcal{A}}\text{–elementary formula}\big\}.$$

and call $\kappa^{\mathcal{A}}$ the *(inductive) closure ordinal* of the structure $\mathcal{A}$. Our aim is to characterize $\kappa^{\mathbb{N}}$. But before doing that we give some abstract consequences of the Stage Comparison Theorem.

7.2.1 Lemma *Let $\varphi(X,\vec{x})$ be an elementary X–positive formula. Then I^ξ_φ is hyperelementary for any $\xi < \kappa^{\mathbb{N}}$. Especially if $||\varphi|| < \kappa^{\mathbb{N}}$ then I_φ is hyperelementary.*

Proof: The proof depends heavily on the Stage Comparison Theorem. The Lemma is true for arbitrary structures $\mathcal{A}$ replacing $\mathbb{N}$. But, since we want to concentrate on $\mathbb{N}$, we only stated it as above. For $\xi < \kappa^{\mathbb{N}}$ we find an elementary inductive definition $\psi(Y,y)$ and an $n \in I_\psi$ such that $|n|_\psi = \xi$. Using stage comparison we get

$$\begin{aligned} \vec{x} \in I^\xi_\varphi \quad &\Leftrightarrow \quad \vec{x} \leq^*_{\varphi,\psi} n \\ &\Leftrightarrow \quad \neg(n <^*_{\psi,\varphi} \vec{x}). \end{aligned}$$

Hence I^ξ_φ is hyperelementary. $\square$

7.2.2 Theorem (Closure Theorem) *The fixed–point of an elementary inductive definition $\varphi(X,\vec{x})$ is hyperelementary iff $||\varphi|| < \kappa^{\mathbb{N}}$.*

Proof: One direction is Lemma 7.2.1. For the other direction let I_φ be hyperelementary and define

$$\begin{aligned} \chi(Z,z,\vec{x}) :\equiv \quad &\big[z = 0 \wedge \varphi(\{\vec{u}|\ (0,\vec{u}) \in Z\},\vec{x})\big] \\ &\vee\ [z = 1 \wedge (\forall \vec{y})(\vec{y} \in I_\varphi \to (0,\vec{y}) \in Z)]\,. \end{aligned}$$

A close look at the proof of Lemma 6.3.7 shows that there is a positively elementary formula θ with $\kappa^{\mathbb{N}} \geq ||\theta|| \geq ||\chi||$, furthermore I_χ is trivially contained in the elementary inductive set I_θ, thus it is elementary inductive, too. First we show

$$I^\xi_\varphi = \big\{\vec{x}|\ (0,\vec{x}) \in I^\xi_\chi\big\} \tag{i}$$

by induction on ξ. From the induction hypothesis we get

$$\begin{aligned} \vec{x} \in I^\xi_\varphi \quad &\Leftrightarrow \quad \varphi(I^{<\xi}_\varphi,\vec{x}) \\ &\Leftrightarrow \quad \varphi(\{\vec{x}|\ (0,\vec{x}) \in I^{<\xi}_\chi,\vec{x})\} \\ &\Leftrightarrow \quad \chi(I^{<\xi}_\chi,0,\vec{x}) \\ &\Leftrightarrow \quad (0,\vec{x}) \in I^\xi_\chi. \end{aligned}$$

As a consequence of (i) we get

$$I_\varphi = \big\{\vec{x}|\ (0,\vec{x}) \in I^{<||\varphi||}_\chi\big\}. \tag{ii}$$

For any $\xi < ||\varphi||$ there is a $\vec{y} \in I_\varphi$ such that $\vec{y} \notin I^\xi_\varphi$, i.e. $(0,\vec{y}) \notin I^\xi_\chi$ by (i). Therefore we have

$$\neg\chi(I_\chi^\xi, 1, \vec{x}) \tag{iii}$$

for any $\vec{x}$ and $\xi < ||\varphi||$. By (ii), however, we have

$$\chi(I_\chi^{<||\varphi||}, 1, \vec{x}) \tag{iv}$$

for all $\vec{x}$. Hence by (iii) and (iv)

$$(1, \vec{x}) \in I_\chi^{||\varphi||} \setminus I_\chi^{<||\varphi||}. \tag{v}$$

From (v) we finally obtain $||\varphi|| = |(1, \vec{x})|_\chi < \kappa^{\mathbb{N}}$. □

As a consequence of the Closure Theorem (Theorem 7.2.2) we obtain a characterization of the closure ordinal $\kappa^{\mathbb{N}}$.

7.2.3 Theorem *The inductive closure ordinal of the structure of natural numbers is ω_1^{CK}.*

Proof: By Theorem 5.4.9 we have

$$\omega_1^{CK} = \sup\{\mathit{otyp}^{\mathit{Tree}}(e) \mid e \in \mathit{WT}\}.$$

However, if T is a decidable well–founded tree, we get by (7.7)

$$\mathit{otyp}(T) + 1 = ||\varphi_T|| \le \kappa^{\mathbb{N}}$$

since φ_T is an elementary formula. Hence

$$\omega_1^{CK} \le \kappa^{\mathbb{N}}.$$

Assume $\omega_1^{CK} < \kappa^{\mathbb{N}}$. Choose some predicate $P \in \Pi_1^1 \setminus \Delta_1^1$. Such P exists by the Analytical Hierarchy Theorem. Now we apply Theorem 7.1.7 to obtain

$$\begin{aligned} P(\vec{x}) &\Leftrightarrow T_P(\vec{x}) \in \mathit{WT} \\ &\Leftrightarrow (\langle\rangle, \vec{x}) \in I_{\varphi_P}. \end{aligned}$$

By (7.7) we have

$$\begin{aligned} ||\varphi_P|| &\le \sup\{\mathit{otyp}^{\mathit{Tree}}(T_P(\vec{x})) + 1 \mid \vec{x} \in P\} \\ &\le \sup\{\mathit{otyp}^{\mathit{Tree}}(e) + 1 \mid e \in \mathit{WT}\} \\ &= \omega_1^{CK} < \kappa^{\mathbb{N}}. \end{aligned}$$

It follows from the Closure Theorem (Theorem 7.2.3) that I_φ is hyperelementary which by Corollary 7.1.10 entails that I_φ is Δ_1^1. But this contradicts the choice of P. □

To obtain further characterizations of $\kappa^{\mathbb{N}}$ — and thus also of ω_1^{CK} — we introduce some notations.

7.2.4 Definition A binary well–founded predicate $\prec$ is a *pre–well–ordering* iff

$$x \prec y \;\Leftrightarrow\; x \in \mathit{field}(\prec) \wedge y \in \mathit{field}(\prec) \wedge \mathit{otyp}_\prec(x) < \mathit{otyp}_\prec(y).$$

Pre–well–orderings are closely connected to norms.

7.2.5 Lemma *Let $\mu\colon S \xrightarrow{\mathit{onto}} \lambda$ be a norm. The predicate $\prec_\mu$ defined by*

$$\vec{x} \prec_\mu \vec{y} \;:\Leftrightarrow\; \vec{x} \in S \wedge \vec{y} \in S \wedge \mu(\vec{x}) < \mu(\vec{y})$$

is a pre–well–ordering such that

$$\mathit{otyp}_{\prec_\mu}(\vec{x}) = \mu(\vec{x})$$

holds for all $\vec{x} \in S$.

Proof: The predicate $\prec_\mu$ is obviously well–founded. So we only have to prove

$$\vec{x} \in S \Rightarrow \mathit{otyp}_{\prec_\mu}(\vec{x}) = \mu(\vec{x}). \tag{i}$$

This is done by induction on $\prec_\mu$. Using the induction hypothesis we compute

$$\begin{aligned} \mathit{otyp}_{\prec_\mu}(\vec{x}) &= \sup\{\mathit{otyp}_{\prec_\mu}(\vec{y}) + 1 \mid \vec{y} \prec_\mu \vec{x}\} \\ &= \sup\{\mathit{otyp}_{\prec_\mu}(\vec{y}) + 1 \mid \mu(\vec{y}) < \mu(\vec{x})\} \\ &= \sup\{\mu(\vec{y}) + 1 \mid \mu(\vec{y}) < \mu(\vec{x})\} \\ &= \mu(\vec{x}). \end{aligned}$$

□

7.2.6 Theorem *We have*

$$\begin{aligned} \kappa^{\mathbb{N}} &= \sup\{\mathit{otyp}(\prec) \mid \prec \text{ is a hyperelementary pre–well–ordering}\} \\ &= \sup\{\mathit{otyp}(\prec) \mid \prec \text{ is a hyperelementary well–founded binary predicate}\} \\ &= \sup\{\mathit{otyp}(\prec) \mid \prec \text{ is a coinductive well–founded binary predicate}\}. \end{aligned}$$

However, none of these suprema is attained.

Proof: Before we start proving the theorem we want to mention that it is true for arbitrary structures. Put

$$\sigma_{\text{hp}} := \sup\{\mathit{otyp}(\prec) \mid \prec \text{ is a hyperelementary pre–well–ordering}\}$$

$$\sigma_{\text{hf}} := \sup\{\mathit{otyp}(\prec) \mid \prec \text{ is a hyperelementary well–founded binary predicate}\}$$

and

$$\sigma_{\text{cf}} := \sup\{\mathit{otyp}(\prec) \mid \prec \text{ is a coinductive well–founded binary predicate}\}.$$

Starting with an elementary X–positive formula $\varphi(X, \vec{x})$ we construct for every $\vec{x}_0 \in I_\varphi$ a hyperelementary pre–well–ordering $\prec_{\vec{x}_0}$ such that

$$|\vec{x}_0|_\varphi + 1 \leq \mathit{otyp}(\prec_{\vec{x}_0}). \tag{i}$$

Then (i) proves $\kappa^{\mathbb{N}} \leq \sigma_{\text{hp}}$. Since

$$\sigma_{\text{hp}} \leq \sigma_{\text{hf}} \leq \sigma_{\text{cf}}$$

holds trivially it then remains to show

$$\sigma_{\text{cf}} \leq \kappa^{\mathbb{N}} \tag{ii}$$

to finish the proof. Let's prove (i). Choose $\vec{x}_0 \in I_\varphi$ and define

$$\begin{aligned} \vec{x} \prec_{\vec{x}_0} \vec{y} :&\Leftrightarrow |\vec{x}|_\varphi < |\vec{y}|_\varphi \leq |\vec{x}_0|_\varphi \\ &\Leftrightarrow x <^*_{\varphi,\varphi} \vec{y} \leq^*_{\varphi,\varphi} \vec{x}_0 \\ &\Leftrightarrow \neg(\vec{y} \leq^*_{\varphi,\varphi} \vec{x}) \wedge \neg(\vec{x}_0 <^*_{\varphi,\varphi} \vec{y}). \end{aligned} \tag{iii}$$

Then it is clear from (iii) that $\prec_{\vec{x}_0}$ is hyperelementary and well–founded. By Lemma 7.2.5 it is also a pre–well–ordering such that

$$\mathit{otyp}_{\prec_{\vec{x}_0}}(\vec{x}) = |\vec{x}|_\varphi.$$

Therefore we obtain

$$\begin{aligned} \mathit{otyp}(\prec_{\vec{x}_0}) &= \sup\{\mathit{otyp}_{\prec_{\vec{x}_0}}(\vec{x})+1 \mid |\vec{x}|_\varphi \le |\vec{x}_0|_\varphi\} \\ &= \sup\{|\vec{x}|_\varphi+1 \mid |\vec{x}|_\varphi \le |\vec{x}_0|_\varphi\} \\ &= |\vec{x}_0|_\varphi+1. \end{aligned}$$

To prove (ii) let $\prec$ be a coinductive well–founded binary predicate. Recall the definition of the accessible part $\mathit{Acc}(\prec)$ of $\prec$ which is the fixed–point of the formula

$$\varphi_\prec(X,x) :\equiv (\forall y)(y \prec x \to y \in X).$$

Denote by $\mathit{Acc}^\xi(\prec)$ the ξ–th stage of this fixed–point. We prove

$$x \in \mathit{Acc}^\xi(\prec) \Rightarrow \mathit{otyp}_\prec(x) \le \xi \tag{iv}$$

by transfinite induction on ξ. For $x \in \mathit{Acc}^\xi(\prec)$ we get $(\forall y)\left[y \prec x \to y \in \mathit{Acc}^{<\xi}(\prec)\right]$ which by induction hypothesis gives

$$\mathit{otyp}_\prec(x) = \sup\{\mathit{otyp}_\prec(y)+1 \mid y \prec x\} \le \xi.$$

Now we prove

$$x \in \mathit{Acc}^{\mathit{otyp}_\prec(x)}(\prec) \tag{v}$$

by induction on $\prec$. From the induction hypothesis we get

$$(\forall y)(y \prec x \Rightarrow y \in \mathit{Acc}^{<\mathit{otyp}_\prec(x)}(\prec))$$

which entails immediately

$$x \in \mathit{Acc}^{\mathit{otyp}_\prec(x)}(\prec).$$

From (iv) and (v), however, we obtain

$$\mathit{otyp}_\prec(x) = |x|_{\mathit{Acc}(\prec)} \tag{7.9}$$

which holds for arbitrary well–founded predicates. From (7.9) and (6.2) we get

$$\begin{aligned} \mathit{otyp}(\prec) &= \sup\{\mathit{otyp}_\prec(x)+1 \mid x \in \mathit{field}(\prec)\} \\ &\le \sup\{|x|_{\mathit{Acc}(\prec)}+1 \mid x \in \mathit{Acc}(\prec)\} \\ &= ||\varphi_\prec||. \end{aligned} \tag{vi}$$

Since $\prec$ is coinductive we get by Lemma 6.3.7 that $\mathit{Acc}(\prec) = I_{\varphi_\prec}$ is inductive. Hence

$$||\varphi_\prec|| \le \kappa^{\mathbb{N}} \tag{vii}$$

and we get from (vi) and (vii)

$$\sigma_{\mathrm{cf}} \le \kappa^{\mathbb{N}}.$$

It remains to show that none of the suprema is attained. For that it suffices to show that σ_{cf} is not attained. This, however, is obvious since for a given coinductive well–founded predicate $\prec$ we define

$$\begin{aligned} x \prec' y \;\;:\Leftrightarrow\;\; & \mathit{Seq}(x) \land \mathit{Seq}(y) \land \mathit{lh}(x) = \mathit{lh}(y) = 2 \\ & \land\ [((x)_0 = (y)_0 = 0 \land (x)_1 \prec (y)_1) \lor (y)_0 = (y)_1 = 1]\,. \end{aligned}$$

Then $\prec'$ is a coinductive well–founded predicate, too, and $\mathit{otyp}(\prec') \ge \mathit{otyp}(\prec)+1$. $\square$

Recalling Theorems 6.3.12, 7.2.6 and 7.2.3 we have shown

7.2.7 Theorem *The ordinal ω_1^{CK} is the supremum of the order–types of Σ^1_1–definable well–orderings. This supremum is not attained, i.e. the order–type of any well–founded Σ^1_1–definable predicate is less than ω_1^{CK}.*

There is an extension of Theorem 7.2.7 to Σ^1_1–definable collections of well–orderings.

7.2.8 Theorem (Boundedness Principle) *Let P be a Σ^1_1–definable subset of WO (or WT). Then*

$$\sup\{\mathit{otyp}^{\mathit{WO}}(e)|\ e \in P\} < \omega_1^{CK}$$

(or $\sup\{\mathit{otyp}^{\mathit{Tree}}(e)|\ e \in P\} < \omega_1^{CK}$*).*
If P is a Σ^1_1–definable subset of $\mathbb{WO}$ (or $\mathbb{WT}$) then

$$\sup\{\mathit{otyp}(\alpha)|\ \alpha \in P\} < \omega_1^{CK}.$$

Proof: Similarly to Theorem 7.2.3, the key to the proof will be the Analytical Hierarchy Theorem. Let $P \subseteq \mathit{WO}$ be Σ^1_1–definable and put

$$\begin{aligned} Q(a,b) \;\;:\Leftrightarrow\;\; & a \in \mathit{LO} \wedge P(b) \\ & \wedge (\exists \alpha)(\forall x)(\forall y)[\{a\}^{2,0}(x,y) = 0 \Rightarrow \{b\}^{2,0}(\alpha(x),\alpha(y)) = 0]. \end{aligned}$$

Then $Q(a,b)$ says that a is the index of an ordering which is order preserving embeddable into an ordering in P. This implies that a is a well–ordering. Hence

$$(\exists b)Q(a,b) \Rightarrow a \in \mathit{WO}. \tag{i}$$

Now assume $\sup\{\mathit{otyp}^{\mathit{WO}}(e)|\ e \in P\} = \omega_1^{CK}$. Then we get for any $a \in \mathit{WO}$ a $b \in P$ such that $\mathit{otyp}^{\mathit{WO}}(a) \le \mathit{otyp}^{\mathit{WO}}(b)$ and therefore also an order–preserving embedding from $\mathit{field}(\{a\}^{2,0})$ into $\mathit{field}(\{b\}^{2,0})$, i.e. we get

$$a \in \mathit{WO} \;\;\Rightarrow\;\; (\exists b)Q(a,b). \tag{ii}$$

From (i) and (ii) we obtain

$$a \in \mathit{WO} \;\;\Leftrightarrow\;\; (\exists b)Q(a,b).$$

For any Π^1_1–predicate R, however, we have $R \le_m \mathit{WT} \le_m \mathit{WO}$ by Theorem 7.1.4 and Lemma 5.4.6. Since $(\exists b)Q(a,b)$ is a Σ^1_1–predicate every Π^1_1–predicate would already be Σ^1_1. This contradicts the Analytical Hierarchy Theorem. The same proof works for WO replaced by WT.
If $P \subseteq \mathbb{WO}$ then we define

$$\begin{aligned} Q(\alpha,\beta) \;\;\Leftrightarrow\;\; & \alpha \in \mathbb{LO} \wedge \beta \in P \\ & \wedge (\exists \eta)(\forall x)(\forall y)[\alpha(\langle x,y\rangle) = 0 \Rightarrow \beta(\langle \eta(x),\eta(y)\rangle) = 0] \end{aligned}$$

which again is Σ^1_1 and copy the above argument. $\square$

In the Closure Theorem we have seen that the complexity of the obtained fixed–point depends on the number of steps which are needed to construct the fixed–point, i.e. on $||\varphi||$. An interesting question to ask is whether $||\varphi||$ depends on the complexity of the defining formula φ or not. Let us regard the formula

$$\varphi_C(X,x,e) :\equiv (\forall y)\left[\{e\}^{1,0}(x^\frown\langle y\rangle) \simeq 0 \Rightarrow (x^\frown\langle y\rangle, e) \in X\right]. \tag{7.10}$$

Then φ_C is Π^0_1. We know from Theorem 7.1.5

$$e \in \mathit{Tree} \Rightarrow (e \in \mathit{WT} \;\;\Leftrightarrow\;\; (\langle\rangle, e) \in I_{\varphi_C}).$$

and for $e \in \mathit{WT}$

$$|(\langle\rangle, e)|_{\varphi_C} = \mathit{otyp}^{\mathit{Tree}}(e).$$

Since $\omega_1^{CK} = \sup\{\mathit{otyp}^{\mathit{Tree}}(e) + 1|\ e \in \mathit{WT}\}$ we obtain for every $\xi < \omega_1^{CK}$ an e such that $\xi \le |(\langle\rangle, e)|_{\varphi_C}$ which shows

$$\sup\{||\varphi||\ |\ \varphi \text{ is an } X\text{–positive } \Pi^0_1\text{–formula}\} = \omega_1^{CK}. \tag{7.11}$$

It follows from (7.11) that restricting the inductive definition to Π^0_1–definable ones does not decrease the inductive closure ordinal. In the next section we are going to study the case of Σ^0_1–definable operators.

