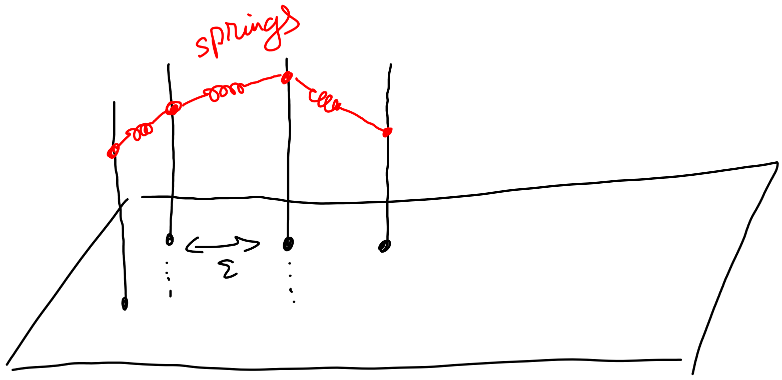
Φ^4_3 MEASURE ON COMPACT RIEMANNIAN MANIFOLDS.

Tô (SORBONNE)

Bailleul (BREST)

Ferdinand (ORSAY)

What is this about? Spin System, discrete Ginzburg - Landau

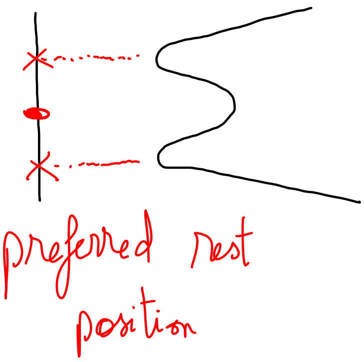


Λ = discrete Box in $(\varepsilon \mathbb{Z})^d$

Configuration of spins, $\sigma : \Lambda \to \mathbb{R}$, $\sigma \in \mathbb{R}^\Lambda$

ACTION: $$S[\sigma] = \underbrace{\sum_{i \sim j} \frac{|\sigma_i - \sigma_j|^2}{2} + \sum_i m^2 \frac{\sigma_i^2}{2}}_{\text{discrete DIRICHLET energy}} + \underbrace{g\left(1 - \sigma_i^2\right)^2}_{\text{Interaction}}$$

Interaction:

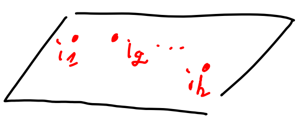


POTENTIAL,

Gibbs measure on $\mathbb{R}^\Lambda$: $$\frac{e^{-S[\sigma]} d\hat{\sigma}}{\int_{\mathbb{R}^\Lambda} e^{-S[\sigma]} d^\Lambda \sigma}$$

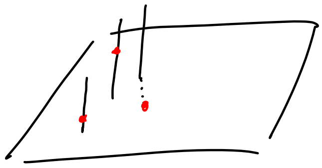
$\int_{\mathbb{R}^\Lambda} e^{-S[\sigma]} d^\Lambda \sigma = Z$ partition function

Correlations:



$$\langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle := \frac{\int_{\mathbb{R}^\Lambda} \sigma_{i_1} \cdots \sigma_{i_k} e^{-S[\sigma]} d^\Lambda \sigma}{\int_{\mathbb{R}^\Lambda} e^{-S[\sigma]} d^\Lambda \sigma}$$

DIGRESSION : 3 Ferromagnetic spin systems



	Config space	Gibbs measure
ISING	$\sigma \in \{\pm 1\}^\Lambda$	$e^{\sum_{i \sim j} + \sigma_i \sigma_j}$
DGFF	$\sigma \in \mathbb{R}^\Lambda$	$e^{\sum_{i \sim j} + \sigma_i \sigma_j - c \sum_i \sigma_i^2}$
Ginzburg - Landau	$\sigma \in \mathbb{R}^\Lambda$	$e^{\sum_{i \sim j} + \sigma_i \sigma_j + \sum_i \underbrace{P(\sigma_i)}_{\text{interaction}}}$

P bounded from below

QUESTION : Why Gibbs $\dfrac{e^{-\beta S[\phi]} \mathcal{D}\phi}{\int e^{-\beta S[\phi]} \mathcal{D}\phi}$ special ?

* Boltzmann - Gibbs variational principle.

$Z_\beta = \int e^{-\beta S[\phi]} \mathcal{D}\phi$ partition function

Free energy $F_\beta = -\log(Z_\beta)$, $\forall\, \nu \in \mathrm{Proba}(\mathbb{R}^\Lambda)$:

$$\beta^{-1} F_\beta \leq \langle S \rangle_\nu + \beta^{-1} \int_{\mathbb{R}^\Lambda} \log(\nu)\, \nu \, d^\Lambda \sigma$$

SMALL when ν charges ground states

SMALL when ν makes spins $(\sigma_i)_{i \in \Lambda}$ i.i.d

Infimum attained when $\nu = \dfrac{e^{-S[\phi]} \mathcal{D}\phi}{\int_{\mathbb{R}^\Lambda} e^{-S[\phi]} \mathcal{D}\phi}$

KRAJEWSKI : Analogy with taxes.

