Master's thesis

On the existence of horizontal geodesics in the shape space of unparameterized Sobolev immersions

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1 Introduction

The study of the existence of geodesics, which are shortest paths between two points, is of fundamental importance in various areas of mathematics and applied sciences. Geodesics possess intrinsic properties that make them optimal in terms of distance, energy, or other geometric measures. Moreover, they play a crucial role in path planning algorithms [28], computer animation [10], and image segmentation [7].

In this thesis we focus on geodesics in the space of closed regular and parameterized Sobolev immersions of order n, denoted by $\mathrm{Imm}^n(S^1, \mathbb{R}^d)$, as well as geodesics in the orbit space under the action of the reparameterization group, denoted by $\mathrm{Imm}^n(S^1, \mathbb{R}^d)/\mathrm{Diff}^n(S^1)$. The latter space holds greater significance as it involves unparameterized curves which eliminate the dependence on specific parameterizations, and facilitate meaningful comparisons and analyses based solely on shape information [26]. To draw conclusions about the quotient space we first need to study the space of immersions itself.

In order to measure the distance between two shapes we equip the above spaces with a Sobolev metric G_c of order two or higher. This kind of metric takes into account not only the first derivative but also higher-order derivatives, resulting in a more comprehensive and informative measure of shape variability as shown in [21]. Thus, higher derivatives provide a powerful framework for analyzing and quantifying shape variations in geometric objects like curves.

A horizontal geodesic is a geodesic, whose derivative lies in the horizontal space, a subspace of the tangent space. Compared to geodesics in the full ambient space, horizontal geodesics are easier to compute as they have reduced dimensionality. This simplification enables the use of more efficient numerical methods and algorithms tailored to the specific plane, leading to faster computations and enhanced numerical stability [13].

In order to prove horizontality of geodesics the shape space has to be a manifold. The manifold structure implies the existence of a principal connenction with parallel transport, which enables a horizontal lift from the quotient space to the original manifold. Unfortunately, the shape space of unparameterized curves is not a manifold, as described in [22]. Hence, we restrict to the dense open subset $\mathrm{Imm}_f(S^1, \mathbb{R}^d)$ of curves upon which the group of diffeomorphisms acts freely and we prove the manifold structure for this space instead.

The primary objectives of this thesis are to investigate the existence of geodesics in the shape space of unparameterized Sobolev immersions of order n and to show that geodesics in $\mathrm{Imm}^n_f(S^1, \mathbb{R}^d)/\mathrm{Diff}^n(S^1)$ are horizontal. By understanding the properties and existence of these geodesics, we aim to deepen our comprehension of shape spaces and contribute to the fields of shape analysis and optimization.

This paper is organized as follows. After recalling the notions of metric space and some basic concepts in differantial geometry, we introduce in Section 3 the concept of the winding number and present the proof of the Whitney-Graustein Theorem which was originally done in [27]. The theorem states that two curves in the plane are regularly homotopic if and only if they have the same winding number. In the context of the existence of geodesics we obtain a positive result for two curves in the same connected component. The theorem can then be used to characterize the connected completeness of $\mathbb{R}^2$.

Afterwards, we prove metric completeness of the space $(\mathrm{Imm}^n(S^1, \mathbb{R}^d), \mathrm{dist}_G)$ in Section 4. To accomplish this, we use the fact that the space $(H^n(S^1, \mathbb{R}^d), \|\cdot\|_{H^n(d\theta)})$ is metrically complete. While it is straightforward to demonstrate the equivalence between the $H^n(ds)$-norm and the distance function induced by a Sobolev metric, proving the equivalence of the $H^n(d\theta)$- and $H^n(ds)$-norm presents a more intricate challenge, forming the primary focus of this section. The metric completeness of the original space is required to show that the quotient space $(\mathrm{Imm}^n(S^1, \mathbb{R}^d)/\mathrm{Diff}^n(S^1), \mathrm{dist}_{I/D})$ is a complete metric space as well, which is a key step towards establishing the existence of geodesics in the latter space.

In Section 5 we prove that any two curves in $\mathrm{Imm}^n(S^1, \mathbb{R}^d)$ lying in the same connected component can be joined by a minimizing geodesic. The proof employs the direct method of the Calculus of Variation to find a minimizer. Then, we transfer the existence result from the space of parameterized curves to the space of geometric curves. Moreover, we show in the last subsection with a simple argument that the existence result also holds in the space of free (geometric) curves.

Subsequently, in Section 6 we show that $\mathrm{Imm}_f(S^1, \mathbb{R}^d)/\mathrm{Diff}$ admits a manifold structure. By studying the behavior of freely immersed curves and their local neighborhoods, we introduce the notion of tubular neighborhoods. We show that the tubular neighborhood of an immersed curve exclusively contains immersed curves. Similarly, within a tubular neighborhood of a freely immersed curve, all curves also remain freely immersed. Furthermore, we discuss the equivalence with the neighborhoods induced by the Banach topology.

In the final section we present the proof that geodesics in the space $\mathrm{Imm}_f(S^1, \mathbb{R}^d)/\mathrm{Diff}$ can be lifted horizontally to the space $\mathrm{Imm}_f(S^1, \mathbb{R}^d)$. This proof relies on a crucial element, namely the manifold structure of the space. We introduce the concept of fiber bundles and transfer the manifold result into the framework of principal fiber bundles. Moreover, we present the definition of horizontal and vertical subspaces of the tangent space and a principal connection. The latter one is needed to define parallel transport, which is a mathematical concept that allows for the transport of vectors or tensors along curves while preserving their properties.

2 Preliminaries

In this section we lay down the necessary preliminaries of differential geometry which are required to understand the content of this thesis. We introduce some basic definitions and concepts from differential geometry, especially the theory of immersed curves and their properties. Unless otherwise indicated , the definitions are derived from [3, Sec. 2] and [19, Sec. 2]. We begin with the notion of smooth parameterized curves and subsequently delve into the exploration of geometric curves.

2.1 The Space of Curves and the Tangent Space

Definition 2.1 (*Space of Curves*)**.** We call

$$\mathrm{Imm}(S^1, \mathbb{R}^d) := \{c \in C^\infty(S^1, \mathbb{R}^d) : c'(\theta) \neq 0\}$$

the space of *parameterized regular immersions*. The space of *Sobolev immersions* of order n is denoted by

$$\mathrm{Imm}^n(S^1, \mathbb{R}^d) := \{c \in H^n(S^1, \mathbb{R}^d) : c'(\theta) \neq 0\}.$$

Since S^1 can be viewed as $\mathbb{R}/(2\pi)$, we note that curves from $S^1 \to \mathbb{R}^d$ can be considered as curves from $\mathbb{R} \to \mathbb{R}^d$ which are 2π-periodic. In the following we assume a curve c to be closed, *i.e.* $c(0) = c(2\pi)$ and $c'(0) = c'(2\pi)$. To shorten notation, we write Imm for $\mathrm{Imm}(S^1, \mathbb{R}^d)$ and Imm^n for $\mathrm{Imm}^n(S^1, \mathbb{R}^d)$.

Definition 2.2 (*Tangent Space*)**.** Let f be a function between two finite dimensional manifolds M and N and let $\pi : TN \to N$ be the tangent bundle map. Then the *tangent space* $T_f C(M, N)$ contains every

$$\tilde{f} : M \to TN,$$

such that $\tilde{f}$ is smooth and $\pi \circ \tilde{f} = f$.

For the above defined spaces the tangent spaces can simply be identified as follows

$$\begin{aligned} T_c\mathrm{Imm}(S^1, \mathbb{R}^d) &\cong C^\infty(S^1, \mathbb{R}^d), \\ T_c\mathrm{Imm}^n(S^1, \mathbb{R}^d) &\cong H^n(S^1, \mathbb{R}^d). \end{aligned}$$

For a curve $c \in \mathrm{Imm}^n$ we encounter two different kinds of derivatives. On the one hand, we have $\partial_\theta c$ with $\theta \in S^1$, which we will denote by c'. On the other hand, we have the derivative with respect to arc length, which we will denote by $D_s c = \partial_\theta c / |c'|$. We define integration with respect to arc length by $ds = |c'|d\theta$. Moreover, we have the unit length vector $V = c'/|c'|$ which is well-defined since c is a C^1-immersion.

Definition 2.3 (*Mean Curvature*)**.** If c is C^2-regular, we define the *mean curvature* H of c as

$$H := \frac{\partial}{\partial s}\frac{\partial}{\partial s}c = \frac{\partial}{\partial s}V.$$

Definition 2.4 (*Normal Vector*)**.** For a planar curve c we establish the *normal vector* N to the curve c by imposing the conditions that $|N| = 1$, N is orthogonal to V and N is rotated $2/\pi$ degree counter clockwise with respect to V .

Definition 2.5 (*Scalar Curvature*)**.** If c is planar and C^2, then we define the signed *scalar curvature* $\kappa = \langle H, N\rangle$, so that

$$\frac{\partial}{\partial s}V = \kappa N = H,$$
$$\frac{\partial}{\partial s}N = -\kappa V.$$

To compute the scalar curvature κ we make use of the angle function $\phi : \mathbb{R} \to \mathbb{R}$ (see Def. 3.1) which is defined such that

$$V(s) = (\cos(\phi(s)), \sin(\phi(s))).$$

Since $N \perp V$, we conclude

$$N(s) = (-\sin(\phi(s)), \cos(\phi(s))).$$

Observe that

$$\frac{\partial}{\partial s}V(s) = \frac{1}{|c'|}\phi'(s)(-\sin(\phi(s)), \cos(\phi(s))) = \frac{\phi'}{|c'|}N.$$

As $\frac{\partial}{\partial s}V = \kappa N$, we obtain

$$\kappa = \frac{\phi'}{|c'|} = \frac{\partial}{\partial s}\phi.$$

Definition 2.6. Consider the curve $c : S^1 \to \mathbb{R}^2$ with scalar curvature κ. We define the constants

$$\delta_c := \frac{\pi}{(3\max|\kappa|)}$$
$$\tau_c := \frac{1}{(2\max|\kappa|)}.$$

Note that since we are considering closed curves, the scalar curvature is not zero.

2.2 Length of Curves

Definition 2.7 (*Length*)**.** Let c_0, c_1 be two curves in Imm^n and let $c(t)$ be a path connecting c_0 and c_1, i.e $c(0) = c_0$ and $c(1) = c_1$. We define the *length* of a path by

$$\mathrm{len}_G(c) := \int_0^1 |\dot{c}(t)|_{c(t)} dt.$$

Here $\dot{c}(t)$ denotes the derivative with respect to time and the induced norm is given by $|\dot{c}(t)|_{c(t)} = \sqrt{G_{c(t)}(\dot{c}(t), \dot{c}(t))}$, where G denotes a Riemannian metric. Then, the induced *geodesic distance* is defined as

$$\mathrm{dist}_G(c_0, c_1) := \inf\{\mathrm{len}_G(c) : c(0) = c_0,\ c(1) = c_1,\ c \text{ piecewise smooth}\},$$

and the *open metric ball* is given by

$$B(c_0, r) = \{c_1 : \mathrm{dist}_G(c_0, c_1) < r\}.$$

Moreover, we define the *path energy* as

$$E(c) := \int_0^1 G_{c(t)}(\dot{c}(t), \dot{c}(t))\, dt.$$

We have $L(c) \leq \sqrt{E(c)}$, since

$$L(c) = \int_0^1 1 * \sqrt{G_{c(t)}(\dot{c}(t), \dot{c}(t))}\, dt \leq \sqrt{\int_0^1 1^2\, dt} \sqrt{\int_0^1 \sqrt{G_{c(t)}(\dot{c}(t), \dot{c}(t))}^2\, dt} = \sqrt{E(c)},$$

where the inequality comes from the Cauchy-Schwarz inequality. For the case that c has constant speed, the Cauchy-Schwarz inequality yields equality, leading to $L(c) = \sqrt{E(c)}$. Thus, in order to identify the minimizers of the path length, we can look for minimizers of the energy functional and the minimizers will have constant speed. We call such local minimizers *geodesics*.

Note that we distinguish between the length of a curve and the length of a path that connects two curves. When computing the length of a curve which is defined over the interval $[0, 2\pi]$, we integrate from 0 to 2π. However, when measuring the length of a path $c : [0, 1] \to \mathrm{Imm}^n$, we integrate from 0 to 1.

Lemma 2.8. *(*Constant Speed Reparameterization, taken from [24, Thm. 2.1]). For every curve $c \in \mathrm{Imm}^n$ there exists a reparameterization $\tilde{c} = c \circ \phi$ with $\phi \in \mathrm{Diff}(S^1)$ such that $\tilde{c}$ has constant speed, i.e. $|\tilde{c}'| \equiv l$ with $l = \mathrm{len}(c)/2\pi$.

Proof: Consider the function $s : [0, 2\pi] \to [0, 2\pi]$ defined as

$$s(t) := \frac{2\pi}{L} \int_0^t |c'|\, d\theta,$$

with $L = \text{len}(c)$. Then we get

$$s'(t) = \frac{2\pi}{L}|c'|.$$

Since c is an immersion, we obtain

$$\frac{ds}{dt} = \frac{2\pi}{L}|c'| > 0.$$

Hence the function s is strictly increasing with positive derivative. Then by the inverse function Theorem there exists an inverse function $t(s)$, so that

$$\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{L}{2\pi}\frac{1}{|c'|}.$$

Since s is a diffeomorphism, $\tilde{c}(s) := c \circ t(s)$ is a reparameterization of c. It follows that

$$|\tilde{c}'(s)| = |c'(t(s))\frac{dt}{ds}| = |c'(t(s))|\frac{L}{2\pi}\frac{1}{|c'(t(s))|} = \frac{L}{2\pi}.$$ □

We say that a curve c is parameterized by *arc parameter*, if $|c'| = 1$. Clearly, a curve can be reparameterized to arc parameter if and only if len(c)= 2π. Note that any curve can be rescaled in order to have a length of 2π.

Definition 2.9 (*Length of Curve Arcs*)**.** Let $c \in \text{Imm}^n$ and $\sigma, \tilde{\sigma} \in S^1$. If $\sigma \neq \tilde{\sigma}$, then there are two arcs in S^1 connecting σ and $\tilde{\sigma}$. By

$$\text{len}\, c_{|\,[\sigma,\tilde{\sigma}]},$$

we will mean the minimum of the lengths of c when restricted to one of the two arcs connecting σ and $\tilde{\sigma}$.

For the periodical extension $c : \mathbb{R} \to \mathbb{R}^d$ and $\sigma, \tilde{\sigma} \in \mathbb{R}$ there exists an unique $k \in \mathbb{Z}$ such that

$$\sigma \leq \tilde{\sigma} + 2\pi k < \sigma + 2\pi.$$

Now, the minimum length of c restricted to one of the two arcs connecting σ and $\tilde{\sigma}$ is given by

$$\text{len}\, c_{|\,[\sigma,\tilde{\sigma}]} = \min\{l_1, l_2\},$$

where

$$l_1 = \int_{\sigma}^{\tilde{\sigma}+2\pi k} |c'(\theta)| d\theta,$$
$$l_2 = \int_{\tilde{\sigma}+2\pi k}^{\sigma+2\pi k} |c'(\theta)| d\theta.$$

Note, that if c is parameterized at constant speed, i.e. $|c'| = l = \frac{\text{len}(c)}{2\pi}$, then we will assume that σ

and $\tilde{\sigma}$ are chosen (up to adding a constant $2\pi k$) such that $|\sigma - \tilde{\sigma}| = d_{S^1}(\sigma, \tilde{\sigma}) \leq \pi$. Then we get

$$\operatorname{len} c_{|[\sigma,\tilde{\sigma}]} = l\,|\sigma - \tilde{\sigma}|.$$

Lemma 2.10. Consider a curve $c \in \mathrm{Imm}^n$. Let

$$M = \max|c'|,$$
$$m = \min|c'|.$$

Then

$$m\, d_{S^1}(\sigma, \tilde{\sigma}) \leq \operatorname{len} c_{|[\sigma,\tilde{\sigma}]} \leq M\, d_{S^1}(\sigma, \tilde{\sigma}) \quad \forall \sigma, \tilde{\sigma} \in S^1,$$

where $d_{S^1}(\sigma, \tilde{\sigma})$ denotes the length of the shortest arc in S^1 connecting σ and $\tilde{\sigma}$.

Proof: Let γ and $\overline{\gamma}$ be the two arcs in S^1 that connect σ and $\tilde{\sigma}$. Then

$$\begin{aligned}\operatorname{len} c_{|[\sigma,\tilde{\sigma}]} &= \min\Big\{\int_\sigma^{\tilde{\sigma}} |c'(t)|\,dt, \int_{\tilde{\sigma}}^{\sigma} |c'(t)|\,dt\Big\} \leq M \min\Big\{\int_\sigma^{\tilde{\sigma}} 1\,dt, \int_{\tilde{\sigma}}^{\sigma} 1\,dt\Big\}\\ &= M \min\{\gamma, \tilde{\gamma}\} = M\, d_{S^1}(\sigma, \tilde{\sigma}).\end{aligned}$$

We have used the fact that for small distances and lengths the arc where $\operatorname{len} c_{|[\sigma,\tilde{\sigma}]}$ is computed, is also the shortest arc connecting σ and $\tilde{\sigma}$. The proof for the first inequality works analogously.□

2.3 Metric Contributions

Previously, we have determined the space of curves for which we are going to prove metric completeness and the existence of geodesics. It remains to identify the metric which we equip the space with and which we use when talking about the length of a curve.

Definition 2.11 (*Sobolev Metric*)**.** Let $h_1, h_2 \in T_c\mathrm{Imm}^n$ and $a_j \geq 0$ for all $j \leq n$. A *Sobolev metric* of order m is then given by

$$G_c(h_1, h_2) = \int_{S^1} a_0 < h_1, h_2 > + a_1 < D_s h_1, D_s h_2 > + \ldots + a_n < D_s^m h_1, D_s^m h_2 > ds.$$

In order to establish metric completeness of the space $(\mathrm{Imm}^n(S^1, \mathbb{R}^d), \mathrm{dist}_G)$ we introduce the following Sobolev norms on $H^n(S^1, \mathbb{R})$ for $n \leq 2$

$$\|h\|^2_{H^n(d\theta)} = \int_{S^1} |h(\theta)|^2 + |\partial_\theta^n h(\theta)|^2 d\theta,$$
$$\|h\|^2_{H^n(ds)} = \int_{S^1} |h(s)|^2 + |D_s^n h(s)|^2 ds.$$

In Lemma 4.5 we will see that these two norms are equivalent on metric balls.

Lemma 2.12. (Taken from [4, Lemma 2.13]). Let $c \in \mathrm{Imm}^2(S^1, \mathbb{R}^d)$ and $h : S^1 \to \mathbb{R}^d$ be absolutely continuous and closed. Then

$$\sup_{\theta\in S^1}\left|h(\theta) - \frac{1}{\operatorname{len}(c)}\int_{S^1} h\,ds\right| \leq \frac{1}{2}\int_{S^1} |D_s h|\,ds.$$

Proof: We have

$$h(\theta) - h(0) = \frac{1}{2}\left(\int_0^\theta h'(\sigma)\,d\sigma - \int_\theta^{2\pi} h'(\sigma)\,d\sigma\right),$$

where we used the fact that h is closed and hence $h(0) = h(2\pi)$. Now, integration on both sides by arc parameter and multiplying by $1/\operatorname{len}(c)$ yields

$$\frac{1}{\operatorname{len}(c)}\int_{S^1} h(\theta) - h(0)\,ds = \frac{1}{2\operatorname{len}(c)}\int_{S^1}\left(\int_0^\theta h'(\sigma)\,d\sigma - \int_\theta^{2\pi} h'(\sigma)\,d\sigma\right)ds.$$

Since $h(0)$ is a constant and $\int_{S^1} 1\,ds = \operatorname{len}(c)$, we get for the left-hand side

$$\frac{1}{\operatorname{len} c}\int_{S^1} h(\theta) - h(0)\,ds = \frac{1}{\operatorname{len} c}\int_{S^1} h\,ds - h(0).$$

By taking the absolute values on both sides and putting them into the integral on the right-hand side we get

$$\begin{aligned}\left|\frac{1}{\operatorname{len} c}\int_{S^1} h\,ds - h(0)\right| &\leq \frac{1}{2\operatorname{len} c}\int_{S^1}\left(\int_0^\theta |h'(\sigma)|\,d\sigma + \int_\theta^{2\pi} |h'(\sigma)|\,d\sigma\right)ds\\ &\leq \frac{1}{2\operatorname{len} c}\int_0^{2\pi} |h'(\sigma)|\,d\sigma \int_{S^1} 1 ds \leq \frac{1}{2}\int_{S^1} |D_s h| ds.\end{aligned}$$

The proof for any arbitrary point $\theta \in S^1$, instead of 0, can be done similarly, leading to the same result. Thus, we can take the supremum over $\theta \in S^1$, which completes the proof of the lemma. □

Lemma 2.13. (*Poincare Inequalities*, taken from [4, Lemma 2.14, 2.15]). Let $c \in \operatorname{Imm}^n$, $h \in H^n$ and $k \leq n$. Then the following estimates hold

i) $\|D_s h\|^2_{L^\infty} \leq \frac{\operatorname{len}(c)}{4}\|D_s^2 h\|^2_{L^2(ds)}$

ii) $\|D_s h\|^2_{L^2(ds)} \leq \frac{\operatorname{len}(c)^2}{4}\|D_s^2 h\|^2_{L^2(ds)}$

iii) $\|D_s^k h\|^2_{L^2(ds)} \leq \|h\|^2_{L^2(ds)} + \|D_s^n h\|^2_{L^2(ds)}$.