7.3 Σ^0_1–inductive definitions and semi–decidable sets

7.3.1 Lemma (Σ^0_1–Reflection) *Let $\varphi(X,\vec{x})$ be an X–positive Σ^0_1–formula and I_ψ any fixed–point such that $\varphi(I_\psi^{<\omega},\vec{x})$. Then there is some $n<\omega$ such that $\varphi(I_\psi^n,\vec{x})$.*

Proof: We induct on the definition of "$\varphi(X,\vec{x})$ is an X–positive formula". The claim is obvious if X does not occur in $\varphi(X,\vec{x})$. If $\varphi(X,\vec{x}) \equiv t(\vec{x}) \in X$ and $t(\vec{x}) \in I_\psi^{<\omega}$ then there is some $n<\omega$ such that $t(\vec{x}) \in I_\psi^n$. If $\varphi(X,\vec{x}) \equiv \varphi_1(X,\vec{x}) \overset{\wedge}{\vee} \varphi_2(X,\vec{x})$ we find $n_1,n_2<\omega$ such that $\varphi_1(I_\psi^{n_1},\vec{x}) \overset{\wedge}{\vee} \varphi_2(I_\psi^{n_2},\vec{x})$. Putting $n := \max\{n_1,n_2\}$ we get $\varphi(I_\psi^n,\vec{x})$ by the X–positivity of $\varphi_i(X,\vec{x})$. The last possibility is that $\varphi(X,\vec{x}) \equiv (\exists y)\varphi_0(X,\vec{x},y)$. If $\varphi(I_\psi^{<\omega},\vec{x})$ then we find some $y<\omega$ such that $\varphi_0(I_\psi^{<\omega},\vec{x},y)$ and by induction hypothesis an $n<\omega$ such that $\varphi_0(I_\psi^n,\vec{x},y)$. But this implies $\varphi(I_\psi^n,\vec{x})$. $\square$

Observe that the above proof depended heavily on the fact that $\varphi(X,\vec{x})$ was Σ^0_1. The above argument would break down for $\varphi(X,\vec{x}) \equiv (\forall y)\varphi_0(X,\vec{x},y)$. Observe further that the opposite direction in Lemma 7.3.1 holds by monotonicity. Hence

$$\mathbb{N} \models \varphi(I_\psi^{<\omega},\vec{x}) \;\Leftrightarrow\; (\exists n<\omega)\left[\mathbb{N} \models \varphi(I_\psi^n,\vec{x})\right]. \tag{7.12}$$

As a consequence of Lemma 7.3.1 we obtain

7.3.2 Theorem *Let $\varphi(X,\vec{x})$ be an X–positive Σ^0_1–formula. Then $||\varphi|| \le \omega$.*

Proof: By (7.12) we have

$$\begin{aligned} \vec{x} \in I_\varphi^\omega &\Leftrightarrow \mathbb{N} \models \varphi(I_\varphi^{<\omega},\vec{x}) \\ &\Leftrightarrow (\exists n<\omega)\left[\vec{x} \in I_\varphi^{n+1}\right] \\ &\Leftrightarrow \vec{x} \in I_\varphi^{<\omega}. \end{aligned}$$

$\square$

It follows from Theorem 7.3.2 and the Closure Theorem 7.2.2 that every X–positive Σ^0_1–formula has Δ^1_1 fixed–point. This estimate, however, is much too crude. It follows from Theorem 7.3.2 that

$$\vec{x} \in I_\varphi \;\Leftrightarrow\; (\exists n)(\vec{x} \in I_\varphi^n).$$

Thus, if we succeed to show that $\{(\vec{x},n) \mid \vec{x} \in I_\varphi^n\}$ is arithmetical or even Σ^0_1, we get a much lower complexity of the fixed–point. The key here is a restatement of the Recursion Theorem.

7.3.3 Theorem (Recursion Theorem for semi–decidable predicates) *Let $\varphi(X,\vec{x})$ be an X–positive Σ^0_1–formula. There is an index e such that*

$$\vec{x} \in \mathsf{W}_e^{n,0} \;\Leftrightarrow\; \varphi(\mathsf{W}_e^{n,0},\vec{x})$$

Proof: Observe first that substituting a semi–decidable set R into an X–positive Σ^0_1–formula $\varphi(X,\vec{x})$ yields a semi–decidable predicate

$$\{(\vec{x},\vec{y}) \mid \varphi(\{\vec{z} \mid (\vec{z},\vec{y}) \in R\},\vec{x})\}.$$

The proof is by induction on the definition of "$\varphi(X,\vec{x})$ is an X–positive Σ^0_1–formula" and is straight forward using the closure properties of semi–decidable predicates. Now we regard

$$Q = \{(\vec{x}, y) \mid \varphi(\mathsf{W}^{n,0}_{S(y,y)}, \vec{x})\}$$

which is semi–decidable and therefore has an index e_0. Putting $e := S(e_0, e_0)$ we obtain

$$\begin{aligned} \vec{x} \in \mathsf{W}^{n,0}_e &\Leftrightarrow (\vec{x}, e_0) \in \mathsf{W}^{n,0}_{e_0} \\ &\Leftrightarrow \varphi(\mathsf{W}^{n,0}_{S(e_0,e_0)}, \vec{x}) \\ &\Leftrightarrow \varphi(\mathsf{W}^{n,0}_e, \vec{x}) \end{aligned}$$

□

In consequence of the Recursion Theorem for semi–decidable predicates we get that the semi–decidable predicates are closed under inductive definitions.

7.3.4 Theorem *The fixed–point of an X–positive Σ^0_1–formula φ is a Σ^0_1–predicate.*

Proof: We use the Recursion Theorem to obtain an index e such that

$$(\vec{x}, m) \in \mathsf{W}_e \;\Leftrightarrow\; [m = 0 \wedge \varphi(\emptyset, \vec{x})] \vee \big[m = k+1 \wedge \varphi(\{\vec{u} \mid (\vec{u}, k) \in \mathsf{W}_e\}, \vec{x})\big].$$

We prove

$$I^m_\varphi = \{\vec{x} \mid (\vec{x}, m) \in \mathsf{W}_e\}$$

by induction on m and obtain the claim since

$$\begin{aligned} \vec{x} \in I_\varphi &\Leftrightarrow \vec{x} \in I^{<\omega}_\varphi \\ &\Leftrightarrow (\exists n)\left[\vec{x} \in I^n_\varphi\right] \\ &\Leftrightarrow (\exists n)\left[(\vec{x}, n) \in \mathsf{W}_e\right]. \end{aligned}$$

□

7.4 Some properties of Π^1_1– and related predicates

We will apply the theory of inductive sets to pursue the study of Π^1_1–predicates. Recalling (7.10) we put

$$\varphi_{\mathit{Tree}}(X, x, e) :\equiv e \in \mathit{Tree} \wedge (\forall y)(\{e\}^{1,0}(x^\frown\langle y\rangle) = 0 \;\Rightarrow\; (x^\frown\langle y\rangle, e) \in X).$$

Let $I_{\mathit{Tree}} := I_{\varphi_{\mathit{Tree}}}$ and put

$$\mathit{WT}_\sigma := \{e \mid (\langle\rangle, e) \in I^\sigma_{\mathit{Tree}}\}. \tag{7.13}$$

For $\sigma < \omega_1^{CK}$ the set WT_σ is Δ^1_1 by Theorem 7.2.2 and Corollary 7.1.9. We prove

$$\mathit{WT}_\sigma = \{e \in \mathit{WT} \mid \mathit{otyp}^{\mathit{Tree}}(e) \le \sigma\}. \tag{7.14}$$

Assume $e \in \mathit{WT}_\sigma$ and put $T_e := \{s \mid \{e\}^{1,0}(s) = 0\}$. By (7.5) we get

$$(\langle\rangle, e) \in I^\sigma_{\mathit{Tree}} \Rightarrow T_e\restriction\langle\rangle \in \mathit{WT} \wedge \mathit{otyp}(T_e\restriction\langle\rangle) \le \sigma.$$

Hence $e \in \mathit{WT}$ and $\mathit{otyp}^{\mathit{Tree}}(e) \le \sigma$.
For the converse inclusion assume $e \in \mathit{WT}$ and $\mathit{otyp}^{\mathit{Tree}}(e) \le \sigma$. Then by (7.3) $(\langle\rangle, e) \in I^\sigma_{\mathit{Tree}}$.
As a consequence of the Boundedness Principle (Theorem 7.2.8) we get

7.4.1 Lemma *Let $S \subseteq \mathit{WT}$ be a Σ^1_1–set. Then there is an ordinal $\sigma < \omega_1^{CK}$ such that $S \subseteq \mathit{WT}_\sigma$.*

Proof: By the Boundedness Principle there exists a $\sigma < \omega_1^{CK}$ such that $\sup\{\mathit{otyp}^{\mathit{Tree}}(e) \mid e \in S\} \le \sigma$. This implies $S \subseteq \mathit{WT}_\sigma$. □

From Lemma 7.4.1 we get a characterization of the Δ^1_1–sets.

7.4.2 Theorem *Let H be a Δ^1_1–set. Then there is a $\sigma < \omega_1^{CK}$ such that $H \leq_m \mathsf{WT}_\sigma$.*

Proof: Since $H \in \Pi^1_1$ and WT is Π^1_1–complete we have

$$H \leq_m \mathsf{WT} \tag{i}$$

say via f. Because H is also Σ^1_1 we get

$$M := f[H] = \{f(x) \mid x \in H\} \subseteq \mathsf{WT} \tag{ii}$$

as a Σ^1_1–subset of WT. Hence $M \subseteq \mathsf{WT}_\sigma$ for some $\sigma < \omega_1^{CK}$ by Lemma 7.4.1. By (i) and (ii), however, we get

$$H \leq_m \mathsf{WT}_\sigma$$

via f. □

7.4.3 Theorem (Reduction Theorem) *Let P and Q be Π^1_1–predicates. Then there are Π^1_1–predicates $P_1 \subseteq P$ and $Q_1 \subseteq Q$ such that*

$$P_1 \cap Q_1 = \emptyset$$

and

$$P_1 \cup Q_1 = P \cup Q.$$

Cf. Figure 7.4.1.

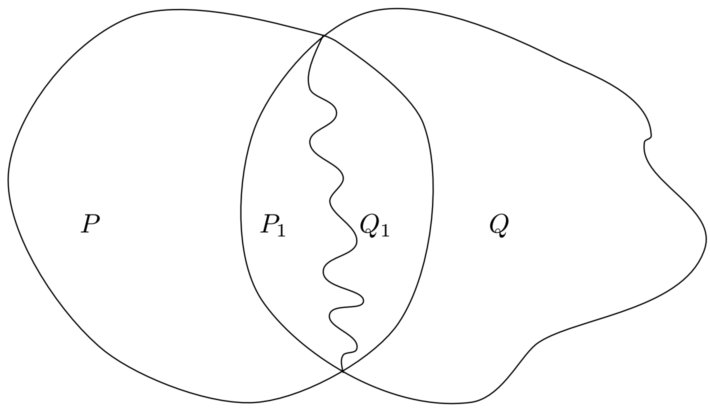

Figure 7.4.1: Reducing sets P_1 and Q_1 for P and Q

Proof: The theorem is a consequence of the Stage Comparison Theorem. Put

$$R(z, \vec{x}) \;\;:\Leftrightarrow\;\; [z = 0 \wedge P(\vec{x})] \;\vee\; [z = 1 \wedge Q(\vec{x})]\,.$$

Thus R is Π^1_1 and hence inductive. Thus R admits an inductive norm $|\;|_R$ by Theorem 6.4.5. Put

$$P_1 := \{\vec{x} \mid (0, \vec{x}) \preceq^*_R (1, \vec{x})\}$$

and

$$Q_1 := \{\vec{x} \mid (1, \vec{x}) \prec^*_R (0, \vec{x})\}$$

where $\preceq^*_R$ and $\prec^*_R$ are the predicates defined in (6.13) and (6.14) on page 73. Then $\preceq^*_R$ as well as $\prec^*_R$ are inductive, i.e. Π^1_1–relations such that

$$P_1 \cap Q_1 = \emptyset.$$

Moreover we have

$$P_1 \cup Q_1 \subseteq \{\vec{x} \mid (\exists z)\,[(z,\vec{x}) \in R]\} \subseteq P \cup Q \tag{i}$$

and for $\vec{x} \in P \cup Q$ we either get $(0,\vec{x}) \in R$ or $(1,\vec{x}) \in R$. Hence $(0,\vec{x}) \preceq^*_R (1,\vec{x})$ or $(1,\vec{x}) \prec^*_R (0,\vec{x})$ which implies $\vec{x} \in P_1$ or $\vec{x} \in Q_1$. This gives also the converse inclusion of (i) and the proof is finished. ☐

As a consequence of the Reduction Theorem we get

7.4.4 Theorem (Separation Theorem) *Let P and Q be two disjoint Σ^1_1–predicates. Then there is a Δ^1_1–predicate H which separates P and Q, i.e. which satisfies*

$$P \subseteq H$$

and

$$H \cap Q = \emptyset.$$

Cf. Figure 7.4.2.

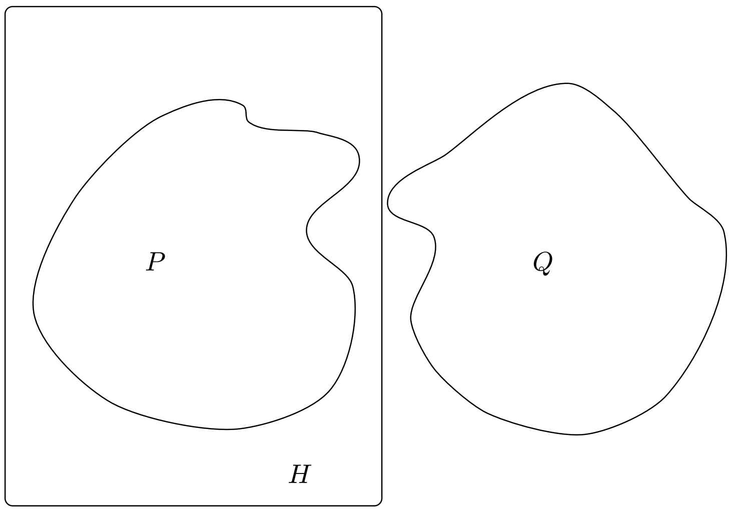

Figure 7.4.2: Separating P and Q by a Δ^1_1– set H

Proof: We regard the complements $\neg P$ and $\neg Q$ and reduce them to $P_1 \subseteq \neg P$ and $Q_1 \subseteq \neg Q$ by the Reduction Theorem. Because of

$$P_1 \cup Q_1 = \neg P \cup \neg Q = \neg(P \cap Q) = \mathbb{N}^n$$

and

$$P_1 \cap Q_1 = \emptyset$$

we get

$$P_1 = \neg Q_1.$$

Putting $H := \neg P_1$ we get H as a Δ^1_1–predicate such that

$$P \subseteq H$$

and

$$Q \cap H = Q \cap Q_1 = \emptyset$$

because $Q_1 \subseteq \neg Q$. $\square$

7.4.5 Theorem (Weak Π^1_1–uniformization) *Let P be an $(m+1,n)$–ary Π^1_1–relation. Then there is a partial functional F_P such that*

$$\mathsf{dom}(F_P) = \{\mathfrak{a} \mid (\exists x) P(\mathfrak{a}, x)\}$$

$$(\forall \mathfrak{a} \in \mathsf{dom}(F_P))\,[P(\mathfrak{a}, F_P(\mathfrak{a}))]$$

The graph of F_P is Π^1_1–definable.

