Loop Vertex Representation for Random Matrices with Higher Order Interactions

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joint work with Vincent Rivasseau and Vasily Sazonov

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Stochastic Analysis meets QFT - critical theory
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Motivation : Divergence of perturbative expansions

Perturbative expansion in QFT over **Feynman graphs**

$$\log Z = \log \int [\mathcal{D}\phi] \exp - \int \left\{ \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 \right\}$$

$$" = " \sum_{G \text{ Feynman graph}} \mathcal{A}(G)\, g^{\#\text{vertices}}$$

The perturbative expansion is a **divergent** power series (otherwise Z defined for $\mathrm{Re}(g) < 0$, $g = 0$ boundary of analyticity domain).

Perturbative expansion only valid as an **asymptotic series** for $g \to 0$ but does not allow for a definition of a QFT.

Origins of the divergence : $\sum_{G \text{ order } n} \mathcal{A}(G) \sim n!$

- **too many graphs** of given order (instantons)
- **too large graph amplitudes** at given order (renormalons)

Construction of QFT from its perturbative expansion usually addressed using **Borel summation**.

Factorial growth of the number of Feynman graphs

Consider a simple integral analogue to the functional integral in quantum field theory ($\mathrm{Re}(g) \geq 0$) with **asymptotic expansion**

$$Z = \int_{\mathbb{R}} \frac{d\phi}{\sqrt{2\pi}} \exp -\left\{ \frac{\phi^2}{2} + \frac{g\phi^4}{4!} \right\} \quad " = " \quad \sum_{n=0}^{+\infty} (-1)^n a_n g^n \qquad (1)$$

This integral counts **Feynman graphs** with **factorial growth**, thus impeding the convergence of the series (two many graphs)

$$a_n = \sum_{\substack{G \text{ 4-valent graph} \\ \text{with } n \text{ vertices}}} \frac{1}{\#\,\mathrm{aut}(G)} = \int_{\mathbb{R}} \frac{d\phi}{\sqrt{2\pi}} \exp -\left\{ \frac{\phi^2}{2} \right\} \frac{\phi^{4n}}{(4!)^n n!} \qquad (2)$$

$$\underset{n \to +\infty}{\sim} C \left(2/3\right)^n n! \qquad (3)$$

From a physical viewpoint, the integral defines an analytic function for $\mathrm{Re}(g) > 0$ such that the origin lies on the boundary of analyticity domain. If the series were convergent, it would make sense for $\mathrm{Re}(g) < 0$ leading to an **unstable model**. Alternatively, $g\phi^4$ cannot be treated as small for large field ϕ.

Borel summability and instanton singularity

Starting with a possibly divergent series $\sum (-1)^n a_n g^n$ asymptotic to a function $F(g)$, we can attempt at recovering F using

$$F(g) = \frac{1}{g} \int_0^{+\infty} ds\ B(s) \exp(-s/g) \tag{4}$$

with $B(s) = \sum_{n=0}^{+\infty} \frac{(-1)^n a_n}{n!} s^n$ the **Borel transform**.

This requires $B(s)$ to be free of singularities on the positive real axis. After rescaling the field $\phi \to \phi/\sqrt{g}$

$$Z = \int_{\mathbb{R}} \frac{d\phi}{\sqrt{2\pi}} \exp -\left\{ \frac{1}{g} \left(\frac{\phi^2}{2} + \frac{\phi^4}{4!} \right) \right\} \tag{5}$$

$$= \frac{1}{\sqrt{g}} \int_0^{+\infty} ds\ \exp(-s/g) \int_{\mathbb{R}} \frac{d\phi}{\sqrt{2\pi}} \delta\left\{ \frac{\phi^2}{2} + \frac{\phi^4}{4!} - s \right\} \tag{6}$$

The composed Dirac distribution $\delta(F(\phi)) = \sum_{F(\phi_i)=0} \frac{\delta(\phi - \phi_i)}{|F'(\phi_i)|}$ leads to **instanton singularities** for classical solutions of the equations of motion.