Proof: i) Replacing h in Lemma 2.12 by $D_s h$ and noting that $\int_{S^1} D_s h\,ds = \int_{S^1} h'\,d\theta = 0$, we obtain

$$\sup_{\theta\in S^1}|D_s h(\theta)| \leq \frac{1}{2}\int_{S^1} |D_s^2 h|\,ds.$$

Taking squares this equation and using the fact that

$$\Big(\int_{S^1} |D_s^2 h| \cdot 1 ds\Big)^2 \overset{\text{C.S}}{\leq} \int_{S^1} |D_s^2 h|^2 \, ds \cdot \int_{S^1} 1^2 \, ds,$$

we get

$$\|D_s h\|_{L^\infty}^2 \leq \Big(\frac{1}{2} \int_{S^1} |D_s^2 h| \, ds\Big)^2 \leq \frac{\text{len}(c)}{4} \|D_s^2 h\|^2_{L^2(ds)}.$$

ii) The second statement can be easily shown by using the first one

$$\|D_s h\|^2_{L^2(ds)} \leq \|D_s h\|^2_{L^\infty} \cdot \int_{S^1} 1 \, ds \overset{i)}{\leq} \frac{\text{len}(c)^2}{4} \|D_s^2 h\|^2_{L^2(ds)}.$$

iii) By Lemma 2.8 we can reparameterize c such that $|c'| = \text{len}(c)/2\pi$. Then we have

$$D_s^k h = \left(\frac{1}{|c'|}\right)^k \partial_\theta^k h = \left(\frac{2\pi}{\text{len}(c)}\right)^k \partial_\theta^k h.$$

In order to prove iii) we need to show

$$\int_0^{2\pi} \left(\frac{2\pi}{\text{len}(c)}\right)^{2k-1} |h^{(k)}(\theta)|^2 \, d\theta \leq \int_0^{2\pi} \frac{\text{len}(c)}{2\pi} |h(\theta)|^2 + \left(\frac{2\pi}{\text{len}(c)}\right)^{2n-1} |h^{(n)}(\theta)|^2 \, d\theta.$$

Note that we switched from ds to $|c'| \, d\theta$ and therefore we have $\frac{1}{|c'|}$ once less. Let $\varphi(x) = \frac{2\pi}{\text{len}(c)} x$. By a change of variables in the previous one we get

$$\int_0^{\text{len}(c)} |(h \circ \varphi)^{(k)}(x)|^2 dx \overset{!}{\leq} \int_0^{\text{len}(c)} |h \circ \varphi(x)|^2 + |(h \circ \varphi)^{(n)}(x)|^2 \, dx.$$

Define $f := h \circ \varphi \in L^2([0, \text{len}(c)], \mathbb{R})$ and $f_k(x) := \text{len}(c)^{-1/2} \exp(i \frac{2\pi k}{\text{len}(c)} x)$. Due to Fourier analysis, we can conclude that the set of functions $\{f_k(x)\}_{k \in \mathbb{Z}}$ is an orthonormal basis of $L^2([0, \text{len}(c)], \mathbb{R})$. Hence we can write f as

$$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) f_k(x).$$

Plugging this into the previous one and recalling that $\exp(i \, 2\pi k) = 1$, we get

$$\sum_{k \in \mathbb{Z}} \Big(\frac{2\pi k}{\text{len}(c)}\Big)^{2k} |\hat{f}(k)|^2 \overset{!}{\leq} \sum_{k \in \mathbb{Z}} \left(1 + \Big(\frac{2\pi k}{\text{len}(c)}\Big)^{2n}\right) |\hat{f}(k)|^2.$$

Observe that $a^k \leq 1 + a^n$ for $a \geq 0$ and $k \leq n$. Setting $a = (\frac{2\pi k}{\text{len}(c)})^2 \geq 0$, proves the above inequality. □

The following content will be important in Section 5 for the existence of geodesics. In particular, Proposition 2.16 plays a crucial role since it gives a criterion that provides the existence of geodesics in a complete metric space.

Lemma 2.14. (Taken from [6, Prop. 1.5.9]). Suppose X is a metric space and X' a dense subset of X. Let Y be a complete space and $f : X' \to Y$ a Lipschitz map. Then there exists a unique continuous map $\tilde{f} : X \to Y$ such that $\tilde{f}|_{X'} = f$.

Proof: For every $x \in X$ we define $\tilde{f}$ as follows: Since X' is dense in X, we can always find sequence $\{x_n\}_{n\in\mathbb{N}}$ in X' such that $x_n \to x$ for $n \to \infty$. The Lipschitz continuity of f implies that

$$|f(x_i) - f(x_j)| \leq C|x_i - x_j| \quad \text{for } i, j \in \mathbb{N}.$$

Then $\{f(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, since $\{x_n\}_{n\in\mathbb{N}}$ converges. As Y is a complete space, $\{f(x_n)\}_{n\in\mathbb{N}}$ converges. Then, define $\tilde{f}(x) = \lim_{n\to\infty} f(x_n)$. □

Definition 2.15 (*Intrinsic Metric*)**.** A metric is said to be *intrinsic* if it coincides with the induced intrinsic metric which we denoted by the distance function in Definition 2.7. We call a metric *strictly* intrinsic if for every two points x,y there exists an admissible path joining them, whose length is equal to the distance function. So there exists a geodesic between any two points.

Proposition 2.16. (Taken from [6, Prop. 2.4.16]). Let (X, d) be a complete metric space. If for every $x, y \in X$ there exists a midpoint, that is a point D such that

$$d(x, D) = d(D, y) = \tfrac{1}{2} d(x, y),$$

then d is strictly intrinsic.

Proof: To construct a path $\gamma : [0, 1] \to X$ that connects x, y, i.e. $\gamma(0) = x$ and $\gamma(1) = y$ with $\mathrm{len}(\gamma) = d(x, y)$, we assign values of γ for all dyadic rationals (these are rational numbers of the form $k/2^m$ for $k, m \in \mathbb{N}$). Since by assumption midpoints exist between any elements in X we can choose a midpoint of x and y and assign it to be the image of $\gamma(\frac{1}{2})$. Then we assign $\gamma(\frac{1}{4})$ to be the midpoint of $\gamma(0)$ and $\gamma(\frac{1}{2})$ and $\gamma(\frac{3}{4})$ to be the midpoint between $\gamma(\frac{1}{2})$ and $\gamma(1)$. Following this procedure, we define γ for all dyadic rationals between 0 and 1. We then have

$$\tfrac{1}{2} d(x, y) = d(\gamma(0), \gamma(\tfrac{1}{2})) = d(\gamma(0), \gamma(\tfrac{1}{4})) + d(\gamma(\tfrac{1}{4}), \gamma(\tfrac{1}{2})).$$

Since $d(\gamma(0), \gamma(\frac{1}{4})) = d(\gamma(\frac{1}{4}), \gamma(\frac{1}{2}))$, we conclude

$$\tfrac{1}{4} d(x, y) = d(\gamma(0), \gamma(\tfrac{1}{4})) = d(\gamma(\tfrac{1}{4}), \gamma(\tfrac{1}{2})).$$

In general, we have for every two dyadic rationals t_i, t_j

$$d(\gamma(t_i), \gamma(t_j)) = |t_i - t_j| \cdot d(x, y).$$

With this equality we see that the map γ is Lipschitz on the set of dyadic rationals. Since this set is dense in $[0, 1]$ and X is complete by assumption, we can use Lemma 2.14 and extend the map to the entire interval [0,1]. Now, γ satisfies all the assumptions which we stated in the beginning. □

2.4 Geometric Curves

Definition 2.17 (*Diffeomorphism*)**.** The set of all *diffeomorphisms* of S^1 is given by

$$\mathrm{Diff}(S^1) := \{\phi : S^1 \to S^1 : \phi, \phi^{-1} \in C^\infty(S^1)\}.$$

Similarly, the set of all diffeomorphisms of order n is given by

$$\mathrm{Diff}^n(S^1) := \{\phi : S^1 \to S^1 : \phi, \phi^{-1} \in C^n(S^1)\}.$$

The group has two connected components

$$\mathrm{Diff}(S^1) = \mathrm{Diff}^+(S^1) \cup \mathrm{Diff}^-(S^1),$$

where $\mathrm{Diff}^+(S^1)$ denotes the set of orientation-preserving diffeomorphisms which respect the winding number of a curve and $\mathrm{Diff}^-(S^1)$ is the set of orientation-reversing diffeomorphisms, that map curves of winding number p to curves of winding number $-p$ (see Def. 3.2 for the definition of the winding number). If $\phi \in \mathrm{Diff}^+(S^1)$, then $\phi' > 0$ and if $\phi \in \mathrm{Diff}^-(S^1)$, then $\phi' < 0$. To shorten notation, we simply write $\mathrm{Diff} = \mathrm{Diff}(S^1)$, since we always consider diffeomorphisms on S^1.

So far we focused on parametric curves. These are maps $c : S^1 \to \mathbb{R}^d$ such that we can identify each point on the curve with a point on S^1. However, when discussing *geometric curves*, we are not concerned with this one-to-one correspondence, but rather with the image of the map. Thus, we consider two curves to be equal if they only differ in their parameterization. This leads to the use of equivalence classes, where the quotient space is given by

$$\mathrm{Imm}(S^1, \mathbb{R}^d)/\mathrm{Diff}(S^1).$$

Reparameterization acts on the curves by composition from the right. In other words, two parametric curves c_1 and c_2 are the same geometric curve within $\mathrm{Imm}(S^1, \mathbb{R}^d)/\mathrm{Diff}(S^1)$ if there exists $\phi \in \mathrm{Diff}(S^1)$ such that $c_1 = c_2 \circ \phi$.

Unfortunately, this space is not a manifold, it is an orbifold and has singularities at any curve that has a non trivial isotropy group (see [22, Sec. 2.5]). In order to obtain a manifold structure on the quotient space, we take a look at the space of freely immersed curves.

2.5 Free Immersions

Definition 2.18 (*Isotropy Group*)**.** Consider the action above of diffeomorphisms on S^1 acting on Imm. The *isotropy group* $\mathcal{G}_c$ of a curve $c \in \mathrm{Imm}$ is given by the subgroup that leaves c invariant, i.e,

$$\mathcal{G}_c = \{\phi \in \mathrm{Diff}(S^1) : c \circ \phi = c\}.$$

Definition 2.19 (*Free Immersions*)**.** An immersion c is called *free*, if its isotropy group only contains the identity. We denote the space of free immersions by

$$\mathrm{Imm}_f(S^1, \mathbb{R}^d) = \{c \in \mathrm{Imm}(S^1, \mathbb{R}^d) : c = c \circ \phi \Rightarrow \phi = Id_{S^1}, \text{ for } \phi \in \mathrm{Diff}(S^1)\}.$$

In Section 6 we will give a detailed proof that the quotient space

$$\mathrm{Imm}_f(S^1, \mathbb{R}^d)/\mathrm{Diff}(S^1)$$

is indeed a manifold.

In order to get a better understanding of free immersions, we provide an example of a non-free immersion:

Example 2.20 (*The doubly traversed circle in* $\mathbb{R}^2$)**.** Since we can identify $S^1 = \mathbb{R}/(2\pi)$, we can define the doubly traversed circle as follows:

$$c(\theta) = (\cos(2\theta), \sin(2\theta)) \quad \text{for } \theta \in \mathbb{R}/2\pi.$$

Then, setting $\phi(t) = t + \pi$, we get $c \circ \phi(t) = (\cos(2(t+\pi)), \sin(2(t+\pi))) = (\cos(2t), \sin(2t)) = c(t)$. Hence c is not freely immersed.

3 The Whitney-Graustein Theorem

In this section we will prove the Whitney-Graustein Theorem which states that two curves in the plane have the same winding number if and only if they are regularly homotopic. We will use this theorem to characterize the connected components of $\mathbb{R}^2$, in particular when discussing the existence of geodesics in Section 5.

Definition und Lemma 3.1 (*Angle Function*, taken from [11])**.** Let $c \in \mathrm{Imm}^n(S^1, \mathbb{R}^2)$. Set $V = c'/|c'|$, which is a smooth parameterized curve in the plane with values in the unit circle. We can write V as $V(t) = (a(t), b(t))$, where $a(t)$ and $b(t)$ depend on c. Then there exists a smooth function $\phi : [0, 2\pi) \to \mathbb{R}$ such that $a(t) = \cos\phi(t)$ and $b(t) = \sin\phi(t)$. We call $\phi(t)$ the *angle function* for the curve c. It is unique up to adding the constant $2\pi k$.

Proof: Let $\phi_0 \in [0, 2\pi)$ be such that $a(0) = \cos\phi_0 =$ and $b(0) = \sin\phi_0$. Define $\phi : [0, 2\pi) \to \mathbb{R}$ as follows:

$$\phi(t) = \phi_0 + \int_0^t (ab' - ba')d\tau.$$

Now, let $F(t) = (a(t) - \cos\phi(t))^2 + (b(t) - \sin\phi(t))^2$. Then one can show by differentiation that $F'(t) = 0$. This implies that $F(t)$ is a constant function and since $F(0) = 0$ we have $F(t) = 0$. Hence we can write $V(t) = (\cos\phi(t), \sin\phi(t))$. □

Definition 3.2 (*Winding Number*, taken from [11])**.** The *winding number* of a closed parameterized curve $c : [0, 2\pi] \to S^1$ is defined as

$$\gamma(c) := \frac{1}{2\pi}(\phi(2\pi) - \phi(0)),$$

where ϕ is the angle function from above. For an arbitrary curve $c(t) = (a(t), b(t))$ which takes values in $\mathbb{R}^2 \setminus \{0\}$, we write $c(t) = r(t)\varphi(t)$ with $r = \sqrt{a^2 + b^2} > 0$ and $\varphi(t)$ taking values in the unit circle and then set $\gamma(c) = \gamma(\varphi)$. Note that for closed curves the winding number is always an integer.

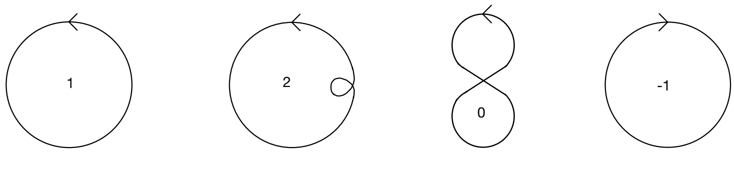

Figure 3.1: Examples of different winding numbers.

In general, the concept of winding numbers can be extended to higher dimensions, as explained in [18]. However, for our specific purpose, it is sufficient to focus solely on the case $d = 2$, since the Whitney-Graustein Theorem is only valid for curves in the plane.

Definition 3.3 (*Regular Homotopy*)**.** Let X, Y be topological spaces and $f, g : X \to Y$ continuous maps. A regular homotopy from f to g is a continuous function $F : X \times [0, 1] \to Y$ with continuous derivative such that

i) $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in X$.

ii) If we set $f_u(x) = F(x, u)$, then f_u is a regular immersion for each $u \in [0, 1]$.

If such a homotopy exists, we say that f is homotopic to g.

Theorem 3.4. (*Whitney-Graustein Theorem*, taken from [27, Thm. 1]). Two immersions in the plane have the same winding number if and only if they are regularly homotopic.

Proof: First, let f_u be a regular homotopy between $c_1, c_2 \in \mathrm{Imm}(S^1, \mathbb{R}^2)$. Then f'_u is continuous in u. Since the winding number is an integer, the number of times f'_u wraps around S^1 is constant for all u. Hence $\gamma(c_1) = \gamma(c_2)$.

Conversely, assume that $\tilde{c}_0$ and c_1 have the same winding number $\hat{\gamma}$. By Lemma 2.8 we can reparameterize $\tilde{c}_0$ and c_1 such that

$$\left|\tilde{c}'_0\right| = \frac{\mathrm{len}(\tilde{c}_0)}{2\pi} =: l_0 \quad \text{and} \quad \left|c'_1\right| = \frac{\mathrm{len}(c_1)}{2\pi} =: l_1.$$

Now, define the following homotopy

$$\tilde{c}_u(t) := \tilde{c}_0(0) + \Big(u\frac{l_1}{l_0} + (1-u)\Big)(\tilde{c}_0(t) - \tilde{c}_0(0)),$$

which connects $\tilde{c}_0(t)$ and $\tilde{c}_1(t) = \tilde{c}_0(0) + \frac{l_1}{l_0}(\tilde{c}_0(t) - \tilde{c}_0(0))$. Set $c_0 = \tilde{c}_1$. Now, we need to prove that c_0 and c_1 are homotopic. The fact that they both have constant speed l_1 simplifies the proof. Let K be the circle of radius l_1. Define the vector function

$$V(t) := (l_1 \cos(t), l_1 \sin(t)),$$

which gives an angular coordinate t in K. Without loss of generality we may alter c_0 and c_1 such that $c'_0(0) = c'_1(0) = V(0) = (l_1, 0)$. Since $c'_i(t)$ lies on K, we get by Lemma 3.1 the existence of functions ϕ_i such that

$$c'_i(t) = V(\phi_i(t)) = \big(l_1 \cos\phi_i(t), l_1 \sin\phi_i(t)\big) \;\; \text{with} \quad \phi_i(0) = 0 \text{ for } i = 0, 1.$$

By the definition of the winding number we can conclude that $\phi_i(2\pi) = 2\pi\hat{\gamma}$ for $i = 0, 1$. Now, define

$$\phi_u(t) := u\,\phi_1(t) + (1-u)\,\phi_0(t) \quad \text{and} \quad h_u(t) := V(\phi_u(t)).$$

Then h_u is a homotopy connecting c_0' and c_1', since $h_i(t) = V(\phi_i(t)) = c_i'(t)$ for $i = 0, 1$.
Now, we alter each h_u by translation to obtain the maps

$$\psi_u(t) := h_u(t) - \frac{1}{2\pi}\int_0^{2\pi} h_u(s)ds.$$

The average of these maps lies at zero, which can be seen as follows:

$$\int_0^{2\pi} \psi_u(s)\,ds = \int_0^{2\pi}\Big(h_u(s) - \frac{1}{2\pi}\int_0^{2\pi} h_u(s)\,ds\Big)ds = \int_0^{2\pi} h_u(s)\,ds - 2\pi \cdot \frac{1}{2\pi}\int_0^{2\pi} h_u(s)\,ds = 0.$$

Define

$$c_u(t) := c_0(0) + u\big(c_1(0) - c_0(0)\big) + \int_0^t \psi_u(s)\,ds,$$

then $c_u(t)$ connects $c_0(t)$ and $c_1(t)$. It remains to show the second condition of Definition 3.3 which states that each c_u has to be a regular (closed) immersion. Therefore, we need to check the following property: $c_u(0) = c_u(2\pi)$ is trivial, since $\psi_u(t)$ has zero average. Moreover, we have to verify whether $c_u'(2\pi) = c_u'(0)$ is true. Observe that $c_u'(t) = \psi_u(t)$ and hence

$$\begin{aligned} c_u'(2\pi) - c_u'(0) &= h_u(2\pi) - \frac{1}{2\pi}\int_0^{2\pi} h_u(s)ds - h_u(0) + \frac{1}{2\pi}\int_0^{2\pi} h_u(s)ds \\ &= V(\phi_u(2\pi)) - V(\phi_u(0)) = V(2\pi\hat{\gamma}) - V(0) = (l_1, 0) - (l_1, 0) = 0, \end{aligned}$$

where we have used that $\phi_u(0) = 0$ for all $u \in [0, 1]$ since $\phi_i(0) = 0$ for $i = 0, 1$ by definition. Also we applied the fact that $\cos(2\pi\hat{\gamma}) = 1$ and $\sin(2\pi\hat{\gamma}) = 0$. Hence we have $c_u'(2\pi) = c_u'(0)$.
Next, we prove that $c_u'(t) \neq 0$ for $u \in [0, 1]$: By the Schwarz inequality we have

$$\left|\int_0^{2\pi} h_u(s)\,ds\right|^2 \leq \int_0^{2\pi} |h_u(s)|^2 ds.$$

If $\hat{\gamma} \neq 0$, then $h_u(t)$ passes over all of K and thus $h_u(t)$ cannot be constant. If $\hat{\gamma} = 0$, $\phi_u(t)$ could be constant just as well as $h_u(t)$. In order to avoid this case we alter $\phi_u(t)$. Therefore, choose t_0 such that $\phi_1(t_0) \neq 0$. Then deform $\phi_0(t)$ to $\phi_1(t)$ in a sufficiently small neighborhood of t_0. Now, define $\phi_u(t)$ as above but with the deformed $\phi_0(t)$. Then we get

$$\begin{aligned} &\phi_u(t_0) = u\,\phi_1(t_0) + (1-u)\,\phi_0(t_0) = \phi_1(t_0) \neq 0, \\ &\phi_u(0) = 0. \end{aligned}$$

Thus, $h_u(t)$ is not constant for any $u \in [0, 1]$. For this reason the above Schwarz inequality becomes a strict inequality.
Furthermore, since

$$|h_u(s)|^2 = |V(\phi_u(t))|^2 = |(L_1 \cos\phi_u(t), L_1 \sin\phi_u(t))|^2 = l_1,$$

we get

$$\frac{1}{2\pi}\left|\int_0^{2\pi} h_u(s)ds\right|^2 < l_1.$$

Thus, we observe that the average of $h_u(t)$ lies in the interior of K. Hence, there exists no t such that

$$h_u(t) = \frac{1}{2\pi}\int_0^{2\pi} h_u(s)ds.$$

By the definition of $\psi_u(t)$, this implies $c_u'(t) = \psi_u(t) \neq 0\ \forall t$, which proves the last condition of $c_u(t)$ being a regular curve for each $u \in [0, 1]$. By the continuity of $c_u(t)$ in u it follows that $c_u(t)$ is a valid deformation of c_0 into c_1, which ends the proof. □

4 Metric Completeness

In this section our objective is to establish metric completeness of the space $(\mathrm{Imm}^n, \mathrm{dist}_G)$, which will subsequently allow us to deduce the metric completeness of $(\mathrm{Imm}^n/\mathrm{Diff}^n, \mathrm{dist}_{I/D})$. The metric completeness of the latter space is needed to apply Lemma 2.16 for proving the existence of geodesics in the quotient space.