Cf. Figure 7.4.3.

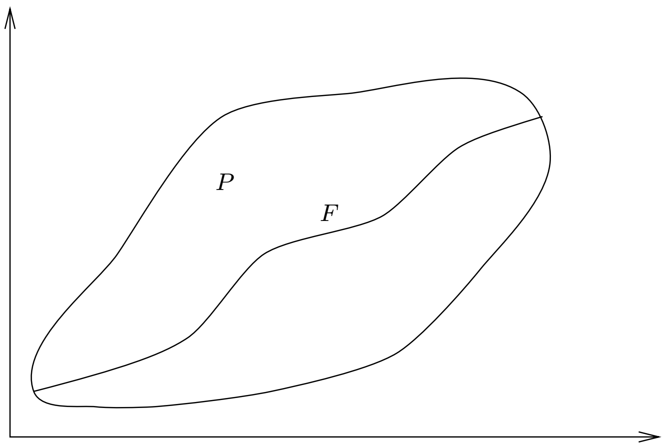

Figure 7.4.3: Uniformizing P by F

Proof: The naive try to put

$$F_P(\mathfrak{a}) :\simeq \mu x\,.\,P(\mathfrak{a}, x)$$

fails, because expressing that x is the least element such that $P(\mathfrak{a}, x)$ requires to say $(\forall y < x)\neg P(\mathfrak{a}, y)$ which is not necessarily a Π^1_1–relation. However, using Stage Comparison we can first select an x of minimal $|\ |_P$ norm and then select the least among those elements having the same $|\ |_P$ norm. I.e. we put

$$\begin{aligned} F_P(\mathfrak{a}) \simeq y \quad :\Leftrightarrow \quad & P(\mathfrak{a}, y) \\ & \wedge (\forall z)\,[(\mathfrak{a}, y) \preceq^*_P (\mathfrak{a}, z)] \\ & \wedge (\forall z < y)\,[(\mathfrak{a}, y) \prec^*_P (\mathfrak{a}, z)]\,. \end{aligned}$$

Since $\preceq^*_P$ as well as $\prec^*_P$ are Π^1_1 we easily check that F_P satisfies the claim. $\square$

There is, however, an even stronger version of the Uniformization Theorem — due to KONDO and ADDISON — which says that there is even a function–valued selection functional for Π^1_1–relations. This is obviously much harder to prove because it is by far not clear how to pick a function out of those having the same Π^1_1–norm.

7.4.6 Theorem (Strong Π^1_1–Uniformization) *Let P be an $(m, n+1)$–ary Π^1_1–relation. Then there is an $(m, n+1)$–ary Π^1_1–relation Q such that*

$$(\forall \mathfrak{a})(\forall \alpha)\,[Q(\mathfrak{a}, \alpha) \Rightarrow P(\mathfrak{a}, \alpha)] \tag{i}$$

$$(\forall \mathfrak{a})(\forall \alpha)(\forall \beta)\,[Q(\mathfrak{a}, \alpha) \wedge Q(\mathfrak{a}, \beta) \;\;\Rightarrow\;\; \alpha = \beta] \tag{ii}$$

$$(\forall \mathfrak{a})\,[(\exists \alpha)P(\mathfrak{a}, \alpha) \Rightarrow (\exists \alpha)Q(\mathfrak{a}, \alpha)]\,. \tag{iii}$$

Proof: Fix $\mathfrak{a}$. If $\neg(\exists \alpha)P(\mathfrak{a}, \alpha)$ we trivially put $Q := \emptyset$. Thus assume $(\exists \alpha)P(\mathfrak{a}, \alpha)$. By Theorem 7.1.3 we have a computable functional F such that

$$P(\mathfrak{a}, \alpha) \;\;\Leftrightarrow\;\; \lambda x\,.\,F(\mathfrak{a}, \alpha, x) \in \mathbb{WT}. \tag{iv}$$

Let

$$T_\alpha := \big\{s \in \mathit{Seq} \,\big|\; F(\mathfrak{a}, \alpha, s) = 0\big\}$$

be the associated tree. Put

$$\sigma := \min\big\{\mathit{otyp}(T_\alpha) \,\big|\; P(\mathfrak{a}, \alpha)\big\}$$

and let

$$Q_0 := \big\{\alpha \,\big|\; P(\mathfrak{a}, \alpha) \wedge \mathit{otyp}(T_\alpha) = \sigma\big\}. \tag{v}$$

We are going to define relations Q_n by induction on n and assume that Q_n is already defined. We put

$$s_n := \min\big\{\overline{\alpha}(n) \,\big|\; P(\mathfrak{a}, \alpha)\big\},$$

$$\sigma_n := \min\big\{\mathit{otyp}(T_\alpha{\restriction}n) \,\big|\; P(\mathfrak{a}, \alpha) \wedge \overline{\alpha}(n) = s_n\big\}$$

and define

$$Q_{n+1} := \big\{\alpha \in Q_n \,\big|\; \overline{\alpha}(n) = s_n \wedge \mathit{otyp}(T_\alpha{\restriction}n) = \sigma_n\big\}. \tag{vi}$$

Let

$$Q := \bigcap_{n \in \omega} Q_n.$$

From (v) and (vi) we get

$$(\forall n < \omega)[\alpha \in Q_n \;\;\Rightarrow\;\; P(\mathfrak{a}, \alpha)]$$

by induction on n. By $Q \subseteq Q_0$ and (v) we have

$$\alpha \in Q \;\;\Rightarrow\;\; P(\mathfrak{a}, \alpha). \tag{vii}$$

Another immediate consequence is

$$\begin{aligned} Q(\alpha) \wedge Q(\beta) \;\;&\Rightarrow\;\; (\forall n \in \omega)[\overline{\alpha}(n) = s_n = \overline{\beta}(n)] \\ &\Rightarrow\;\; \alpha = \beta. \end{aligned} \tag{viii}$$

By (vii) and (viii) we obtain claims (i) and (ii) of the theorem. The real work is to prove (iii) and the fact that Q is Π^1_1–definable. Since we assumed $(\exists \alpha)P(\mathfrak{a}, \alpha)$ it suffices to prove

$$(\exists \alpha)Q(\alpha)$$

to show (iii). Since $Q_{n+1} \subseteq Q_n$ we have $s_n \subseteq s_{n+1}$. Hence

$$m \leq n \;\;\Rightarrow\;\; s_m \subseteq s_n.$$

Therefore there is a unique function, say γ, such that

$$(\forall n \in \omega)[\overline{\gamma}(n) = s_n]. \tag{ix}$$

We claim

$$Q(\gamma). \tag{x}$$

In a first step we prove

$$m <^*_{T_\gamma} n \;\Rightarrow\; \sigma_m < \sigma_n. \tag{xi}$$

Since the functional F in (iv) is computable its value $F(\mathfrak{a}, \gamma)$ depends only on an initial segment of γ. Therefore there is a $k \in \mathbb{N}$ such that

$$(\forall \alpha)[\overline{\alpha}(k) = \overline{\gamma}(k) \;\Rightarrow\; (\{n, m\} \subseteq T_\gamma \;\Leftrightarrow\; \{n, m\} \subseteq T_\alpha)]. \tag{xii}$$

We may choose k bigger than m and n. Pick $\alpha \in Q_{k+1}$. Then $\overline{\alpha}(n) = s_n = \overline{\gamma}(n)$ as well as $\overline{\alpha}(m) = s_m = \overline{\gamma}(m)$ and by (xii) we get $m, n \in T_\alpha$. But then $m <^*_{T_\gamma} n$ implies $m <^*_{T_\alpha} n$ and we obtain $\mathit{otyp}(T_\alpha{\upharpoonright}m) < \mathit{otyp}(T_\alpha{\upharpoonright}n)$. But since $\alpha \in Q_{k+1} \supseteq Q_{i+1}$ for $i = m, n$ we finally obtain $\sigma_m = \mathit{otyp}(T_\alpha{\upharpoonright}m) < \mathit{otyp}(T_\alpha{\upharpoonright}n) = \sigma_n$. This terminates the proof of (xi).
By a similar argument we also obtain

$$m \in T_\gamma \;\Rightarrow\; \sigma_m < \sigma. \tag{xiii}$$

We choose $k > m$ such that (xii) and pick $\alpha \in Q_{k+1}$. But then $\sigma_m = \mathit{otyp}(T_\alpha{\upharpoonright}m) < \mathit{otyp}(T_\alpha) = \sigma$ since $\alpha \in Q_{k+1} \subseteq Q_{m+1} \subseteq Q_0$.
It follows from (xi) that T_γ is well–founded. Hence

$$P(\mathfrak{a}, \gamma). \tag{xiv}$$

Next we prove

$$n \in T_\gamma \;\Rightarrow\; \mathit{otyp}(T_\gamma{\upharpoonright}n) \le \sigma_n \tag{xv}$$

by induction on $<^*_{T_\gamma}$. We have

$$\begin{aligned}
\mathit{otyp}(T_\gamma{\upharpoonright}n) &= \sup\{\mathit{otyp}_{T_\gamma{\upharpoonright}n}(m) + 1 \mid m \in T_\gamma{\upharpoonright}n\} \\
&= \sup\{\mathit{otyp}_{T_\gamma}(n^\frown m) + 1 \mid n^\frown m \in T_\gamma\} \\
&= \sup\{\mathit{otyp}_{T_\gamma}(m) + 1 \mid m <^*_{T_\gamma} n\} \\
&= \sup\{\mathit{otyp}(T_\gamma{\upharpoonright}m) + 1 \mid m <^*_{T_\gamma} n\} \\
&\le \sup\{\sigma_m + 1 \mid m <^*_{T_\gamma} n\} \le \sigma_n
\end{aligned}$$

where we used the induction hypothesis to come from the last but one line to the last line and (xi) for the inequality in the last line.
Now we show

$$(\forall n)[\gamma \in Q_n] \tag{xvi}$$

by induction on n. From (xiii) we get $\mathit{otyp}(T_\gamma) \le \sigma$ which together with (xiv) shows $\gamma \in Q_0$. If $\gamma \in Q_n$ then we obtain from (ix) and (xv) $\gamma \in Q_{n+1}$.
Now (x) follows from (xvi) and it remains to show that Q is Π^1_1–definable. First observe that for $T \in \mathbb{WT}$ the relation

$$\mathit{otyp}(S) \le \mathit{otyp}(T) \tag{xvii}$$

as well as

$$\mathit{otyp}(S) < \mathit{otyp}(T)$$

are both Σ^1_1–definable. To see this recall the formula φ_T in (7.2) and assume $T \in \mathbb{WT}$. Then

$$\begin{aligned} \mathit{otyp}(S) \le \mathit{otyp}(T) &\Leftrightarrow |\langle\rangle|_{\varphi_S} \le |\langle\rangle|_{\varphi_T} \\ &\Leftrightarrow \langle\rangle \notin I_{\varphi_T} \vee (\langle\rangle \in \varphi_S \wedge \neg|\langle\rangle|_{\varphi_T} < |\langle\rangle|_{\varphi_S}) \\ &\Leftrightarrow \neg(\langle\rangle <^*_{\varphi_S,\varphi_T} \langle\rangle) \end{aligned}$$

and the last line is Σ^1_1 by Stage Comparison. Analogously we also obtain

$$T \in \mathbb{WT} \;\Rightarrow\; (\mathit{otyp}(S) < \mathit{otyp}(T) \;\Leftrightarrow\; \neg(\langle\rangle \le^*_{\varphi_T,\varphi_S} \langle\rangle)).$$

Regard that according to the definition (vi) of Q_n we have

$$\beta \in Q_n \;\Leftrightarrow\; \mathit{otyp}(T_\beta) \le \sigma \wedge (\forall m < n)[\overline{\beta}(m) \le s_m \wedge \mathit{otyp}(T_\beta\restriction m) \le \sigma_m].$$

Thus, if we assume $\alpha \in Q_n$,

$$\begin{aligned} \beta \in Q_n \;\Leftrightarrow\; & \mathit{otyp}(T_\beta) \le \mathit{otyp}(T_\alpha) \\ & \wedge (\forall m < n)\left[\overline{\beta}(m) \le \overline{\alpha}(m) \wedge \mathit{otyp}(T_\beta\restriction m) \le \mathit{otyp}(T_\alpha\restriction m)\right]. \end{aligned} \tag{xviii}$$

According to (xvii) the right hand side in (xviii) is a Σ^1_1–relation, say $R_0(\alpha,\beta,n)$ (where we suppress the parameters $\mathfrak{a}$ which are hidden in Q_n). Still assuming $\alpha \in Q_n$ we thus get

$$\begin{aligned} \alpha \notin Q_{n+1} \;&\Leftrightarrow\; (\exists\beta)\{\beta \in Q_n \wedge [\overline{\beta}(n) < \overline{\alpha}(n) \vee (\overline{\beta}(n) = \overline{\alpha}(n) \\ &\qquad\qquad \wedge \mathit{otyp}(T_\beta\restriction n) < \mathit{otyp}(T_\alpha\restriction n))]\} \\ &\Leftrightarrow\; (\exists\beta)\{R_0(\alpha,\beta,n) \wedge [\overline{\beta}(n) < \overline{\alpha}(n) \vee (\overline{\beta}(n) = \overline{\alpha}(n) \\ &\qquad\qquad \wedge \mathit{otyp}(T_\beta\restriction n) < \mathit{otyp}(T_\alpha\restriction n))]\} \\ &\Leftrightarrow:\; R_1(\alpha,n). \end{aligned} \tag{xix}$$

By (xix) we see that $R_1(\alpha,n)$ is a Σ^1_1–relation. Using (xix) we finally get

$$\begin{aligned} \alpha \in Q \;&\Leftrightarrow\; \alpha \in Q_0 \wedge (\forall n)\neg R_1(\alpha,n) \\ &\Leftrightarrow\; P(\mathfrak{a},\alpha) \wedge (\forall\beta)[(\mathfrak{a},\alpha) \preceq^*_P (\mathfrak{a},\beta)] \wedge (\forall n)\neg R_1(\alpha,n) \end{aligned}$$

where $\preceq^*_P$ is the relation defined in (6.13). Since P is Π^1_1 and thus inductive we get by Theorem 6.4.5 that $\preceq^*_P$ is inductive and thus Π^1_1–definable. $\square$

7.5 Basis Theorems

Let P be an $(0,1)$–ary relation, i.e. P is a collection of functions. Even if P can be classified in the arithmetical or analytical hierarchy we cannot hope to get some information about the members of P. Regard for example the collection of **all** functions which is decidable but contains functions of arbitrary complexity. All we can say is that there are computable functions among all functions. We are going to prove that in many cases we have a similar situation. If P is a collection having a simple classification then some functions in P can be classified in a simple way. This is made precise in the following definition.

7.5.1 Definition Let $\mathcal{C}$ be a collection of $(0,1)$–ary relations. A class B of functions is called a *basis for* $\mathcal{C}$ if for every P in $\mathcal{C}$ we have

$$(\exists\alpha)P(\alpha) \Rightarrow (\exists\alpha \in B)P(\alpha).$$

As an example we regard the collection $\mathcal{C}$ of all Σ^0_1–classes of functions. Let $P \in \mathcal{C}$ and $P \neq \emptyset$. Then $\alpha \in P \Leftrightarrow (\exists x)R(\overline{\alpha}(x))$ for some decidable predicate R. Since $P \neq \emptyset$ there is some $s \in \mathit{Seq}$ such that $R(s)$. Defining

$$\beta(x) := \begin{cases} (s)_x & \text{if } x < \mathit{lh}(s) \\ 0 & \text{otherwise} \end{cases}$$

we get $\beta \in P$ and see that the class of functions which have value 0 almost everywhere form a basis for the collection of Σ^0_1–classes of functions.

7.5.2 Lemma *Let B be a basis for the collection of Π^0_1–classes of functions. Then $\{(\gamma)_0 \mid \gamma \in B\}$ is a basis for the collection of Σ^1_1–classes of functions.*

Proof: Let P be in Σ^1_1. Then

$$P(\alpha) \;\Leftrightarrow\; (\exists\beta)Q(\alpha,\beta) \tag{i}$$

for some Π^0_1–relation $Q(\alpha,\beta)$. From $(\exists\alpha)P(\alpha)$ it follows $(\exists\gamma)Q((\gamma)_0,(\gamma)_1)$ and, since B is a basis for the collection of Π^0_1–classes of functions, we obtain a $\gamma \in B$ such that $Q((\gamma)_0,(\gamma)_1)$. But then $P((\gamma)_0)$. □

By literally the same proof we obtain also

7.5.3 Lemma *Let B be a basis for the collection of Π^1_n–classes of functions. Then $\{(\gamma)_0 \mid \gamma \in B\}$ is a basis for the collection of Σ^1_{n+1}–classes of functions.*

Let P be a class of functions. We define

$$\mathit{In}(P) = \{s \in \mathsf{Seq} \mid (\exists\alpha)\,[P(\alpha) \wedge \overline{\alpha}(\mathsf{lh}(s)) = s]\}, \tag{7.15}$$

i.e. $\mathit{In}(P)$ is the set of initial segments of functions in P. Generalizing our above example we obtain

7.5.4 Lemma *If P is a nonempty Π^0_1–class of functions then $P(\beta)$ for some $\beta \leq_T \mathit{In}(P)$.*

Proof: We define

$$F(n) :\simeq \mu x\,.\,(\overline{F}(n)^\frown\langle x\rangle \in \mathit{In}(P))\,.$$

Then F is computable from $\mathit{In}(P)$. We show

$$(\forall n)\,\big[\overline{F}(n) \in \mathit{In}(P)\big]$$

by induction on n. $\overline{F}(0) = \langle\rangle \in \mathit{In}(P)$ follows from the hypothesis $(\exists\alpha)P(\alpha)$. Now assume $\overline{F}(n) \in \mathit{In}(P)$. But then

$$F(n) = \min\{\alpha(n) \mid P(\alpha) \wedge \overline{\alpha}(n) = \overline{F}(n)\}$$

is defined and $\overline{F}(n+1) = \overline{F}(n)^\frown\langle F(n)\rangle \in \mathit{In}(P)$. Since P is Π^0_1 we get

$$P(\alpha) \;\Leftrightarrow\; (\forall x)R(\overline{\alpha}(x))$$

for some decidable predicate R. Hence

$$\mathit{In}(P) \subseteq R$$

and we get $(\forall x)R(\overline{F}(n))$. This proves $P(F)$. □

As a consequence we obtain the first half of KLEENE's Basis Theorem.