Only the rich pay a lot ! VS We all pay the same !

Why does Stochastic quantization gives the correct measure ?

G. Rivière (NANTES)

STOCHASTIC QUANTIZATION (Parisi-Wu 80's)

Idea : operator $P = \underbrace{-\Delta_{\mathbb{R}^n}}_{\text{Diffusion term}} - \underbrace{\langle \nabla S, \nabla \rangle}_{\text{transport by gradient of } S}$

L^2-adjoint $P^* = \nabla \cdot (e^{-S} \nabla e^{S})$, note $\underline{P^* e^{-S} = 0}$.

If $\rho(t)$ solves $\quad \partial_t \rho + P^* \rho = 0 \;, \; \rho_0 = \rho(0)$ $\quad$ Fokker-Planck

r.h.s of Boltzmann-Gibbs for $\gamma = \rho(t)\, d\hat{\sigma}$

$$\int_{\mathbb{R}^\Lambda} \left(S + \log \rho(t)\right) \rho(t)\, d\hat{\sigma}$$

then $\quad \displaystyle \frac{d}{dt} \int_{\mathbb{R}^\Lambda} \left(S + \log \rho(t)\right) \rho(t)\, d\hat{\sigma} \leq 0$

$\Rightarrow \quad t \to \displaystyle\int_{\mathbb{R}^\Lambda} \left(S + \log \rho(t)\right) \rho(t)\, d\hat{\sigma} \quad \searrow$ decreases!

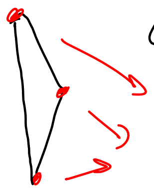
Expect $\quad \rho(t)\, d\hat{\sigma} \xrightarrow[t \to +\infty]{} \dfrac{e^{-S[\sigma]}\, d\hat{\sigma}}{\int e^{-S[\sigma]}\, d\hat{\sigma}}$

Moreover: $\quad \mu_{Gibbs}\left(e^{-tP} f\right) = \mu_{Gibbs}(f) \;, \; \forall t \geq 0$

μ_{Gibbs} invariant measure of $\left(e^{-tP}\right)_{t \geq 0}$ $\frac{1}{2}$-group

Question: 1) Uniqueness?

Inv measures



CONVEX

EXTREMAL POINTS = Ergodic invariant measures

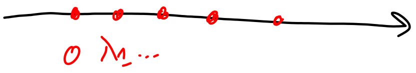
If μ_1, μ_2 ergodic, $\left.\begin{array}{l} P^* \mu_1 = 0 \\ P^* \mu_2 = 0 \end{array}\right\} \Rightarrow \mu_1, \mu_2 \in \mathcal{C}^\infty$ by elliptic regularity

μ_1, μ_2 ergodic and non mutually singular $\Rightarrow \mu_1 = \mu_2$

2) Speed of cnv ? If S enough convex at ∞

$$e^{\frac{S}{2}}\left(-\Delta_{\mathbb{R}^\Lambda} - \langle \nabla S, \nabla \rangle\right) e^{-\frac{S}{2}} = -\Delta_{\mathbb{R}^\Lambda} \underbrace{- \frac{\mathrm{div}(S) + \|\nabla S\|^2}{2}}_{\text{elliptic operator + confining potential}}$$

WITTEN



$\sigma(P)$, $L^2(e^{-S/2} d^\Lambda \sigma)$

$\Rightarrow$ spectral gap

$\forall \sigma_0 \in \mathbb{R}^\Lambda$, $\forall F \in \mathcal{C}^0(\mathbb{R}^\Lambda)$

$$\left(e^{-tP} F\right)(\sigma_0) \xrightarrow[t \to +\infty]{} \langle F \rangle_{\mu \text{Gibbs}}$$

and

$$\left|\left(e^{-tP} F\right)(\sigma_0) - \langle F \rangle\right| \lesssim e^{-t\lambda_1}$$

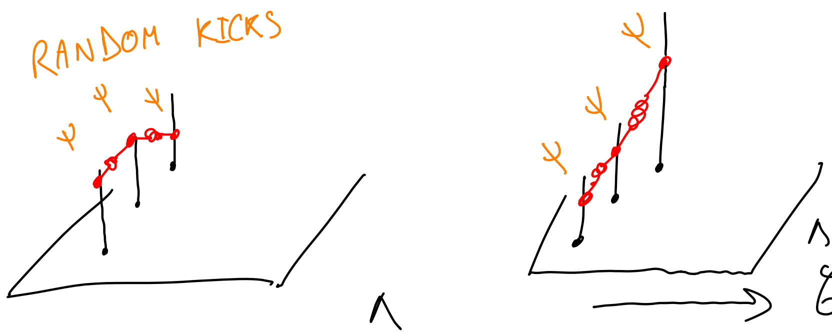
Exponential fast convergence

Another way, represent $\frac{1}{2}$-group $\left(e^{-tP}\right)_{t \geq 0}$ by process.

