Convergence of the renormalised model for the generalised KPZ equation via preparation maps

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Generalised KPZ and local expansion

Generalised KPZ equation:

$$\partial_t u = \partial_x^2 u + f(u)\,(\partial_x u)^2 + g(u)\xi.$$

A Taylor-type expansion (Regularity Structures):

$$u = \sum_{\tau \in \mathcal{T}} c_{\tau,x}\, \Pi_x \tau + R_{\mathcal{T},x}.$$

- Decorated trees $\mathcal{T}$.
- Stochastic recentered iterated integrals $\Pi_x \tau$.

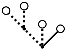
Decorated and stochastic iterated integrals

Let ξ^ε stand for a regularization of ξ by convolution with a smooth function of the form

$$\rho^\varepsilon(z) = \varepsilon^{-3}\rho(\varepsilon^{-2}t, \varepsilon^{-1}x),$$

Example of stochastic iterated integrals:

Π^ε  $= \xi^\varepsilon (K \star \xi^\varepsilon)$, Π^ε  $= (\partial_x K \star \xi^\varepsilon)^2$

Π^ε  $= \left(\partial_x K \star (\partial_x K \star \xi^\varepsilon)^2\right) (\partial_x K \star \xi^\varepsilon) (K \star \xi^\varepsilon)$

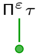
where the kernel K is a well-chosen truncation of the heat kernel having the norm

$$\|K\|_{1,m} = \sup_{|k|_\mathfrak{s} \leq m} \sup_z \|z\|_\mathfrak{s}^{|k|_\mathfrak{s}+1} \left|\partial^k K(z)\right|$$

Tree-like diagrams

The function $\Pi^\varepsilon\tau$ is integrated against some smooth test functions φ:

$$\langle \Pi^\varepsilon\tau, \varphi\rangle = \int (\Pi^\varepsilon\tau)(y)\varphi(y)dy =$$



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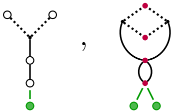
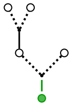
Mirror graph when computing $Var(\langle \Pi^\varepsilon\tau, \varphi\rangle)$



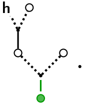
A purple • represents $(\rho^\varepsilon \star \rho^\varepsilon)(z-z') = \mathbb{E}\left(\xi^\varepsilon(z)\xi^\varepsilon(z')\right)$.

Malliavin derivatives

For $0 \leq k \leq |\tau|_\xi - 1$ denote by $d^k \Pi^\varepsilon \tau$ the kth order Malliavin derivative of $\Pi^\varepsilon \tau$; this is an element of $L\big(H^{\otimes k}, L^2(\Omega, \mathbb{P})\big)$.

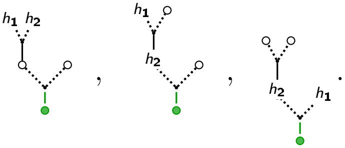
$\tau =$ 

here is a piece of $\big\langle \big(d\Pi^\varepsilon \tau\big)(h), \varphi \big\rangle$

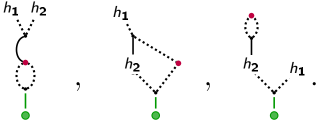


Malliavin derivatives

Here are pieces of $\big\langle \big(d^2\Pi^\varepsilon\tau\big)(h_1, h_2), \varphi\big\rangle$ for the preceding tree



As an example here are pieces of $\mathbb{E}\big[\big\langle \big(d^2\Pi^\varepsilon\tau\big)(h_1, h_2), \varphi\big\rangle\big]$, for the same tree τ as above



Spectral Gap inequality

We follow Linares, Otto, Templmayr and Tsatsoulis approach

$$\left\| \left\langle \Pi_z^{\varepsilon}\tau - \Pi_z^{\varepsilon'}\tau \, , \, \varphi_z^{\lambda} \right\rangle \right\|_{L^2(\Omega)} \leq \left| \mathbb{E}\left[\left\langle \Pi_z^{\varepsilon}\tau - \Pi_z^{\varepsilon'}\tau \, , \, \varphi_z^{\lambda} \right\rangle \right] \right| + \left\| \left\langle d\left(\Pi_z^{\varepsilon}\tau - \Pi_z^{\varepsilon'}\tau \right) , \, \varphi_z^{\lambda} \right\rangle \right\|_{L^2(\Omega)}$$

- First: choice of renormalisation constants.
- Second term: we iterate.