7.5.5 Theorem *The functions which are computable in the class of Σ^1_1–predicates are a basis for the collection of Π^0_1–classes of functions and hence also for the collection of Σ^1_1–classes of functions.*

Proof: For a Π^0_1–class P of functions we see from (7.15) that $\mathit{In}(P)$ is Σ^1_1. By Lemma 7.5.4 it follows that the class of functions computable in the class of Σ^1_1–predicates is a basis for the collection of Π^0_1–classes of functions and by Lemma 7.5.2 also for the collection of Σ^1_1–classes of functions. $\square$

As already remarked Theorem 7.5.5 is only one half of KLEENE's Basis Theorem which will be our Theorem 8.2.3. The second half says that the class of Δ^1_1–definable functions is not a basis for the collection of Π^0_1–classes of functions. We have to postpone this part until we have a better characterization of the Δ^1_1–definable functions.
Recall that we identify sets with their characteristic functions. Therefore we may talk about bases for collections of classes of sets. A remarkable result is

7.5.6 Theorem (KREISEL's Basis Theorem) *The class of Δ^0_2–functions is a basis for the collection of Π^0_1–classes of sets.*

To prepare the proof we formulate a lemma which on its turn is an easy consequence of the Finiteness Theorem (Theorem 5.2.6).

7.5.7 Lemma *Let P be a Π^0_1–relation and define*

$$Q(\mathfrak{a}) \;\;:\Leftrightarrow\;\; (\exists\alpha^*)P(\mathfrak{a},\alpha^*).$$

Then Q is also Π^0_1.

Proof: Since $P \in \Pi^0_1$ we have

$$P(\mathfrak{a},\alpha) \;\;\Leftrightarrow\;\; (\forall x)R(\overline{\mathfrak{a}}(x),\overline{\alpha}(x))$$

for some decidable relation R. The tree

$$\big\{s\,\big|\; (\forall i<\mathit{lh}(s))\,[(s)_i \le 1] \;\wedge\; (\forall s_0{\subseteq}s)\,[R(\overline{\mathfrak{a}}(\mathit{lh}(s)_0),s_0)]\big\}$$

is boundedly branching. Hence

$$\begin{aligned}(\exists\alpha^*)P(\mathfrak{a},\alpha^*) \;&\Leftrightarrow\; (\exists\alpha^*)(\forall x)R(\overline{\mathfrak{a}}(x),\overline{\alpha}^*(x))\\ &\Leftrightarrow\; (\forall n)(\exists s)[\mathit{Seq}(s) \wedge \mathit{lh}(s)=n\\ &\qquad\qquad \wedge (\forall i<n)((s)_i \le 1) \wedge (\forall s_0{\subseteq}s)R(\overline{\mathfrak{a}}(\mathit{lh}(s_0)),s_0)]\end{aligned}$$

by (5.26) in Theorem 5.2.6. Both quantifiers $(\exists s)$ and $(\forall s_0{\subseteq}s)$ can obviously be bounded. Hence $(\exists\alpha^*)P(\mathfrak{a},\alpha^*) \in \Pi^0_1$. $\square$

For the proof of Theorem 7.5.6 observe that for every Π^0_1–class of sets P

$$x \in \mathit{In}(P) \;\;\Leftrightarrow\;\; \mathit{Seq}(x) \wedge (\exists\alpha^*)\,[P(\alpha^*) \wedge \overline{\alpha}^*(\mathit{lh}(x)) = x]$$

holds. Thus $\mathit{In}(P)$ is Π^0_1 by Lemma 7.5.7. The functions which are computable in the Π^0_1–classes of functions are therefore by Lemma 7.5.4 a basis for the collection of Π^0_1–classes of sets. By POST's Theorem (Theorem 3.2.6) these are the functions which are $\Delta^0_1[\Pi^0_1]$, i.e. Δ^0_2. $\square$

To obtain even further reaching basis theorems we introduce some notations.

7.5.8 Definition A $(0,1)$–ary relation P defines a function γ *implicitly* if

$$(\forall\alpha)(\forall\beta)\,[P(\alpha) \wedge P(\beta) \Rightarrow \alpha=\beta]$$

and

$$P(\gamma).$$

The function γ is called a *singleton*.

7.5.9 Lemma *Let P define a function γ implicitly. Then*

$$P \in \Delta_n^1 \;\;\Leftrightarrow\;\; \gamma \in \Delta_n^1$$

for all n.

Proof: Assume first $P \in \Delta_n^1$. Then

$$\begin{aligned}\gamma(x) \simeq y \;\;&\Leftrightarrow\;\; (\exists\alpha)\,[P(\alpha) \wedge \alpha(x) = y]\\ &\Leftrightarrow\;\; (\forall\alpha)\,[P(\alpha) \Rightarrow \alpha(x) = y]\,.\end{aligned}$$

If $\gamma \in \Delta_n^1$ we get

$$P = \big\{\alpha \,\big|\; (\forall x)(\forall y)\,[\alpha(x) = y \Leftrightarrow \gamma(x) = y]\big\}. \qquad \square$$

As an immediate consequence we get

7.5.10 Corollary *Let $\mathcal{C}$ be a collection of classes of functions such that every nonempty class in $\mathcal{C}$ has a Δ_n^1–subclass which contains exactly one function. Then the class of Δ_n^1–functions is a basis for the collection $\mathcal{C}$.*

From the strong Π_1^1–uniformization and Corollary 7.5.10 we get the following theorem.

7.5.11 Theorem *The class of Δ_2^1–functions is a basis for the collection of Π_1^1–classes of functions.*

By Lemma 7.5.3 we get

7.5.12 Theorem *The class of Δ_2^1–functions is a basis for the collection of Σ_2^1–classes of functions.*

7.6 The complexity of KLEENE's $\mathcal{O}$

We will now settle the still open question for the complexity of KLEENE's $\mathcal{O}$ within the analytical hierarchy. We defined $\mathcal{O}$ in Definition 5.4.1 by a rather complicated simultaneous inductive definition. Now we are going to unravel this definition into single steps.

7.6.1 Definition We define inductively the binary predicate $<'_{\mathcal{O}}$ by the following clauses.

1) If $a \in \big\{2^b \,\big|\; b \neq 0\big\} \cup \big\{3 \cdot 5^e \,\big|\; e \in \mathbb{N}\big\}$ then $1 <'_{\mathcal{O}} a$.
2) If $a \leq'_{\mathcal{O}} b$ then $a <'_{\mathcal{O}} 2^b$.
3) If $a \leq'_{\mathcal{O}} \{e\}^{1,0}(n)$ for some $n \in \mathbb{N}$ then $a <'_{\mathcal{O}} 3 \cdot 5^e$.

Here $a \leq'_{\mathcal{O}} b$ stands for $a <'_{\mathcal{O}} b \vee a = b$. Observe that the operator associated to the inductive definition in Definition 7.6.1 is defined by the formula

$$\begin{aligned}\varphi(X,x,y) \;\;:\Leftrightarrow\;\; &(x = 1 \wedge (\exists z)\,[(y = 2^z \wedge z \neq 0) \vee y = 3 \cdot 5^z])\\ &\vee (\exists z)\,[((x,z) \in X \vee x = z) \wedge y = 2^z]\\ &\vee (\exists e)(\exists n)(\exists u)(\exists z)\,[\mathsf{T}(e,n,u) \wedge U(u) = z \wedge\; [(x,z) \in X \vee x = z] \;\wedge y = 3 \cdot 5^e]\,.\end{aligned}$$

This shows that $<'_{\mathcal{O}}$ is defined by a Σ_1^0–formula.

By Theorem 7.3.4 we therefore obtain

7.6.2 Lemma *The predicate $<'_{\mathcal{O}}$ is Σ_1^0–definable.*

In the next step we show

7.6.3 Lemma

1) $<'_{\mathcal{O}}$ *is a transitive predicate*

2) $a <'_{\mathcal{O}} b \wedge b \in \mathcal{O} \Rightarrow a \in \mathcal{O} \wedge a <_{\mathcal{O}} b$

3) $a \in \mathcal{O} \wedge b \in \mathcal{O} \wedge a <_{\mathcal{O}} b \Rightarrow a <'_{\mathcal{O}} b$

Proof: We prove $a <'_{\mathcal{O}} b <'_{\mathcal{O}} c \Rightarrow a <'_{\mathcal{O}} c$ by induction on the definition of $b <'_{\mathcal{O}} c$. The case $b = 1$ is excluded since $a <'_{\mathcal{O}} b$.
If $b <'_{\mathcal{O}} c$ because of $c = 2^y \neq 1$ and $b \leq'_{\mathcal{O}} y$ then $a \leq'_{\mathcal{O}} y$ by the induction hypothesis. Hence $a \leq'_{\mathcal{O}} b$ by clause 2) in Definition 7.6.1.
If $c = 3 \cdot 5^e$ and $b \leq'_{\mathcal{O}} \{e\}^{1,0}(n)$ we get $a \leq'_{\mathcal{O}} \{e\}^{1,0}(n)$ by the induction hypothesis and $a <'_{\mathcal{O}} c$ by clause 3) in Definition 7.6.1.
We show 2) by induction on $|b|_{\mathcal{O}}$. For $b = 1$, i.e. $|b|_{\mathcal{O}} = 0$, there is nothing to show.
Assume that $b = 2^y \neq 1$. Then $y \in \mathcal{O}$, $|y|_{\mathcal{O}} < |b|_{\mathcal{O}}$ and $a \leq'_{\mathcal{O}} y$ and we have either $a = y \in \mathcal{O}$ or $a <'_{\mathcal{O}} y$ and hence $a \in \mathcal{O}$ by the induction hypothesis. By the induction hypothesis for the second claim we also get $a \leq_{\mathcal{O}} y$ which implies $a <_{\mathcal{O}} b$.
If $b = 3 \cdot 5^e$ and $a <'_{\mathcal{O}} b$ we have an $n \in \mathbb{N}$ such that $a \leq'_{\mathcal{O}} \{e\}^{1,0}(n)$. But $\{e\}^{1,0}(n) \in \mathcal{O}$ and $|\{e\}^{1,0}(n)|_{\mathcal{O}} < |b|_{\mathcal{O}}$. From the induction hypothesis we immediately get $a \in \mathcal{O}$ and $a <_{\mathcal{O}} \{e\}(n)$. Hence $a <_{\mathcal{O}} 3 \cdot 5^e$.
Finally we prove 3) by induction on $|b|_{\mathcal{O}}$. The claim is clear for $b = 1$. For $b = 2^y \neq 1$ we get $a \leq_{\mathcal{O}} y$ which implies $a \leq'_{\mathcal{O}} y$ by the induction hypothesis. Hence $a <'_{\mathcal{O}} b$.
For $b = 3 \cdot 5^e$ we get $a \leq_{\mathcal{O}} \{e\}^{1,0}(n)$ for some $n \in \mathbb{N}$ and $a \leq'_{\mathcal{O}} \{e\}^{1,0}(n)$ by the induction hypothesis. Hence $a <'_{\mathcal{O}} b$. $\square$

The idea is now to get $\mathcal{O}$ as the accessible part of $<'_{\mathcal{O}}$.

7.6.4 Definition We define inductively the set $\mathcal{O}'$ by the following clauses.

1) $1 \in \mathcal{O}'$

2) $a \in \mathcal{O}' \Rightarrow 2^a \in \mathcal{O}'$

3) $(\forall n)\left[\{e\}^{1,0}(n) \in \mathcal{O}'\right] \wedge (\forall n)\left[\{e\}^{1,0}(n) <'_{\mathcal{O}} \{e\}^{1,0}(n+1)\right] \Rightarrow 3 \cdot 5^e \in \mathcal{O}'$.

Then $\mathcal{O}'$ is positively arithmetically inductive, hence a Π^1_1–predicate. We show that $\mathcal{O}$ and $\mathcal{O}'$ coincide.

7.6.5 Lemma *We have $\mathcal{O} = \mathcal{O}'$ and $<_{\mathcal{O}} = <'_{\mathcal{O}} \restriction \mathcal{O} \times \mathcal{O}$.*

Proof: We show

$$x \in \mathcal{O} \;\Leftrightarrow\; x \in \mathcal{O}' \tag{i}$$

simultaneously by induction on the definition of $x \in \mathcal{O}$ and $x \in \mathcal{O}'$, respectively. Claim (i) is obvious for $x = 1$ and immediate from the induction hypothesis in case that $x = 2^y \neq 1$. Thus let $x = 3 \cdot 5^y$. If $x \in \mathcal{O}$ we get $\{y\}^{1,0}(n) \in \mathcal{O}$ for all $n \in \mathbb{N}$ and therefore $\{y\}^{1,0}(n) \in \mathcal{O}'$ for all $n \in \mathbb{N}$. We moreover have $(\forall n)\left[\{y\}^{1,0}(n) <_{\mathcal{O}} \{y\}^{1,0}(n+1)\right]$. By clause 3) of Lemma 7.6.3 this implies

$$(\forall n)\left[\{y\}^{1,0}(n) <'_{\mathcal{O}} \{y\}^{1,0}(n+1)\right]$$

and we obtain $3 \cdot 5^y \in \mathcal{O}'$ by clause 3) of Definition 7.6.4.
If $3 \cdot 5^y \in \mathcal{O}'$ we get

$$(\forall n)\left[\{y\}^{1,0}(n) \in \mathcal{O}\right] \wedge (\forall n)\left[\{y\}^{1,0}(n) <'_{\mathcal{O}} \{y\}^{1,0}(n+1)\right]$$

by the induction hypothesis and Definition 7.6.4. Hence

$$(\forall n)\left[\{y\}^{1,0}(n) \in \mathcal{O}\right] \wedge (\forall n)\left[\{y\}^{1,0}(n) <_{\mathcal{O}} \{y\}^{1,0}(n+1)\right]$$

by Lemma 7.6.3. The second claim follows from (i) and Lemma 7.6.3. □

It follows from Lemma 7.6.5 that $\mathcal{O}$ is a Π^1_1–set. We show even a bit more.

7.6.6 Theorem *The set $\mathcal{O}$ is Π^1_1–complete.*

Proof: By Theorem 7.1.7 there is a formula φ_P (the formula in (7.8)) such that

$$P \leq_m I_{\varphi_P}.$$

Thus it suffices to show

$$I_{\varphi_P} \leq_m \mathcal{O}.$$

We want to get a computable function G such that

$$(s,x) \in I_{\varphi_P} \;\Leftrightarrow\; G(s,x) \in \mathcal{O}. \tag{i}$$

First we define a function

$$G_0(e,s,x) = \begin{cases} 1 & \text{if } \{T_P(x)\}^{1,0}(s) \simeq 1 \\ 3 \cdot 5^z & \text{if } \{T_P(x)\}^{1,0}(s) \simeq 0 \end{cases}$$

where z is an index of the function F defined by

$$F(0) = 1$$
$$F(n+1) = F(n) +_{\mathcal{O}} \{e\}^{2,0}(s^\frown\langle n\rangle, x) +_{\mathcal{O}} 2.$$

Note that the case distinction in the defintion of G_0 is decidable because $T_P(x) \in$ *Tree*. Using the Recursion Theorem we get an index e_0 such that

$$\{e_0\}^{2,0}(s,x) \simeq G_0(e_0,s,x)$$

and we put $G := \{e_0\}^{2,0}$. By definition G is computable. We show that G satisfies (i) and start to prove

$$(s,x) \in I_{\varphi_P} \Rightarrow G(s,x) \in \mathcal{O}$$

by induction on $|(s,x)|_{\varphi_P}$. With B_x we denote the tree given by $T_P(x)$,

$$B_x := \left\{s \,\middle|\, \{T_P(x)\}^{1,0}(s) \simeq 0\right\}.$$

If $s \notin B_x$ then $G(s,x) = 1 \in \mathcal{O}$. Now let $s \in B_x$ and $n \in \mathbb{N}$. If we have $s^\frown\langle n\rangle \in B_x$ then $|(s^\frown\langle n\rangle, x)|_{\varphi_P} < |(s,x)|_{\varphi_P}$ and we obtain $G(s^\frown\langle n\rangle, x) \in \mathcal{O}$ by the induction hypothesis. If on the other hand $s^\frown\langle n\rangle \notin B_x$ then $G(s^\frown\langle n\rangle, x) = 1 \in \mathcal{O}$. Hence

$$(\forall n)\left[G(s^\frown\langle n\rangle, x) \in \mathcal{O}\right]. \tag{ii}$$

Since $F(n+1) = F(n) +_{\mathcal{O}} G(s^\frown\langle n\rangle, x) +_{\mathcal{O}} 2$ and $F(0) = 1$ we get from Lemma 5.4.5 and (ii)

$$(\forall n)\left[F(n) \in \mathcal{O}\right]$$

as well as

$$(\forall n)\left[F(n) <_{\mathcal{O}} F(n+1)\right].$$

Because z is an index of F we obtain

$$G(s,x) = 3 \cdot 5^z \in \mathcal{O}.$$

For the opposite direction we have to prove

$$G(s,x) \in \mathcal{O} \Rightarrow (s,x) \in I_{\varphi_P}$$

by induction on $|G(s,x)|_{\mathcal{O}}$. For $s \notin B_x$ we have $\varphi_P(\emptyset, s, x)$, thus $(s,x) \in I^0_{\varphi_P} \subseteq I_{\varphi_P}$. If $s \in B_x$ then $G(s,x) = 3 \cdot 5^z$ and

$$(\forall n)\left[\{z\}^{1,0}(n+1) = \{z\}^{1,0}(n) +_{\mathcal{O}} G(s^\frown\langle n\rangle, x) +_{\mathcal{O}} 2\right].$$

From Lemma 5.4.5 we can infer $(\forall n)\,[G(s^\frown\langle n\rangle, x) \in \mathcal{O}]$, hence $|G(s^\frown\langle n\rangle, x)|_{\mathcal{O}} < |G(s,x)|_{\mathcal{O}}$ for all $s^\frown\langle n\rangle \in B_x$. By induction hypothesis this implies

$$(\forall n)\,[s^\frown\langle n\rangle \in B_x \Rightarrow (s^\frown\langle n\rangle, x) \in I_{\varphi_P}]$$

which is

$$\varphi_P(I_{\varphi_P}, s, x).$$

Hence $(s,x) \in I_{\varphi_P}$. □

As a consequence of Theorem 7.6.6 and the Analytical Hierarchy Theorem we get the following corollary.