$$du + \nabla S(u) dt = \sqrt{2} \left(dB_i\right)_{i \in \Lambda}, \quad (\xi_i dt = dB_i) : \mathbb{R}^\Lambda\text{-valued white noise}$$

Then

$$\frac{1}{T} \int_0^T \mathbb{E}_0\left(F(u(t))\right) dt \xrightarrow[T \to +\infty]{} \langle F \rangle_{\mu \text{ Gibbs}}$$



$d=3$, Interested in

$$\langle \sigma_{i_1} \dots \sigma_{i_k} \rangle_{\Lambda_\varepsilon} \xrightarrow[\varepsilon \to 0^+]{} \ ?$$

Other question: $\varepsilon^{\gamma} \sum_{i \in \Lambda_\varepsilon} \sigma_i \, \delta_\varepsilon(x_i) \xrightarrow{?}$ in law in distributions

Answer: Yes! Glimm-Jaffe Φ^4_3 on $(\mathbb{R}^3, \mathbb{T}^3)$ 70's

Nelson, Segal $P(\phi)_2$ on $(\mathbb{R}^2, \mathbb{T}^2)$ 60's

Many others: Balaban, Brydges, Feldman, Fröhlich, Gawedzki, Kupiainen, Magnen, Rivasseau, Sénéor, Slade, Spencer ...

ϕ exists only as singular distribution, dimensional analysis

$$S(\phi) = \int_{\mathbb{R}^d} |\nabla \phi|^2 \, d^d x \ , \quad [S] = 0 \ , \quad d + 2([\phi] - 1) = 0$$

$$[\phi] = \frac{2-d}{2}$$

Expect $\phi_{GFF} \in \mathcal{C}^{\frac{2-d}{2} - 0}$ a.s

$d=2$, $\phi \in \mathcal{C}^{-0}$

$d=3$, $\phi \in \mathcal{C}^{-\frac{1}{2} - 0}$

Question: Φ^4_3 on manifolds ?

Perturbative QFT on Mfds: Kopper-Müller, Costello, Brunetti-Fredenhagen, Hollands-Wald

One would anticipate that the same is true nonperturbatively, for theories such as QCD. If a theory exists perturbatively in curved spacetime, and nonperturbatively in flat spacetime, one would expect that it works nonperturbatively in curved spacetime. Unfortunately, not much is available in terms of rigorous theorems, except for special models like two-dimensional conformal field theories. That reflects the general mathematical difficulty of understanding quantum field theory rigorously. One would think that rigorous results for a superrenormalizable theory in curved spacetime might be relatively accessible, but such results are not available.

Witten (2021)

Constructive QFT on mfd?

$P(\phi)_2$	Φ^4_3	GFF_d	Liouville CFT
Dimock, Pickrell (2007)	This talk	known	Guillarmou - Rhodes - Vargas (2019)

Goal: Φ^4_3 on closed, compact Riemannian mfd (M,g).

SDE $\longrightarrow$ SPDE

$$\partial_t \phi + (1-\Delta_g)\phi = -\phi^3 + \xi, \quad \xi(t,x) \text{ white Noise on } \mathbb{R}\times M$$

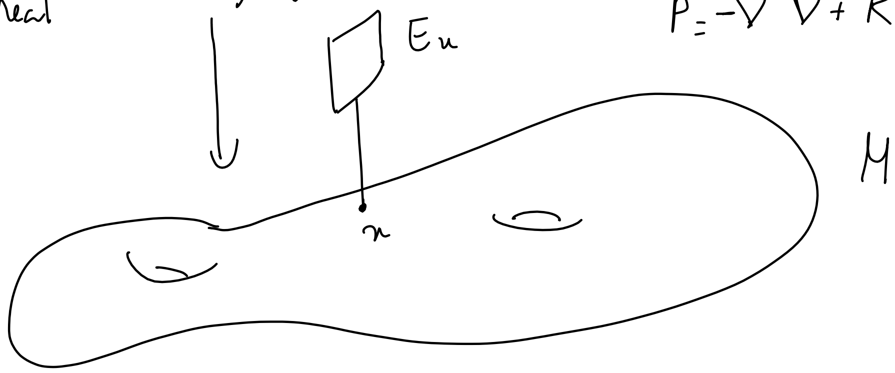
Construct Φ^4_3 invariant measure

Albeverio, Barashkov, Gubinelli, Hairer, Hofmanova, Kusuoka, Mourrat, Perkowski, Weber, Rinaldi

Duch, Kupiainen with RG

Our approach based on Jagannath - Perkowski.