4.1 Metric Completeness of the Space $(\mathbf{Imm^n}(S^1, \mathbb{R}^d), \mathbf{dist_G})$

To show metric completeness of $(\mathrm{Imm}^n, \mathrm{dist}_G)$, we present some estimates relating to the induced geodesic distance of Riemannian metrics. Specifically, we will show that the functions $\sqrt{\mathrm{len}(c)}$, $\sqrt{\mathrm{len}(c)}^{-1}$, $|c'|$ and $|c'|^{-1}$ are bounded on metric balls. This helps us to prove our main estimate for Sobolev metrics which states that the $H^n(d\theta)$- and $H^n(ds)$-norms are equivalent. Moreover, we will show that the $H^n(d\theta)$-norm and the distance function induced by a Sobolev metric are equivalent. Then we make use of the fact that the space $(H^n(S^1, \mathbb{R}^d), \|\cdot\|_{H^n(d\theta)})$ is metrically complete to show that $(\mathrm{Imm}^n, \mathrm{dist}_G)$ is complete as well. Proposition 4.2, 4.3, 4.4 and 4.5 are oriented to [4, Sec. 4] and Proposition 4.6 and 4.7 are taken from [3, Sec. 4].

In the following we assume that G_c is a Sobolev metric of order $n \geq 2$ with constant coefficients a_j and that the order of the regularity of the Sobolev curve c is n as well. Moreover, we remark that the constants C may change during the computations, but we still denote all constants by C, as their exact values are not a matter of interest in this context.

Lemma 4.1. There exists a constant $C > 0$ such that

$$C^{-1}\|h\|_{H^n(ds)} \leq \sqrt{G_c(h,h)} \leq C\|h\|_{H^n(ds)}$$

is satisfied for all $c \in \mathrm{Imm}$ and all $h \in H^n$.

Proof: Let $h \in H^n$. Then

$$\|h\|^2_{H^n(ds)} = \int_{S^1} |h|^2 + |D_s^n h|^2 ds \leq \int_{S^1} |h|^2 + |D_s h|^2 + \ldots + |D_s^n h|^2 ds \leq C\, G_c(h,h),$$

where the last inequality is trivial, if $a_i \geq 1$ for all $i \leq n$. Otherwise, choose $C = \frac{1}{\min(a_0,\ldots,a_n)}$. This proves the first inequality of the lemma. For the second one, consider

$$\begin{aligned} G_c(h,h) &= a_0\|h\|^2_{L^2(ds)} + \sum_{i=1}^n a_i\|D_s^i h\|^2_{L^2(ds)} \overset{2.13\,iii)}{\leq} a_0\|h\|^2_{L^2(ds)} + \sum_{i=1}^n a_i(\|h\|^2_{L^2(ds)} + \|D_s^n h\|^2_{L^2(ds)}) \\ &\leq C\,\|h\|^2_{H^n(ds)}, \end{aligned}$$

with $C = \max(a_0, \ldots, a_n)$. □

Proposition 4.2. The function

$$\sqrt{\operatorname{len}(c)} : (\operatorname{Imm}(S^1, \mathbb{R}^d, \operatorname{dist}_G)) \to \mathbb{R}_{>0},$$

is Lipschitz continuous. Moreover, the function $\operatorname{len}(c)$ is bounded.

Proof: For proving Lipschitz continuity of $\sqrt{\operatorname{len}(c)}$, we first have to show Lipschitz continuity of $\sqrt{|c'|}$. For this let c_1, c_2 be two curves in Imm^n and let $c(t,\theta)$ be a smooth path connecting them, i.e., $c(0,\theta) = c_1(\theta)$ and $c(1,\theta) = c_2(\theta)$. Then we have

$$\sqrt{|c_2'|}(\theta) - \sqrt{|c_1'|}(\theta) = \int_0^1 \partial_t\Big(\sqrt{|c'|}\Big)(t,\theta)\, dt \quad \text{pointwise } \forall \theta \in S^1.$$

For computing the derivative of $\sqrt{|c'|}$, we apply two times the chainrule and use the fact that $D_s c_t = \partial_t c'/|c'|$. This yields

$$\partial_t \sqrt{|c'|} = \frac{1}{2}\frac{1}{\sqrt{|c'|}} \cdot \frac{c'}{|c'|} \cdot \partial_t c' = \frac{1}{2} \cdot V \cdot D_s c_t \cdot \sqrt{|c'|} = \frac{1}{2}\langle D_s c_t, V\rangle \sqrt{|c'|}.$$

If we plug this into the previous equation, we get

$$\left\| \sqrt{|c_2'|} - \sqrt{|c_1'|} \right\|_{L^2(d\theta)} \leq \frac{1}{2}\int_0^1 \left\| \langle D_s c_t, V\rangle \sqrt{|c'|} \right\|_{L^2(d\theta)} dt \leq \frac{1}{2}\int_0^1 \left\| \langle D_s c_t, V\rangle \right\|_{L^2(ds)} dt,$$

where we used Jensen's inequality in the first estimate. Observe that

$$\left\| \langle D_s c_t, V\rangle \right\|_{L^2(ds)} \leq \left\| D_s c_t \right\|_{L^2(ds)} \leq \left\| c_t \right\|_{H^n(ds)} \overset{4.1}{\leq} C \sqrt{G_c(c_t, c_t)}.$$

Moreover, we then have

$$\left\| \sqrt{|c_2'|} - \sqrt{|c_1'|} \right\|_{L^2(d\theta)} \leq \frac{C}{2}\int_0^1 \sqrt{G_c(c_t, c_t)}\, dt \leq \frac{C}{2}\operatorname{len}(c).$$

Now, take the infimum over all paths c that connect c_1 and c_2, then we obtain

$$\left\| \sqrt{|c_2'|} - \sqrt{|c_1'|} \right\|_{L^2(d\theta)} \leq \frac{C}{2}\inf_c \operatorname{len}(c) = \frac{C}{2}\operatorname{dist}_G(c_1, c_2).$$

This proves Lipschitz continuity of $\sqrt{|c'|}$. The Lipschitz continuity of $\sqrt{\operatorname{len}(c)}$ follows immediately by using

$$\operatorname{len}(c) = \int_{S^1} |c'(\theta)|\, d\theta = \left\| \sqrt{|c'|} \right\|^2_{L^2(d\theta)}.$$

We obtain,

$$\begin{aligned}\left|\sqrt{\operatorname{len}(c_1)}-\sqrt{\operatorname{len}(c_2)}\right| &= \left|\left\|\sqrt{|c_1'|}\right\|_{L^2(d\theta)}-\left\|\sqrt{|c_2'|}\right\|_{L^2(d\theta)}\right|\\ &\leq \left\|\sqrt{|c_1'|}-\sqrt{|c_2'|}\right\|_{L^2(d\theta)} \leq \frac{C}{2}\operatorname{dist}_G(c_1,c_2). \qquad \square\end{aligned}$$

Proposition 4.3. Given $c_0 \in \mathrm{Imm}$ and $N > 0$, there exists a constant $C = C(c_0, N)$ such that for all $c_1, c_2 \in \mathrm{Imm}$ with $\operatorname{dist}_G(c_0, c_i) < N$ we have

$$\left|\operatorname{len}(c_1)^{-1/2} - \operatorname{len}(c_2)^{-1/2}\right| \leq C \operatorname{dist}_G(c_1, c_2).$$

In particular, the function

$$\frac{1}{\sqrt{\operatorname{len}(c)}} : (\mathrm{Imm}(S^1, \mathbb{R}^d, \operatorname{dist}_G)) \to \mathbb{R}_{>0},$$

is Lipschitz continuous and bounded on metric balls.

Proof: Fix $c_0 \in \mathrm{Imm}$ and let $c_1, c_2 \in B_N(c_0)$ and $c(t,\theta)$ be a path connecting c_1 and c_2 such that $\operatorname{dist}_G(c_0, c(t)) < 2N$. In order to calculate the derivative of $\operatorname{len}(c)^{-1/2}$, we first have to examine the derivative of $\operatorname{len}(c)$:

$$\partial_t \operatorname{len}(c) = \int_{S^1} \partial_t |c'(t,\theta)|\, d\theta = \int_{S^1} \frac{c'(t,\theta)\cdot\partial_t c'(t,\theta)}{|c'(t,\theta)|}\, d\theta = \int_{S^1} \langle D_s c_t, V\rangle \cdot |c'|\, d\theta.$$

Using this for the derivative of $\operatorname{len}(c)^{-1/2}$, we get

$$\partial_t(\operatorname{len}(c)^{-1/2}) = -\frac{1}{2}\operatorname{len}(c)^{-3/2}\int_{S^1} \langle D_s c_t, V\rangle \cdot |c'|\, d\theta.$$

Now, by taking the absolute values and removing the minus by a plus, we get

$$\begin{aligned}\left|\partial_t(\operatorname{len}(c)^{-1/2})\right| &\leq \frac{1}{2}\operatorname{len}(c)^{-3/2}\int_{S^1} |\langle D_s c_t, V\rangle| \cdot |c'|\, d\theta\\ &\overset{C.S.}{\leq} \frac{1}{2}\operatorname{len}(c)^{-3/2}\sqrt{\int_{S^1}|c'|\, d\theta}\,\sqrt{\int_{S^1}|\langle D_s c_t, V\rangle^2|\cdot|c'|\, d\theta}.\end{aligned}$$

Taking into account that

$$\sqrt{\int_{S^1}|c'|\, d\theta} = \sqrt{\operatorname{len}(c)},$$

and since $V^2 = 1$ and $ds = |c'|\, d\theta$ implies

$$\sqrt{\int_{S^1}|\langle D_s c_t, V\rangle^2|\cdot|c'|\, d\theta} = \|D_s c_t\|_{L^2(ds)},$$

we get

$$\begin{aligned}\left|\partial_t(\mathrm{len}(c)^{-1/2})\right| &\leq \frac{1}{2}\mathrm{len}(c)^{-1}\|D_s c_t\|_{L^2(ds)} \overset{2.13\,ii)}{\leq} \frac{1}{2}\mathrm{len}(c)^{-1}\frac{\mathrm{len}(c)}{2}\|D_s^2 c_t\|_{L^2(ds)}\\ &\overset{2.13\,ii)}{\leq} \dots \overset{2.13\,ii)}{\leq} \frac{1}{2}\mathrm{len}(c)^{-1}\left(\frac{\mathrm{len}(c)}{2}\right)^{n-1}\|D_s^n c_t\|_{L^2(ds)} \overset{4.1}{\leq} 2^{-n}\mathrm{len}(c)^{n-2}\,C\,\sqrt{G_c(c_t,c_t)}.\end{aligned}$$

We have $n \geq 2$, so by the previous Lemma $\mathrm{len}(c)^{n-2}$ is bounded on $B_{2N}(c_0)$. Then

$$\left|\mathrm{len}(c_1)^{-1/2} - \mathrm{len}(c_2)^{-1/2}\right| \leq \int_0^1 \left|\partial_t(\mathrm{len}(c)^{-1/2})\right| dt \leq C\int_0^1 \sqrt{G_c(c_t,c_t)}.$$

Now, take the infimum over all paths c that connect c_1 and c_2, then we obtain

$$\left|\mathrm{len}(c_1)^{-1/2} - \mathrm{len}(c_2)^{-1/2}\right| \leq C\,\mathrm{dist}_G(c_1,c_2).$$

This proves Lipschitz continuity on metric balls. For the boundedness we have

$$\mathrm{len}(c)^{-1/2} \leq \mathrm{len}(c_0)^{-1/2} + |\mathrm{len}(c)^{-1/2} - \mathrm{len}(c_0)^{-1/2}| \leq \mathrm{len}(c_0) + C\,\mathrm{dist}_G(c_0,c) \leq C\cdot 2N,$$

which shows that $\mathrm{len}(c)^{-1/2}$ is bounded on metric balls and so is $\mathrm{len}(c)^{-1}$. □

The following proposition provides an upper and lower bound on $|c'|$. The lower bound ensures that the geodesic c stays in the space of immersions, since the derivative will not equal zero.

Proposition 4.4. Given $c_0 \in \mathrm{Imm}$ and $N > 0$, there exists a constant $C = C(c_0, N)$ such that for all $c_1, c_2 \in \mathrm{Imm}$ with $\mathrm{dist}_G(c_0, c_i) < N$ we have

$$\left\|\log|c_2'| - \log|c_1'|\right\|_{L^\infty} \leq \mathrm{dist}_G(c_1,c_2).$$

In particular, the function

$$\log|c'| : (\mathrm{Imm}(S^1,\mathbb{R}^d,\mathrm{dist}_G)) \to L^\infty(S^1,\mathbb{R})$$

is Lipschitz continuous on every metric ball. Moreover, there exists another constant $C_{c_0} > 0$ such that

$$\|c'\|_{L^\infty} \leq C \quad \text{and} \quad \left\|\frac{1}{|c'|}\right\|_{L^\infty} \leq C$$

is satisfied for all $c \in \mathrm{Imm}$ with $\mathrm{dist}_G(c_0, c(t)) < 2N$.

Proof: Let $\theta \in S^1$ be fixed. Suppose that $c_0 \in \mathrm{Imm}$ and $c_1, c_2 \in B_N(c_0)$. Let $c(t,\theta)$ be a path connecting c_1 and c_2 such that $\mathrm{dist}_G(c_0, c(t)) < 2N$. We compute the derivative of $\log|c'|$:

$$\partial_t(\log|c'(t,\theta)|) = \frac{1}{|c'(t,\theta)|}\cdot\partial_t c'(t,\theta)\cdot\frac{c'(t,\theta)}{|c'(t,\theta)|} = \langle D_s c_t(\theta), V(\theta)\rangle.$$

Since $|V| = 1$, we get

$$\left|\log|c_2'(\theta)| - \log|c_1'(\theta)|\right| \leq \int_0^1 |\partial_t(\log|c'(t,\theta)|)|\, dt = \int_0^1 |D_s c_t(t,\theta)|\, dt.$$

Consider,

$$\begin{aligned}\|D_s c_t\|_{L^\infty} &\overset{2.13\,i)}{\leq} \frac{\sqrt{\operatorname{len}(c)}}{2}\|D_s^2 c_t\|_{L^2(ds)} \overset{2.13\,iii)}{\leq} \frac{\sqrt{\operatorname{len}(c)}}{2}\sqrt{\|c_t\|^2_{L^2(ds)} + \|D_s^n c_t\|^2_{L^2(ds)}}\\ &= \frac{\sqrt{\operatorname{len}(c)}}{2}\|c_t\|_{H^n(ds)} \overset{4.1}{\leq} C\sqrt{G_c(c_t,c_t)}.\end{aligned}$$

Again, we have used the fact that $c \in B_{2N}(c_0)$, so $\operatorname{len}(c)$ is bounded. Then, taking the L^∞-norm and the infimum over all paths c between c_1 and c_2, we get

$$\left\|\log|c_2'| - \log|c_1'|\right\|_{L^\infty} \leq C \inf_c \int_0^1 \sqrt{G_c(c_t,c_t)}\, dt = C \inf_c \operatorname{len}(c) = C \operatorname{dist}_G(c_1,c_2).$$

This shows Lipschitz continuity. For the boundedness consider

$$\log|c'(\theta)| \leq \log|c_0'(\theta)| + \left\|\log|c'| - \log|c_0'|\right\|_{L^\infty} \leq C \cdot 2N.$$

Furthermore, apply the exponential function and take the L^∞-norm. Then $\|c'\|_{L^\infty} \leq C$. Similarly, to establish $\left\||c'|^{-1}\right\|_{L^\infty} \leq C$, consider

$$-\log|c'(\theta)| \leq -\log|c_0'(\theta)| + \left\|\log|c_0'| - \log|c'|\right\|_{L^\infty} \leq C \cdot 2N$$

and observe that $\exp(-\log|c'|) = |c'|^{-1}$. □

The next proposition would be trivial, if we knew that the constant C of the previous proposition depends on the curve c. So the key aspect of this proposition pertains to the uniformity of the constant. Hence, if c remains in a metric ball than we can choose C independently of c.

Proposition 4.5. For a given metric ball $B_r(c_0)$ in Imm there exists a constant C such that

$$C^{-1}\|h\|_{H^n(d\theta)} \leq \|h\|_{H^n(ds)} \leq C\|h\|_{H^n(d\theta)}$$

holds for all $c \in B_r(c_0)$ and $h \in H^n$.

Proof: Observe, that for $k \leq n$, we have

$$\begin{aligned}\|h\|^2_{H^k(d\theta)} &= \|h\|^2_{L^2(d\theta)} + \|\partial_\theta^k h\|^2_{L^2(d\theta)},\\ \|h\|^2_{H^k(ds)} &= \|h\|^2_{L^2(ds)} + \|D_s^k h\|^2_{L^2(ds)}.\end{aligned}$$

For the $L^2(d\theta)$- and $L^2(ds)$-norm we have the following estimates on bounded metric balls:

$$\|h\|_{L^2(ds)} = \int_{S^1} |h|^2\, ds = \int_{S^1} |h|^2\, |c'|\, d\theta \leq \|c'\|_{L^\infty} \int_{S^1} /h/^2\, d\theta = \|c'\|_{L^\infty} \|h\|^2_{L^2(d\theta)} \leq C\, \|h\|^2_{L^2(d\theta)},$$
$$\|h\|_{L^2(ds)} = \int_{S^1} |h|^2\, ds = \int_{S^1} |h|^2\, |c'|\, d\theta \geq \min_{\theta\in S^1} |c'|\, \|h\|^2_{L^2(d\theta)} \geq C\, \|h\|^2_{L^2(d\theta)}.$$

In the last inequalities we used that $|c'|$ is bounded from above and bounded away from 0 on metric balls by Proposition 4.4. This proves the equivalence of the $L^2(d\theta)$- and $L^2(ds)$-norm. It remains to show that there exists a $C > 0$ such that

$$C^{-1} \|\partial_\theta^k h\|^2_{L^2(d\theta)} \leq \|D_s^k h\|^2_{L^2(ds)} \leq C\, \|\partial_\theta^k h\|^2_{L^2(d\theta)}.$$

To this end, we need a computation of $\partial_\theta^k h$ with $k \leq n$. For $k = 2$ we have

$$D_s^2 h = D_s\Big(\frac{1}{|c'|} \cdot \partial_\theta h\Big) = \frac{1}{|c'|}\partial_\theta\Big(\frac{1}{|c'|}\partial_\theta h\Big) = \frac{1}{|c'|} \cdot \frac{1}{|c'|}\partial_\theta^2 h + \frac{1}{|c'|} \cdot \partial_\theta\Big(\frac{1}{|c'|}\Big) \cdot \partial_\theta h$$

By multiplying both sides with $|c'|^2$ we get

$$|c'|^2 D_s^2 h = \partial_\theta^2 h + |c'|^2 \partial_\theta\Big(\frac{1}{|c'|}\Big) D_s h,$$

which is equivalent to

$$\partial_\theta^2 h = |c'|^2 D_s^2 h + \partial_\theta(|c'|) D_s h.$$

Inductively, we get

$$\begin{aligned}
\partial_\theta h &= |c'| D_s h \\
\partial_\theta^2 h &= |c'|^2 D_s^2 h + \partial_\theta(|c'|) D_s h \\
\partial_\theta^3 h &= |c'|^3 D_s^3 h + 3|c'|\, \partial_\theta(|c'|) D_s^2 h + \partial_\theta^2(|c'|) D_s h \\
\partial_\theta^4 h &= |c'|^4 D_s^4 h + 6|c'|^2\, \partial_\theta(|c'|) D_s^3 h + \Big(3\partial_\theta(|c'|)^2 + 4|c'|\partial_\theta^2(|c'|)\Big) D_s^2 h + \partial_\theta^3(|c'|) D_s h \\
&\;\;.\\
&\;\;.\\
&\;\;.\\
\partial_\theta^k h &= \sum_{j=1}^{k} \sum_{\alpha\in A_j} c_{j,\alpha} \prod_{i=0}^{k-1} \Big(\partial_\theta^i |c'|\Big)^{\alpha_i} D_s^j h,
\end{aligned}$$

where $c_{j,\alpha}$ are constants and $\alpha = (\alpha_0, ..., \alpha_{k-1})$ are multi-indices that are given by

$$A_j := \Big\{\alpha : \sum_{i=0}^{k-1} i\alpha_i = k - j,\ \sum_{i=0}^{k-1} \alpha_i = j\Big\}.$$

It is shown in [3, Sec. 3] that

$$\|\partial_\theta^k |c'|\,\|_{L^\infty} \text{ for } 0 \le k \le n-2 \text{ and}$$
$$\|\partial_\theta^{n-1} |c'|\,\|_{L^2(d\theta)}$$

are bounded on metric balls. Since the boundedness only holds for $k \le n-2$ and $k = n-1$, we treat the cases where $k \le n-1$ and $k = n$, separately. For the case $k \le n-1$ we use induction. Let $k = 0$, then we have no derivative. This equivalence has been shown in the beginning. Now, assume that the equivalence has been shown for $k-1$, i.e.,

$$C^{-1}\|\partial_\theta^j h\|_{L^2(d\theta)} \le \|D_s^j h\|_{L^2(ds)} \le C\|\partial_\theta^j h\|_{L^2(d\theta)} \text{ for } j \le k-1.$$

Moreover, we have

$$\|\partial_\theta^k h\|_{L^2(d\theta)}^2 \le \sum_{j=1}^{k}\sum_{\alpha\in A_j} c_{j,\alpha}\left\|\prod_{i=0}^{k-1}\left(\partial_\theta^i |c'|\right)^{\alpha_i} D_s^j h\right\|_{L^2(d\theta)}^2 \le \sum_{j=1}^{k}\sum_{\alpha\in A_j} c_{j,\alpha}\left\|\prod_{i=0}^{k-1}\left(\partial_\theta^i |c'|\right)^{\alpha_i}\right\|_{L^\infty}^2 \|D_s^j h\|_{L^2(d\theta)}^2.$$