7.6.7 Corollary *There is no Σ^1_1–definition of $\mathcal{O}$.*

We can even strengthen the statement of Corollary 7.6.7 to get the Boundedness Principle for $\mathcal{O}$.

7.6.8 Lemma *Let P be a Σ^1_1–definable subset of $\mathcal{O}$. Then*

$$\sup\{|a|_{\mathcal{O}} \mid a \in P\} < \omega_1^{CK}.$$

Proof: By Theorem 5.4.8 there is a computable function g such that

$$a \in \mathcal{O} \;\;\Rightarrow\;\; g(a) \in \mathit{WT} \wedge |a|_{\mathcal{O}} = \mathit{otyp}^{\mathit{Tree}}(g(a)). \tag{i}$$

Thus $g[P]$ is a Σ^1_1–definable subset of WT. Hence the Boundedness Principle (Theorem 7.2.8) and (i) yield

$$\sup\{|a|_{\mathcal{O}} \mid a \in P\} = \sup\{\mathit{otyp}^{\mathit{Tree}}(g(a)) \mid a \in P\} < \omega_1^{CK}. \qquad \square$$

The tree–like structure of $\mathcal{O}$ leads to the following definition.

7.6.9 Definition A set $P \subseteq \mathcal{O}$ which is linearly ordered by $<_{\mathcal{O}}$ is called a *path in $\mathcal{O}$*. If P is a path in $\mathcal{O}$ and $\sup\{|a|_{\mathcal{O}} \mid a \in P\} = \omega_1^{CK}$ then P is called a *path through $\mathcal{O}$*.

As a consequence of Theorem 5.4.9 and Lemma 7.6.8 we get

7.6.10 Corollary *There are no Σ^1_1–definable paths through $\mathcal{O}$.*

However, as we will see in section 9.1, there are Π^1_1–definable paths through $\mathcal{O}$.

8. The Hyperarithmetical Hierarchy

8.1 Hyperarithmetical sets

We are now prepared for the study of infinite iterations of the jump operator.

8.1.1 Definition For $a \in \mathcal{O}$ we put

$$H_a := \begin{cases} \emptyset & \text{if } a = 1\text{, i.e. } |a|_{\mathcal{O}} = 0 \\ j(H_b) & \text{if } a = 2^b \neq 1\text{, i.e. } |a|_{\mathcal{O}} = |b|_{\mathcal{O}} + 1 \\ \{\langle x, y\rangle \mid y <_{\mathcal{O}} a \wedge x \in H_y\} & \text{if } a = 3 \cdot 5^e\text{, i.e. } |a|_{\mathcal{O}} \in \mathsf{Lim}. \end{cases}$$

We say that a set $S \subseteq \mathbb{N}$ is *hyperarithmetical* if there is an $a \in \mathcal{O}$ such that $S \leq_T H_a$.
The class

$$\mathit{Hyp} := \{H_a \mid a \in \mathcal{O}\}$$

is the *hyperarithmetical hierarchy*.

The definition of the set H_a depends heavily on the ordinal notation $a \in \mathcal{O}$. It will take some effort to obtain the independence of the hyperarithmetical hierarchy from the ordinal notation. This will be achieved as soon as we are able to prove

8.1.2 Theorem *For $a, b \in \mathcal{O}$ such that $|a|_{\mathcal{O}} = |b|_{\mathcal{O}}$ we have $H_a \equiv H_b$.*

The proof needs some effort and is done in several steps. We first prove

8.1.3 Lemma *Let $a \leq_{\mathcal{O}} b$. Then $H_a \leq_m H_b$. This holds uniformly in a and b, i.e. an index for the reducing function can be computed from a and b.*

Proof: Each b consists of an m-fold ($m \geq 0$) iteration of exponentiations by 2 starting at a $b' \in \mathbb{N}$ which is not of the form 2^z. We descend this tower of exponentiations until we reach a or until we cannot descend any further. Let c be the element of $\mathcal{O}$ we reached and let n be the number of steps we took. By $H_b = H_c^{(n)}$ and (3.2) of Lemma 3.1.3 there exists a computable function f with

$$H_c \leq_m H_b \text{ via } \{f(0, n)\}^{1,0}. \tag{i}$$

If $a = c$ we are done. Otherwise we have $a <_{\mathcal{O}} c$ and $|c|_{\mathcal{O}} \in \mathsf{Lim}$. By (i)

$$\begin{aligned} x \in H_a &\Leftrightarrow \langle x, a\rangle \in H_c \\ &\Leftrightarrow \{f(0, n)\}^{1,0}(\langle x, a\rangle) \in H_b \end{aligned}$$

holds.
Observe that the algorithm described above terminates even if $a \notin \mathcal{O}$. □

8.1.4 Lemma *For $a \in \mathcal{O}$ we put*

$$\mathcal{O}_a := \{x \in \mathcal{O} \mid |x|_{\mathcal{O}} < |a|_{\mathcal{O}}\}.$$

Then $\mathcal{O}_a$ is computable in H_{2^a} uniformly in a, i.e. a H_{2^a}–index for $\chi_{\mathcal{O}_a}$ is computable from a.

Proof: We use the Recursion Lemma along $<_{\mathcal{O}}$ to define a computable function, say g, such that for $a \in \mathcal{O}$ its value $g(a)$ is a H_{2^a}–index for $\chi_{\mathcal{O}_a}$. The recursion hypothesis gives

$$(\forall b <_{\mathcal{O}} a)\left[\chi_{\mathcal{O}_b} = \{\{e\}(b)\}^{H_{2^b}}\right]$$

and we look for a computable function G such that

$$\chi_{\mathcal{O}_a} = \{G(a,e)\}^{H_{2^a}}.$$

We distinguish the following cases.
$a = 1$. Then $\mathcal{O}_a = \emptyset$ and we choose $G(a,e)$ to be an H_{2^a}–index of the empty set.
$a = 2$. Then $\mathcal{O}_a = \{1\}$ and we choose $G(a,e)$ to be an H_{2^a}–index of $\{1\}$.
$a = 2^b$ and $b = 2^c \neq 1$. Then

$$\mathcal{O}_a = \mathcal{O}_b \cup \left\{2^x \mid x \in \mathcal{O}_b\right\},$$

so $\mathcal{O}_a$ is decidable in $\mathcal{O}_b$ and $\{e\}^{1,0}(b)$ is an H_{2^b}–index for $\mathcal{O}_b$. By Lemma 8.1.3 and $b <_{\mathcal{O}} a$ we can compute an H_{2^a}–index of $\mathcal{O}_b$ from e and b, which in turn easily gives an H_{2^a}–index of $\mathcal{O}_a$. We let $G(a,e)$ be such an index.
$a = 2^b$ and $b = 3 \cdot 5^z$. Then

$$\begin{aligned}\mathcal{O}_a = \mathcal{O}_b \cup \{3 \cdot 5^u \mid\ \{u\}^{1,0} \text{ is total } & \wedge (\forall n)\left[\{u\}^{1,0}(n) \in \mathcal{O}_b\right] \\ & \wedge (\forall n)\left[\{u\}^{1,0}(n) <'_{\mathcal{O}} \{u\}^{1,0}(n+1)\right]\}.\end{aligned}$$

The statements "$\{u\}^{1,0}$ is total" and "$(\forall n)\left[\{u\}^{1,0}(n) <'_{\mathcal{O}} \{u\}^{1,0}(n+1)\right]$" are Π^0_2, hence decidable in H_{2^2}. For total $\{u\}^{1,0}$ the set $\left\{n \mid\ \{u\}^{1,0}(n) \in \mathcal{O}_b\right\}$ is decidable in $\mathcal{O}_b$ and $\{e\}^{1,0}(b)$ is an H_{2^b}–index for $\mathcal{O}_b$. Since $b > 2$ we obtain $\mathcal{O}_a$ as Π^0_1 in H_{2^b}, hence decidable in H_{2^a} and an H_{2^a}–index for $\mathcal{O}_a$ is computable from e and a.
$a = 3 \cdot 5^b$. Then

$$\mathcal{O}_a = \left\{x \mid\ (\exists n)[x \in \mathcal{O}_{\{b\}(n)}]\right\}.$$

By recursion hypothesis we obtain $\{e\}(\{b\}(n))$ as an $H_{2^{(\{b\}(n))}}$–index for $\mathcal{O}_{\{b\}(n)}$. Using Lemma 8.1.3 we obtain $\mathcal{O}_a$ as semi–decidable in H_a and hence decidable in H_{2^a}. An H_{2^a}–index for $\mathcal{O}_a$ depends computably on e and a.
If a is of any other shape then we put $G(a,e) := 0$.
A close look at our construction shows that $G(a,e)$ is defined even if $a \notin \mathcal{O}$. Thus the function g given by the Recursion Lemma is computable. □

As an easy consequence of Lemma 8.1.4 we obtain the next lemma.

8.1.5 Lemma *For $a \in \mathcal{O}$*

$$\left\{x \in \mathcal{O} \mid\ |x|_{\mathcal{O}} = |a|_{\mathcal{O}}\right\}$$

is decidable in $H_{2^{2^a}}$ uniformly in a, i.e. an $H_{2^{2^a}}$–index for $\left\{x \in \mathcal{O} \mid\ |x|_{\mathcal{O}} = |a|_{\mathcal{O}}\right\}$ is computable from a.

Proof: We get

$$x \in \mathcal{O} \wedge |x|_{\mathcal{O}} = |a|_{\mathcal{O}} \ \Leftrightarrow\ (x \in \mathcal{O} \wedge |x|_{\mathcal{O}} < |2^a|_{\mathcal{O}}) \wedge \neg(x \in \mathcal{O} \wedge |x|_{\mathcal{O}} < |a|_{\mathcal{O}}).$$

By Lemma 8.1.4 the first conjunct is decidable in $H_{2^{2^a}}$ and the second in H_{2^a}. Both statements hold uniformly in a. Thus their conjunction is decidable in $H_{2^{2^a}}$ uniformly in a. □

8.1.6 Lemma *If $a \in \mathcal{O}$ and $b \in \mathcal{O}$ such that $|a|_{\mathcal{O}} = |b|_{\mathcal{O}}$ then $H_a \leq_T H_b$.*

Proof: We define a well–founded predicate $<_P$ by

$$\begin{aligned}\langle c,d\rangle <_P \langle a,b\rangle \;\;:\Leftrightarrow\;\; & \{a,b,c,d\} \subseteq \mathcal{O} \wedge |c|_{\mathcal{O}} \leq |d|_{\mathcal{O}} \wedge |a|_{\mathcal{O}} \leq |b|_{\mathcal{O}} \\ & \wedge\; [|c|_{\mathcal{O}} < |a|_{\mathcal{O}} \vee (|c|_{\mathcal{O}} = |a|_{\mathcal{O}} \wedge |d|_{\mathcal{O}} < |b|_{\mathcal{O}})]\end{aligned}$$

and use the Recursion Lemma along $<_P$ to define a computable function g such that for $\langle a,b\rangle \in$ *field*$(<_P)$ (i.e. for $a,b \in \mathcal{O}$ with $|a|_{\mathcal{O}} \leq |b|_{\mathcal{O}}$)

$$\chi_{H_a} = \{g(a,b)\}^{H_b} \tag{i}$$

holds. Let $\langle a,b\rangle \in$ *field*$(<_P)$. The recursion hypothesis gives

$$\langle c,d\rangle <_P \langle a,b\rangle \;\;\Rightarrow\;\; \chi_{H_c} = \{\{e\}(c,d)\}^{H_d}$$

and we search for a computable function G such that

$$\chi_{H_a} = \{G(e,a,b)\}^{H_b}.$$

We distinguish the following cases:
$a = 1$. Let $G(e,a,b)$ be an e_0 with $\{e_0\}^X = \chi_\emptyset$ for all $X \subseteq \mathbb{N}$.
$a = 2^c \neq 1$ and $b = 2^d$. Then $\langle c,d\rangle <_P \langle a,b\rangle$, and so the recursion hypothesis gives $\chi_{H_c} = \{\{e\}(c,d)\}^{H_d}$. By clause 2) of Theorem 3.1.1 we can compute an e_0 with $\chi_{H_a} = \{e_0\}^{H_b}$ from $\{e\}(c,d)$ and put $G(e,a,b) := e_0$.
$a = 2^c \neq 1$ and $b = 3 \cdot 5^u$. Then $|a|_{\mathcal{O}} < |b|_{\mathcal{O}}$, and so there exist n with

$$|a|_{\mathcal{O}} < |\{u\}(n)|_{\mathcal{O}}. \tag{ii}$$

By Lemma 8.1.4 "$|a|_{\mathcal{O}} < |\{u\}(n)|_{\mathcal{O}}$" is uniformly decidable in $H_{2^{\{u\}(n)}}$, which in turn is uniformly decidable in H_b by Lemma 8.1.3. Thus an n satisfying (ii) is uniformly computable in H_b. Because of $\langle a, \{u\}(n)\rangle <_P \langle a,b\rangle$ the recursion hypothesis gives

$$\chi_{H_a} = \{\{e\}(a,\{u\}(n))\}^{H_{\{u\}(n)}}.$$

By Lemma 8.1.3 $H_{\{u\}(n)}$ is uniformly decidable in H_b, and so, with some considerable effort, $G(e,a,b)$ cen be defined appropriately.
$a = 3 \cdot 5^u$. As $\langle c,b\rangle <_P \langle a,b\rangle$ for $c <_{\mathcal{O}} a$ the recursion hypothesis implies

$$\begin{aligned} y \in H_a \;\;&\Leftrightarrow\;\; y = \langle x,c\rangle \wedge c <_{\mathcal{O}} a \wedge x \in H_c \\ &\Leftrightarrow\;\; y = \langle x,c\rangle \wedge c <'_{\mathcal{O}} a \wedge \{\{e\}(c,b)\}^{H_b}(x) = 0.\end{aligned}$$

Because of $|a|_{\mathcal{O}} \leq |b|_{\mathcal{O}}$ the Σ^0_1–predicate $<'_{\mathcal{O}}$ is uniformly decidable in H_b.
In the usual way we see that it is possible to turn G into a total function. So the g satisfying (i) given by the Recursion Lemma is computable. □

Theorem 8.1.2 is an easy consequence of the last lemma.
In the next step we want to show that the hyperarithmetical hierarchy exhausts the Δ^1_1–sets. Recall the concept of Δ^1_1–indices for sets as introduced in Theorem 4.2.6. We will prove that every hyperarithmetical set is Δ^1_1 in a pretty strong sense.