Also treat $E, \nabla, \|\cdot\|_E$



$P = -\nabla^*\nabla + R$ generalized Laplacian acting on $\mathcal{C}^\infty(E)$

$P \geq 0$

Solutions ϕ are E-valued:

$$\partial_t \phi + P\phi = -\phi\|\phi\|_E^2 + \xi_E, \quad \xi_E = \sum_{\lambda\in\sigma(P)} \xi_\lambda(t)\, e_\lambda(x)$$

$(e_\lambda)_{\lambda\in\sigma(P)}$ ONB of $L^2(E)$

$(\xi_\lambda)_\lambda$ iid white noise on $\mathbb{R}$

ϕ is bundle $O(N)$-model.

Solve equation graphically: $\mathcal{L}\phi = \xi - \phi^3$

$$\phi = \underset{\mathcal{C}^{-\frac{1}{2}-}}{|} - \underset{\mathcal{C}^{\frac{1}{2}-}}{\Psi} + \underbrace{v}_{\mathcal{C}^{1-}}, \quad \text{formally } \phi = \sum_{\tau \in \text{Trees}} c_\tau \tau$$

Equation satisfied by v:

$$\mathcal{L} v = \dots - 3 \underset{\mathcal{C}^{1-0}}{\underset{\cap}{v}} \underset{\mathcal{C}^{-1-0}}{\underset{\cap}{V}} \qquad \text{PROBLEM!}$$

Recall <u>Thm (YOUNG)</u>: $u \in \mathcal{C}^\alpha$, $v \in \mathcal{C}^\beta$, $\alpha+\beta>0$

$$\Rightarrow \quad uv \in \mathcal{C}^{\alpha \wedge \beta}$$

does not apply for $v\, V$

Key insight of JP:

$$\phi = | - \Psi + e^{-3Y} u, \quad u \text{ new unknown}$$

New parametrization of ϕ, u solves

$$\mathcal{L} u = -e^{3Y} u^3 + Z_0 + Z_1 u + Z_2 u^2 - 6 \langle \nabla Y, \nabla u \rangle$$

Z_0, Z_1, Z_2 $\mathcal{C}^{-\frac{1}{2}-}$ depend on

$$\left(|, V, \Psi, R(V \circ Y), R(V \circ Y), R(\| \nabla Y \|^2) \right)$$

Bundle case, random bundle map:

$$V : s \in C^\infty(E) \longrightarrow \langle \phi, \phi \rangle_E\, s + 2 \langle \phi, s \rangle_E\, \phi \in$$

$$\cap\; \mathcal{D}'(M, \mathrm{End}(E))$$

V local, in the sense $C^\infty(M)$-module morphism

Vectorial Cole-Hopf: $\quad e^{-3Y} = e^{-3\mathcal{L}^{-1}(V)} \in C^{1-0}(M, \mathrm{End}$

also $C^\infty(M)$-module morphism.

Thm (Bailleul - D - Ferdinand - Tô)

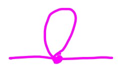
$(E, \nabla, \|\cdot\|_E)$

$\xi \rightsquigarrow \xi_r := e^{-rP}$

Heat mollify

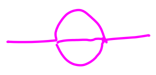
$$\partial_t \phi + P\phi + \big(2 + rk(E)\big)\big(C_1 - C_2\big)\phi = -\phi \|\phi\|_E^2$$