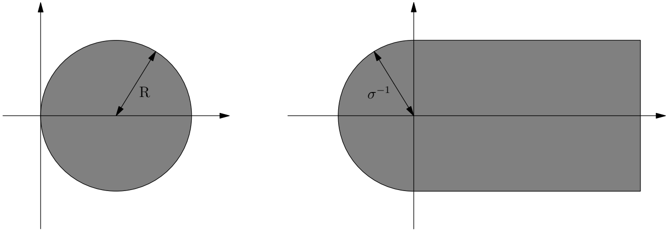
The Nevanlinna-Sokal theorem on Borel resummation

Recall the Nevanlinna-Sokal theorem : If F is a analytic function in the disk $\mathrm{Re}(1/g) > R^{-1}$ and $\sum_n a_n g^n$ a formal power series such that

$$\left| F(g) - \sum_{k=1}^{n} a_k g^k \right| \leq C\sigma^{-(n+1)} |g|^{n+1} (n+1)! \tag{7}$$

then F can be reconstructed from $B(s) = \sum_n \frac{a_n}{n!} s^n$

$$F(g) = \frac{1}{g} \int_0^{+\infty} \exp(-s/g)\, B(s) \tag{8}$$



Combinatorial approach : Loop Vertex Representation

Basic idea (V. Rivasseau, `arxiv 0706.1224`) : **expand the partition function over forests** (= not necessarily connected graphs without loops) over instead of graphs and **logarithm expanded over trees** (connected components)

$$Z = \sum_{F \text{ forest}} \mathcal{A}_F(g) \qquad \Leftrightarrow \qquad \log Z = \sum_{T \text{ tree}} \mathcal{A}_T(g)$$

Convergence of the expansion possible because of power law growth (solving the "too many graphs" issue)

$$\#\begin{pmatrix}\text{trees of} \\ \text{order } n\end{pmatrix} \underset{n\to+\infty}{\sim} \kappa^n \quad \text{vs} \quad \#\begin{pmatrix}\text{graphs of} \\ \text{order } n\end{pmatrix} \underset{n\to+\infty}{\sim} n!$$

and power law bounds on tree amplitudes $|\mathcal{A}_T(g)| \leq C^n |g|^n$

Usual perturbative expansion recovered by further expanding $\mathcal{A}_T(g)$ in powers of g (addition of loops to T)

Open question in QFT but interesting results for random matrices.

Random Matrices

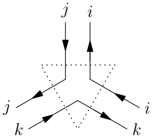
Topological **ribbon graph expansion** of matrix integral

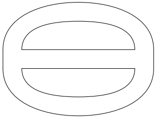
$$\frac{1}{N^2}\log\int DM\exp -N\left\{\mathrm{Tr}\,M^2+g\,\mathrm{Tr}\,M^{2p}\right\}=\sum_{G\text{ ribbon graph}}\mathcal{A}_G\,g^{\#(\text{vertices})}N^{\chi(G)}$$

with $\chi = 2 - \text{genus} = \#(\text{vertices}) - \#(\text{edges}) + \#(\text{faces})$

Ribbon Feynman graph (double line) dual to trianagulations

$$\mathrm{Tr}\,M^3=\sum_{i,j,k}M_{ij}M_{jk}M_{ki}\rightarrow$$





Multiple occurence in physics as random Hamiltonians (spectra of heavy nuclei, JT gravity in the Schwarzian limit, ..) or topological expansion (large N QCD, 2d gravity, ...).

Main result : Uniform analyticity in a "Pac-Man" domain

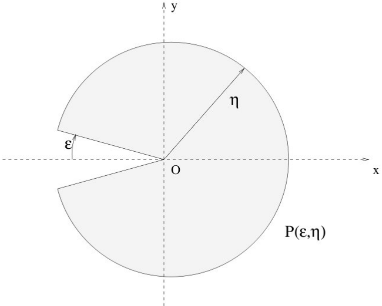
For any $\epsilon > 0$ there exists $\eta > 0$ such that the LVE expansion

$$\frac{1}{N^2} \log \int DM \exp -N\Big\{\mathrm{Tr}\, M^2 + g^{p-1}\mathrm{Tr}\, M^{2p}\Big\} = \sum_{T \text{tree}} \mathcal{A}_T(g,N) \quad (9)$$

is convergent and defines an analytic function for

$$g \in \Big\{0 < |g| < \eta, |\arg g| < \frac{\pi}{2} + \frac{\pi}{p-1} - \epsilon\Big\} \quad (10)$$

It is bounded by a constant independent of N and Borel summable in g, uniformly in N (with a cut for $p = 2$).