We write a *Stroock type formula*:

$$\left\| \left\langle \Pi_z^{\varepsilon}\tau - \Pi_z^{\varepsilon'}\tau \, , \, \varphi_z^{\lambda} \right\rangle \right\|_{L^2(\Omega)} \leq \sum_{k=0}^{|\tau|_{\xi}-1} \left\| \mathbb{E}\left[\langle d^k (\Pi_z^{\varepsilon}\tau - \Pi_z^{\varepsilon'}\tau), \varphi_z^{\lambda} \rangle \right] \right\| + \left\| \langle d^{|\tau|_{\xi}} (\Pi_z^{\varepsilon}\tau - \Pi_z^{\varepsilon'}\tau), \varphi_z^{\lambda} \rangle \right\| .$$

Main results

Proposition (Bailleul-B 23')

One has for all $0 < \lambda \leq 1$ *and every* $\tau \in \mathcal{B}^-$ *the estimates*

$$\| \langle d^{|\tau|_\xi} \Pi_z^\varepsilon \tau, \varphi_z^\lambda \rangle \| \lesssim \lambda^{|\tau|},$$

Proposition (Bailleul-B. 23')

One has for all $\tau \in \mathcal{B}^-$ *and* $0 \leq k \leq |\tau|_\xi - 1$,

$$\| \mathbb{E}[\langle d^k \overline{\Pi}_z^\varepsilon \tau, \varphi_z^\lambda \rangle] \| \lesssim \lambda^{|\tau|}.$$

where $\overline{\Pi}_z^\varepsilon$ *is a suitable renormalised model.*

Labelled graphs

Except from the green edges each edge $e = (e_+, e_-)$ in our mirror graphs represents

$$L_e(z_{e_-}, z_{e_+}) = K_e(z_{e_+} - z_{e_-}) - \sum_{|j|_{\mathfrak{s}} < r_e} \frac{(z_{e_+} - z_{v_e})^j}{j!} \partial^j K_e(z_{v_e} - z_{e_-}),$$

- $z_{v_e} = z$, recentering around the point z.
- $z_{v_e} \neq z$, renormalisation of order r_e.

The terms (K_e, a_e) are chosen among:

$$(K, 1), \quad (\partial K, 2), \quad (\rho^\varepsilon \star \rho^\varepsilon, 3)$$

where one has for a_e

$$\|K\|_{a_e, m} = \sup_{|k|_{\mathfrak{s}} \leq m} \sup_z \|z\|_{\mathfrak{s}}^{|k|_{\mathfrak{s}} + a_e} |\partial^k K(z)|$$

Power counting Assumptions

Renormalisation (convergence):

$$\sum_{e\in E_{\text{int}}(V)} a_e + \sum_{e\in E^{\uparrow}_{r>0}(V)} 1_{v_e\in V}(a_e + r_e - 1) - \sum_{e\in E^{\downarrow}(V)} 1_{v_e\in V} r_e < 3(|V| - 1).$$

Recentering:

$$\sum_{E_{\text{int}}(V)} a_e + \sum_{e\in E^{\downarrow}(V)} \left(1_{\{v_e\in V\}\cup\{r_e=0\}}(a_e + r_e - 1) - (r_e - 1)\right) + \sum_{e\in E^{\uparrow}(V)} \left(a_e + r_e 1_{v_e\notin V}\right) > 3|V|.$$

Extension of [Hairer-Quastel 18'] given in [B. 15'], [B.-Nadeem 22']

Bounds on Feynman diagrams

Let $\mathcal{E}$ the set of non-green edges of our graph G and write

$$\mathscr{I}^G(\lambda) = \int \prod_{e\in\mathcal{E}} L_e(z_{e_-}, z_{e_+})\, \varphi^\lambda(z_1)\varphi^\lambda(z_2)\, dz$$

We denote by V_0 the set of non-green vertices of G.