8.1.7 Lemma *There is a computable function h such that for every $a \in \mathcal{O}$ the value $h(a)$ is a Δ^1_1–index for the set H_a.*

Proof: We use the Recursion Lemma (Lemma 5.4.7) along $<_{\mathcal{O}}$ to show the existence of h. For $a \in \mathcal{O}$ the recursion hypothesis says

$$(\forall b <_{\mathcal{O}} a)\left[H_b = \left\{x \,\middle|\, \mathsf{U}^{\Delta^1_1}_{\{e\}(b)}(x)\right\}\right]$$

where $\mathsf{U}^{\Delta^1_1}$ is the universal predicate for Δ^1_1–sets as defined in Theorem 4.2.6. By this theorem we obtain

$$H_b = \{x|\ \mathsf{U}^{\Sigma^1_1}_{(\{e\}^{1,0}(b))_0}(x)\} = \{x|\ \mathsf{U}^{\Pi^1_1}_{(\{e\}^{1,0}(b))_1}(x)\}.$$

The recursion step consists in defining a partial computable function G such that

$$H_a = \{x|\ \mathsf{U}^{\Delta^1_1}_{G(a,e)}(x)\}.$$

We distinguish the following cases:
$a = 1$. Then $H_a = \emptyset$ and we define $G(a,e)$ to be a Δ^1_1–index of the empty set.
$a = 2^c \neq 1$. Then $c <_{\mathcal{O}} a$ and $H_a = j(H_c)$. Hence

$$x \in H_a \ \Leftrightarrow\ (\exists z)R(x,z,\overline{\chi_{H_c}}(z))$$

for a well–known semi–decidable predicate R. So we obtain

$$\begin{aligned} x \in H_a \ \Leftrightarrow\ & (\exists z)(\exists s)[\mathsf{Seq}(s) \wedge \mathsf{lh}(s) = z \wedge (\forall i < z)((s)_i \leq 1) \\ & \qquad\qquad \wedge (\forall i < z)((s)_i = 0 \ \Leftrightarrow\ i \in H_c) \\ & \qquad\qquad \wedge R(x,z,s)] \\ \Leftrightarrow\ & (\exists z)(\exists s)[\mathsf{Seq}(s) \wedge \mathsf{lh}(s) = z \wedge (\forall i < z)((s)_i \leq 1) \\ & \qquad\qquad \wedge (\forall i < z)((s)_i = 0 \Rightarrow \mathsf{U}^{\Sigma^1_1}_{(\{e\}(c))_0}(i)) \\ & \qquad\qquad \wedge (\forall i < z)((s)_i = 1 \Rightarrow \neg\mathsf{U}^{\Pi^1_1}_{(\{e\}(c))_1}(i)) \\ & \qquad\qquad \wedge R(x,z,s)] \end{aligned} \tag{i}$$

and, completely analogous,

$$\begin{aligned} x \in H_a \ \Leftrightarrow\ & (\exists z)(\exists s)[\mathsf{Seq}(s) \wedge \mathsf{lh}(s) = z \wedge (\forall i < z)((s)_i \leq 1) \\ & \qquad\qquad \wedge (\forall i < z)((s)_i = 0 \Rightarrow \mathsf{U}^{\Pi^1_1}_{(\{e\}(c))_1}(i)) \\ & \qquad\qquad \wedge (\forall i < z)((s)_i = 1 \Rightarrow \neg\mathsf{U}^{\Sigma^1_1}_{(\{e\}(c))_0}(i)) \\ & \qquad\qquad \wedge R(x,z,s)]. \end{aligned} \tag{ii}$$

From (i) we see that H_a is Σ^1_1 and a Σ^1_1–index e_1 for H_a can be computed from e and c which in turn is computable from a. Analogously we see from (ii) that H_a is Π^1_1 and a Π^1_1–index e_2 for H_a can be computed from e and a. Hence H_a is Δ^1_1 and we put $G(a,e) = \langle e_1, e_2\rangle$.
$a = 3 \cdot 5^c$. Then

$$x \in H_a \ \Leftrightarrow\ \mathsf{Seq}(x) \wedge \mathsf{lh}(x) = 2 \wedge (x)_1 <_{\mathcal{O}} a \wedge (x)_0 \in H_{(x)_1}. \tag{iii}$$

Using Lemma 7.6.3 we infer from (iii)

$$\begin{aligned} x \in H_a \ \Leftrightarrow\ & \mathsf{Seq}(x) \wedge \mathsf{lh}(x) = 2 \wedge (x)_1 <'_{\mathcal{O}} a \wedge (x)_0 \in H_{(x)_1} \\ \Leftrightarrow\ & \mathsf{Seq}(x) \wedge \mathsf{lh}(x) = 2 \wedge (x)_1 <'_{\mathcal{O}} a \wedge \mathsf{U}^{\Sigma^1_1}_{(\{e\}((x)_1))_0}((x)_0) \\ \Leftrightarrow\ & \mathsf{Seq}(x) \wedge \mathsf{lh}(x) = 2 \wedge (x)_1 <'_{\mathcal{O}} a \wedge \mathsf{U}^{\Pi^1_1}_{(\{e\}((x)_1))_1}((x)_0). \end{aligned}$$

This shows that H_a is Δ^1_1 and a Σ^1_1–index e_1 as well as a Π^1_1–index e_2 for H_a can be computed from e and a. We put $G(a,e) := \langle e_1, e_2\rangle$.
Yet again, note that G is total, and so the g given by the Recursion Lemma is total, too. □

To obtain also the opposite direction we are going to use Theorem 7.4.2 according to which every Δ^1_1–set is many–one reducible to some WT_σ. It will therefore suffice to show that WT_σ is hyperarithmetical for any $\sigma < \omega_1^{CK}$. We prove

8.1.8 Lemma *There is a computable function d such that*

$$\mathit{WT}_{|a|_{\mathcal{O}}} = \{d(a)\}^{H_{2^{2^a}},1,0}$$

for all $a \in \mathcal{O}$.

Proof: By (7.13) we have

$$x \in \mathit{WT}_\sigma \;\Leftrightarrow\; (\langle\rangle, x) \in I^\sigma_{\mathit{Tree}},$$

hence it suffices to show that there is a computable function g such that for all $a \in \mathcal{O}$ we have

$$\chi_{I^{|a|_\mathcal{O}}_{\mathit{Tree}}} = \{g(a)\}^{H_{2^{2^a}},2,0}. \tag{i}$$

We are going to prove (i) by the recursion lemma along $|a|_\mathcal{O}$. Therefore we have the recursion hypothesis

$$(\forall b <_\mathcal{O} a)\left[\chi_{I^{|b|_\mathcal{O}}_{\mathit{Tree}}} = \{\{e\}(b)\}^{H_{2^{2^b}},2,0}\right].$$

We have to define a partial computable function G such that

$$\chi_{I^{|a|_\mathcal{O}}_{\mathit{Tree}}} = \{G(e,a)\}^{H_{2^{2^a}},2,0}.$$

We distinguish the following cases:
$a = 1$. We have

$$I^0_{\mathit{Tree}} = \big\{(s,x) \mid\; x \in \mathit{Tree} \wedge \{x\}^{1,0}(s) = 0 \wedge (\forall y)\left[\{x\}^{1,0}(s^\frown\langle y\rangle) = 1\right]\big\}.$$

This shows that I^0_{Tree} is Π^0_2 and hence decidable in $H_{2^{2^a}}$. We define $G(e,a)$ to be an $H_{2^{2^a}}$–index of I^0_{Tree}.
$a = 2^c \neq 1$. Then, using $I^{|a|_\mathcal{O}}_{\mathit{Tree}} = I^{|c|_\mathcal{O}+1}_{\mathit{Tree}}$, we obtain

$$\begin{aligned}(s,x) \in I^{|a|_\mathcal{O}}_{\mathit{Tree}} \;&\Leftrightarrow\; x \in \mathit{Tree} \wedge (\forall y)\left[\{x\}^{1,0}(s^\frown\langle y\rangle) = 0 \Rightarrow (s^\frown\langle y\rangle, x) \in I^{|c|_\mathcal{O}}_{\mathit{Tree}}\right]\\ &\Leftrightarrow\; x \in \mathit{Tree} \wedge (\forall y)\left[\{x\}^{1,0}(s^\frown\langle y\rangle) = 0 \Rightarrow \{\{e\}(c)\}^{H_{2^{2^c}},2,0}(s^\frown\langle y\rangle, x) = 0\right].\end{aligned} \tag{ii}$$

The formula "$x \in \mathit{Tree}$" is Π^0_2, the second conjunct in (ii) is Π^0_1 in $H_{2^{2^c}}$. Thus $\neg I^{|a|_\mathcal{O}}_{\mathit{Tree}}$ is Σ^0_1 in $H_{2^{2^c}}$ and by Theorem 3.1.1 it follows that $\neg I^{|a|_\mathcal{O}}_{\mathit{Tree}}$ is m–reducible to $j(H_{2^{2^c}})$. Hence $I^{|a|_\mathcal{O}}_{\mathit{Tree}} \leq_T j(H_{2^{2^c}}) \leq_T H_{2^{2^a}}$ and an $H_{2^{2^a}}$–index for $I^{|a|_\mathcal{O}}_{\mathit{Tree}}$ is computable from the $H_{2^{2^c}}$–index $\{e\}(c)$. Since c is computable from a we get a computable function G such that

$$\{G(e,2^c)\}^{H_{2^{2^a}},2,0} = I^{|2^c|_\mathcal{O}}_{\mathit{Tree}}.$$

$a = 3\cdot 5^c$. Then $|a|_\mathcal{O} \in \mathsf{Lim}$ and we get

$$\begin{aligned}(s,x) \in I^{|a|_\mathcal{O}}_{\mathit{Tree}} \;&\Leftrightarrow\; x \in \mathit{Tree} \wedge (\forall y)\left[\{x\}^{1,0}(s^\frown\langle y\rangle) = 0 \Rightarrow (s^\frown\langle y\rangle, x) \in I^{<|a|_\mathcal{O}}_{\mathit{Tree}}\right]\\ &\Leftrightarrow\; x \in \mathit{Tree} \wedge (\forall y)(\exists v)\left[v <'_\mathcal{O} a \wedge (\{x\}^{1,0}(s^\frown\langle y\rangle) = 0 \Rightarrow (s^\frown\langle y\rangle, x) \in I^{|v|_\mathcal{O}}_{\mathit{Tree}})\right]\\ &\Leftrightarrow\; x \in \mathit{Tree} \wedge (\forall y)(\exists v)\left[v <'_\mathcal{O} a \wedge (\{x\}^{1,0}(s^\frown\langle y\rangle) = 0\right.\\ &\qquad\qquad\qquad \left.\Rightarrow \{\{e\}(v)\}^{H_{2^{2^v}},2,0}(s^\frown\langle y\rangle, x) = 0)\right].\end{aligned} \tag{iii}$$

But observe that for $v <'_\mathcal{O} a$ we have $H_{2^{2^v}} \leq_m H_a$ since $x \in H_{2^{2^v}} \Leftrightarrow \langle x, 2^{2^v}\rangle \in H_a$. Therefore we get from (iii) that $I^{|a|_\mathcal{O}}_{\mathit{Tree}}$ is Π^0_2 in H_a. Hence $I^{|a|_\mathcal{O}}_{\mathit{Tree}} \leq_T H_{2^{2^a}}$ by Theorem 3.1.1 and an $H_{2^{2^a}}$–index for $I^{|a|_\mathcal{O}}_{\mathit{Tree}}$ is effectively computable from e and a. Letting $G(e,a)$ be this index we get

$$\chi_{I^{|a|_\mathcal{O}}_{\mathit{Tree}}} = \{G(e,a)\}^{H_{2^{2^a}},2,0}.$$

By the Recursion Lemma we get a partial–computable function g such that, for all $a \in \mathcal{O}$, $g(a)$ is an $H_{2^{2^a}}$–index for $\chi_{I^{|a|_\mathcal{O}}_{\mathit{Tree}}}$ and we define $d(a)$ as an index for the set

$$\{x \mid \{g(a)\}^{H_{2^{2^a}},2,0}(\langle\rangle, x) = 0\}.$$

Observe that d is computable. $\square$

8.1.9 Theorem (Characterization Theorem for Δ^1_1–sets) *The hyperarithmetical sets are the Δ^1_1–sets.*

Proof: It is an easy exercise to show that the Δ^1_1–sets are closed under $\leq_T$. From this and Lemma 8.1.7 it follows that every hyperarithmetical set is Δ^1_1. Conversely, if a set S is Δ^1_1 then, according to Theorem 7.4.2, $S \leq_m \mathsf{WT}_\sigma$ for some $\sigma < \omega_1^{CK}$. By Lemma 8.1.8 there is some $a \in \mathcal{O}$ such that $S \leq_m \mathsf{WT}_{|a|_\mathcal{O}} \leq_T H_{2^{2^a}}$. Hence S is hyperarithmetical. $\square$

8.2 Hyperarithmetical functions

8.2.1 Definition A function $\alpha\colon \mathbb{N} \longrightarrow \mathbb{N}$ is *hyperarithmetical* if its graph G_α is a hyperarithmetical predicate.

Since we are talking about total functions we have

$$\alpha(x) \neq y \;\Leftrightarrow\; (\exists z)\,[\alpha(x) = z \wedge z \neq y]$$

which implies

$$G_\alpha \in \Delta^1_1 \;\Leftrightarrow\; G_\alpha \in \Pi^1_1 \;\Leftrightarrow\; G_\alpha \in \Sigma^1_1.$$

Therefore a function is hyperarithmetical if it possesses a Π^1_1–graph. This opens the possibility to define indices for hyperarithmetical functions via the weak Π^1_1–uniformization Theorem (Theorem 7.4.5). Though we did not emphasize it in the proof of Theorem 7.4.5 it should be clear that a Π^1_1–index of the uniformizing function is computable from a Π^1_1–index of the original predicate via a computable function, say k. Then we define

$$\{e\}^I(x) \simeq y \;\Leftrightarrow\; \mathsf{U}^{\Pi^1_1}_{k(e)}(x, y) \tag{8.1}$$

and call $\{e\}^I$ a *hyperarithmetical index*. Note that $\{e\}^I$ is not necessarily total.
We denote by $\mathfrak{H}$ the class of hyperarithmetical functions. Then we obtain

$$\{e\}^I \in \mathfrak{H} \;\Leftrightarrow\; (\forall x)(\exists y)\mathsf{U}^{\Pi^1_1}_{k(e)}(x, y) \tag{8.2}$$

which is a Π^1_1–statement.
We are going to prove that $\mathfrak{H}$ is a genuine Π^1_1–relation.

8.2.2 Lemma *The relation $\mathfrak{H}$ is Π^1_1 but not Σ^1_1.*

Proof: Because of

$$\alpha \in \mathfrak{H} \;\Leftrightarrow\; (\exists e)[\{e\}^I \in \mathfrak{H} \wedge (\forall x)(\alpha(x) = \{e\}^I(x))]$$

and (8.2) we obtain $\mathfrak{H}$ as a Π^1_1–relation.
Now assume $\mathfrak{H} \in \Sigma^1_1$. Define

$$P(\alpha, a) \;:\Leftrightarrow\; (\alpha \in \mathfrak{H} \wedge a \in \mathcal{O} \wedge \alpha \leq_m \mathsf{WT}_{|a|_\mathcal{O}}) \vee (a = 1 \wedge \alpha \notin \mathfrak{H}). \tag{i}$$

By Lemma 8.1.8 and Lemma 8.1.7 the predicate Q defined by

$$Q(x, a) \;:\Leftrightarrow\; a \in \mathcal{O} \wedge x \in \mathsf{WT}_{|a|_\mathcal{O}}$$

is Π^1_1. Since

$$\alpha \leq_m \mathit{WT}_{|a|_{\mathcal{O}}} \;\Leftrightarrow\; (\exists e)(\forall x)(\forall y)\left[\alpha(x) = y \Leftrightarrow (\exists z)[\mathsf{T}^{1,0}(e, \langle x, y\rangle, z) \wedge U(z) \in \mathit{WT}_{|a|_{\mathcal{O}}}]\right]$$

for $a \in \mathcal{O}$ the relation $P(\alpha, a)$ is Π^1_1. Using weak Π^1_1 uniformization we obtain a functional F_P whose graph is Π^1_1–definable. By Theorem 7.4.2 we get

$$(\forall\alpha)(\exists a)P(\alpha, a)$$

which shows that F_P is a total functional, hence the graph of F_P is Δ^1_1–definable.
On the other hand, for every $a \in \mathcal{O}$ there is an $\alpha \in \mathfrak{H}$ such that $\alpha \not\leq_m \mathit{WT}_{|a|_{\mathcal{O}}}$: For $\sigma < \omega_1^{CK}$ we have $\mathit{WT}_\sigma \in \Delta^1_1$, hence $Y := j(\mathit{WT}_\sigma) \in \Delta^1_1$ with $Y \not\leq_m \mathit{WT}_\sigma$. Putting $\alpha := \chi_Y$ we get $\alpha \in \mathfrak{H}$ and $\alpha \not\leq_m \mathit{WT}_\sigma$.
Therefore $\mathsf{rng}(F_P)$ is Σ^1_1–definable and unbounded in $\mathcal{O}$. This, however, contradicts Lemma 7.6.8. □

8.2.3 Theorem (Kleene's Basis Theorem) *The functions which are computable in the class of Σ^1_1–predicates are a basis for the collection of Σ^1_1–classes of functions.*
The class of Δ^1_1–definable functions is not a basis for this collection and hence not even a basis for the collection of Π^0_1–classes of functions.

Proof: The first part is Theorem 7.5.5. For the second part we define a relation P by

$$P(\alpha) \;:\Leftrightarrow\; \alpha \notin \mathfrak{H}.$$

Thus P is a nonempty Σ^1_1–relation for which

$$P(\alpha) \;\Leftrightarrow\; \alpha \notin \Delta^1_1$$

holds. Obviously there is no $\beta \in \Delta^1_1$ with $P(\beta)$. Thus the class of Δ^1_1–definable functions is not a basis for the collection of Σ^1_1–classes of functions. The rest follows from Lemma 7.5.2. □

This theorem has a surprising consequence.