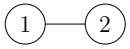


Forest Formula (Abdesselam, Brydges, Kennedy, Rivasseau)

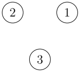
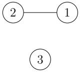
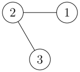
ϕ function of $\frac{n(n-1)}{2}$ variables $x_{ij} \in [0,1]$(edges between n vertices)

$$\phi(1,\ldots,1) = \sum_{\substack{F \text{ forest} \\ \text{on } n \text{ vertices}}} \int_0^1 \prod_{(i,j)\in F} du_{ij} \left(\frac{\partial^{\#(\text{edges in } F)}\phi}{\prod_{(i,j)\in F} \partial x_{ij}} \right) (v_{ij}) \ ,$$

where v_{ij} is the infimum of u_{kl} along the unique path from from i to j in F if it exists and 0 otherwise

- $n = 2$: 2 forests , ,

$$\phi(1) = \phi(0) + \int_0^1 du_{12} \left(\frac{\partial\phi}{\partial x_{12}} \right) (u_{12})$$

- $n = 3$: , , ... , ...

$$\phi(1,1,1) = \phi(0,0,0) + \int_{[0,1]} du_{12} \left(\frac{\partial\phi}{\partial x_{12}} \right) (u_{12},0,0) + \text{perm.}$$

$$+ \int_{[0,1]^2} du_{12} du_{23} \left(\frac{\partial^2\phi}{\partial x_{12}\partial x_{23}} \right) (u_{12}, u_{23}, \inf(u_{12},u_{23})) + \text{perm.}$$

Tree expansion of the partition function

Then, one can rewrite the integral of the exponential over a variable ϕ as a sum of multiple integrals over multiple variable $\phi_1, \ldots, \phi_n$

$$Z = \int d\mu_C(\phi) \exp\{-V(\phi)\} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \int d\mu_{C_{ij}=1}(\phi_1, \ldots, \phi_n) V(\phi_1) \cdots V(\phi_n) \quad (11)$$

In this formula, $d\mu_{C_{ij}=1}(\phi)$ is a Gaussian measure with a covariance matrix whose entries are all equal to 1 $\Leftrightarrow$ sets $\phi_1 = \cdots = \phi_n$

Replacing $C_{ij} = u_{ij}$ for $i \neq j$, we can apply the forest formula

$$Z = \int d\mu_{C(u)}(\phi) \exp\{-V(\phi)\}\Big|_{u_{ij}=1} = \sum_{\text{forests } \mathcal{F}} \mathcal{A}_{\mathcal{F}} \quad (12)$$

Since $\mathcal{A}_{\mathcal{F}} = \mathcal{A}_{\mathcal{T}_1} \cdots \mathcal{A}_{\mathcal{T}_c}$, the logarithm reduces to a sum over trees

$$\log Z = \sum_{\text{trees } \mathcal{T}} \mathcal{A}_{\mathcal{F}} \quad (13)$$

Tree expansion of the matrix integral

Let us apply the **forest formula** after rescaling the matrix as $M \to M/g$

$$Z = \int dM \exp -\frac{N}{g}\left\{ \mathrm{Tr}\, M^2 + V(M) \right\} \tag{14}$$

$$= \sum_n \frac{(-1)^n}{n!} \int dM_1 \cdots dM_n \exp -\frac{N}{g}\left\{ \sum_{1\leq i,j\leq n} C_{ij}^{-1} \mathrm{Tr}(M_i M_j) \right\} V(M_1)\cdots V(M_1) \Bigg|_{\underbrace{C_{ij}=1}_{\text{sets } A_i = A_j}} \tag{15}$$

$$= \sum_{F \text{ embedded forest}} \mathcal{A}_F \tag{16}$$

Then the free energy $\log Z$ is a sum over **embedded** trees.

It remains to bound the amplitudes and the number of trees.

Morse-Palais change of variables

The **Morse-Palais lemma** states that any functional can be reduced to a quadratic one in the vicinity of its extrema. For the matrix integral, we set

$$K = M\sqrt{1 + M^{2p-2}} \qquad \Leftrightarrow \qquad M = K\sqrt{T(-K^{2p-2})} \tag{17}$$

with T the **Fuß-Catalan function** such that $T(z) = 1 + zT^p(z)$.