Theorem

If our graph G satisfies the previous assumptions then

$$|\mathscr{I}^G(\lambda)| \le c\lambda^\alpha \prod_{e\in\mathcal{E}} \|K_e\|_{a_e,1}, \quad \alpha = 3|V_0| - \sum_{e\in\mathcal{E}} a_e$$

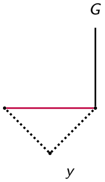
Strategy for treating bounds on renormalised model

We construct diagrams out of $\left\|\mathbb{E}\left[\left\langle d^k \overline{\Pi}_z^\varepsilon \tau, \varphi_z^\lambda \right\rangle\right]\right\|$

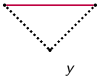
- At the beginning, $z_{v_e} = z$, recentering bounds are satisfied.
- Convergence bounds may fail.
- Perform local transformations (moving edges that preserve recentering and cure subdivergence).
- Renormalisation constants appear and need to be removed (encoded in $\overline{\Pi}_z^\varepsilon$).

Example

One considers



The subdivergent diagram is given by

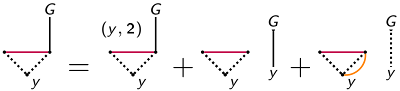


$$= \int K'(y - z_1)K'(y - z_2)(\rho^{\varepsilon} \star \rho^{\varepsilon})(z_1 - z_2)dz.$$

Convergence bounds fail: $2 + 2 + 3 > 3 + 3$.

Telescopic sum

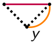
We use a telescopic sum to make appear this renormalisation



The decoration $(z, 2)$ means a kernel of the form

$$K(z_1 - z_3) - K(y - z_3) - (x_1 - x)\partial K(y - z_3)$$

that behaves like $(x_1 - x)^2$ when z_1 is close to $y = (t, x)$. The orange line is encoding a term of the form $(x_1 - x)$



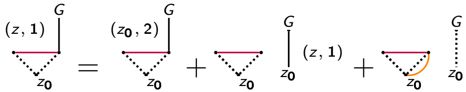
$$= \int (x_1 - x)\partial K(y - z_1)\partial K(y - z_2)(\rho^{\varepsilon} \star \rho^{\varepsilon})(z_1 - z_2)dz_1 dz_2.$$

Moving the recentering

Previous recentering in z gives a new telescopic sum

$$
\begin{aligned}
&K(z_1 - z_3) - K(z - z_3) = \\
&K(z_1 - z_3) - K(z_0 - z_3) - (x_1 - x_0)\partial K(z_0 - z_3) \\
&+ K(z_1 - z_3) - K(z - z_3) \\
&+ (x_1 - x_0)\partial K(z_0 - z_3)
\end{aligned}
$$

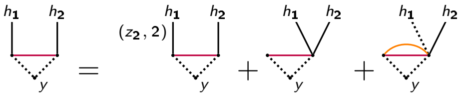
Graphically, one gets



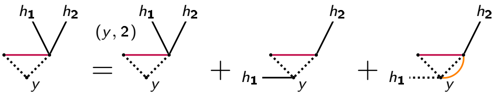
where z is a green vairable coming from φ_z^λ.

More complex diagrams

Below, one has a diagram not covered by [Hairer-Quastel 18']:



Then,



which transpose into the previous case.

Renormalised model

The model $\overline{\Pi}_z^{\varepsilon} = \Pi_z^R$ satisfies the following identities

$$(\Pi_z^R \tau)(y) = \Big(\Pi_z^{R,\times}(R\tau)\Big)(y), \quad \Pi_z^{R,\times}(\tau\sigma) = (\Pi_z^{R,\times}\tau)(\Pi_z^{R,\times}\sigma),$$

$$(\Pi_z^{R,\times}(\mathcal{I}_a\tau))(y) = (D^a K \star \Pi_z^R \tau)(y) - \sum_{|k| \leq |\mathcal{I}_a \tau|} \frac{(y-z)^k}{k!} (D^{a+k} K \star \Pi_z^R \tau)(z).$$

Introduced in [B. 18'] and used in [Bailleul-B. 21']. Abstract Malliavin derivatives:

$$R D_{\Xi_j} = D_{\Xi_j} R, \quad d^k \overline{\Pi}_z^{\varepsilon} \tau = \overline{\Pi}_z^{\varepsilon} \big(D_{\Xi_k} \dots D_{\Xi_1} \tau \big).$$

Introduced in [B.-Nadeem 22'].

Perspectives

- Generalised KPZ a specific case (one subdivergence, four noises trees) but quite complex aorund 40 trees.
- Discrete setting for generalised KPZ [B.-Nadeem; 22'].
- Toward a general convergence theorem with preparation maps.
- Application to Malliavin calculus.
- Similar ideas in other problems where Feynman diagrams are needed ?