Since k is maximal $n-1$, the highest derivative of $|c'|$ is $\partial_\theta^{n-2}|c'|$ which is bounded in L^∞. Using the equivalence of the $L^2(d\theta)$- and $L^2(ds)$-norm we get

$$\|\partial_\theta^k h\|_{L^2(d\theta)}^2 \le C\sum_{j=1}^{k}\sum_{\alpha\in A_j} c_{j,\alpha}\|D_s^j h\|_{L^2(ds)}^2 \overset{2.13\, ii)}{\le} C\|D_s^k h\|_{L^2(ds)}^2.$$

In the last inequality we used the second Poincare inequality multiple times and the fact that $\operatorname{len}(c)$ is bounded. This proves one part of the equivalence. For the other part we observe that in the formula for $\partial_\theta^k h$ the last term, where $j = k$, is always given by $|c'|^k D_s^k h$. Thus, we can write

$$D_s^k h = |c'|^{-k}\partial_\theta^k h - |c'|^{-k}\sum_{j=1}^{k-1}\sum_{\alpha\in A_j} c_{j,\alpha}\prod_{i=0}^{k-1}\left(\partial_\theta^i |c'|\right)^{\alpha_i} D_s^j h.$$

Since the sum only goes up to $k-1$, we can use the induction assumption

$$\|D_s^k h\|_{L^2(ds)} \le \|\,|c'|^{-k}\|_{L^\infty}\|\partial_\theta^k h\|_{L^2(d\theta)} + \|\,|c'|^{-k}\|_{L^\infty}\sum_{j=1}^{k-1}\sum_{\alpha\in A_j} c_{j,\alpha}\left\|\prod_{i=0}^{k-1}\left(\partial_\theta^i |c'|\right)^{\alpha_i}\right\|_{L^\infty}\|\partial_\theta^j h\|_{L^2(d\theta)}.$$

Now, use the the boundedness of $|c'|^{-k}$, the boundedness of $\|\partial_\theta^k |c'|\,\|_{L^\infty}$ for $k \le n-2$ and the second Poincare inequality multiple times for $\|\partial_\theta^j h\|_{L^2(d\theta)}$ to get

$$\|D_s^k h\|_{L^2(ds)} \le C\,\|\partial_\theta^k h\|_{L^2(d\theta)}.$$

This proves the equivalence of the two norms for $k \le n-1$. Now, consider the case, $k = n$. The problem is that we have no bound on $\partial_\theta^{n-1}|c'|$ in the L^∞-norm. However, we see that if $\alpha_{n-1} \ne 0$, then $\alpha_{n-1} = 1$ and $\alpha_i = 0$ for $i \ne n-1$ and if $j \ge 2$, then $\alpha_{n-1} = 0$. Thus, the only term where

$\partial_\theta^{n-1}(|c'|)$ appears, is just given by $\partial_\theta^{n-1}(|c'|)\, D_s h$. This term can be estimated in the $L^2(d\theta)$-norm as follows:

$$\|\partial_\theta^{n-1}(|c'|) D_s h\|^2_{L^2(d\theta)} \leq \|\partial_\theta^{n-1}(|c'|)\,\|^2_{L^2(d\theta)} \|D_s h\|^2_{L^\infty}.$$

In the beginning we mentioned that we can bound $\|\partial_\theta^{n-1}|c'|\,\|^2_{L^2(d\theta)}$ and for $\|D_s h\|^2_{L^\infty}$ we can use the first Poincare inequality and then the second one to get

$$\|\partial_\theta^{n-1}(|c'|) D_s h\|^2_{L^2(d\theta)} \leq C\,\|D_s^n h\|^2_{L^2(ds)}.$$

Furthermore, we have

$$\|\partial_\theta^{n-1}(|c'|) D_s h\|^2_{L^2(d\theta)} \leq \|\partial_\theta^{n-1}(|c'|)\,\|^2_{L^2(d\theta)} \|D_s h\|^2_{L^\infty} \leq C\,\|\,|c'|^{-1}\|^2_{L^\infty} \|\partial_\theta h\|^2_{L^\infty}.$$

Again, using the Poincare inequalities we get

$$\|\partial_\theta^{n-1}(|c'|) D_s h\|^2_{L^2(d\theta)} \leq C\,\|\partial_\theta^n h\|_{L^2(d\theta)}.$$

By proceeding as in the case for $k \leq n-1$ and plugging these two estimates in the formula for $\partial_\theta^k h$ or $D_s^k h$, we get the equivalence of $\|\partial_\theta^n h\|^2_{L^2(d\theta)}$ and $\|D_s^n h\|^2_{L^2(ds)}$. Together with the equivalence of the $L^2(d\theta)$- and $L^2(ds)$-norm we can conclude that the $H^k(d\theta)$- and $H^k(ds)$-norm are equivalent as well. □

So far we considered C^∞-immersions, but unfortunately we cannot prove metric completeness for this space. It is shown in [3, Prop. A.2] that the geodesic distance on the space of Sobolev immersions Imm^n restricted to Imm coincides with the geodesic distance on Imm. Hence, we can extend the previous results from Imm to Imm^n and continue with showing the metric completeness for Imm^n.

Lemma 4.6. 1) Given a metric ball $B_r(c_0)$ in Imm^n there exists a constant $C > 0$ such that

$$\|c_1 - c_2\|_{H^n(d\theta)} \leq C \mathrm{dist}_G(c_1, c_2),$$

holds for all $c_1, c_2 \in B_r(c_0)$.
2) Given $c_0 \in \mathrm{Imm}^n$, there exists $r > 0$ and a constant $C > 0$ such that

$$\mathrm{dist}_G(c_1, c_2) \leq C\|c_1 - c_2\|_{H^n(d\theta)},$$

holds for all $c_1, c_2 \in B_r(c_0)$.

Proof: 1) Let $c_1, c_2 \in B_r(c_0)$ and $c(t,\theta)$ be a piecewise smooth path such that c connects c_1 and c_2 and $\mathrm{len}(c) < 2r$. Then by Jensen's inequality, Proposition 4.5 and Lemma 4.1 we have

$$\|c_1 - c_2\|_{H^n(d\theta)} \leq \int_0^1 \|c_t(t)\|_{H^n(d\theta)}\, dt \leq C \int_0^1 \|c_t(t)\|_{H^n(ds)}\, dt \leq C \int_0^1 \sqrt{G_c(c_t, c_t)}\, dt = C\,\mathrm{len}(c).$$

Taking the infimum over all paths c between c_1 and c_2 yields the inequality.
2) Let $c_0 \in \mathrm{Imm}^n$ and U be a convex, open neighborhood of c_0. It is shown in [17, Prop. 6.1] that

for a smooth, strong Riemannian metric the geodesic distance induces the manifold topology. Hence, we can find an open ball $B_r(c_0)$ with respect to the geodesic distance such that this ball is contained in the open set U of the manifold topology. Define the path

$$c(t) := c_1 + t(c_2 - c_1)$$

to be the linear interpolation between $c_1, c_2 \in \mathrm{Imm}^n$. Then, c is a convex combination and hence $c \in B_r(c_0)$. Moreover, we have

$$\mathrm{dist}_G(c_1, c_2) \leq \mathrm{len}(c) = \int_0^1 \sqrt{G_c(c_2 - c_1, c_2 - c_1)}\, dt \overset{4.5}{\leq} C\, \|c_2 - c_1\|_{H^n(d\theta)}. \qquad \square$$

Theorem 4.7. $(\mathrm{Imm}^n(S^1, \mathbb{R}^d), \mathrm{dist}_G)$ is a complete metric space.

Proof: Consider a Cauchy sequence $(c^j)_{j\in\mathbb{N}}$ with respect to the geodesic distance. Choose $R > 0$ large enough such that the sequence is contained in a metric ball with radius R. By the previous Lemma we obtain

$$\|c^j - c^i\|_{H^n(d\theta)} \leq C\, \mathrm{dist}_G(c^j, c^i), \quad \forall j, i \in \mathbb{N}.$$

Thus, the sequence $(c^j)_{j\in\mathbb{N}}$ is also a Cauchy sequence with respect to $\|\cdot\|_{H^n(d\theta)}$. Since the space $(H^n(S^1, \mathbb{R}^d), \|\cdot\|_{H^n(d\theta)})$ is complete, we can conclude that there exists a $c^* \in H^n(S^1, \mathbb{R}^d)$ such that

$$\|c^j - c^*\|_{H^n(d\theta)} \overset{j\to\infty}{\to} 0.$$

By Proposition 4.4, $\partial_\theta c^j$ is bounded away from 0 and hence

$$\|\partial_\theta c^j\|_{H^n(d\theta)} \geq C > 0.$$

Since this inequality holds for all $j \in \mathbb{N}$, we see that $(\partial_\theta c^j)_{j\in\mathbb{N}}$ does not converge to zero. Hence, we have for the limit c^*

$$\|\partial_\theta c^*\|_{H^n(d\theta)} \geq C > 0,$$

which implies that $c^* \in \mathrm{Imm}^n$. Moreover, the second statement of Lemma 4.6 gives the existence of constants $r > 0$ and $C > 0$ such that

$$\mathrm{dist}_G(c^j, c^*) \leq \|c^j - c^*\|_{H^n(d\theta)} \quad \forall c^j \in B_r(c^*).$$

This converges to zero for all c^j that are close to c^* in the $H^n(d\theta)$-norm. But the inequality holds for all c^j that are close to c^* in the metric distance. With the first statement of Lemma 4.6 we can always find a $H^n(d\theta)$-ball that is contained in a metric ball. So the above holds for all c^j in that $H^n(d\theta)$-ball and hence converges to zero. Thus, we have shown that an arbitrary Cauchy sequence in Imm^n with respect to the geodesic distance converges. This proves metric completeness of Imm^n. $\square$

4.2 Metric Completeness of the Space $(\mathbf{Imm^n}((\mathbf{S^1}), \mathbb{R}^\mathbf{d})/\mathbf{Diff^n}(\mathbf{S^1}), \mathbf{dist_{I/D}})$

Lemma 4.8. (Taken from [3, Lemma 6.5]). Consider a metric space (X, d) upon which the group G acts by isometries. If the quotient space X/G is Hausdorff, then

$$d_{X/G}(G.x, G.y) := \inf_{g,h \in G} d(g.x, h.y) = \inf_{h \in G} d(x, h.y)$$

defines a metric on X/G which coincides with the quotient topology. Moreover, the metric $d_{X/G}$ is intrinsic, if d is intrinsic.

Proof: At first, we check that the properties of a metric are satisfied for $d_{X/G}$. Since d is a metric on X, the symmetry of $d_{X/G}$ is clear. Moreover, we have $d_{X/G}(G.x, G.x) = \inf_{g,h \in G} d(g.x, h.x) = 0$, take for example $h = g = e$, where e denotes the neutral element. Since we will show in the following that the two topologies coincide, we know that the quotient space with the topology induced by $d_{X/G}$ is Hausdorff as well. The Hausdorff property ensures that it is possible to separate two distinct points by disjoint neighborhoods. Consequently, it is not possible to find two distinct points $G.x$ and $G.y$ such that $d_{X/G}(G.x, G.y) = 0$. Hence, $d_{X/G}(G.x, G.y) = 0$ implies $G.x = G.y$. Since G acts on X by isometries, i.e. $d(g.x, h.y) = d(x, g^{-1}h.y)$, we can write $\inf_{g,h \in G} d(g.x, h.y) = \inf_{h \in G} d(x, h.y)$. Moreover we have

$$d_{X/G}(G.x, G.z) = \inf_{g \in G} d(x, g.z) \leq d(x, h.y) + \inf_{g \in G} d(h.y, g.z) = d(x, h.y) + d_{X/G}(G.y, G.z).$$

Taking the infimum over all $h \in G$, we get the triangle inequality.

To prove that the metric $d_{X/G}$ is compatible with the quotient topology on X/G, we need to check that both topologies provide the same open sets. To accomplish this, let $B_X(x, \epsilon)$ be an open ball in X and $B_{X/G}(G.x, \epsilon)$ be an open ball in X/G with respect to the topology induced by $d_{X/G}$. Denote by $\pi : X \to X/G$ the canonical projection. A set is open in the quotient topology if the union of its orbits is open in X. For $B_{X/G}(G.x, \epsilon)$ we have

$$\pi^{-1}(B_{X/G}(G.x, \epsilon)) = \left\{y : \inf_{h \in G} d(x, h.y) < \epsilon\right\} = \left\{g.y : g \in G, y \in B_X(x, \epsilon)\right\} = G.B_X(x, \epsilon).$$

Since $G.B_X(x, \epsilon)$ is open in X, we get that $B_{X/G}(G.x, \epsilon)$ is open in the quotient topology. Conversely, let $U \subseteq X/G$ be open in the quotient topology and $G.x \in U$. Since U is open we can find an ϵ such that $B_X(x, \epsilon) \subseteq \pi^{-1}(U)$. For $G.y \in B_{X/G}(G.x, \epsilon)$ we have $d_{X/G}(G.x, G.y) < \epsilon$ and hence $d(x, g.y) < \epsilon$ for some $g \in G$. This implies $g.y \in B_X(x, \epsilon)$ and thus $G.y = \pi(g.y) \in U$. This shows that $B_{X/G}(G.x, \epsilon) \subseteq U$. Hence, we have proven that for an arbitrary element $G.x \in U$ we can find a neighborhood $B_{X/G}(G.x, \epsilon)$ which is open with respect to the metric $d_{X/G}$ and completely contained in U. Consequently, U is also open with respect to $d_{X/G}$. Thus, the open sets of the topologies coincide.

Next, we show that the metric $d_{X/G}$ is identical to the metric induced on the quotient space which is given by

$$\begin{aligned} dist_{X/G}(G.x, G.y) &= \inf_{\gamma} \Big\{ \text{len}(\gamma) : \gamma(0) = G.x,\ \gamma(1) = G.y \Big\} \\ &= \inf_{\gamma} \Big\{ \sup_{\{t_1,...,t_N\}} \sum_{i=1}^{N} d_{X/G}(\gamma(t_i), \gamma(t_{i-1})) : \gamma(0) = G.x,\ \gamma(1) = G.y \Big\}, \end{aligned}$$

where we have taken the supremum over all finite partitions $0 = t_1 < t_1 < ... < t_N = 1$ of the interval $[0, 1]$. On the one hand, we have $dist_{X/G}(G.x, G.y) \geq d_{X/G}(G.x, G.y)$ due to the triangle inequality

$$\begin{aligned} dist_{X/G}(G.x, G.y) &= \inf_{\gamma} \Big\{ \sup_{\{t_1,...,t_N\}} \sum_{i=1}^{N} d_{X/G}(\gamma(t_i), \gamma(t_{i-1})) : \gamma(0) = G.x,\ \gamma(1) = G.y \Big\} \\ &\geq d_{X/G}(G.x, G.y). \end{aligned}$$

On the other hand, we have $\text{dist}_{X/G}(G.x, G.y) \leq d_{X/G}(G.x, G.y)$, which can be seen as follows. Let γ be a path in X between $G.x$ and $G.y$ such that $\text{len}(\gamma) \leq d_{X/G}(G.x, G.y) + \epsilon/2$ for $\epsilon > 0$. Let π be the projection into the quotient space and define $\pi(\gamma) := \hat{\gamma}$. Choose $\{t_1, ..., t_N\} \in [0, 1]$ such that $\text{len}(\hat{\gamma}) = \sum_{i=2}^{N} d_{X/G}(\hat{\gamma}(t_i), \hat{\gamma}(t_{i-1})) + \epsilon/2$. Thus, $\gamma(t_i) \in G.\gamma(t_i) = \hat{\gamma}(t_i)$. Then

$$\begin{aligned} \text{dist}_{X/G}(G.x, G.y) \leq \text{len}(\hat{\gamma}) &= \sum_{i=2}^{N} d_{X/G}(\hat{\gamma}(t_i), \hat{\gamma}(t_{i-1})) + \epsilon/2 = \sum_{i=2}^{N} \inf_{g \in G} d(\gamma(t_i), g.\gamma(t_{i-1})) + \epsilon/2 \\ &\overset{e \in G}{\leq} \sum_{i=2}^{N} d(\gamma(t_i), \gamma(t_{i-1})) + \epsilon/2 = \text{len}(\gamma) + \epsilon/2 \leq d_{X/G}(G.x, G.y) + \epsilon, \end{aligned}$$

where we have used in the second last step that γ is optimal and the fact that d is in intrinsic metric. For $\epsilon \to 0$ we get $\text{dist}_{X/G}(G.x, G.y) \leq d_{X/G}(G.x, G.y)$ and hence $\text{dist}_{X/G}(G.x, G.y) = d_{X/G}(G.x, G.y)$. □

Lemma 4.9. (Taken from [3, Lemma 6.5]). Let (X, d) be a metric space and $d_{X/G}$ a metric on the quotient space, defined as above. If (X, d) is complete, then so is $(X/G, d_{X/G})$.

Proof: Consider a Cauchy sequence $(G.x_n)_{n\in\mathbb{N}}$ in X/G. By the Cauchy property we can choose a subsequence such that $d_{X/G}(G.x_n, G.x_{n+1}) < 2^{-n}$ holds for all $n \in \mathbb{N}$. Moreover, we can find $\tilde{x}_n \in G.x_n$ and $\tilde{x}_{n+1} \in G.x_{n+1}$ such that $d(\tilde{x}_n, \tilde{x}_{n+1}) < d_{X/G}(G.x_n, G.x_{n+1}) + 2^{-n}$. Consider

$$d(\tilde{x}_n, \tilde{x}_{n+k}) \leq \sum_{i=n}^{n+k-1} d(\tilde{x}_i, \tilde{x}_{i+1}) \leq \sum_{i=n}^{n+k-1} d_{X/G}(G.x_i, G.x_{i+1}) + 2^{-i} \leq 2^{2-n}(1 - 2^{-k}).$$

Thus $d(\tilde{x}_n, \tilde{x}_{n+k}) \overset{n\to\infty}{\to} 0$ and hence $(\tilde{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete by assumption, we can find $\tilde{x} \in X$ such that $\tilde{x}_n \overset{n\to\infty}{\to} \tilde{x}$. Then

$$\lim_{n\to\infty} G.x_n = \lim_{n\to\infty} \pi(\tilde{x}_n) = \pi(\lim_{n\to\infty} \tilde{x}_n) = \pi(\tilde{x}) = G.\tilde{x}.$$

Hence we found a limit $G.\tilde{x}$ for the Cauchy sequence $(G.x_n)_{n\in\mathbb{N}}$ and hence $(X/G, d_{X/G})$ is complete. $\square$

Corollary 4.10. The space $(\mathrm{Imm}^n/\mathrm{Diff}^n, d_{I/D})$ with the quotient metric induced by the geodesic distance on $(\mathrm{Imm}^n, \mathrm{dist}_G)$ is a complete metric space.

Proof: In Theorem 4.7 we proved the metric completeness of $(\mathrm{Imm}^n, \mathrm{dist}_G)$ and it is shown in [8, Thm. 2.1] that $\mathrm{Imm}^n/\mathrm{Diff}^n$ is Hausdorff. Thus, we can apply Lemma 4.8 and Lemma 4.9 and get that $(\mathrm{Imm}^n/\mathrm{Diff}^n, d_{I/D})$ with the quotient metric induced by the geodesic distance on $(\mathrm{Imm}^n, \mathrm{dist}_G)$ is a complete metric space. $\square$

From now on we leave the subscript G in the notation of the dist-function. Instead, we write dist_I for the induced distance function on Imm^n and $\mathrm{dist}_{I/D}$ for the induced distance function on $\mathrm{Imm}^n/\mathrm{Diff}^n$. Since dist_I is intrinsic by definition, it follows by Lemma 4.8 that $d_{I/D}$ is intrinsic as well. Thus, we also get metric completeness of $(\mathrm{Imm}^n/\mathrm{Diff}^n, \mathrm{dist}_{I/D})$.

5 Existence of Geodesics

As described in [12] the Hopf-Rinow Theorem asserts that in a complete finite-dimensional Riemannian manifold, it is possible to connect any two points using a minimal geodesic. However, this property does not hold in the infinite-dimensional case, as noted in [1] by Atkin. Hence, we have to show the existence of geodesics in a different way. First, we show via the direct method of Calculus of Variation that any two curves in Imm^n in the same connected component can be joined by a geodesic. Then we transfer the existence result from the space of parameterized curves to the shape space of unparameterized curves. Unless stated otherwise, the structure of the proofs mainly follows from [3, Sec. 5,6].

5.1 Existence of Geodesics in $\mathbf{Imm^n}(\mathbf{S^1}, \mathbb{R}^\mathbf{d})$

In the following we will denote the unit interval by $I = [0, 1]$. For brevity we write

$$H_t^1 H_\theta^n = H_t^1 H_\theta^n(I \times (S^1, \mathbb{R}^d)) \cong H^1(I, H^n(S^1, \mathbb{R}^d)),$$

and similarly we write $C_t H_\theta^n, L_t^2 L_\theta^2$, etc. The norm $\|\cdot\|_{H_t^1 H^n(d\theta)}$ is given by

$$\|c\|^2_{H_t^1 H^n(d\theta)} = \int_0^1 \|c(t)\|^2_{H^n(d\theta)} + \|\dot{c}(t)\|^2_{H^n(d\theta)} \, dt, \quad \text{for } c \in H_t^1 H_\theta^n.$$

Theorem 5.1. Let G be a Sobolev metric of order $n \geq 2$ with constant coefficients. Fix a curve $c_0 \in \mathrm{Imm}^n$. Further suppose $A \subseteq \mathrm{Imm}^n$ is a weakly closed set concerning the H^n-topology such that at least one curve in A belongs to the same connected component as c_0. Then there exists $c_1 \in A$ and a geodesic $c(t)$ with $c(0) = c_0$ and $c(1) = c_1$ such that

$$L(c) = \mathrm{dist}_I(c_0, c_1) = \mathrm{dist}_I(c_0, A) = \inf_{\tilde{c} \in A} \mathrm{dist}_I(c_0, \tilde{c}).$$

So the geodesic c(t) realizes the minimal distance between c_0 and A. Moreover, the energy is minimized by c as well.