For the matrix integral, it leads to

$$\int dM \exp -\frac{N}{g}\Big\{\mathrm{Tr}\, M^2 + \mathrm{Tr}\, M^{2p}\Big\} = \tag{18}$$

$$\int dK \exp -\Big\{N\mathrm{Tr}\, K^2 + V_{eff}(K)\Big\} \tag{19}$$

with an **effective potential** computed from the Jacobian

$$V_{eff}(K) = -\log\det\frac{\delta M}{\delta K} = -\mathrm{Tr}_{\otimes}\log\frac{\delta M}{\delta K} \tag{20}$$

The derivative of the logarithm is a resolvent and analytic properties of $T(z)$ lead to useful bounds on tree amplitudes .

Effective potential and matrix derivative

For any single matrix function $M = \sum_n a_K^n$, the **matrix derivative** acting on matrices $M_N(\mathbb{C}) \sim \mathbb{C}^N \otimes \mathbb{C}^N$ by left and right multiplication is defined as

$$\delta M = \sum_n \sum_{k=0}^{n-1} a_n K^k \, \delta K \, K^{n-1-k} \Leftrightarrow \frac{\delta M}{\delta K} = \sum_n \sum_{k=0}^{n-1} a_n K^k \otimes K^{n-1-k} \tag{21}$$

In our case with Fuß-Catalan function T

$$\frac{\delta M}{\delta K} = \frac{K\sqrt{T(-K^{2p-2})} \otimes 1 - 1 \otimes K\sqrt{T(-K^{2p-2})}}{K \otimes 1 - 1 \otimes K} \tag{22}$$

if e_i diagonalises K, $Ke_i = \nu_i e_i$, then $e_i^\dagger e_j$ diagonalises $\frac{\delta M}{\delta K}$ and

$$V_{\text{eff}}(K) = \sum_{i,j} \log \left| \frac{\nu_i \sqrt{T(-\nu_i^{2p-2})} - \nu_j \sqrt{T(-\nu_j^{2p-2})}}{\nu_i - \nu_j} \right| \tag{23}$$

Fuß-Catalan generating function

Lagrange inversion formula leads to Fuß-Catalan numbers

$$T(z) = 1 + zT^p(z) \quad \Rightarrow \quad T(z) = \sum_n \frac{(np)!}{n!(np-n+1)!} z^n \quad (24)$$

ordinary Catalan numbers for $p = 2$ (counting p-ary trees)

Some useful properties :

- $T(z)$ analytic on the cut plane $\mathbb{C} - \left[\frac{(p-1)^{p-1}}{p^p}, +\infty\right[$
- behavior at infinity $T(z) \sim -\left(\frac{1}{z}\right)^{1/p}$ ‘
- $T(z) \neq 0$ for finite z

On any domain Ω staying at a finite distance from the cut

$$\left|T(z)\right| \leq \frac{C_\Omega}{(1+|z|)^{1/p}}, \quad \left|T'(z)\right| \leq \frac{C'_\Omega}{(1+|z|)^{1/p+1}}, \ldots \quad (25)$$

Analyticity from counting trees

The number of **labelled and embedded trees** on n vertices is

$$\sum_{\substack{r_1-1+\cdots+r_n-1\\=n-2}} \overbrace{\frac{(n-2)!}{(r_1-1)!\cdots(r_n-1)!}}^{\text{trees with coordination } r_1,\ldots,r_n} \times \overbrace{(r_1-1)!\cdots(r_n-1)!}^{\text{embeddings}}$$

$$= \frac{(2n-3)!}{(n-1)!} \qquad (26)$$

Bounding each tree amplitude leads to a **convergent series**

$$|F| \leq \sum_T |\mathcal{A}_T| \leq \frac{(2n-3)!}{(n-1)!} \times \frac{|\lambda|^n \left[\kappa(\arg\lambda)\right]^n}{n!} \qquad (27)$$

The factorial growth cancel so that the series has a finite radius of convergence, with $\kappa(\arg\lambda) \propto \frac{1}{|\cos(\arg\lambda)|}$ for $\left|\arg\lambda\right| < \pi/2$ (positivity of Gaußian measure) $\Rightarrow$ **not enough for Borel summation**.