8.2.4 Theorem *There is a non well–founded decidable tree without infinite hyperarithmetical path (i.e. $\mathfrak{H}$ thinks that the tree is well–founded).*

Proof: By the second part of the last theorem there is a nonempty $(0,1)$-ary Π^0_1-relation P with

$$(\forall\alpha \in \mathfrak{H})\neg P(\alpha).$$

As P is Π^0_1 there is a decidable predicate R such that

$$P(\alpha) \;\Leftrightarrow\; (\forall x)R(\overline{\alpha}(x))$$

holds. The tree

$$T := \left\{s \in \mathit{Seq} \,\middle|\, (\forall s_0)(s_0 \subsetneq s \Rightarrow R(s_0))\right\}$$

is the one we are looking for. □

A somehow more constructive proof of the last theorem is given on page 107.
One further goal of the present section is to show that the class $\mathfrak{H}$ is a model of the scheme

$$(\Pi^1_1 - AC^{01}) \qquad (\forall x)(\exists\alpha)P(x, \alpha) \Rightarrow (\exists\beta)(\forall x)P(x, (\beta)_x)$$

where P is a $(1,1)$–ary Π^1_1–relation. We call $(\Pi^1_1 - AC^{01})$ the Π^1_1–axiom of choice of type $(0,1)$.
By the weak Π^1_1–uniformization theorem (Theorem 7.4.5) we get for a Π^1_1–predicate P

$$(\forall x)(\exists y)P(x, y) \Rightarrow (\exists\beta \in \mathfrak{H})(\forall x)P(x, \beta(x)). \tag{8.3}$$

This shows that $\mathfrak{H}$ is a model of the Π^1_1–axiom of choice of type $(0,0)$

$$(\Pi^1_1 - AC^{00}) \qquad (\forall x)(\exists y)P(x,y) \Rightarrow (\exists \beta)(\forall x)P(x,\beta(x)).$$

8.2.5 Theorem *The class $\mathfrak{H}$ of hyperarithmetical functions is a model of $(\Pi^1_1 - AC^{01})$, i.e. for a $(1,1)$–ary Π^1_1–relation P we have*

$$(\forall x)(\exists \alpha \in \mathfrak{H})P(x,\alpha) \Rightarrow (\exists \beta \in \mathfrak{H})(\forall x)P(x,(\beta)_x).$$

Proof: Using indices for hyperarithmetical functions we get

$$(\forall x)(\exists \alpha \in \mathfrak{H})P(x,\alpha) \;\;\Leftrightarrow\;\; (\forall x)(\exists e)\left[\{e\}^I \in \mathfrak{H} \wedge P(x,\{e\}^I)\right]. \tag{i}$$

It follows from (8.1) that $\{e\}^I \in \mathfrak{H}$ is a Π^1_1–statement. But we also have for total $\{e\}^I$

$$P(x,\{e\}^I) \;\;\Leftrightarrow\;\; (\forall \alpha)\left[((\forall z)(\forall y)(\{e\}^I(z) = y \Rightarrow \alpha(z) = y)) \Rightarrow P(x,\alpha)\right]$$

which shows that the expression in square brackets in (i) is Π^1_1. Thus starting with

$$(\forall x)(\exists \alpha \in \mathfrak{H})P(x,\alpha)$$

we get by (i) and (8.3) a hyperarithmetical function γ such that

$$(\forall x)\left[\{\gamma(x)\}^I \in \mathfrak{H} \wedge P(x,\{\gamma(x)\}^I)\right].$$

We define a total function β by

$$\beta(u) := \begin{cases} \{\gamma((u)_0)\}^I((u)_1) & \text{if } \mathit{Seq}(u) \wedge \mathit{lh}(u) = 2 \\ 0 & \text{otherwise} \end{cases}$$

and easily see

$$(\beta)_x = \{\gamma(x)\}^I.$$

Furthermore we obtain

$$\begin{aligned} \beta(\langle a,b\rangle) \simeq y \;\;&\Leftrightarrow\;\; \{\gamma(a)\}^I(b) \simeq y \\ &\Leftrightarrow\;\; \mathsf{U}^{\Pi^1_1}_{k(\gamma(a))}(b,y) \end{aligned}$$

which shows that β has a Π^1_1–graph. Hence $\beta \in \mathfrak{H}$ and

$$(\forall x)P(x,(\beta)_x). \qquad \square$$

The next goal is to show the class of hyperarithmetical functions is characterized by $(\Pi^1_1 - AC^{01})$. This needs some preparation.
Our first observation is that the stages H_a can be defined implicitly. Let $a \in \mathcal{O}$. Then

$$\begin{aligned} x \in H_a \;\;\Leftrightarrow\;\; & (a = 1 \wedge x \neq x) \\ & \vee (\exists z)\,[a = 2^z \neq 1 \wedge x \in j(H_z)] \\ & \vee (\exists z)\left[a = 3 \cdot 5^z \wedge (x)_1 <'_{\mathcal{O}} a \wedge (x)_0 \in H_{(x)_1} \wedge \mathit{Seq}(x) \wedge \mathit{lh}(x) = 2\right] \\ \Leftrightarrow\;\; & (a = 1 \wedge x \neq x) \\ & \vee (\exists z)\left[a = 2^z \neq 1 \wedge (\exists u)R(\overline{\chi}_{H_z}(u),x)\right] \\ & \vee (\exists z)\left[a = 3 \cdot 5^z \wedge (x)_1 <'_{\mathcal{O}} a \wedge (x)_0 \in H_{(x)_1} \wedge \mathit{Seq}(x) \wedge \mathit{lh}(x) = 2\right] \end{aligned}$$

$$\begin{aligned} \Leftrightarrow\;\; & (a = 1 \wedge x \neq x) \\ & \vee (\exists z)[a = 2^z \neq 1 \wedge (\exists u)(\exists s)(\mathit{Seq}(s) \wedge \mathit{lh}(s) = u \\ & \qquad\qquad \wedge (\forall i < u)((s)_i = \chi_{H_z}(i) \wedge R(s,x)] \\ & \vee (\exists z)\left[a = 3 \cdot 5^z \wedge (x)_1 <'_{\mathcal{O}} a \wedge (x)_0 \in H_{(x)_1} \wedge \mathit{Seq}(x) \wedge \mathit{lh}(x) = 2\right] \end{aligned} \tag{8.4}$$

for some decidable predicate R. Putting *Hyp*(b,α) as

$$\begin{aligned}
&(\forall x)(\alpha(x) \leq 1 \wedge (\alpha(x) = 0 \Rightarrow \textit{Seq}(x) \wedge \textit{lh}(x) = 2)) \\
&\wedge (\forall a)(\neg a \leq'_{\mathcal{O}} b \Rightarrow (\forall x)(\alpha(\langle x, a\rangle) = 1)) \\
&\wedge (\forall a)(\forall z)(a \leq'_{\mathcal{O}} b \Rightarrow \\
&\qquad [a = 1 \Rightarrow (\forall x)(\alpha(\langle x, a\rangle) = 1) \\
&\qquad \wedge (a = 2^z \neq 1 \Rightarrow (\forall x)(\alpha(\langle x, a\rangle) = 0 \\
&\qquad\qquad\qquad \Leftrightarrow (\exists u)(\exists s)(\textit{Seq}(s) \wedge \textit{lh}(s) = u \\
&\qquad\qquad\qquad\qquad \wedge (\forall i < u)((s)_i = \alpha(\langle i, z\rangle) \wedge R(s, x))))) \\
&\qquad \wedge (a = 3 \cdot 5^z \Rightarrow (\forall x)((\textit{Seq}(x) \wedge \textit{lh}(x) = 2) \Rightarrow \alpha(\langle x, a\rangle) = \alpha(\langle (x)_0, (x)_1\rangle))))])
\end{aligned} \tag{8.5}$$

we recognize *Hyp* as an $(1,1)$–ary arithmetical relation. Let

$$H_{\leq b} := \chi_{\{\langle x, a\rangle |\ a \leq_{\mathcal{O}} b \wedge x \in H_a\}}.$$

It follows from (8.4) that for $b \in \mathcal{O}$ we have

$$\textit{Hyp}(b, H_{\leq b}). \tag{8.6}$$

On the other hand if $b \in \mathcal{O}$ then we have

$$\textit{Hyp}(b, \alpha) \Rightarrow \alpha = H_{\leq b}. \tag{8.7}$$

To prove (8.7) we show

$$\alpha(\langle x, a\rangle) = 0 \;\; \Leftrightarrow \;\; a \leq_{\mathcal{O}} b \wedge x \in H_a \tag{8.8}$$

by induction on $|a|_{\mathcal{O}}$. But (8.8) is more or less obvious from the induction hypothesis, the definition (8.5) and (8.4). Summarizing we get

8.2.6 Lemma *There is an $(1,1)$–ary arithmetical relation Hyp such that for $b \in \mathcal{O}$ we have*

$$\textit{Hyp}(b, \alpha) \;\; \Leftrightarrow \;\; \alpha = H_{\leq b}.$$

8.2.7 Lemma *Let $\mathcal{M}$ be a nonempty collection of functions which is closed under $\leq_T$ and satisfies $(\Delta^1_0 - AC^{01})$. Then $b \in \mathcal{O}$ implies $H_{\leq b} \in \mathcal{M}$.*

Proof: We prove

$$b \in \mathcal{O} \Rightarrow H_{\leq b} \in \mathcal{M}$$

by induction on $|b|_{\mathcal{O}}$.
For $b = 1$ we have $H_{\leq b} = \chi_\emptyset$. Hence $H_{\leq b}$ is computable. But since $\mathcal{M}$ is nonempty and closed under $\leq_T$ it contains all computable functions.
Let $b = 2^c \neq 1$. Then

$$H_{\leq b}(\langle x, a\rangle) = 0 \;\; \Leftrightarrow \;\; (a = 2^c \wedge x \in j(H_c)) \vee (H_{\leq c}(\langle x, a\rangle) = 0). \tag{i}$$

It follows from (i) that $H_{\leq b}$ is semi–decidable in $H_{\leq c}$. Therefore there is a decidable relation R such that

$$H_{\leq b}(x) = 0 \;\; \Leftrightarrow \;\; (\exists z) R(H_{\leq c}, x, z).$$

Define

$$Q(\alpha, x, y) \;\; :\Leftrightarrow \;\; \textit{Hyp}(c, \alpha) \wedge y \leq 1 \wedge \; [y = 0 \;\; \Leftrightarrow \;\; (\exists z) R(\alpha, x, z)] \,.$$

By Lemma 8.2.6 and $b \in \mathcal{O}$ we obtain

$$(\forall x)(\forall y) \, [(\exists \alpha) Q(\alpha, x, y) \Rightarrow H_{\leq b}(x) = y] \,. \tag{ii}$$

Since $|c|_{\mathcal{O}} < |b|_{\mathcal{O}}$ we get by the induction hypothesis and Lemma 8.2.6

$$(\exists\alpha \in \mathcal{M})\,[\mathit{Hyp}(c,\alpha)]$$

which implies

$$(\forall x)(\exists y)(\exists\alpha \in \mathcal{M})Q(\alpha,x,y).$$

Since $\mathcal{M}$ is closed under $\leq_T$ we get by contraction of quantifiers

$$(\forall x)(\exists\beta \in \mathcal{M})Q((\beta)_0,x,(\beta)_1(0)). \tag{iii}$$

As $\mathcal{M} \models (\Delta^1_0 - AC^{01})$ and Q is arithmetical we obtain from (iii)

$$(\exists\gamma \in \mathcal{M})(\forall x)Q((\gamma)_{x0},x,(\gamma)_{x1}(0)) \tag{iv}$$

and by (ii) and (iv)

$$(\forall x)\,[H_{\leq b}(x) = (\gamma)_{x1}(0)]\,.$$

Hence $H_{\leq b} = \lambda x\,.\,(\gamma)_{x1}(0)$ and $H_{\leq b} \in \mathcal{M}$ since $\mathcal{M}$ is closed under $\leq_T$.
Let $b = 3 \cdot 5^e$. Then

$$\begin{aligned} H_{\leq b}(\langle z,a\rangle) = 0 \;&\Leftrightarrow\; [a = b \wedge z \in H_b] \;\vee\; [a <_{\mathcal{O}} b \wedge z \in H_a] \\ &\Leftrightarrow\; \big[a = b \wedge \mathit{Seq}(z) \wedge \mathit{lh}(z) = 2 \wedge (\exists n)(H_{\leq\{e\}(n)}(z) = 0)\big] \\ &\quad\;\vee\; \big[a <'_{\mathcal{O}} b \wedge (\exists n)(H_{\leq\{e\}(n)}(\langle z,a\rangle) = 0)\big]\,. \end{aligned}$$

Now we put

$$\begin{aligned} R(\alpha,x) \;:\Leftrightarrow\; (\exists z)(\exists a)(x = \langle z,a\rangle \\ \wedge\, ([a = b \wedge \mathit{Seq}(z) \wedge \mathit{lh}(z) = 2 \wedge (\exists n)((\alpha)_n(z) = 0)] \\ \vee\; [a <'_{\mathcal{O}} b \wedge (\exists n)((\alpha)_n(x) = 0)])) \end{aligned} \tag{v}$$

and define

$$Q(\alpha,x,y) \;:\Leftrightarrow\; (\forall n)\,[\mathit{Hyp}(\{e\}(n),(\alpha)_n)] \;\wedge\, y \leq 1 \wedge (y = 0 \Leftrightarrow R(\alpha,x)).$$

By Lemma 8.2.6 and (v) we get

$$(\forall x)(\forall y)\,[(\exists\alpha)Q(\alpha,x,y) \Rightarrow H_{\leq b}(x) = y]\,. \tag{vi}$$

The induction hypothesis yields

$$(\forall n)(\exists\alpha \in \mathfrak{H})\,[\mathit{Hyp}(\{e\}(n),\alpha)] \tag{vii}$$

which entails by $\mathcal{M} \models (\Delta^1_0 - AC^{01})$

$$(\exists\alpha \in \mathfrak{H})(\forall n)\,[\mathit{Hyp}(\{e\}(n),(\alpha)_n)]\,. \tag{viii}$$

From (viii), however, we get

$$(\forall x)(\exists\alpha \in \mathcal{M})(\exists y)Q(\alpha,x,y)$$

which, analogous to the previous case, yields

$$(\forall x)(\exists\beta \in \mathcal{M})Q((\beta)_0,x,(\beta)_1(0)).$$

Using $\mathcal{M} \models (\Delta^1_0 - AC^{01})$ we obtain

$$(\exists\gamma \in \mathcal{M})(\forall x)Q((\gamma)_{x0},x,(\gamma)_{x1}(0))$$

and finally we get from (vi)

$$(\forall x)[H_{\leq b}(x) = \gamma_{x1}(0)],$$

i.e.

$$H_{\leq b} = \lambda x\,.\,(\gamma)_{x1}(0).$$

Hence $H_{\leq b} \in \mathcal{M}$. $\square$

Summing up we have shown

8.2.8 Theorem *The collection $\mathfrak{H}$ of hyperarithmetical functions is the with respect to set inclusion smallest nonempty class of functions which is closed under "computable in" and satisfies $(\Delta^1_0 - AC^{01})$. We even have $\mathfrak{H} \models (\Pi^1_1 - AC^{01})$.*

8.3 The hyperarithmetical quantifier theorem

If we regard all ordinals below ω_1^{CK} as given, i.e. we allow bounded search over ω_1^{CK}, then all arithmetical predicates are decidable and so are all the sets H_a. In that sense we may regard the collection $\mathfrak{H}$ of hyperarithmetical functions as computable and Δ^1_1–sets as decidable. The aim of the present section is to show that in that interpretation the Π^1_1–sets play the role of semi–decidable sets.
We introduce some notations. If φ is an analytical formula we denote by $\varphi^{\mathfrak{H}}$ the formula which is obtained from φ by restricting all function quantifiers to functions in $\mathfrak{H}$. Then

$$\Sigma^{1,\mathfrak{H}}_n = \big\{\varphi^{\mathfrak{H}} \,\big|\; \varphi \in \Sigma^1_n\big\}$$

and dually

$$\Pi^{1,\mathfrak{H}}_n = \big\{\varphi^{\mathfrak{H}} \,\big|\; \varphi \in \Pi^1_n\big\}.$$

It is quite easy to see that

$$\Sigma^{1,\mathfrak{H}}_1 \subseteq \Pi^1_1. \tag{8.9}$$

This follows by induction from

$$\begin{aligned}(\exists\alpha \in \mathfrak{H})(\forall x)P(\mathfrak{a},\alpha,x) \quad &\Leftrightarrow \quad (\exists e)(\forall x)\left[\{e\}^I \in \mathfrak{H} \wedge P(\mathfrak{a},\lambda y\,.\,\{e\}^I(y),x)\right] \\ &\Leftrightarrow \quad (\exists e)(\forall\alpha)(\forall x)\left[\{e\}^I \in \mathfrak{H} \wedge ((\forall y)(\forall z)(\{e\}^I(y) = z \Rightarrow \alpha(y) = z)\right. \\ &\qquad\qquad\qquad \left.\Rightarrow P(\mathfrak{a},\alpha,x))\right].\end{aligned}$$

We can now give an alternative proof of Theorem 8.2.4 where we showed that there is a non well–founded decidable tree without infinite hyperarithmetical path.
Proof: We show that there is a decidable predicate R such that

$$(\exists\alpha)(\forall x)R(\overline{\alpha}(x)) \wedge \neg(\exists\alpha \in \mathfrak{H})(\forall x)R(\overline{\alpha}(x)).$$