Remark 5.2. The connected components of a space are the subsets in where every pair of points can be connected by a continuous path that lies entirely within the subset. For $d \geq 3$, the space Imm^n is connected, i.e.it has only one component. Thus, the restriction from Theorem 5.1 that A has to belong to the same connected component as c_0 can be disregarded. However, if $d = 2$, this is not the case. Nevertheless, as shown in Section 3, the connected components for immersions in $\mathbb{R}^2$, which are also homotopy classes, consist of curves with the same winding number. Hence, we can characterize the connected components by the winding number of the curves lying in them.

In the next subsection we will also consider the connected components of the quotient space $\mathrm{Imm}^n/\mathrm{Diff}^n$. Once again, for $d \geq 3$ there is only one component. For $d = 2$ we have the following decomposition

$$\mathrm{Imm}^n/\mathrm{Diff}^n = \bigcup_{p>0} \mathrm{Imm}^n_p/\mathrm{Diff}^{n,+} \cup \mathrm{Imm}^n_0/\mathrm{Diff}^n,$$

where Imm^n_p denotes the space of curves with winding number p and $\mathrm{Diff}^{n,+}$ represents orientation-preserving diffeomorphisms of order n as described in Definition 2.17.

For the proof of Theorem 5.1 we need the following lemma which describes the behavior of weak convergence of arc length derivatives.

Lemma 5.3. Let $s \in \mathbb{R}, s > \frac{3}{2}$ and $k \in \mathbb{N}, 0 \leq k \leq s$. If $c^j, c \in H^1_t \mathrm{Imm}^s_\theta$ and $h^j, h \in L^2_t H^k_\theta$, then

$$\left.\begin{array}{l} c^j \rightharpoonup c \text{ weakly in } H^1_t \mathrm{Imm}^s_\theta \\ h^j \rightharpoonup h \text{ weakly in } L^2_t H^k_\theta \\ (h^j)_{j\in\mathbb{N}} \text{ is bounded in } L^2_t H^k_\theta \end{array}\right\} \quad \Rightarrow \quad D^k_{c^j} h^j \rightharpoonup D^k_c h \text{ weakly in } L^2_t L^2_\theta.$$

A proof of this lemma can be found in [3, Lemma 6.9].

Proof of Theorem 5.1: It is shown in [15, Lemma 2.4.3], that a minimizer of the energy

$$E(c) = \int_0^1 G_c(\dot{c}, \dot{c})\, dt,$$

on the set

$$\Omega := \left\{ c \in H^1(I, \mathrm{Imm}^n) : c(0) = c_0, c(1) \in A \right\}$$

is a minimizing geodesic between c_0 and $c(1) \in A$. To show the existence of a minimizer of the energy functional we use the direct method of the Calculus of Variation (see [16, Sec. 3] for details on the direct method). Therefore, we need to show compactness of any minimizing sequence. That is, if $(c^j) \subset \Omega$ is a minimizing sequence for E, i.e.

$$\lim_{j\to\infty} E(c^j) = \inf_{c\in\Omega} E(c),$$

then there exists a subsequence (c^{j_k}) such that $c^{j_k} \rightharpoonup c^*$ weakly in Ω to some $c^* \in \Omega$. Also we need lower semicontinuity of the functional E, i.e. we need that

$$E(c) \leq \liminf_j E(c^j) \quad \text{for all } c \in \Omega \text{ and } (c^j) \subset \Omega \text{ s.t. } (c^j \rightharpoonup c) \text{ weakly in } \Omega,$$

in order to apply the direct method.

We start by showing compactness. Let $(c^j) \in \Omega$ be a minimizing sequence. Then we can bound the energy $E(c^j)$ by r^2 for some $r > 0$. Since each $c^j(t)$ starts in c_0 and $L(c^j) \leq \sqrt{E(c^j)}$, as shown in Definition 2.7, we have

$$\mathrm{dist}_I(c_0, c^j(t)) \leq \sqrt{E(c^j)} \leq r.$$

So every $c^j(t)$ lies in a metric ball around c_0 with radius r. Hence, we can apply Proposition 4.5 and Lemma 4.1 to get the existence of a constant $C > 0$ such that

$$C^{-1}\|h\|_{H^n(d\theta)} \leq \sqrt{G_{c^j(t)}(h,h)} \leq C\|h\|_{H^n(d\theta)}$$

is satisfied for all $h \in H^n$. Then

$$\|c^j(t)\|_{H^n(d\theta)} \leq \|c_0\|_{H^n(d\theta)} + \|c^j(t) - c_0\|_{H^n(d\theta)} \leq \|c_0\|_{H^n(d\theta)} + C_1 \mathrm{dist}_I(c_0, c^j(t)) \leq \|c_0\|_{H^n(d\theta)} + C_1 r,$$

where C_1 is the constant from Lemma 4.6. Furthermore, we have

$$\begin{aligned}\|c^j\|^2_{H^1_t H^n(d\theta)} = \int_0^1 \|c^j(t)\|^2_{H^n(d\theta)} + \|\dot{c}^j(t)\|^2_{H^n(d\theta)}\, dt &\leq (\|c_0\|_{H^n(d\theta)} + C_1 r)^2 + C^2 \int_0^1 G_{c^j(t)}(\dot{c}^j, \dot{c}^j)\, dt \\ &= (\|c_0\|_{H^n(d\theta)} + C_1 r)^2 + C^2 E(c^j).\end{aligned}$$

Since $E(c^j)$ is bounded, we conclude that $\|c^j\|_{H^1_t H^n_\theta}$ is bounded as well. By Banach-Alaoglu it follows that there exists a subsequence which converges weakly to $c^* \in H^1_t H^n_\theta$. We denote the subsequence again by (c^j). Consider the embedding $H^1_t H^n_\theta \hookrightarrow C_t H^{n-\epsilon}_\theta$ with $n - \epsilon > 3/2$ and $0 < \epsilon < 1$. Since this embedding is compact by [2, Thm. 1], we can conclude that $c^j \to c^*$ converges strongly in $C_t H^{n-\epsilon}_\theta$. It remains to show that $c^* \in \Omega$. We use Lemma 4.4 to obtain a constant $C_2 > 0$ such that

$$|\partial_\theta c^j(t,\theta)| \geq C_2 \quad \forall \theta \in S^1, \forall t \in [0,1], \forall j \in \mathbb{N}.$$

Due to the strong convergence this bound also holds for the limit c^* and hence $c^*(t)$ is indeed an immersion for all $t \in I$. Since $c^j \rightharpoonup c^*$ weakly in $H^1_t H^n_\theta$, we have $c^j(1) \rightharpoonup c^*(1)$ weakly in H^n_θ. As the minimizing sequence $c^j(t)$ was chosen such that $c^j(0) = c_0$ and $c^j(1) \in A$ and by assumption A is weakly closed concerning the H^n−topology we get that $c^*(0) = c_0$ and $c^*(1) \in A$. Thus, $c^* \in \Omega$ and we have shown compactness of minimizing sequences.

Next, we show lower semicontinuity of the functional E. Note that we can write E as

$$E(c) = \sum_{k=0}^n a_k \left\| D_c^k \dot{c} \sqrt{|c'|} \right\|^2_{L^2_t L^2_\theta}.$$

Since the squared norm-function $h \mapsto \|h\|^2_{L^2_t L^2_\theta}$ is weakly sequentially lower semicontinuous, we observe that if $D^k_{c^j} \dot{c}^j \sqrt{|\partial_\theta c^j|} \rightharpoonup D^k_{c^*} \dot{c}^* \sqrt{|\partial_\theta c^*|}$ weakly in $L^2_t L^2_\theta$, then

$$\sum_{k=0}^n a_k \left\| D^k_{c^*} \dot{c}^* \sqrt{|\partial_\theta c^*|} \right\|^2_{L^2_t L^2_\theta} \leq \liminf_{j\to\infty} \sum_{k=0}^n a_k \left\| D^k_{c^j} \dot{c}^j \sqrt{|\partial_\theta c^j|} \right\|^2_{L^2_t L^2_\theta}.$$

Hence, we have to prove the following implication

$$c^j \rightharpoonup c^* \text{ weakly in } H^1_t H^n_\theta \;\Rightarrow\; D^k_{c^j}\dot{c}^j \sqrt{|\partial_\theta c^j|} \rightharpoonup D^k_{c^*}\dot{c}^* \sqrt{|\partial_\theta c^*|} \text{ weakly in } L^2_t L^2_\theta \;\text{ for } k = 1, ..., n.$$

The boundedness of the (c^j) in $H^1_t H^n_\theta$ and Lemma 5.3 imply that $D^k_{c^j}\dot{c}^j \rightharpoonup D^k_{c^*}\dot{c}^*$ weakly in $L^2_t L^2_\theta$. Moreover, we have $c^j \to c^*$ in $C_t H^n_\theta$ and thus $\sqrt{|\partial_\theta c^j|} \rightharpoonup \sqrt{|\partial_\theta c^*|}$ in $C_t H^{n-1-\epsilon}_\theta$. As ϵ was chosen such that $n - 1 - \epsilon > 1/2$, we get that the pointwise product $D^k_{c^j}\dot{c}^j \sqrt{|\partial_\theta c^j|}$ converges weakly in $L^2_t L^2_\theta$, which proves the above implication. Now, if we put all the results together, we get that if $c^j \rightharpoonup c^*$ weakly in $H^1_t H^n_\theta$, then

$$E(c^*) = \sum_{k=0}^{n} a_k \left\| D^k_{c^*}\dot{c}^* \sqrt{|\partial_\theta c^*|} \right\|^2_{L^2_t L^2_\theta} \leq \liminf_{j\to\infty} \sum_{k=0}^{n} a_k \left\| D^k_{c^j}\dot{c}^j \sqrt{|\partial_\theta c^j|} \right\|^2_{L^2_t L^2_\theta} = \liminf_{j\to\infty} E(c^j).$$

This shows the lower semicontinuity of the functional E. Moreover, as we have seen in the first step of the proof that $c^* \in \Omega$, we conclude that c^* is indeed a minimizer for E. □

5.2 Existence of Geodesics in $\mathrm{Imm}^n(S^1, \mathbb{R}^d)/\mathrm{Diff}^n(S^1)$

In Section 2.4 we have introduced the concept of geometric curves in the quotient space. Here we show the existence of geodesics also in the space of geometric curves. To do so, we make use of the previous subsection and the fact that $(\mathrm{Imm}^n/\mathrm{Diff}^n, \mathrm{dist}_{I/D})$ is metrically complete.

Theorem 5.4. For $C_1, C_2 \in \mathrm{Imm}^n/\mathrm{Diff}^n$ in the same connected component, there exist $c_1, c_2 \in \mathrm{Imm}^n$ with $c_1 \in \pi^{-1}(C_1)$ and $c_2 \in \pi^{-1}(C_2)$ such that

$$\mathrm{dist}_{I/D}(C_1, C_2) = \mathrm{dist}_I(c_1, c_2).$$

In other words: The infimum in

$$\mathrm{dist}_{I/D}(\pi(c_1), \pi(c_2)) = \inf_{\phi\in\mathrm{Diff}^n} \mathrm{dist}_I(c_1, c_2 \circ \phi)$$

is attained.

Proof: Fix $c_1, c_2 \in \mathrm{Imm}^n$. As stated in Section 2, a minimizer of the energy functional is also a minimizer for the path length with constant speed. So again, we consider the energy functional

$$E(c) = \int_0^1 G_c(\dot{c}, \dot{c})dt,$$

on the set

$$\tilde{\Omega} := \{c \in H^1 : c(0) = c_1, c(1) \in c_2 \circ \mathrm{Diff}^n\}.$$

As we do not know if the orbit $c_2 \circ \mathrm{Diff}^n$ is weakly closed, we are unable to proceed in the same way as in the proof of Theorem 5.1. Nevertheless, we can adopt a similar approach as used in the proof of that theorem. We choose a minimizing sequence $(c^j) \in \tilde{\Omega}$ for E, that is

$$\lim_{j\to\infty} E(c^j) = \operatorname{dist}_{I/D}(\pi(c_1), \pi(c_2)).$$

Again, we can pass to a subsequence which converges weakly $c^j \rightharpoonup c^*$ in $H^1_t H^n_\theta$ and strongly in $C_t H^{n-\epsilon}_\theta$ with ϵ such that $n - \epsilon > 3/2$ and $0 < \epsilon < 1$. Analogous to the proof of Theorem 5.1, we can show that $E(c^*) \leq \liminf_{j\to\infty} E(c^j)$ and that c^* is an immersion. What remains to show, is that $c^*(1) \in c_2 \circ \mathrm{Diff}^n$, since in the previous proof this was guaranteed by the fact that A is weakly closed. Since $(c^j) \in \tilde{\Omega}$, we have $c^j(1) \in c_2 \circ \mathrm{Diff}^n$ for all $j \in \mathbb{N}$. Using the fact that $\mathrm{Imm}^n/\mathrm{Diff}^n$ is Hausdorff and the strong convergence $c^j(1) \to c^*(1)$ in $H^{n-\epsilon}_\theta$, we get $c^*(1) \in c_2 \circ \mathrm{Diff}^{n-\epsilon}$. So we can write $c^*(1) = c_2 \circ \phi$ with $\phi \in \mathrm{Diff}^{n-\epsilon}$. According to Lemma 2.8 we can assume that c_2 has constant speed and hence

$$|c^*(1)'| = |c_2'| \circ \phi \cdot \phi' = \frac{\operatorname{len}(c_2)}{2\pi} \phi'.$$

Since $c^* \in H^n_\theta$, it follows $c^{*\prime} \in H^{n-1}_\theta$. This implies that $\phi' \in H^{n-1}_\theta$ and hence $\phi \in \mathrm{Diff}^n$. Thus, $c^*(1) \in c_2 \circ \mathrm{Diff}^n$, so c^* is indeed a minimizer of the functional E on the set $\tilde{\Omega}$. □

Finally, we are able to prove the main theorem of this section which provides the existence of geodesics in the quotient space of Sobolev immersions.

Theorem 5.5. The space $(\mathrm{Imm}^n/\mathrm{Diff}^n, \operatorname{dist}_{I/D})$ with the induced metric is a length space and any two shapes in the same connected component can be joined by a minimizing geodesic.

Proof: As already mentioned, $(\mathrm{Imm}^n/\mathrm{Diff}^n, \operatorname{dist}_{I/D})$ is a complete metric space. Hence, we can apply Proposition 2.16 and it remains to show that for every C_0 and C_1 in the same connected component there exists a midpoint. By Theorem 5.4 we know that there exist $c_0, c_1 \in \mathrm{Imm}^n$ lying in the same connected component and such that $C_i = \pi(c_i)$ and

$$\operatorname{dist}_{I/D}(C_0, C_1) = \operatorname{dist}_I(c_0, c_1).$$

Using Theorem 5.1 we get the existence of a geodesic $c(t)$ with constant speed, connecting c_0 and c_1. Hence, we have

$$\operatorname{dist}_I(c_0, c(\tfrac{1}{2})) = \operatorname{dist}_I(c(\tfrac{1}{2}), c_1).$$

Set $C(t) = \pi(c(t))$. If we can show that

$$\operatorname{dist}_{I/D}(C_0, C(\tfrac{1}{2})) = \operatorname{dist}_I(c_0, c(\tfrac{1}{2})) \text{ and } \operatorname{dist}_{I/D}(C(\tfrac{1}{2}), C_1) = \operatorname{dist}_I(c(\tfrac{1}{2}), c_1),$$

then $\pi(c(\frac{1}{2})) = C(\frac{1}{2})$ is a midpoint between C_0 and C_1 and we are done. So let us assume the converse:

$$\operatorname{dist}_{I/D}(C_0, C(\tfrac{1}{2})) < \operatorname{dist}_I(c_0, c(\tfrac{1}{2})) \text{ or } \operatorname{dist}_{I/D}(C(\tfrac{1}{2}), C_1) < \operatorname{dist}_I(c(\tfrac{1}{2}), c_1).$$

Then

$$\begin{aligned}\mathrm{dist}_{I/D}(C_0, C_1) &\leq \mathrm{dist}_{I/D}(C_0, C(\tfrac{1}{2})) + \mathrm{dist}_{I/D}(C(\tfrac{1}{2}), C_1)\\ &< \mathrm{dist}_I(c_0, c(\tfrac{1}{2})) + \mathrm{dist}_I(c(\tfrac{1}{2}), c_1) = \mathrm{dist}_I(c_0, c_1),\end{aligned}$$

which yields a contradiction. □

5.3 Existence of Geodesics in $\mathbf{Imm_f^n(S^1, \mathbb{R}^d)}$ and $\mathbf{Imm_f^n(S^1, \mathbb{R}^d)/Diff^n}$

Unfortunately, we cannot argue as in the previous subsections for the spaces Imm_f^n and $\mathrm{Imm}_f^n/\mathrm{Diff}^n$, since Imm_f^n is not a complete metric space. Nevertheless, Imm_f^n is contained in Imm^n and hence two curves $c_0, c_1 \in \mathrm{Imm}_f^n$ in the same connected component can be joined by a minimizing geodesic $c(t)$. It remains to show that the geodesic itself is a free immersion for each $t \in (0, 1)$. This can be done with the help the following proposition.

Proposition 5.6. (Taken from [25, Sec. 2.2.3.5]). Let $c_0, c_1 \in \mathrm{Imm}_f^n$ be in the same connected component and $c(t) \in \mathrm{Imm}^n$ a geodesic connecting c_0 and c_1. Then $c(t) \in \mathrm{Imm}_f^n$ $\forall t \in (0, 1)$.

Proof: Let $c_0, c_1 \in \mathrm{Imm}_f^n$ and let the group of diffeomorphisms act on Imm^n. According to Theorem 5.1 there exists a geodesic $c(t) \in \mathrm{Imm}^n$ such that $c(0) = c_0$ and $c(1) = c_1$. The isotropy groups G_{c_0} and G_{c_1} are trivial, since c_0 and c_1 are free immersions. We have to prove that for every $t \in (0, 1)$ the isotropy group $G_{c(t)}$ is trivial as well. Choose $a \in G_{c(t)} \setminus G_{c_0}$ for a fixed but arbitrary $t \in (0, 1)$ and define the path

$$\tilde{c}(s) := \begin{cases} ac(s) & \text{if } s \leq t, \\ c(s) & \text{else.} \end{cases}$$

Then $\tilde{c}(s)$ is a piecewise geodesic which connects ac_0 and c_1 and has the same length as c, since the distance is invariant under the isometric action of Diff^n. Observe that $\tilde{c}$ has a discontinuity at t (otherwise $\tilde{c}$ would be equal to c in every point t, which would imply that $a \in G_{c_0}$). This discontinuity contradicts with the opitmality of $\tilde{c}$, since we can find a small neighborhood of t, where we can shorten the path. Hence, there exists no $a \in G_{c(t)} \setminus G_{c_0}$. Thus, $G_{c(t)} \subset G_{c_0}$ for every $t \in (0, 1)$. As every isotropy group contains the neutral element, the isotropy groups of $c(t)$ are also trivial. Hence, the geodesic $c(t) \in \mathrm{Imm}_f^n$ for every $t \in (0, 1)$. □

Theorem 5.7. Any two orbits $C_0, C_1 \in \mathrm{Imm}_f^n/\mathrm{Diff}^n$ in the same connected component can be joined by a geodesic.

Proof: Taking into account Theorem 5.5 where we proved the existence of geodesics in the quotient space, we can run the same argumentation as above. □

6 The Manifold of free geometric Curves

The goal of this section is to prove the following theorem.

Theorem 6.1. The quotient space of the space of free immersions under the action of diffeomorphisms $\mathrm{Imm}_f/\mathrm{Diff}$ admits a manifold structure.

This theorem will be important for showing horizontality of geodesics in the next section. The main work of the proof lies in finding open neighborhoods in the space of immersions such that the properties of a manifold are satisfied. To do so, we need to introduce the concept of tubular neighborhoods. The structure and the proofs of this section are mainly based on [19].

Before we start, we need to give two remarks. In the following we consider only curves in the plane. In general, all the statements apply to higher dimensions as well, but for the sake of simplicity we focus on curves in the plane. In this context a curve c is always given by $c : S^1 \to \mathbb{R}^d$ and has regularity at least C^1. Moreover, in the proofs of the following lemmata and propositions we will often reparameterize the curve c by arc parameter. This slightly changes the assumptions of the theorems regarding the length of the curves which will depend on δ_c, and distance to other curves which will depend on τ_c, in the following way:

1. Rotating and translating c does not affect τ_c, δ_c and $\operatorname{len} c_{|[\sigma,\tilde{\sigma}]}$ for $\sigma, \tilde{\sigma} \in S^1$.
2. Scaling c by a factor λ will result in the values τ_c, δ_c and $\operatorname{len} c_{|[\sigma,\tilde{\sigma}]}$ being multiplied by λ as well.
3. If we reparameterize c, then τ_c and δ_c remain unchanged. For $\psi \in \mathrm{Diff}$ and $\tilde{c} = c \circ \psi$ we get $\operatorname{len} \tilde{c}_{|[\psi(\sigma),\psi(\tilde{\sigma})]} = \operatorname{len} \tilde{c}_{|[\sigma,\tilde{\sigma}]}$.