$$C_1 = \frac{1}{8\sqrt{2}\,\pi^{\frac{3}{2}}\sqrt{r}}$$



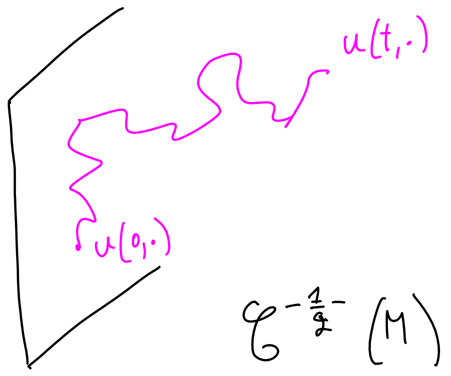
Local covariance

$$C_2 = \frac{|\log(r)|}{128\pi^2}$$

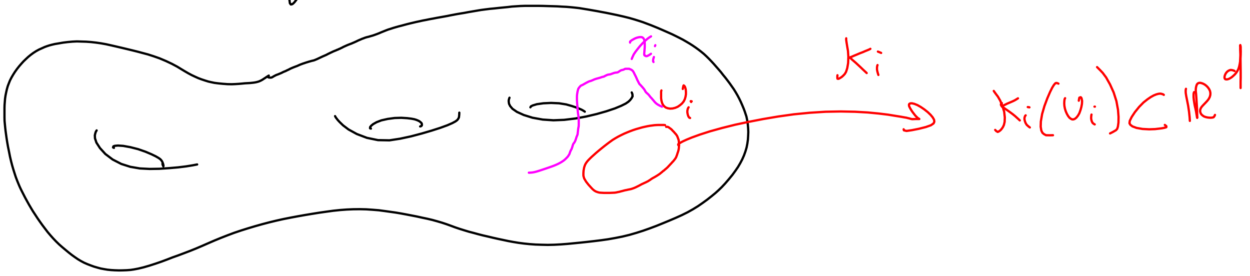


has solution ϕ when $r \to 0^+$, for all times ≥ 0.

$t \to u(t, \cdot) \in C^{-\frac{1}{2}-}(M)$ Markov



New ingredients: paradifferential calculus on Mfd



$$\sum \chi_i = 1$$

$$uv = \sum \chi_i u v = \sum \chi_i u \tilde{\chi}_i v$$

$$= \sum \kappa_i^* \Big(\underbrace{\kappa_{i*}(\chi_i u)} \quad \underbrace{\kappa_{i*}(\tilde{\chi}_i v)} \Big)$$

on $\mathbb{R}^d$

$$= \sum \kappa_i^* \Big(\kappa_{i*}(\chi_i u) \underbrace{\begin{smallmatrix}\odot \\ \prec \\ \succ\end{smallmatrix}}_{\text{on } \mathbb{R}^d} \kappa_{i*}(\tilde{\chi}_i v) \Big)$$

$$= \sum_i u \odot_i v + u \prec_i v + u \succ_i v$$

$\odot_i, \prec_i, \succ_i$ = localized resonant + paraproducts

same analytical properties as $\mathbb{R}^d$!

Localize singularities via multiple commutators.

$$\underbrace{F(\Upsilon)}_{\mathcal{C}^{1-}} \underbrace{(\Psi \prec V)}_{\mathcal{C}^{-1-}}$$

borderline

paralinearization

$$= (F'(\Upsilon) \prec \Upsilon)(\Psi \prec V) + \text{nice}$$

$$= (F'(\Upsilon) \prec \Upsilon) \odot (\Psi \prec V) + \text{nice} = F'(\Upsilon)\big(\Upsilon \odot (\Psi \prec V)\big) + \dots$$

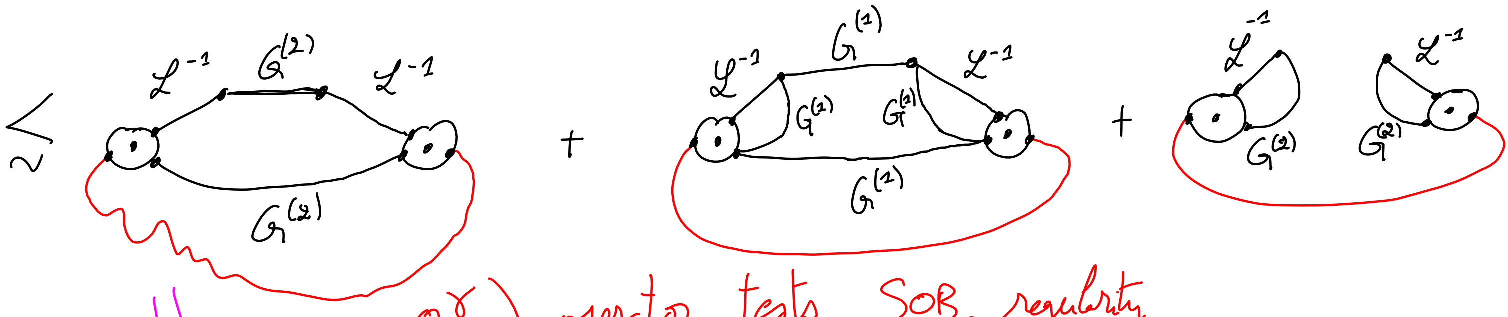
$$= F'(\Upsilon)\Psi \underbrace{(\Upsilon \odot V)} + \dots = F'(\Upsilon)\Psi\big(R(\Upsilon \odot V) + c_2\big)$$

Second ingredient: Stochastic estimates on Mfd.