Analytic continuation from contour rotation

To accommodate a large range for $\arg\lambda$ let us rotate the matrix integral by and angle α $M \to \exp(i\alpha)M$ as well as all Cauchy contours with two constraints :

- positivity of the Gaußian measure $\exp -\dfrac{\mathrm{tr}\,K^2}{g}$

$$-\pi/2 < \arg\lambda - 2\alpha < \pi/2 \tag{28}$$

- singularity of the Fuß-Catalan function $T\big(-K^{2p-2}\big)$

$$-\pi < (2p-2)\alpha < \pi \tag{29}$$

The maximal opening of the domain of analyticity is therefore

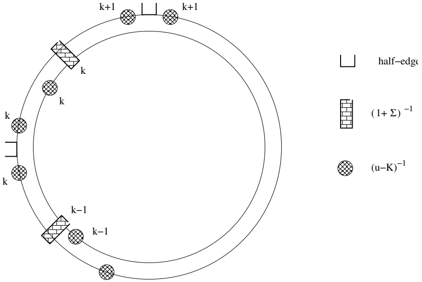
$$-\frac{\pi}{2} - \frac{\pi}{p-1} + \epsilon \leq \arg\lambda \leq \frac{\pi}{2} + \frac{\pi}{p-1} - \epsilon \tag{30}$$

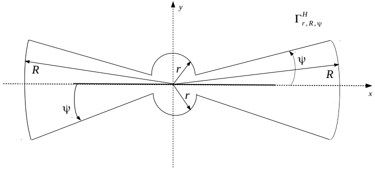
Bounds on the tree amplitude

Writing the effective potential as $V_{\text{eff}}(K) = \text{Tr}_{\otimes} \log(1 - \Sigma)$ (acting in $\mathbb{C}^N \otimes \mathbb{C}^N$), every vertex is represented as a double line graph with insertions of $(1-\Sigma)^{-1}$ or $\frac{\partial \Sigma}{\partial K}$, written as a contour integral. The tree amplitude $\mathcal{A}_T$ involves $E(T)+2$ faces and is a trace in $\mathbb{C}^{\otimes(E(T)+2)}$. Using $|\text{Tr}(A)| \leq N^{\otimes(E(T)+2)} \|A\|$, it can be bounded as

$$|\mathcal{A}_T| \leq N^2 \prod_i \oint_{\Gamma_i} |du_i| \prod_j \|\mathcal{O}_j\| \tag{31}$$

where $\mathcal{O}_j$ are the operators encountered around the vertices and contracted along the edges.





Borel summability of the matrix model partition function

Borel summability follows from the **Nevanlinna-Sokal** theorem checking the two hypothesis :

- Analyticity on the circle tangent to the positive axis follows from the analyticity in the Pac-Man domain.
- The bound on the remainder can be obtained by recursively adding edges to the trees

$$\log Z = \sum_{k=0}^{n} (-1)^n a_n g^n + R_{n+1}(g, N) \tag{32}$$

The remainder is a over trees with at least $n+1-k$ edges on which k edges have been added using the representation

$$\int d\mu_C(K)\, f(K) = \exp\left(\frac{1}{2} \sum_{ij} C_{ij} \frac{\partial^2}{\partial K_i \partial K_j} \right) f(K) \Big|_{K=0} \tag{33}$$

Towards a similar approach in Quantum Field Theory

Change of variables from **Morse-Palais lemma** : reduction of a functional around a critical point in Hilbert space to a quadratic form $S[\phi] = \langle \chi(\phi), \chi(\phi) \rangle$

$$\int \left\{ \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 \right\} = \int \left\{ \frac{1}{2}(\partial\chi)^2 + \frac{m^2}{2}\chi^2 \right\}$$

leading to the non local effective potential (Jacobian)

$$V_{\text{eff}}[\chi] = \log \det \frac{\delta\phi}{\delta\chi} = \text{Tr} \log \frac{\delta\phi}{\delta\chi}$$

Difficulty : find suitable **cut-off independent bounds**.

Matrix model with kinetic term (Grosse-Wulkenhaar model)

$$\int DM \exp - \left\{ \text{Tr}\, AM^2 + g \text{Tr}\, M^4 \right\}$$

2d case by V. Rivasseau and Z.T. Wang `arxiv1805.06365`.