In what follows we disregard these transformations, in order simplify the proofs, which work analogously if we do the transformations, as described above. This means that if we assume c to have constant speed, then the assumptions on c still hold.

6.1 Tubular Neighborhoods of Immersions

Proposition 6.2. Let c be in $C^2, a, b \in S^1$ and $L := \operatorname{len}(c)_{|[a,b]}$. Assume that $L \leq 2\delta_c$, where δ_c is defined as in Definition 2.6. Then $|c(b) - c(a)| \geq L/2$.

Proof: For simplicity, we extend c to a periodic function $c : \mathbb{R} \to \mathbb{R}^2$ and we identify the interval in $\mathbb{R}$ that is associated with the arc of the curve, where the length $\operatorname{len}(c)_{|[a,b]}$ is computed (see

Def. 2.9). We call this interval $[a, b]$ again. By Lemma 2.8 we can assume c to have constant speed, i.e., $|c'| = l = \frac{\mathrm{len}(c)}{2\pi}$. As noted in Definition 2.9 we then have

$$\mathrm{len}(c)_{|[a,b]} = l\,(b - a).$$

Consider the scalar curvature κ from Definition 2.5. We derived the following formula

$$|\kappa| = \frac{|\phi'|}{|c'|} = \frac{|\phi'|}{l},$$

where ϕ is the angle function from Definition 3.1. Plugging this into the definition of δ_c, we obtain

$$\delta_c = \frac{l\,\pi}{(3\max|\phi'|)}.$$

Let $m = (a + b)/2$ be the middle point of $[a, b]$. As $|c'| = l$ and $|V| = 1$, we can rotate c such that $c'(m) = (l, 0)$ and $V(m) = (1, 0)$. Consequently, we can assume that $\phi(m) = 0$. By assumption we have $L = l\,(b - a) \leq 2\delta_c$. As m is the middle point of $[a, b]$, we get for $\theta \in [a, b]$

$$l\,|m - \theta| \leq \delta_c.$$

Plugging this in the above formula for δ_c, we get

$$|\theta - m| \leq \frac{\pi}{(3\max|\phi'|)}.$$

Since $\phi(m) = 0$, we obtain for all $a \leq \theta \leq b$

$$|\phi(\theta)| = |\phi(\theta) - \phi(m)| = \Big|\int_m^\theta \phi'(\sigma)\,d\sigma\Big| \leq |\theta - m| \max|\phi'| \leq \frac{\pi}{3}.$$

Hence, we can conclude that $\cos(\phi(\theta)) \geq 1/2$. Finally, if $a \leq \theta_1 \leq \theta_2 \leq b$, then we have for the abscissa

$$c_1(\theta_2) - c_1(\theta_1) = \int_{\theta_1}^{\theta_2} c_1'(\theta)\,d\theta = \int_{\theta_1}^{\theta_2} V_1 l\,d\theta = l\int_{\theta_1}^{\theta_2} \cos(\phi(\theta))\,d\theta \geq l\,(\theta_2 - \theta_1)\frac{1}{2} = \mathrm{len}(c)_{|[\theta_1,\theta_2]}/2,$$

where we have used that $c'/l = V = (\cos(\phi), \sin(\phi))$. Note that it is sufficient to prove the inequality only for the x−coordinate since $|c(b) - c(a)| \geq c_1(b) - c_1(a)$ and we rotated c' in such a way that it is reasonable to consider the distance in the direction of the x-axis. □

The above proposition states that the curve c restricted to $[a, b]$ is an embedding, since $|c(b) - c(a)| \geq L/2$ implies that there a no intersection points in c.

Lemma 6.3 (*Tubular Neighborhood*)**.** Let $a, b \in S^1$ such that $\mathrm{len}(c)_{|[a,b]} \leq 2\delta_c$. Define

$$\Psi : [a, b] \times [-\tau_c, \tau_c] \to \mathbb{R}^2,$$
$$\Psi(s, t) = c(s) + tN(s).$$

Then Ψ is a diffeomorphism with its image and if the arc $[s_1, s_2]$ is contained in the arc $[a, b]$, then

$$|\Psi(s_1, t_1)\Psi(s_2, t_2)| \geq \frac{1}{4}\operatorname{len} c_{|[s_1,s_2]},$$

whereas

$$|\Psi(s, t_1) - \Psi(s, t_2)| = |t_2 - t_1|.$$

Proof: With no loss of generality, we may rescale and reparameterize c such that c has length 2π and is parameterized in arc parameter, i.e., $|c'| = 1$. As in the previous proof we extend c to a periodic function $c : \mathbb{R} \to \mathbb{R}^2$ and identify the interval in $\mathbb{R}$ that is associated with the arc of the curve, where the length $\operatorname{len}(c)_{|[a,b]}$ is computed. Again, we denote this interval by $[a, b]$.

As $\frac{\partial}{\partial s} c = \frac{c'}{|c'|} = V$ and $\frac{\partial}{\partial s} N = -\kappa V$, we obtain for the Jacobian of Ψ

$$\frac{\partial}{\partial s}\Psi = V(1 - \kappa t)$$
$$\frac{\partial}{\partial t}\Psi = N.$$

By assumption we have $|t| \leq \tau_c = 1/(2\max|\kappa|)$,. Thus, we get for the determinant $(1 - \kappa t) \geq 1/2$. Applying the inverse function Theorem we get that Ψ is a local diffeomorphism. For a global diffeomorphism, we must additionally show bijectivity. Surjectivity is clear, since we are considering a diffeomorphism with its image, so what remains to prove is injectivity.

To accomplish this, we choose (s_1, t_1) and (s_2, t_2) with $a \leq s_1 < s_2 \leq b$ and $|t_1|, |t_2| \leq \tau_c$. Similar to the proof of Proposition 6.2, we set $m = (s_1 + s_2)/2$ and up to rotation we may assume that $V(m) = (1, 0)$ and $\phi(m) = 0$. Again, we can conclude that $\cos(\phi(s)) \geq 1/2$ for all $s_1 \leq s \leq s_2$. Since $|c'| = 1$, we have $V = c'$. This yields

$$\Psi(s, t) = c(s) + tN(s) = c(m) + \int_m^s V(\theta)\, d\theta + tN(s).$$

With $V(s) = (\cos(\phi(s)), \sin(\phi(s)))$ and $N(s) = (-\sin(\phi(s)), \cos(\phi(s)))$ we get for the x-coordinate

$$\Psi(s, t)_1 = c(m)_1 + \int_m^s \cos(\phi(\theta))\, d\theta - t\sin(\phi(s)).$$

Computing the derivative we get

$$\frac{\partial}{\partial s}\Psi(s, t)_1 = \cos(\phi(s))(1 - t\phi'(s)).$$

As c is parameterized in arc parameter, we have $|\kappa| = |\phi'|$. Now, we use the facts that $\cos(\phi(s)) \geq 1/2$ and $|t| \leq \tau_c = \frac{1}{2\max|\kappa|}$ to obtain

$$\cos(\phi(s)(1 - t\phi'(s))) \geq \frac{1}{4}.$$

Hence, $\frac{\partial}{\partial s}\Psi(s,t)_1 \geq \frac{1}{4}$. Then we observe that

$$\Psi(s_2,t_2)_1 - \Psi(m,t_2)_1 \geq \frac{1}{4}(s_2 - m),$$
$$\Psi(m,t_1)_1 - \Psi(s_1,t_1)_1 \geq \frac{1}{4}(m - s_1).$$

Adding these two equations and noting that $\Psi(m,t_1)_1 = \Psi(m,t_2)_1 = c(m)_1$, since $\phi(m) = 0$, we obtain

$$\Psi(s_2,t_2)_1 - \Psi(s_1,t_1)_1 \geq \frac{1}{4}(s_2 - s_1).$$

Since c has unit speed parameterization, we have $\mathrm{len}(c)_{|[s_1,s_2]} = 1(s_2 - s_1)$. By taking the norm on both sides, we have proven the first inequality of the lemma. With the same argument as in the previous proof, it is enough to show the inequality only for the abscissa. The second inequality follows by

$$|\Psi(s,t_1) - \Psi(s,t_2)| = |c(s) + t_1 N(s) - c(s) - t_2 N(s)| = |t_1 - t_2|\,|N| = |t_1 - t_2|.$$

Moreover, we have shown injectivity of Ψ. Indeed, if $\Psi(s_2,t_2) = \Psi(s_1,t_1)$, then we get

$$0 \geq \frac{1}{4}(s_2 - s_1),$$

which implies $s_2 = s_1$. Now, using the second inequality we get

$$0 = |\Psi(s_1,t_1) - \Psi(s_1,t_2)| = |t_1 - t_2|$$

and thus $t_1 = t_2$. □

If $\tilde{c}(s) = \Psi(s,t) = c(s) + tN(s)$, then we say that we can write $\tilde{c}$ in *tubular coordinates* around c. We define the set of *nearby points* of c as $U_\tau := \Psi(S^1 \times [-\tau,\tau])$.

Lemma 6.4. The points in U_τ have distance at most τ from the trace $c(S^1)$.

Proof: The distance function is given by

$$d_{c(S^1)}(x) := \inf_{y \in c(S^1)} |x - y| = \inf_{\theta \in S^1} |x - c(\theta)|.$$

Consider an element $x \in U_\tau$, then there exist $(\tilde{\theta}, t) \in S^1 \times [-\tau,\tau]$ such that

$$x = c(\tilde{\theta}) + tN(\tilde{\theta}).$$

Then we have

$$|x - c(\tilde{\theta})| = |t| \leq \tau.$$

By geometric arguments the minimum of $\inf_{\theta \in S^1} |x - c(\theta)|$ is attained in $\tilde{\theta}$, since the segment of the minimal distance of x to a point on c is orthogonal to the tangent at this point. Hence,

$$d_{c(S^1)}(x) = |x - c(\tilde{\theta})| \leq \tau. \qquad \square$$

We have seen that for $x \in U_\tau$ we can always find $(\theta, t) \in S^1 \times [-\tau, \tau]$ such that

$$x = c(\theta) + tN(\theta).$$

Now consider a curve $\tilde{c}$, instead of x, whose trace is contained in U_τ. One might think that we will always find a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ and $a : \mathbb{R} \to [-\tau, \tau]$ such that

$$\tilde{c}(\tilde{\theta}) = \Psi(\varphi(\theta), a(\theta)) = c(\varphi(\theta)) + a(\theta)N(\varphi(\theta)).$$

This would imply that if the trace of $\tilde{c}$ is contained in a tube of size 2τ around c, then we could write each point of $\tilde{c}$ in tubular coordinates around c with the same φ and a. But this is not true. A counterexample is shown in Figure 6.1. Here the two curves are close to each other, but we cannot find φ and a such that the above equation holds. The problem is that we are considering the trace of a curve and say that two curves are close to each other if every point on the one curve is close to *any* point of the other curve. A suggestion of improvement would be to consider the parameterization of the curves and say that two curves are close to each other if

$$|\tilde{c}(\theta) - c(\theta)| \leq \tau \quad \forall \theta \in S^1.$$

The next lemma acts on this idea.

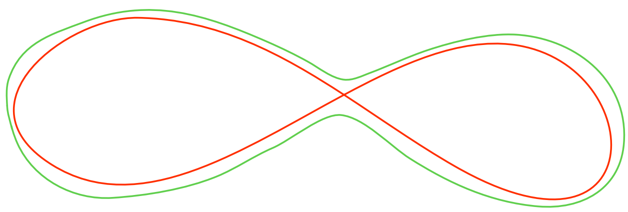

Figure 6.1: The green curve is all contained in the set of nearby points U_τ of the red curve.

Lemma 6.5 (*Nearby Projection*)**.** Let c be a fixed C^R-curve with $R \geq 2$. Then:

1. For $x \in \mathbb{R}^2$ and $\tilde{\sigma} \in S^1$ such that

$$d := |x - c(\tilde{\sigma})| < \frac{\delta_c}{4},$$

there exists an $a \in \mathbb{R}$ with $|a| \leq d$ and $\sigma \in S^1$ with

$$\text{len}(c)_{|[\sigma, \tilde{\sigma}]} \leq 4d,$$

such that

$$x = c(\sigma) + aN(\sigma).$$

Observe that a is uniquely identified by σ.

2. They are unique in the family of σ, a such that $|a| \le \tau_c$ and

$$\mathrm{len}(c)_{|[\sigma,\tilde{\sigma}]} \le \delta_c.$$

3. Let $\tilde{x} \in \mathbb{R}^2$ and $\tilde{\sigma} \in S^1$ such that

$$d := |\tilde{x} - c(\tilde{\sigma})| < \frac{\tau_c}{2}.$$

Let $\epsilon > 0$ small so that $\epsilon + d < \delta_c/4$ and define $B = B(\tilde{x}, \epsilon) = \{x \in \mathbb{R}^2 : |x - \tilde{x}| < \epsilon\}$. Then there exist functions $a, \varphi : B \to \mathbb{R}$ of class C^{R-1} such that

$$x = \Psi(\varphi(x), a(x)) = c(\varphi(x)) + a(x)N(\varphi(x))$$

for all $x \in B$ and they are unique as specified above.

Proof: Without loss of generality, we may assume that the curve c has length 2π and hence can be parameterized in arc parameter. Thus, we can write $|a - b|$ for $\mathrm{len}(c)_{|[a,b]}$.

For proving 1) we choose $x \in \mathbb{R}^2$ and $\tilde{\sigma} \in S^1$ such that $d := |x - c(\tilde{\sigma})| < \frac{\delta_c}{4}$. Define $J_{\tilde{\sigma}} := [\tilde{\sigma} - \delta_c, \tilde{\sigma} + \delta_c]$. Let $\hat{\theta}$ be a minimum of

$$\min_{\theta \in J_{\tilde{\sigma}}} |x - c(\theta)|.$$

Then we have $|x - c(\hat{\theta})| \le d$ and hence

$$|c(\tilde{\sigma}) - c(\hat{\theta})| \le |c(\tilde{\sigma}) - x| + |x - c(\hat{\theta})| \le 2d.$$

By the definition of $J_{\tilde{\sigma}}$ we get $|\tilde{\sigma} - \hat{\theta}| \le \delta_c$. Thus, we may apply Proposition 6.2 and obtain

$$|c(\tilde{\sigma}) - c(\hat{\theta})| \ge \frac{1}{2}|\hat{\theta} - \tilde{\sigma}|.$$

These two inequalities yield

$$4d \ge |\hat{\theta} - \tilde{\sigma}|.$$

As $4d < \delta_c$, we see that $|\hat{\theta} - \tilde{\sigma}| < \delta_c$ and hence the minimum is not attained at the boundary of $J_{\tilde{\sigma}}$. Then by geometric arguments the minimal segment from x to $c(\hat{\theta})$ is orthogonal to the tangent at $c(\hat{\theta})$, so we can find an a such that

$$x = c(\hat{\theta}) + aN(\hat{\theta}) = \Psi(\hat{\theta}, a),$$

which proves 1). Note that $|a| = |a|\,|N(\hat{\theta})| = |x - c(\hat{\theta})| \le d$. In the proof of 6.3 we have seen that

Ψ is injective for $|a| \le \tau_c$ and $|\hat{\theta} - \tilde{\sigma}| \le 2\delta_c$. So if $|a| \le \tau_c$ and $\hat{\theta} \in J_{\tilde{\sigma}}$, the uniqueness is given by the injectivity of Ψ. This proves the second statement.

For the last one, consider $x \in B$. Then we obtain

$$|x - c(\tilde{\sigma})| \le |x - \tilde{x}| + |\tilde{x} - c(\tilde{\sigma})| < \epsilon + d < \frac{\delta_c}{4}.$$

By 1) and 2) there exist a unique $\sigma \in J_{\tilde{\sigma}}$ and a with $|a| \le \tau_c$ such that

$$x = c(\sigma) + aN(\sigma).$$

As they are depending on x, we denote them by $\sigma = \varphi(x), a = a(x)$. Furthermore, we have shown in the proof of Proposition 6.3 that Ψ is a diffeomorphism with its image in C^{R-1}. Hence, we can invert the function and write

$$\Psi^{-1} : \mathbb{R}^2 \to J_{\tilde{\sigma}} \times [-\tau_c, \tau_c]$$
$$\Psi^{-1}(x) = (\varphi(x), a(x)),$$

which proves that $\varphi, a \in C^{R-1}$. □

The main point of this lemma is that we fix $\tilde{\sigma} \in S^1$ in the beginning and assume that x has to be close to this point on the curve and not just close to any point on that curve. Hence, the lemma provides some necessary conditions whether a curve c can be lifted into another curve $\tilde{c}$, which is explained in detail in the following proposition.

Proposition 6.6 (*Global Lifting*)**.** Let $c : S^1 \to \mathbb{R}^2$ be in C^R and $\tilde{c} : S^1 \to \mathbb{R}^2$ be in C^{R-1}, with $R \ge 2$. Suppose that we have $|\tilde{c}(\theta) - c(\theta)| \le \tau$ for all $\theta \in S^1$ and for a fixed $\tau < \delta_c/4$. Then there exists a choice of $a : S^1 \to \mathbb{R}$ and $\varphi : S^1 \to S^1$ such that

$$\tilde{c}(\sigma) = \Psi(\varphi(\sigma), a(\sigma)) = c(\varphi(\sigma)) + a(\sigma)N_c(\varphi(\sigma)), \quad \forall \sigma \in S^1,$$

with $|a(\sigma)| \le \tau$ and

$$\operatorname{len}(c)_{|[\sigma,\varphi(\sigma)]} \le 4\tau$$

is satisfied for all $\sigma \in S^1$. Moreover, they are unique in the class of C^{R-1} functions such that $|a| \le \tau_c$ and

$$\operatorname{len}(c)_{|[\sigma,\varphi(\sigma)]} \le \delta_c.$$

Proof: We have demonstrated in the previous lemma that the statement is valid for all $x \in S^1$ which are close enough to $c(\tilde{\sigma})$. By substituting $x = \tilde{c}(\sigma)$ we see that the statement is also true for every point that is on the curve $\tilde{c}$. Uniqueness and regularity follow by the second and third point of the previous lemma. □

Remark 6.7. If we can lift c_1 into c_2 with (a, φ), as explained in the proposition above, then we have the following relation between c_1 and c_2 and (a, φ). If we translate or rotate the curves, we can still lift c_1 to c_2 with the same (a, φ). By rescaling both curves with $\lambda > 0$, we can use $(\lambda a, \varphi)$ to lift the rescaled curve. If we reparameterize both curves at the same time by $\phi \in \mathrm{Diff}$, i.e., $\tilde{c}_1 = c_1 \circ \phi, \tilde{c}_2 = c_2 \circ \phi$, then we can lift $\tilde{c}_1$ to $\tilde{c}_2$ by

$$\tilde{c}_2(s) = c_1(\tilde{\varphi}(s)) + \tilde{a}(s) N_{\tilde{c}_1}(\tilde{\varphi}(s)), \quad \forall s \in S^1,$$

with

$$\begin{aligned} \tilde{a} &= a \circ \phi \\ \tilde{\varphi} &= \phi^{-1} \circ \varphi \circ \phi. \end{aligned}$$

Moreover, let $\alpha > 0$ and consider a third curve c_3 such that

$$|c_2'(\theta) - c_3'(\theta)| \leq \alpha |c_1'(\theta)|.$$

If we now reparameterize all the curves at the same time by $\tilde{c}_i = c_i \circ \phi$, then

$$|\tilde{c}_2'(\theta) - \tilde{c}_3'(\theta)| \leq \alpha |\tilde{c}_1'(\theta)|.$$

This relation also holds for rescaling, rotation and translation.

It seems that we have found some sufficient hypotheses so that we can present $\tilde{c}$ in tubular coordinates around c. However, the function φ of Proposition 6.6 is not necessarily a diffeomorphism, which is required in the definition of the tubular coordinates. An example where φ fails to be a diffeomorphism is shown in Figure 6.2. In order to avoid such cases, we need to control the difference of the derivatives of c' and $\tilde{c}'$. This is done in the next lemma.

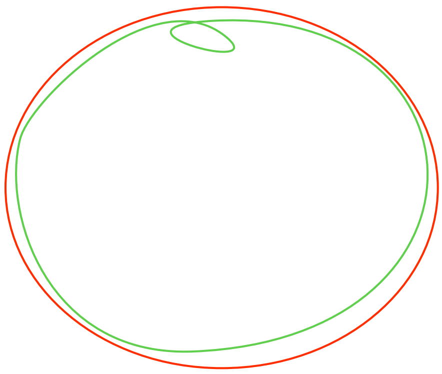

Figure 6.2: The curve c is presented in red and $\tilde{c}$ in green.