Need to control $\mathbb{E}\left(\| \mathsf{Y} \odot \mathsf{V} \|^2_{H^\gamma} \right)$

Mirror Feynman graphs

$\lesssim$



Q^γ) operator tests SOB. regularity

$\|$

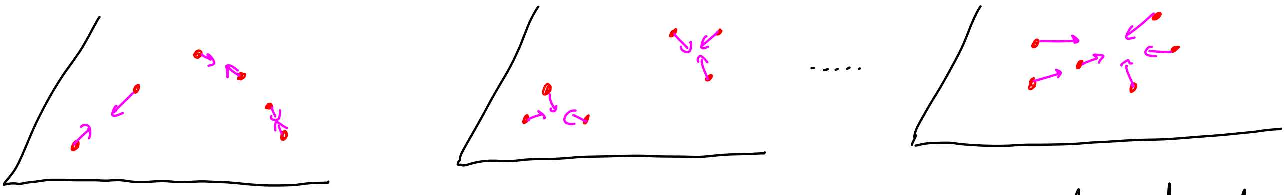
$$\int_{M^8} \int_{\mathbb{R}^4} G^{(2)}(x_1,x_2)\, \mathcal{L}^{-1}(x_3,x_1)\, \mathcal{L}^{-1}(x_2,x_4)\, [\odot](x_3,x_5,x_6)\, [\odot](x_4,x_7,x_8)\, G^{(2)}(x_5,x_7)\, Q^\gamma(x_6,x_8)$$

$$G^{(p)}(t-s,x,y) = \left[\left(\frac{e^{-|t-s|P}}{2P} \right)(x,y) \right]^p ,\quad [\odot] \text{ Schwartz kernel}$$

Question: find γ for which $\mathbb{E}\left(\| \mathsf{Y} \odot \mathsf{V} \|^2_{H^\gamma} \right) < +\infty$?

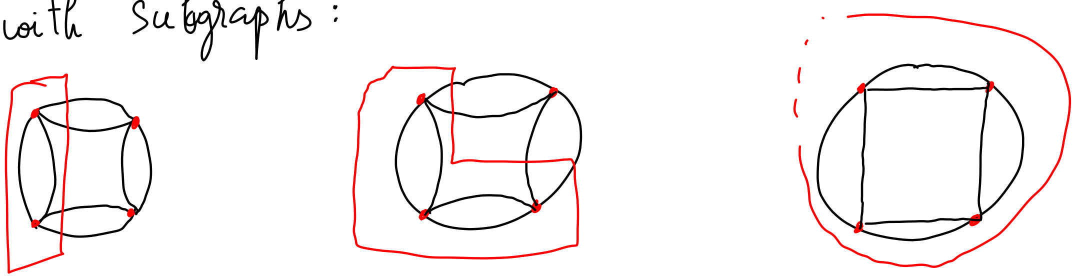
Idea, big distribution $\mathfrak{B}$ on $\mathbb{R}^4 \times M^8$, singularities at coinciding points

Recursive extension when 2pts, 3pts, 4pts, ... 8pts collide.

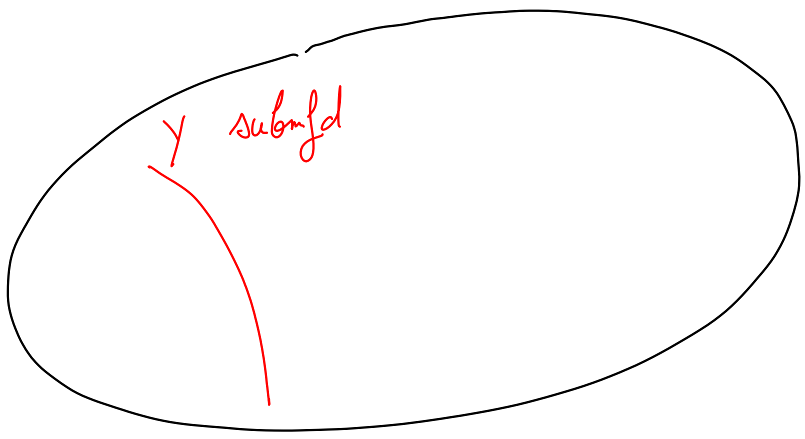


2 pts   3 pts   deepest diagonal

Deal with subgraphs:



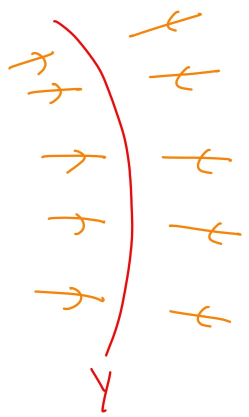
Metaprinciple behind proof:



X ambient mfd

$u \in \mathcal{D}'(X \setminus Y)$ <u>outside</u> Y. Extend u to whole X ?