Putting

$$T := \big\{s \,\big|\; (\forall s_0)\,[s_0 \subsetneq s \Rightarrow R(s_0)]\big\}$$

we have a tree as desired. To construct R we define

$$K_{\Sigma^1_1} := \big\{x \,\big|\; x \in \mathsf{U}^{\Sigma^1_1,1,0}_x\big\} = \big\{x \,\big|\; (\exists\alpha)\left[(\alpha,x) \notin \mathsf{W}^{1,1}_x\right]\big\}.$$

Now let

$$M := \big\{x \,\big|\; (\exists\alpha \in \mathfrak{H})\left[(\alpha,x) \notin \mathsf{W}^{1,1}_x\right]\big\}.$$

Then $M \subseteq K_{\Sigma^1_1}$ and $M \in \Sigma^{1,\mathfrak{H}}_1 \subseteq \Pi^1_1$ by (8.9). Let e be a Π^1_1–index for M. Then we obtain

$$\begin{aligned} e \in \mathsf{U}_e^{\Pi_1^1} &\Leftrightarrow e \in M \\ &\Rightarrow e \in K_{\Sigma_1^1} \\ &\Leftrightarrow e \in \mathsf{U}_e^{\Sigma_1^1,1,0} \\ &\Leftrightarrow e \notin \mathsf{U}_e^{\Pi_1^1,1,0} \end{aligned}$$

since we defined $\mathsf{U}_e^{\Pi_1^1}$ as the complement of $\mathsf{U}_e^{\Sigma_1^1}$. Hence $e \notin \mathsf{U}_e^{\Pi_1^1}$ which entails $e \in \mathsf{U}_e^{\Sigma_1^1}$. Therefore

$$e \notin M \wedge e \in K_{\Sigma_1^1}.$$

Let P be a decidable predicate such that

$$(\alpha, e) \in \mathsf{W}_e^{1,1} \;\Leftrightarrow\; (\exists z)P(\overline{\alpha}(z)).$$

From $e \notin M$ it follows

$$\neg(\exists \alpha \in \mathfrak{H})(\forall z)\neg P(\overline{\alpha}(z))$$

and from $e \in K_{\Sigma_1^1}$

$$(\exists \alpha)(\forall z)\neg P(\overline{\alpha}(z)).$$

Choosing $R :\!= \neg P$ the proof is terminated. $\square$

In order to obtain also the opposite inclusion in (8.9) we need some preparations.
It is obvious that Lemma 8.2.2 relativizes. I.e. we introduce the class

$$\mathfrak{H}^A := \{\alpha \mid G_\alpha \in \Delta_1^1[A]\}$$

and obtain

$$\mathfrak{H}^A \in \Pi_1^1[A] \setminus \Sigma_1^1[A]. \tag{8.10}$$

Another obvious observation is that $\mathfrak{H}^A$ is closed under relativizations, i.e.

$$\alpha \in \mathfrak{H}^A \;\Rightarrow\; \mathfrak{H}^{A,\alpha} = \mathfrak{H}^A. \tag{8.11}$$

This holds since we have $\mathfrak{H}^A \subseteq \mathfrak{H}^{A,\alpha}$ and for $\beta \in \mathfrak{H}^{A,\alpha}$ the graph G_β is a $\Delta_1^1[A,\alpha]$–predicate. But α has a $\Delta_1^1[A]$ graph and the $\Delta_1^1[A]$–predicates are closed under substitution with functions having $\Delta_1^1[A]$ graphs. Hence $\beta \in \mathfrak{H}^A$.
Let

$$\Sigma_1^{1,\mathfrak{H}^A}[A] := \{\varphi^{\mathfrak{H}^A} \mid \varphi \in \Sigma_1^1[A]\}$$

and

$$\Pi_1^{1,\mathfrak{H}^A}[A] := \{\varphi^{\mathfrak{H}^A} \mid \varphi \in \Pi_1^1[A]\}.$$

There are universal predicates

$$\begin{aligned} \mathsf{U}_e^{\Sigma_1^{1,\mathfrak{H}^A}[A]} &:= \{x \mid (\exists \alpha \in \mathfrak{H}^A)[(x,\alpha) \notin \mathsf{W}_e^{A,1,1}]\} \\ \mathsf{U}_e^{\Pi_1^{1,\mathfrak{H}^A}[A]} &:= \{x \mid (\forall \alpha \in \mathfrak{H}^A)[(x,\alpha) \in \mathsf{W}_e^{A,1,1}]\} \end{aligned}$$

and we introduce $\Delta_1^{1,\mathfrak{H}^A}[A]$–indices as pairs of $\Sigma_1^{1,\mathfrak{H}^A}[A]$– and $\Pi_1^{1,\mathfrak{H}^A}[A]$–indices which describe the same sets. We show the following lemma.

8.3.1 Lemma *For* $a \in \mathit{WT}^A$

$$\mathit{WT}^B_{\mathit{otyp}^{\mathit{Tree}^A}(a)} := \{x \in \mathit{WT}^B \mid \mathit{otyp}^{\mathit{Tree}^B}(x) \le \mathit{otyp}^{\mathit{Tree}^A}(a)\}$$

is a $\Delta_1^{1,\mathfrak{H}^{A,B}}[A,B]$ *set. This holds uniformly, i.e. there is a computable function* g *such that* $g(a)$ *is a* $\Delta_1^{1,\mathfrak{H}^{A,B}}[A,B]$*–index for* $\mathit{WT}^B_{\mathit{otyp}^{\mathit{Tree}^A}(a)}$. *Moreover,* g *is independent of* A *and* B.

Proof: We use the Recursion Lemma along $\omega_1^{CK}[A]$. Let $\mathit{rest}^A(a,n)$ be an A–index of the restriction of the tree $\{a\}^A$ to the node $\langle n\rangle$, i.e.

$$\{\mathit{rest}^A(a,n)\}^A = \chi_{\{s \mid \{a\}^A(\langle n\rangle^\frown s)=0\}}.$$

We obtain

$$\begin{aligned} x \in \mathit{WT}^B_{\mathit{otyp}^{\mathit{Tree}^A}(a)} \quad\Leftrightarrow\quad & x \in \mathit{Tree}^B \\ & \wedge (\forall z)[\{x\}^B(\langle z\rangle)=0 \\ & \quad \Rightarrow (\exists m)(\{a\}^A(\langle m\rangle)=0 \wedge \mathit{rest}^B(x,z)\in \mathit{WT}^B_{\mathit{otyp}^{\mathit{Tree}^A}(\mathit{rest}^A(a,m))})]. \end{aligned} \tag{i}$$

Because of $\mathit{otyp}^{\mathit{Tree}^A}(\mathit{rest}^A(a,m)) < \mathit{otyp}^{\mathit{Tree}^A}(a)$ we get by the recursion hypothesis

$$\begin{aligned} u \in \mathit{WT}^B_{\mathit{otyp}^{\mathit{Tree}^A}(\mathit{rest}^A(a,m))} \quad & \Leftrightarrow \quad u \in \mathsf{U}^{\Sigma_1^{1,\mathfrak{H}^{A,B}}[A,B]}_{(\{e\}(\mathit{rest}^A(a,m)))_0} \\ & \Leftrightarrow \quad u \in \mathsf{U}^{\Pi_1^{1,\mathfrak{H}^{A,B}}[A,B]}_{(\{e\}(\mathit{rest}^A(a,m)))_1} \\ & \Leftrightarrow \quad (\exists\alpha\in\mathfrak{H}^{A,B})\left[(u,\alpha)\notin \mathsf{W}^{A,B,1,1}_{(\{e\}(\mathit{rest}^A(a,m)))_0}\right] \\ & \Leftrightarrow \quad (\forall\alpha\in\mathfrak{H}^{A,B})\left[(u,\alpha)\in \mathsf{W}^{A,B,1,1}_{(\{e\}(\mathit{rest}^A(a,m)))_1}\right]. \end{aligned} \tag{ii}$$

Inserting (ii) into (i) and remembering that $\mathfrak{H}^{A,B}$ is a model of $(\Pi_1^1\!-\!AC^{01})$ shows that $\mathit{WT}^B_{\mathit{otyp}^{\mathit{Tree}^A}(a)}$ is a $\Delta_1^{1,\mathfrak{H}^{A,B}}[A,B]$ set whose index can be computed from e and a. Note that the computable function g given by the Recursion Lemma is independent of A and B. $\square$

As a consequence of Lemma 8.3.1 we obtain

8.3.2 Theorem

$$\Delta_1^{1,\mathfrak{H}^A}[A] = \Delta_1^1[A]$$

Proof: The inclusion $\Delta_1^{1,\mathfrak{H}^A}[A]\subseteq\Delta_1^1[A]$ follows from (8.9). The converse inclusion is a consequence of Lemma 8.3.1 and the relativization of Theorem 7.4.2 which says that every $\Delta_1^1[A]$–set is m–reducible to WT^A_σ for some $\sigma<\omega_1^{CK}[A]$. The result now follows from the fact that $\Delta_1^{1,\mathfrak{H}^A}[A]$ is closed under m–reducibility. $\square$

Now we have all the ingredients for one of the main results of this lecture.

8.3.3 Theorem (Hyperarithmetical Quantifier Theorem)

$$\Pi_1^1[A] = \Sigma_1^{1,\mathfrak{H}^A}[A].$$

Proof: The easy direction from right to left is (8.9).
Because WT^A is $\Pi_1^1[A]$–complete it suffices to show

$$\mathit{WT}^A \in \Sigma_1^{1,\mathfrak{H}^A}[A] \tag{i}$$

to obtain also the converse inclusion, as $\Sigma_1^{1,\mathfrak{H}^A}[A]$ is obviously closed under m-reducibility. Since $\mathfrak{H}^A\in\Pi_1^1[A]$ there is a computable function f and an $e\in\mathbb{N}$ such that

$$\begin{aligned} \alpha\in\mathfrak{H}^A \quad & \Leftrightarrow \quad \lambda x.\, f(\alpha,x)\in\mathbb{WT}^A \\ & \Leftrightarrow \quad e\in \mathit{WT}^{A,\alpha}, \end{aligned} \tag{ii}$$

where e is a uniform A, α–index for $\lambda x.\, f(\alpha, x)$. Note that $\mathit{otyp}^{\mathit{Tree}^{A,\alpha}}(e)$ varies with $\alpha \in \mathfrak{H}^A$. We show

$$(\forall \sigma < \omega_1^{CK}[A])(\exists \alpha \in \mathfrak{H}^A)[\sigma \leq \mathit{otyp}^{\mathit{Tree}^{A,\alpha}}(e)] \tag{iii}$$

indirectly and assume

$$(\exists \sigma < \omega_1^{CK}[A])(\forall \alpha \in \mathfrak{H}^A)[\mathit{otyp}^{\mathit{Tree}^{A,\alpha}}(e) < \sigma].$$

But this entails

$$\mathfrak{H}^A = \{\alpha \mid\ \lambda x.\, f(\alpha, x) \in \mathbb{WT}_\sigma^A\}$$

which contradicts (8.10) since $\mathbb{WT}_\sigma^A$ is a $\Delta_1^1[A]$–relation.
Note that for $\alpha \in \mathfrak{H}^A$ we have $\mathit{otyp}^{\mathit{Tree}^{A,\alpha}}(e) < \omega_1^{CK}[A]$. Thus we obtain by (iii) and Lemma 8.3.1

$$\begin{aligned} a \in \mathit{WT}^A \ &\Leftrightarrow\ (\exists \sigma < \omega_1^{CK}[A])\,\big[a \in \mathit{WT}_\sigma^A\big] \\ &\Leftrightarrow\ (\exists \alpha \in \mathfrak{H}^A)[a \in \mathit{WT}^A_{\mathit{otyp}^{\mathit{Tree}^{A,\alpha}}(e)}] \\ &\Leftrightarrow\ (\exists \alpha \in \mathfrak{H}^A)(\exists \beta \in \mathfrak{H}^{A,\alpha}[(\beta, a) \notin \mathsf{W}^{A,\alpha}_{(g(e))_0}]). \end{aligned} \tag{iv}$$

But since $\alpha \in \mathfrak{H}^A$ we have $\mathfrak{H}^{A,\alpha} = \mathfrak{H}^A$ and (iv) yields a $\Sigma_1^{1,\mathfrak{H}^A}[A]$ definition for WT^A. $\square$

9. Appendix

9.1 A Π^1_1–path through $\mathcal{O}$

From the Boundedness Principle we inferred in Corollary 7.6.10 that there are no Σ^1_1–paths through $\mathcal{O}$. We want to show that there are indeed Π^1_1–paths through $\mathcal{O}$ and present a construction which is – as far as we know – due to SPECTOR and FEFERMAN.

9.1.1 Theorem *There is a Π^1_1–path through $\mathcal{O}$.*

Proof: We introduce the set

$$Pd_{<'_{\mathcal{O}}}(a) := \{b \mid b <'_{\mathcal{O}} a\} \tag{9.1}$$

of $<'_{\mathcal{O}}$–predecessors of a. Furthermore we put

$$\varphi(a) \;:\Leftrightarrow\; <'_{\mathcal{O}}\restriction (Pd_{<'_{\mathcal{O}}}(a) \times Pd_{<'_{\mathcal{O}}}(a)) \in \mathbb{WO}$$

and

$$\begin{aligned}\psi(a) \;:\Leftrightarrow\; (\forall x \in Pd_{<'_{\mathcal{O}}}(a))[x = 1 \vee (\exists c)(x = 2^c) \\ \vee (\exists e)(x = 3 \cdot 5^e \wedge \{e\}^{1,0} \text{ is total} \\ \wedge (\forall n)[\{e\}^{1,0}(n) <'_{\mathcal{O}} \{e\}^{1,0}(n+1)])].\end{aligned}$$

We claim

$$a \in \mathcal{O} \;\Leftrightarrow\; \varphi(a) \wedge \psi(a). \tag{9.2}$$

The direction from left to right is Lemma 7.6.5. For the opposite direction we assume the right hand side of (9.2) and prove first

$$b \in Pd_{<'_{\mathcal{O}}}(a) \;\Rightarrow\; b \in \mathcal{O} \tag{i}$$

by induction on the definition of $<'_{\mathcal{O}}$. This is obvious for $b = 1$ and follows for $b = 2^c \neq 1$ immediately from the induction hypothesis. If $b = 3 \cdot 5^e$ then $\{e\}^{1,0}$ is total and we have $\{e\}^{1,0}(n) \in \mathcal{O}$ for all n by induction hypothesis. We moreover get $\{e\}^{1,0}(n) <'_{\mathcal{O}} \{e\}^{1,0}(n+1)$ which by the induction hypothesis and Lemma 7.6.3 entails $\{e\}^{1,0}(n) <_{\mathcal{O}} \{e\}^{1,0}(n+1)$. But then $b = 3 \cdot 5^e \in \mathcal{O}$. From (i) and the right hand side of (9.2) we first get $a \in \mathcal{O}'$ which by Lemma 7.6.5 entails $a \in \mathcal{O}$.
By weakening the right hand side of (9.2) we define

$$\begin{aligned}a \in \mathcal{O}^\dagger \;&:\Leftrightarrow\; (\varphi(a) \wedge \psi(a))^{\mathfrak{H}} \\ &\Leftrightarrow\; \varphi(a)^{\mathfrak{H}} \wedge \psi(a).\end{aligned} \tag{9.3}$$

Observe that $\mathfrak{H}$ thinks that $\mathcal{O}^\dagger$ is $\mathcal{O}$ (note the analogy to Theorem 8.2.4). Because of $\varphi(a) \Rightarrow \varphi(a)^{\mathfrak{H}}$ we have $\mathcal{O} \subseteq \mathcal{O}^\dagger$. By (the contraposition of) the Hyperarithmetical Quantifier Theorem $\mathcal{O}^\dagger \in \Sigma^1_1$ holds. Hence

$$\mathcal{O} \subsetneq \mathcal{O}^\dagger. \tag{ii}$$

We may therefore pick an $a \in \mathcal{O}^\dagger \setminus \mathcal{O}$ and show that

$$P := Pd_{<'_{\mathcal{O}}}(a) \cap \mathcal{O}$$

is a path through $\mathcal{O}$. Towards an indirect proof we assume

$$P \subseteq \mathcal{O}_{|b|_{\mathcal{O}}} := \big\{ c \in \mathcal{O} \big| \ |c|_{\mathcal{O}} \leq |b|_{\mathcal{O}} \big\} \tag{iii}$$

for some $b \in \mathcal{O}$. But then $P = \mathit{Pd}_{<'_{\mathcal{O}}}(a) \cap \mathcal{O}_{|b|_{\mathcal{O}}}$ which shows that P is a Δ^1_1 set. This implies that

$$P' := \mathit{Pd}_{<'_{\mathcal{O}}}(a) \setminus \mathcal{O} = \mathit{Pd}_{<'_{\mathcal{O}}}(a) \setminus P$$

is a nonempty Δ^1_1 set. Thus P' has a $<'_{\mathcal{O}}$–least element, say o. Because of $o \notin \mathcal{O}$ we have $o \neq 1$. If $o = 2^c \neq 1$ we get $c \in \mathcal{O}$ by the minimality of o. But this entails $o \in \mathcal{O}$. Finally if $o = 3 \cdot 5^e$ then we obtain $\{e\}^{1,0}(n) \in \mathcal{O}$ for all $n \in \mathbb{N}$ as well as $(\forall n) \big[\{e\}^{1,0}(n) <'_{\mathcal{O}} \{e\}^{1,0}(n+1)\big]$. Hence $o \in \mathcal{O}$ which shows the absurdity of our assumption. So P is a path through $\mathcal{O}$ and P is obviously Π^1_1–definable (using the parameter a). $\square$

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