Lemma 6.8. Suppose all assumptions of Proposition 6.6 hold. Moreover, assume that $\tau \leq \tau_c/4$ and

$$|\tilde{c}'(\sigma) - c'(\sigma)| \leq \frac{1}{2}|c'(\sigma)|, \quad \forall \sigma \in S^1.$$

Then $\tilde{c}$ is an immersed curve and φ is a diffeomorphism. For c being parameterized at constant speed, we get

$$\frac{1}{5} \leq \varphi' \leq 3.$$

Proof: As always, we may reparameterize c to have unit speed and do the same transformations with $\tilde{c}$. Then the assumption $|\tilde{c}'(\sigma) - c'(\sigma)| \leq \frac{1}{2}|c'(\sigma)|$ still holds, as mentioned in Remark 6.7. Let $\theta = \varphi(\sigma)$. Consider the angle β between $\tilde{c}'(\theta)$ and $c'(\theta)$. Again by Remark 6.7, we have $|\tilde{c}'(\theta) - c'(\theta)| \leq \frac{1}{2}|c'(\theta)|$. This inequality implies

$$\beta \leq \arcsin(1/2) \leq \pi/6.$$

Since c has unit speed, we obtain that $|\kappa| = |\phi'|$, where ϕ is the angle function as in Definition 2.5. For the angle γ between $c'(\sigma)$ and $c'(\varphi(\sigma))$ we have

$$|\gamma| = |\phi(\sigma) - \phi(\theta)| = \Big|\int_\theta^\sigma \phi'(s)\,ds\Big| = \Big|\int_\theta^\sigma \kappa(s)\,ds\Big| \leq |\sigma - \theta|\max|\kappa|.$$

By assumption $\mathrm{len}(c)_{|[\sigma,\varphi(\sigma)]} \leq \delta_c$ and $|c'| = 1$, hence $|\sigma - \theta| \leq \delta_c \leq 4\tau \leq \tau_c$. This yields

$$|\sigma - \theta|\max|\kappa| \leq \tau_c \max|\kappa| = \frac{1}{2\max|\kappa|}\max|\kappa| = \frac{1}{2}.$$

So we have shown that $\gamma \leq 1/2$. Combining the two estimates, we obtain for the angle between $c'(\sigma)$ and $\tilde{c}'(\varphi(\sigma))$

$$\beta + \gamma \leq \frac{\pi}{6} + \frac{1}{2} < \frac{\pi}{2}.$$

Next, we compute the derivative of $\tilde{c}(\sigma) = c(\varphi(\sigma)) + a(\sigma)N(\varphi(\sigma))$ and recall that $|c'| = 1$. Then

$$\begin{aligned}
\tilde{c}'(\sigma) &= c'(\varphi(\sigma))\varphi'(\sigma) + a'(\sigma)N(\varphi(\sigma)) + a(\sigma)N'(\varphi(\sigma))\varphi'(\sigma)\\
&= V\varphi'(\sigma) + a'(\sigma)N + a(\sigma)(-\kappa V)\varphi'(\sigma)\\
&= V\varphi'(\sigma)(1 - \kappa a) + a'N,
\end{aligned}$$

where V, N and κ are evaluated at $\varphi(\sigma)$. Now taking the scalar product on both sides with V and using the fact that $N \perp V$, then we get

$$\tilde{c}' \cdot V = \varphi'(\sigma)(1 - \kappa a).$$

Note that $\tilde{c}' \cdot V > 0$ and hence $\varphi' > 0$. By the assumption of this lemma we have

$$|\tilde{c}'(\sigma)| - 1 = |\tilde{c}'(\sigma)| - |c'(\sigma)| \leq |\tilde{c}'(\sigma) - c'(\sigma)| \leq \frac{1}{2}|c'(\sigma)| = \frac{1}{2}.$$

Moreover, we have

$$1 - |\tilde{c}'(\sigma)| = |c'(\sigma)| - |\tilde{c}'(\sigma)| \leq |c'(\sigma) - \tilde{c}'(\sigma)| \leq \frac{1}{2}|c'(\sigma)| = \frac{1}{2}.$$

Consequently, we get

$$\frac{1}{2} \leq |\tilde{c}'| \leq \frac{3}{2}.$$

For $|a| \leq \tau_c$ we observe as in the proof of Lemma 6.3

$$\frac{1}{2} \leq (1 - \kappa a) \leq \frac{3}{2}.$$

Now all this together yields

$$\varphi'(\sigma) \geq \frac{\tilde{c}'(\theta) \cdot V}{3/2} = \frac{\tilde{c}'(\theta) \cdot c'(\sigma)}{3/2} = \frac{\cos(\beta + \gamma) \cdot |\tilde{c}'| \cdot |c'|}{3/2} \geq \frac{6/10 \cdot 1/2 \cdot 1}{3/2} = \frac{1}{5}$$

and

$$\varphi'(\sigma) \leq \frac{\tilde{c}'(\theta) \cdot V}{1/2} = \frac{\tilde{c}'(\theta) \cdot c'(\sigma)}{1/2} = \frac{\cos(\beta + \gamma) \cdot |\tilde{c}'| \cdot |c'|}{1/2} \leq \frac{1 \cdot 3/2 \cdot 1}{1/2} = 3.$$

Note that we have proven the estimates for a reparameterized $\tilde{\varphi} = \phi^{-1} \circ \varphi \circ \phi$, as described in Remark 6.7. But since $\tilde{\varphi}$ is a diffeomorphism, this implies that the original function φ from Proposition 6.6 is a diffeomorphism as well. □

We now have all the results necessary to formulate a theorem concerning the representation of $\tilde{c}$ in tubular coordinates around c.

Theorem 6.9 (*Representation Theorem*)**.** Let $c : S^1 \to \mathbb{R}^2$ be in C^R and $\tilde{c} : S^1 \to \mathbb{R}^2$ be in C^{R-1}, with $R \geq 2$. Assume that

$$|\tilde{c}(\sigma) - c(\sigma)| \leq \tau$$

and

$$|\tilde{c}'(\sigma) - c'(\sigma)| \leq \frac{1}{2}|c'(\sigma)|$$

for all $\sigma \in S^1$ and fixed $\tau \leq \tau_c/4$. Then there exists a choice of $a : S^1 \to [-\tau, \tau]$ and $\varphi \in \mathrm{Diff}^+$ such that

$$\tilde{c}(\varphi(\sigma)) = \Psi(\sigma, a(\sigma)) = c(\sigma) + a(\sigma)N_c(\sigma), \quad \forall \sigma \in S^1$$

with $|a(\sigma)| \leq \tau$ and

$$\mathrm{len}(c)_{|\,[\sigma,\varphi(\sigma)]} \leq 4\tau, \quad \forall \sigma \in S^1.$$

Moreover, they are unique in the class of C^{R-1} functions such that $|a| \leq \tau_c$ and

$$\mathrm{len}(c)_{|\,[\sigma,\varphi(\sigma)]} \leq \delta_c.$$

Proof: Taking into account Remark 6.7, we can rescale and and reparameterize c and $\tilde{c}$ at the same time and the assumptions of this theorem still hold. The proof then follows immediately from Propositon 6.6 and Lemma 6.8. □

After establishing the conditions that allow us to express $\tilde{c}$ in terms of tubular coordinates around c, we can now define two kinds of neighborhoods of the curve c. To do so, we need to look at the topology on the manifold of immersions. There are two ways on how to identify open neighborhoods around an immersed curve c:

1. The first one is the usual way for Banach spaces where a neighborhood U_{c,ϵ_1} around c includes all $\tilde{c}$ such that the distance between c and $\tilde{c}$ in the norm which comes with the Banach space is smaller than ϵ_1, i.e.,

$$U = U_{c,\epsilon_1} = \left\{\tilde{c} : S^1 \to \mathbb{R}^2 : ||c - \tilde{c}||_{C^R} < \epsilon_1\right\}$$

   where $\epsilon_1 > 0$ determines the size of the neighborhood and $||.||_{C^R}$ is given by

$$||c||_{C^R} := \max_{\sigma \in S^1} |c(\sigma)| + |c'(\sigma)| + ... + |c^{(R)}(\sigma)|.$$

2. In the "geometric way" a neighborhood V_{c,ϵ_2} around c includes all $\tilde{c}$ such that we can write $\tilde{c}$ in tubular coordinates around c, i.e.,

$$\tilde{c}(\sigma) = \Psi(\varphi(\sigma), a(\sigma)) = c(\varphi(\sigma)) + a(\sigma)N(\varphi(\sigma)),$$

   with $a : S^1 \to \mathbb{R}$ and $\varphi \in \text{Diff}$ such that

$$||a||_{C^R} < \epsilon_2$$
$$||\varphi - \text{Id}||_{C^R} < \epsilon_2,$$

   where

$$||\varphi - \text{Id}||_{C^R} = \max_{\sigma \in S^1} d_{S^1}(\varphi(\sigma), \sigma) + |\varphi'(\sigma) - 1| + ... + |\varphi^{(R)}(\sigma) - 1|.$$

Note that these two neighborhoods are equivalent for $c \in C^{R+1}$ in the following sense: For any neighborhood U_{c,ϵ_1} we can find a neighborhood V_{c,ϵ_2} such that $V_{c,\epsilon_2} \subseteq U_{c,\epsilon_1}$. Conversely, for each V_{c,ϵ_2} there exists a neighborhood U_{c,ϵ_1} such that $U_{c,\epsilon_1} \subseteq V_{c,\epsilon_2}$. The fact that the equivalence of C^R-neighborhoods only holds for curves in C^{R+1} justifies the study of the manifold of smooth immersions.

6.2 Neighborhoods of free Immersions

So far, we have focused on parameterized curves where each point on the curve corresponds to a unique point on the unit circle. This correspondence provided a condition, to be able to represent curves in tubular coordinates in a unique way.

However, for geometric curves, we do not a priori have this correspondence, since we "reduce" curves to their image and disregard the parameterization. Nevertheless, the following lemma will show that for a freely immersed curve c we can always find a neighborhood around c such that for $\tilde{c}$ in this neighborhood every reparameterization of $\tilde{c}$ is given by the same tubular coordinates. Consequently, even for geometric curves we can find a unique representation in tubular coordinates, if c is freely immersed.

Lemma 6.10 (*Local Injectivity*)**.** Let c be a free immersion which is in C^2. Then there exists a $r = r_c > 0$ such that if

$$\begin{aligned} \tilde{c}(s) &:= c(s) + a(s)N(s), \\ \tilde{c}(\varphi(s)) &= c(s) + b(s)N(s), \end{aligned}$$

with

$$\begin{aligned} \|a\|_\infty &\leq r, \\ \|b\|_\infty &\leq r, \\ \left\|\frac{\partial a}{\partial c}\right\|_\infty &\leq \frac{1}{2}, \\ \left\|\frac{\partial a}{\partial c}\right\|_\infty &\leq \frac{1}{2}, \end{aligned}$$

where $(\partial/\partial c)$ denotes the arc derivative, then $a \equiv b$ and $\varphi = Id_{S^1}$.

Note that here, $\varphi(s)$ is not the same function as in Theorem 6.9. It simply denotes another reparameterization of $\tilde{c}$.

Proof: With no loss of generality we can parameterize c by arc parameter. If we rescale r by the same factor, then the hypotheses remain unchanged.

First, we start with an estimate for the derivative of φ. Unfortunately, we cannot use the estimate of Lemma 6.8 as the assumption is not satisfied. However, we have here the similar condition $|a'| \leq 1/2$. Consider

$$\tilde{c}(s) = c(s) + a(s)N(s).$$

The derivative is given by

$$\tilde{c}' = T(1 - \kappa a) + a'N.$$

From the previous proofs we know that for $|a| \leq \tau_c$ we obtain

$$\frac{1}{2} \leq (1 - \kappa a) \leq \frac{3}{2}.$$

Then together with $|a'| \leq 1/2$ we get

$$\frac{1}{2} \leq |\tilde{c}'| \leq \left|\frac{3}{2} + \frac{1}{2}\right| = \sqrt{\frac{9}{4} + \frac{1}{4}} \leq 2.$$

Similarly, for

$$\tilde{c}(\varphi(s)) = c(s) + b(s)N(s)$$

with $|b| \leq \tau_c$ and $|b'| \leq 1/2$ we have

$$\frac{1}{2} \leq |\tilde{c}'|\varphi' \leq 2.$$

Together with the estimate for $|\tilde{c}'|$ we get

$$\frac{1}{4} \leq \varphi' \leq 4.$$

Next, we assume by contradiction that there exist a sequence $(\varphi_n) \neq Id_{S^1}$ and sequences $(\tilde{c}_n)$, (a_n) and (b_n) such that

$$\begin{aligned} \tilde{c}_n(s) &= c(s) + a_n(s)N(s) \\ \tilde{c}_n(\varphi_n(s)) &= c(s) + b_n(s)N(s) \end{aligned}$$

where

$$\begin{aligned} \|a_n\|_\infty &\leq \frac{1}{n}, \\ \|b_n\|_\infty &\leq \frac{1}{n}, \\ \|a_n'\|_\infty &\leq \frac{1}{2}, \\ \|b_n'\|_\infty &\leq \frac{1}{2}. \end{aligned}$$

If $1/n < \tau_c/4$ and $\operatorname{len} \tilde{c}_{n|[s_n,\varphi_n(s_n)]} \leq \delta_c$, then we could write $\tilde{c}_n$ in a unique way in tubular coordinates around c. But since φ_n is not the identity, the uniqueness is obviously violated and one of the uniqueness conditions is not fulfilled. If we choose n large enough, then $1/n < \tau_c/4$ is satisfied and hence there must exist a s_n such that

$$\operatorname{len} \tilde{c}_{n|[s_n,\varphi_n(s_n)]} \geq \delta_c.$$

Now set $\varphi_n(s_n) = \theta_n$. Then by Lemma 2.10, we have

$$M d_{S^1}(\varphi_n^{-1}(\theta_n), \theta_n) = M d_{S^1}(s_n, \varphi_n(s_n)) \geq \operatorname{len} \tilde{c}_{n|[s_n,\varphi_n(s_n)]} \geq \delta_c,$$

with $M = \max|\tilde{c}'|$. This yields

$$\liminf_{n\to\infty} d_{S^1}(\varphi_n^{-1}(\theta_n), \theta_n) > 0.$$

Since S^1 is compact, every sequence has a convergent subsequence. So up to a subsequence we have $\theta_n \to \tilde{\theta}$. Moreover, we have by the earlier arguments that

$$\frac{1}{4} \leq \varphi_n' \leq 4.$$

The boundedness of φ_n' and the theorem of Arzelà-Ascoli imply that up to a subsequence $\varphi_n \to \varphi$ and $\varphi_n^{-1} \to \varphi^{-1}$ uniformly, so that φ is a bi-Lipschitz homeomorphism. Since the distance function and φ_n are continuous, we obtain

$$\lim_{n\to\infty} d_{S^1}(\varphi_n^{-1}(\theta_n), \theta_n) = d_{S^1}(\varphi^{-1}(\tilde{\theta}), \tilde{\theta}) =: \tilde{d} > 0.$$

Furthermore, since $a_n \to 0$ and $b_n \to 0$, we get $\tilde{c}_n \to c$ and $\tilde{c}_n(\varphi_n) \to c$. This yields

$$\tilde{c} = \tilde{c} \circ \varphi,$$

which implies that $a = b$. Moreover, we have $\tilde{c}_n(\varphi_n) \to c(\varphi)$ and hence

$$c = c \circ \varphi.$$

By assumption c is a free immersion, which implies that $\varphi = Id_{S^1}$. But this cannot be true, since the distance between $\varphi^{-1}(\tilde{\theta})$ and $\tilde{\theta}$ is strictly positive. Thus we have shown a contradiction, so that the assumption $(\varphi_n) \neq Id_{S^1}$ was false. □

Theorem 6.11. Free immersions are an open subset of immersions.

Proof: This is a simple consequence of the previous lemma. Let $R \geq 2$ and V_{c,ϵ_2} be a tubular neighborhood around c, with $\epsilon_2 < \min\{r_c, 1/2, \tau_c/4\}$ and r_c from Lemma 6.10. Let $\varphi \in \text{Diff}$ be arbitrary and let $\tilde{c}$ and $\tilde{c} \circ \varphi$ be defined as in Lemma 6.10. By the choice of ϵ_2, $\tilde{c}$ is contained in V_{c,ϵ_2}. If $\tilde{c}(s) = \tilde{c}(\varphi(s))$, then we know by Lemma 6.10 that $a = b$ and φ is the identity. Hence any curve $\tilde{c} \in V_{c,\epsilon_2}$ is freely immersed, if c is freely immersed. As mentioned earlier, tubular neighborhoods V_{c,ϵ_2} and neighborhoods U_{c,ϵ_1} induced by the Banach topology of the manifold of smooth immersions are equivalent. Hence, the theorem is proven for both kinds of neighborhoods. □

6.3 The Manifold structure

In this subsection we will finally give the proof of Theorem 6.1 which requires us to show that Imm_f/Diff satisfies the three properties of a manifold.

Definition 6.12. (*Infinite-dimensional manifold*, taken from [5, Def. 1.1]). A smooth manifold modelled on the topological vector space E is a Hausdorff topological space M together with a family of charts $(u_\alpha, U_\alpha)_{\alpha\in A}$ such that

1. $U_\alpha \subseteq M$ are open sets and $\bigcup_{\alpha\in A} U_\alpha = M$.
2. $u_\alpha : U_\alpha \to u_\alpha(U_\alpha) \subseteq E$ are homeomorphisms onto open sets $u_\alpha(U_\alpha)$.
3. $u_\beta \circ u_\alpha^{-1} : u_\alpha(U_\alpha \cap U_\beta) \to u_\beta(U_\alpha \cap U_\beta)$ are C^∞-smooth for all $\alpha, \beta \in A$.

So far, we haven shown that free immersions are open in the space of immersions. For proving the first property of a manifold, we need to find open sets in the quotient topology that cover the quotient space.

Proposition 6.13. Fix a free immersion c_1 and let $\tau \leq \min\{r_{c_1}, \tau_{c_1}/4, 1/2\}$. Then the set

$$\mathcal{U}_{c_1} := \left\{\tilde{c} \in \mathrm{Imm} : |\tilde{c} - c_1| \leq \tau, |\tilde{c}' - c_1'| < \frac{|c_1'|}{3}\right\}$$

is open in C^∞.

Proof: Let $c_2 \in \mathcal{U}_{c_1}$ and define

$$\alpha := \|c_2 - c_1\|_\infty$$
$$\beta := \left\|\frac{\partial}{\partial c_1} c_2 - \frac{\partial}{\partial c_1} c_1\right\|_\infty .$$

By the definition of $\mathcal{U}_{c_1}$ we have $\alpha < \tau$ and $\beta < 1/3$. Consider a smooth curve c_3 which satisfies

$$\|c_3 - c_2\|_\infty < (\tau - \alpha),$$
$$\|c_3' - c_2'\|_\infty < (\frac{1}{3} - \beta)\, m,$$

with $m = \min|c_1'|$. Then we obtain

$$\|c_3 - c_1\|_\infty \leq \|c_3 - c_2\|_\infty + \|c_2 - c_1\|_\infty < (\tau - \alpha) + \alpha = \tau$$
$$\|c_3' - c_1'\|_\infty \leq \|c_3' - c_2'\|_\infty + \|c_2' - c_3'\|_\infty < (\frac{1}{3} - \beta)\, m + \beta = \frac{1}{3} m.$$

Hence, we have $c_3 \in \mathcal{U}_{c_1}$. So for every $c_2 \in \mathcal{U}_{c_1}$ we can find a c_3 close to c_2 such that c_3 is also contained in that set. □

By Theorem 6.9 we know that $\mathcal{U}_{c_1}$ contains all curves $\tilde{c}$ which can be expressed in tubular coordinates around c_1. Moreover, we have seen in the proof of Theorem 6.11 that every curve in $\mathcal{U}_{c_1}$ is a free immersion. Now, consider the set of all curves in $\mathcal{U}_{c_1}$ up to reparameterization, that is

$$\mathcal{W}_{c_1} := \left\{\tilde{c} \circ \varphi : |\tilde{c} - c_1| < \tau, |\tilde{c}' - c_1'| < \frac{|c_1'|}{3}, \varphi \in \mathrm{Diff}\right\}.$$

As already mentioned, the above conditions are invariant under reparameterization. So $\mathcal{W}_{c_1}$ can be written as the union of open sets

$$\mathcal{W}_{c_1} = \bigcup_{c_2 = c_1 \circ \varphi, \varphi \in \mathrm{Diff}} \mathcal{U}_{c_2}.$$

Hence, $\mathcal{W}_{c_1}$ is an open set in C^∞ as well. Now let $\pi : \mathrm{Imm}_f \to \mathrm{Imm}_f/\mathrm{Diff}$ be the canonical projection. Then set

$$\widetilde{\mathcal{W}}_{c_1} := \pi(\mathcal{W}_{c_1}).$$

By the definition of the quotient topology, a set $\mathcal{Z}$ is open if the union of its orbits

$$\pi^{-1}(\mathcal{Z}) = \{c \in \mathrm{Imm}_f : [c] \in \mathcal{Z}\} = \bigcup_{[c] \in \mathcal{Z}} [c]$$

is open in Imm_f, respectively in C^∞. Moreover, we have

$$\pi^{-1}(\widetilde{\mathcal{W}}_{c_1}) = \mathcal{W}_{c_1} = \bigcup_{c_2 = c_1 \circ \varphi, \varphi \in \mathrm{Diff}} \mathcal{U}_{c_2}$$

Hence $\widetilde{\mathcal{W}}_{c_1}$ is open in the quotient topology and

$$\bigcup_{c_1 \in \mathrm{Imm}_f} \widetilde{\mathcal{W}}_{c_1} = \mathrm{Imm}_f/\mathrm{Diff}.$$

This proves the first property of the definition of a manifold. For the second one we need to define a vector space E. The motivation for this space can be seen as follows:

Consider a curve $\overline{c} = \tilde{c} \circ \varphi \in \mathcal{W}_{c_1}$. Then by Theorem 6.9 we can write $\tilde{c}$ in tubular coordinates around c_1

$$\tilde{c} \circ \phi = c_1 + aN,$$

with $|a| \leq \tau$. So for $\overline{c}$ we have the following reparameterization in tubular coordinates

$$\overline{c} \circ \underbrace{\varphi^{-1} \circ \phi}_{=:\psi_1} = c_1 + aN.$$

We want to show that this reparameterization is unique. We know that ϕ is unique by the representation theorem, but the representation of $\bar{c}$ is not, i.e., there are lots of pairs $(\tilde{c}, \varphi)$ such that $\overline{c} = \tilde{c} \circ \varphi$ with $\tilde{c} \in \mathcal{U}_{c_1}$ and $\varphi \in \mathrm{Diff}$. So assume that there exists another reparameterization of $\overline{c}$ in tubular coordinates

$$\overline{c} \circ \psi_2 = c_1 + bN,$$

with $|b| \leq \tau$. Define $\hat{c} := \overline{c} \circ \varphi^{-1} \circ \phi = \overline{c} \circ \psi_1$. Then we have

$$\begin{aligned} \hat{c} &= c_1 + aN, \\ \hat{c} \circ \psi_1^{-1} \circ \psi_2 = \overline{c} \circ \psi_2 &= c_1 + bN. \end{aligned}$$

Since we have chosen τ small enough, we can apply Lemma 6.10. This gives us $\psi_1^{-1} \circ \psi_2 = \mathrm{Id}$. Hence, $\psi_1 = \psi_2$. So we have shown that the reparameterization is uniquely identified for curves in $\mathcal{W}_{c_1}$. Consequently, we will concentrate on a for the space E.