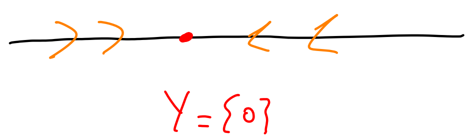
Inspired by Y. Meyer, Brunetti - Fredenhagen, scaling towards Y



<u>Déf</u>: $\rho \in \mathscr{C}^\infty(TX)$ scaling field if $\forall f \in \mathcal{I}_Y$ = ideal of $\mathscr{C}^\infty$ 0 on Y

$$\rho f - f \in \mathcal{I}_Y^2$$

<u>ex</u>:



$X = \mathbb{R}$ $\rho = x\partial_x$ scaling field

$$e^{-t\rho}(x) = e^{-t}x$$

Thm (BDFT, D)

Given $U \in \mathcal{D}'(X \setminus Y)$, growth of U controlled near Y

$$e^{-t\rho *} U = O_{\mathcal{D}'}(e^{-ts}), \quad t \to +\infty$$

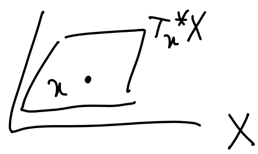
(Weakly homo. degree $s \in \mathbb{R}$ Y. MEYER)

if $s > -\operatorname{codim} Y$ then

U admits unique extension $\bar{U} \in \mathcal{D}'(X)$.

For Feynman amplitudes, U precise singularities.

Cotangent T^*X, $(x; \xi)$ position → x, momentum → ξ



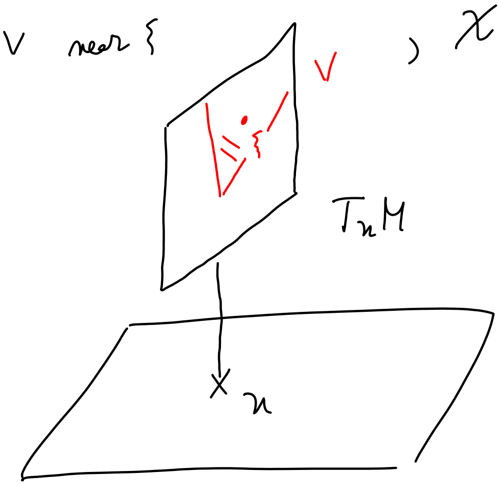
Microlocal version: $\Gamma \subset T^*X$ closed, conic, set

Déf [Hörmander 70's]: U is μlocally $\mathcal{C}^\infty$ at $(x;\xi) \in T^*X \setminus \underline{0}$

if $\exists V$ near ξ, χ near x

$$\widehat{U\chi}(\xi) = O(\langle \xi \rangle^{-\infty})$$

on V



WF(U) = Points in T^*X where U is not μlocally $\mathcal{C}^\infty$

↓

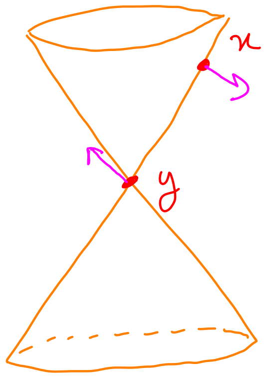
Wave Front Set

Ex: In QFT

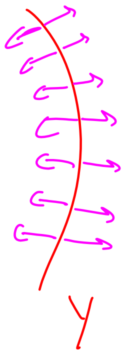
WIGHTMAN

$$\langle \Omega | \Phi(x)\, \Phi(y) | \Omega \rangle = C \, \frac{e^{i(t-s)\sqrt{\Delta}}}{\sqrt{\Delta}} (\vec{x}-\vec{y})$$

$(t,\vec{x}) \qquad (s,\vec{y})$

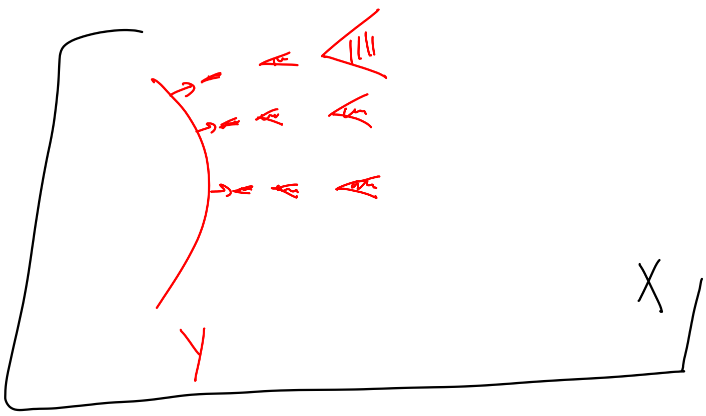


ξ codirection of singularities



δ_Y delta distribution on Y

Thm (BDFT) Γ cone in $T^*(X \setminus Y)$

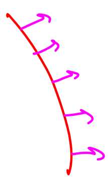


$e^{-t\rho^*} U = O_{\mathcal{D}'_\Gamma}(e^{-ts})$

$s > -\mathrm{codim}\, Y$

$\Rightarrow U$ unique extension $\bar{U}$

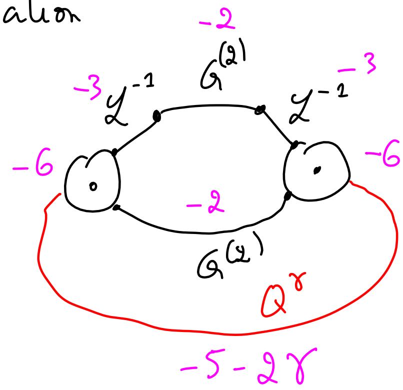
$$WF(\bar{U}) \subset \Gamma \cup \underbrace{N^* Y}$$



When $s = -\mathrm{codim}(Y)$, $\bar{U} = \lim_{\varepsilon \to 0^+} U_\varepsilon - c(\varepsilon)\, \delta_Y$

counterterm

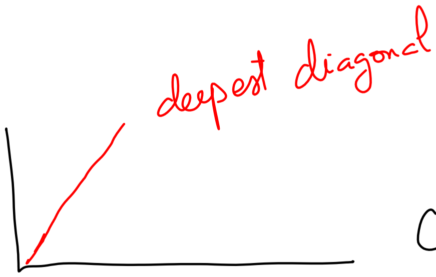
Illustration



weakly homogeneous degree

$= -27 - 2\gamma$

codim $= 3 \times 7 + 2 \times 3 = 27$



Conf $\mathbb{R}^4 \times M^8$

$\Rightarrow$ Amplitude finite if $-27 - 2\gamma > -27$

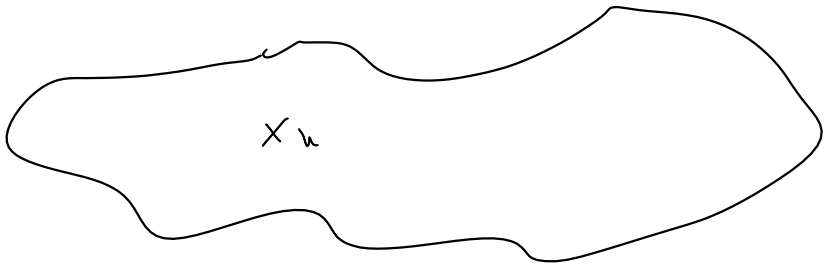
$\Rightarrow \gamma < 0$

LOCAL COVARIANCE , ϕ 3D GFF

Usual Wick: $\phi_r := e^{-r P}\phi$, $:\phi^2_{r}(x): = \phi^2_r(x) - C_r(x,x)$

$C_r(x,x)$ coord indep but not locally covariant!

Indeed $C_r(x,x)$ depends on GLOBAL geometry



Subtract only local invariants of metric g at x (Hollands-Wald, Kandel-Mnev-Wernli)

$$C_r(x,x) \underset{r\to 0^+}{\simeq} \frac{1}{4\pi^{\frac{3}{2}} r^{\frac{3}{2}}} + \mathcal{C}^\infty$$

Covariant WICK : $\phi^2_r(x) - \dfrac{1}{4\pi^{\frac{3}{2}} r^{\frac{3}{2}}}$

$$\lim_{r\to 0^+} \phi^2_r(x) - \frac{1}{4\pi^{\frac{3}{2}} r^{\frac{3}{2}}} \in \mathcal{C}^{-1-0} \quad a.s$$

All renorm steps are locally covariant.

THANKS FOR YOUR

ATTENTION

Recentering = iterated commutator estimates. $d=2$

Δ^{-1} some ΨDO order -2 s.t $\Delta^{-1}(x,x)=+\infty$, $\underbrace{e^{-\varepsilon\Delta}\Delta^{-1}(x,x)}_{\text{mollify}}$

$(\Delta_j)_j$ Littlewood - Paley projector.

Recall $u \odot v = \sum_{|i-j|\leq 1} \Delta_i u \, \Delta_j v$

Compare $\sum_{|i-j|\leq 1} \left(\Delta_i e^{-\varepsilon\Delta}\Delta^{-1}\Delta_j\right)(x,x)$ and $\left(e^{-\varepsilon\Delta}\Delta^{-2}\right)(x,x)$

Then $\sum_{|i-j|\leq 1} \left[\left(\Delta_i e^{-\varepsilon\Delta}\Delta^{-1}\Delta_j\right) - \left(e^{-\varepsilon\Delta}\Delta^{-1}\right)\right](x,x)$

has nice limit, $\forall x$! $[\Delta_i, e^{-\varepsilon\Delta}\Delta^{-1}] \in \Psi^{-3}$ trace class when $d=2$

Consequence, for ϕ GFF $d=2$, ϕ_ε^2 and $\phi_\varepsilon \odot \phi_\varepsilon$ can be renormalized by same counterterm.