Proposition 6.14. Define

$$Q_{c_1} := \{a : S^1 \to \mathbb{R} : \exists\, \tilde{c} \in \mathcal{U}_{c_1}, \exists\, \varphi \in \mathrm{Diff}, \tilde{c} \circ \varphi = c_1 + aN\}.$$

Then Q_{c_1} is open.

Proof: Consider the map

$$(\varphi, a) \mapsto (c_1 + aN) \circ \varphi^{-1},$$

which splits the set $\mathcal{W}_{c_1}$ smoothly as

$$\mathcal{W}_{c_1} \cong Q_{c_1} \times \mathrm{Diff}.$$

Clearly, this map is continuous, so the preimage of an open set is open. Since $\mathcal{W}_{c_1}$ is open and Q_{c_1} is simply the projection on the second component of the preimage of $\mathcal{W}_{c_1}$, we have that Q_{c_1} is open as well. □

Now define the vector space E as follows:

$$E := \bigcup_{c_1 \in \mathrm{Imm}_f} Q_{c_1}.$$

Next, we need to define the charts u_α. Therefore, consider the map

$$\Psi_{c_1} : Q_{c_1} \to \mathcal{W}_{c_1},$$
$$\Psi_{c_1}(a) := c_1 + aN_1.$$

Then the composition with the canonical projection

$$\tilde{\Psi}_{c_1} := \pi \circ \Psi_{c_1} : Q_{c_1} \to \widetilde{\mathcal{W}}_{c_1}$$

is a bijective map, since we have already proven in Lemma 6.10, that Ψ_{c_1} is injective and the surjectivity can be seen as follows: Let $[\tilde{c}] \in \widetilde{\mathcal{W}}_{c_1}$, then

$$\tilde{c} \in \mathcal{U}_{c_1 \circ \varphi}$$

for a unique φ and hence

$$\tilde{c} \circ \varphi^{-1} = c_1 + aN_1$$

for a unique a. So every a gets hit exactly one time. Moreover, $\tilde{\Psi}_{c_1}$ is a smooth mapping and hence a homeomorphism. Now set $u_\alpha := \tilde{\Psi}_{c_1}^{-1}$. We have already seen in Proposition 6.14 that

$$\tilde{\Psi}_{c_1}^{-1}(\widetilde{\mathcal{W}}_{c_1}) = Q_{c_1} \subseteq E$$

is open. Thus the second condition for a manifold structure is satisfied.

For the last condition we consider the atlas given by the charts $(\tilde{\Psi}_{c_1}^{-1}, \widetilde{\mathcal{W}}_{c_1})_{c_1 \in \mathrm{Imm}_f}$. It must be compatible, meaning that any two charts in the atlas must agree on the overlaps of their domains. So let $[\tilde{c}]$ be in the overlap of two charts, i.e.

$$[\tilde{c}] \in \widetilde{\mathcal{W}}_{c_1} \cap \widetilde{\mathcal{W}}_{c_2}.$$

Then by Theorem 6.9 we can write $\tilde{c}$ as

$$\tilde{c}(\psi_1) = c_1(\theta) + \tilde{a}_1(\theta)N_{c_1}(\theta),$$
$$\tilde{c}(\psi_2) = c_2(\theta) + \tilde{a}_2(\theta)N_{c_2}(\theta),$$

where $\psi_i := \varphi_i^{-1} \circ \phi_i$ as described earlier. We can reparameterize c_1 and c_2 to get

$$\tilde{c}(\theta) = c_1(\theta) + \tilde{a}_1(\theta)N_{c_1}(\theta),$$
$$\tilde{c}(\theta) = c_2(\theta) + \tilde{a}_2(\theta)N_{c_2}(\theta).$$

Now we need to show that the map

$$\begin{aligned} \tilde{\Psi}_{c_2}^{-1} \circ \tilde{\Psi}_{c_1} : \tilde{\Psi}_{c_1}^{-1}(\widetilde{\mathcal{W}}_{c_1} \cap \widetilde{\mathcal{W}}_{c_2}) &\to \tilde{\Psi}_{c_2}^{-1}(\widetilde{\mathcal{W}}_{c_2} \cap \widetilde{\mathcal{W}}_{c_1}) \\ \tilde{a}_1 &\mapsto \tilde{a}_2 \end{aligned}$$

is smooth in a neighborhood of $\tilde{a}_1$. Since $\widetilde{\mathcal{W}}_{c_1}$ and $\widetilde{\mathcal{W}}_{c_2}$ are open, there exists a neighborhood around $[\tilde{c}]$ which is contained in both sets. Choose a_1 sufficiently close to $\tilde{a}_1$ such that

$$c_0(\theta) := c_1(\theta) + a_1(\theta)N_{c_1}(\theta)$$

lies within this neighborhood, i.e., $c_0 \in \mathcal{W}_{c_2}$. Again, we obtain by Theorem 6.9 the existence of ψ and a_2 depending on a_1 such that

$$c_0(\theta) = c_1(\theta) + a_1(\theta)N_{c_1}(\theta) = c_2(\psi(\theta)) + a_2(\psi(\theta))N_{c_2}(\psi(\theta)).$$

The proof of the representation Theorem (precisely the proof of Lemma 6.5 where we used the inverse of the tubular coordinates) demonstrates the smooth dependence of a_2 on a_1. This proves the third property for $\mathrm{Imm}_f/\mathrm{Diff}$ being a manifold. $\square$

7 Horizontal Lifting of Geodesics

In section 5 we have seen that we can project geodesics on the space of free immersions to the quotient space with respect to the Diff-action. The goal of this section is to show that these are projections of *horizontal* geodesics on Imm_f. To this end, we use the fact that $\mathrm{Imm}_f/\mathrm{Diff}$ admits a manifold structure. This can be reformulated in the framework of principal fiber bundles, which was done in [8, Thm. 1.5]:

> « *Let i be a free immersion $M \to N$. Then there is an open neighborhood $W(i)$ in* $\mathrm{Imm}(M,N)$ *which is saturated for the* $\mathrm{Diff}(M)$*-action and which splits smoothly as*
>
> $$W(i) = Q(i) \times \mathrm{Diff}(M).$$
>
> *Here $Q(i)$ is a smooth splitting submanifold of* $\mathrm{Imm}(M,N)$*, diffeomorphic to an open neighborhood of 0 in $C^\infty(N(i))$. In particular the space of* $\mathrm{Imm}_{\mathrm{free}}(M,N)$ *is open in $C^\infty(M,N)$. Let π :* $\mathrm{Imm}(M,N) \to \mathrm{Imm}(M,N/)\mathrm{Diff} =$ *$B(M,N)$ be the projection onto the orbit space, which we equip with the quotient topology. Then $\pi|Q(i) : Q(i) \to \pi(Q(i))$ is bijective onto an open subset of the quotient. If i runs trough* $\mathrm{Imm}_{\mathrm{free,prop}}(M,N)$ *of all free and proper immersions these mappings define a smooth atlas for the quotient space, so that*
>
> $$(\mathrm{Imm}_{\text{free, proper}}(M,N), \pi, \mathrm{Imm}_{\text{free, proper}}(M,N)/\mathrm{Diff}(M), \mathrm{Diff}(M))$$
>
> *is a smooth principal fiber bundle with structure group* $\mathrm{Diff}(M)$.»

In particular, we have already proven this statement in Section 6, we just need to express the results in terms of bundles. Therefore, we start with a few definitions on that topic, which are mainly taken from [9, Sec. 1].

7.1 Fiber Bundles

Definition 7.1. (*Bundle*). A *bundle* is a triplet (E, π, B) where E, B are sets and π is a map from E to B. We call E the *total space*, B the *base space* and π the *projection*.

This is the simplest case of a bundle. Normally, we consider bundles with an additional structure, as the *fiber bundle*.

Definition 7.2. (*Fiber Bundle*). A *fiber bundle* (E, π, B, F) consists of manifolds E, B, F and a smooth mapping $\pi : E \to B$ which has to satisfy the following triviality condition. For each $y \in B$ there exists an open neighborhood $U \subseteq B$ such that there is a homeomorphism $\psi : E|U := \pi^{-1}(U) \to U \times F$ such that π agrees with the projection onto the fist factor. In particular the following diagram should commute:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ & U & \end{tikzcd}$$

Here, $\mathrm{proj}_1 : U \times F \to U$ denotes the naturally projection onto the fist factor. If $(U_\alpha)_{\alpha\in A}$ is an open cover of B, then we call the set of all fiber charts $(U_\alpha, \psi_\alpha)_{\alpha\in A}$ a *fiber bundle atlas*. The *fiber* of a point $y \in B$ is given by its preimage $\pi^{-1}(y)$.

Definition 7.3. (*Tangent Bundle*). For every smooth manifold M we define the *tangent bundle* TM as the union of all tangent spaces T_xM at every point $x \in M$. Since the tangent space T_xM consists of all tangent vectors to M at x, the tangent bundle is the collection of all tangent vectors, along with the information of the point to which they are tangent, that is

$$TM = \bigsqcup_{x\in M} T_xM = \bigcup_{x\in M} \{x\} \times T_xM = \{(x, v) : x \in M, v \in T_xM\}.$$

So each element of the tangent bundle is given by (x, v) where x describes the point in the manifold M and v denotes a tangent vector at this point. Moreover, speaking in terms of bundles, as defined above, we have that the triplet (TM, π, M) where $\pi : TM \to M$ is the natural projection defined as $\pi(x, v) = x$, is the tangent bundle over M. Here, the fiber of a point $x \in M$ is given by its tangent space T_xM.

Definition 7.4. (*Normal Bundle*). Consider a Riemannian manifold M with a Riemannian metric g. Let $S \subset M$ be a Riemannian submanifold. The total space of the *normal bundle* to S is defined just as above for the tangent space:

$$NS := \bigsqcup_{p\in S} N_pS = \{(p, n) : p \in S, n \in N_pS\},$$

where N_pS denotes the normal space to S at p which is given by

$$N_pS := \{n \in T_pM : g(n, v) = 0 \ \forall v \in T_pS\}.$$

Hence we can identify the normal bundle as the orthogonal complement of the tangent bundle, $NS = (TS)^\perp$.

In general, we can define the normal bundle for an immersion $i : N \to M$ as the quotient space of the tangent space on M by the tangent space on N. Here N, M do not have to be Riemannian.

Definition 7.5. (*Principal Fiber Bundles*). Let G be a Lie group, E, B manifolds and $\pi : E \to B$. The tuple (E, π, B, G) is called a *G-principal fiber bundle* over B if

1. G acts on E from the right as a Lie transformation group.

2. There exists a bundle atlas $(U_\alpha, \psi_\alpha)_{\alpha\in A}$ consisting of a G-equivariant bundle charts, i.e.

a) $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is a diffeomorphism.

b) $\mathrm{proj}_1 \circ \psi_\alpha = \pi$.

c) $\psi_\alpha(p \cdot g) = \psi_\alpha(p) \cdot g$ for all $p \in \pi^{-1}(U_\alpha)$ and $g \in G$, where G acts on $U_\alpha \times G$ via $(x, a) \cdot g = (x, a \cdot g)$.

7.2 Application to the Space of free Immersions

Now that we have formally defined a principal fiber bundle, we can apply the results from Section 6 and show that $(\mathrm{Imm}_f, \pi, \mathrm{Imm}_f/\mathrm{Diff}, \mathrm{Diff})$ is a principal fiber bundle with structure group Diff. We remark that in the proof of [8] they were missing the two conditions about the size of the neigborhoods ($|\tilde{c}' - c_1'| < |c_1'|/3$ and $\tau \leq \tau_c$). However, we are not going to discuss this since we are more interested in the reformulation of $\mathrm{Imm}_f/\mathrm{Diff}$ being a manifold than showing the differences of the proofs. Moreover, note that in [8] they proved the theorem for immersions $i : M \to N$, where M, N are general finite dimensional manifolds. Since the focus of this work are immersions $c : S^1 \to c(S^1) \subseteq \mathbb{R}^2$, we set $M = S^1$ and $N = c(S^1)$.

We start with the normal bundle of an immersion $c : S^1 \to c(S^1) \subseteq \mathbb{R}^2$. We know that the set of normal vectors of $c(\theta)$ is given by

$$N_{c(\theta)}c(S^1) = \{tN_{c(\theta)} : t \in \mathbb{R}\},$$

where $N_{c(\theta)}$ is defined as in Definition 2.4. Then the tangent bundle $Nc(S^1)$ is provided by

$$Nc(S^1) = \{(\theta, t) \in S^1 \times \mathbb{R} : tN_{c(\theta)} \perp v \ \forall v \in T_{c(\theta)}c(S^1)\}.$$

Hence, the fiber F of the normal bundle is just given by $\mathbb{R}$ for planar immersions.

Now, consider the definition of a G-principal fiber bundle. Set $E = \mathrm{Imm}_f$, $B = \mathrm{Imm}_f/\mathrm{Diff}$, $G = \mathrm{Diff}$ and let π be the canonical projection on the quotient space. Note that by [23, Sec. 1.3.2] one can define a smooth structure on the space of diffeomorphisms such that it becomes a Lie group. As described in the previous section the map

$$\Phi_{c_1}(\varphi, a) = (c_1 + aN) \circ \varphi^{-1}$$

splits open sets W_{c_1} diffeomorphically into

$$W_{c_1} \cong Q_{c_1} \times \mathrm{Diff}.$$

Now let $[c_1] \in \mathrm{Imm}_f/\mathrm{Diff}$. Then there exists an open neighborhood $\widetilde{\mathcal{W}}_{c_1}$ around $[c_1]$ such that $\mathrm{Imm}_f\big|\widetilde{\mathcal{W}}_{c_1} := \pi^{-1}(\widetilde{\mathcal{W}}_{c_1}) = \mathcal{W}_{c_1}$ is diffeomorphic to $\widetilde{\mathcal{W}}_{c_1} \times \mathrm{Diff}$ via a fiber respecting diffeomorphism:

$$\begin{tikzcd}
\mathcal{W}_{c_1} \arrow[rr, "\hat{\tilde{\Psi}}_{c_1} \circ \Phi_{c_1}^{-1}"] \arrow[dr, "\pi"'] & & \widetilde{\mathcal{W}}_{c_1} \times \mathrm{Diff} \arrow[dl, "\mathrm{proj}_1"] \\
& \widetilde{\mathcal{W}}_{c_1} &
\end{tikzcd}$$

where $\hat{\tilde{\Psi}}_{c_1}(a, \varphi) := (\tilde{\Psi}_{c_1}(a), \varphi)$ with the diffeomorphism $\tilde{\Psi}_{c_1} : Q_{c_1} \to \widetilde{\mathcal{W}}_{c_1}$.

Moreover, it is obvious that $\Psi_{c_1}(a \circ f) = \Psi_{c_1}(a) \circ f$ with $a \in Q_{c_1}$ and $f \in \mathrm{Diff}$. Hence, $\tilde{\Psi}_{c_1}(a \circ f) = \tilde{\Psi}_{c_1}(a) \circ f$. Clearly, $\Phi_{c_1}^{-1}(c \circ f) = \Phi_{c_1}^{-1}(c) \circ f$. This proves that $(\mathrm{Imm}_f, \pi, \mathrm{Imm}_f/\mathrm{Diff}, \mathrm{Diff})$ satisfies all of the properties of a G-principal fiber bundle.

7.3 Riemannian Submersions and Connections

The following theory is presented for a general fiber bundle (E, π, B, F) and is principally based on [20, Sec. 17, 24, 26]. In the end of this section we will come back to the case where $E = \mathrm{Imm}_f$ and $B = \mathrm{Imm}_f/\mathrm{Diff}$. In order to define horizontal/vertical vectors in E, we need π to be a submersion, i.e. a function, whose differential $D\pi_p : T_pE \to T_{\pi(p)}B$ is surjective for each $p \in E$. This is true for every fiber bundle (E, π, B, F).

Lemma 7.6. The projection π of the fiber bundle (E, π, B, F) is a submersion.

Proof: By the definition of a fiber bundle, there exists a neighborhood $U \subseteq B$ and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times F$ such that $\pi|_{\pi^{-1}(U)} = \mathrm{proj}_1 \circ \psi$. Thus we need to show that

$$D\pi_p = D(\mathrm{proj}_1 \circ \psi)_p = D\,\mathrm{proj}_1(\psi_p) \circ D\psi_p$$

is a surjection for every $p \in E$. As ψ is a diffeomorphism, it is obviously surjectiv and $D\psi$ is an isomorphism and hence also a surjective map. Moreover, $D\,\mathrm{proj}_1$ is surjectiv, since the projection onto the first factor is a submersion. As the composition of two surjections is a surjection as well, we obtain that $D\pi_p$ is surjectiv. □

Definition 7.7. (*Vertical Bundle*). Let (E, π, B, F) be a fiber bundle. Consider the fiber linear tangent mapping $D\pi : TE \to TB$ which has full rank everywhere by the previous lemma. Then the *vertical bundle* $VE \to E$ is the subbundle of $TE \to E$ defined as

$$VE := \ker D\pi = \{v \in TE : D\pi v = 0\} \subseteq TE.$$

The fibers of the vertical bundle $V_xE \subseteq T_xE$ are called *vertical subspaces*. Since the vertical subspaces are the sets of all vectors in TE that are tangential to any fiber, we have $V_xE = T_x(E_{\pi(x)})$.

Definition 7.8. (*Connection*). A *connection* on the fiber bundle (E, π, B, F) is a vector valued 1-form $\Phi \in \Omega^1(E; VE)$ with values in the vertical bundle VE and such that $\Phi \circ \Phi = \Phi$ and $\mathrm{Im}\,\Phi = VE$. We obtain that Φ is just a projection $TE \to VE$. The kernel $\ker \Phi =: HE$ is a subvector bundle of TE and we call it the *horizontal bundle*. Obviously, we have $TE = HE \oplus VE$ and $T_xE = H_xE \oplus V_xE$. If E is a Riemannian manifold with a metric g, then we can identify the horizontal space H_xE as the orthogonal complement

$$H_xE = V_xE^{\perp} = \{v \in T_xE : g_x(v, w) = 0 \;\forall\; w \in T_xE \text{ with } D\pi(x)\,w = 0\}.$$

Assume that a connection Φ has been chosen. Then consider the mapping

$$(D\pi, p_E) : TE \to TM \times E,$$

where the second component is just the projection to E. Since VE is the kernel of $D\pi$, we obtain $(D\pi, p_E)^{-1}(0_{\pi(x)}, x) = V_xE$. Hence $(D\pi, p_E)$ restricts to a fiber linear isomorphism $(D\pi, p_E)|HE : HE \to TM \times E$. We call its inverse

$$C := ((D\pi, p_E)|HE)^{-1} : TM \times E \to HE \hookrightarrow TE$$

horizontal lift associated to the connection Φ.

For the case of a principal bundle (E, π, B, G) with structure group G, we want to ensure that the connection is "compatible" with the group action of G. More specifically, we say that a connection is a *principal connection* if it is G-equivariant for the right action $r : E \times G \to E$, i.e.

$$T(r^g).\Phi = \Phi.T(r^g)$$

where $T(r^g)$ is the tangent mapping of r for a fixed $g \in G$ and $T(r^g).\Phi$ is given by $(T_xr^g.\Phi_x)(h) = \Phi_x(h \circ r^g)$ for $x \in E$ and $h \in C^\infty(E)$. Additionally we require Φ to be r^g-related to itself, i.e.

$$Tr^g \circ \Phi = \Phi \circ r^g.$$

This means that the connection behaves nicely with respect to the group action, and allows us to define a notion of parallel transport on the bundle. The idea of parallel transport is to transport a vector along a curve in the base space B in such a way that the vector stays "parallel" to itself as it moves along the curve. In other words, the vector should not rotate or change direction as it moves along the curve.

Definition und Proposition 7.9. (*Parallel Transport*). Suppose we have a smooth principal bundle (E, π, B, G) with a regular Lie group as its structure group and a principal connection Φ. Then, the *parallel transport* associated with the connection Φ exists, is globally defined and G-equivariant. This means that for every smooth curve $c : \mathbb{R} \to B$, there is a unique smooth mapping $Pt_c : \mathbb{R} \times E_{c(0)} \to E$ that satisfies the following conditions:

1. $Pt(c, t, x) \in E_{c(t)}$, $Pt(c, 0) = \mathrm{Id}_{E_{c(0)}}$ and $\Phi(\frac{d}{dt}Pt(c, t, x)) = 0$.
2. $Pt(c, t) : E_{c(0)} \to E_{c(t)}$ is G-equivariant, i.e. $Pt(c, t, x.g) = Pt(c, t, x).g$ holds for all $g \in G$ and $x \in E$. Moreover, we have $Pt(c, t)^*(\zeta_X|E_{c(t)}) = \zeta_X|E_{c(0)}$ for all $X \in \mathfrak{g}$.
3. For any smooth function $f : \mathbb{R} \to \mathbb{R}$ we have $Pt(c, f(t), x) = Pt(c \circ f, t, Pt(c, f(0), x))$.

For the sake of completeness we stated all the three properties here, but actually we are only going to use the first one. For this reason we do not specify the fundamental vector field ζ_X and the Lie algebra $\mathfrak{g}$ in the second condition. For further information and a proof of this proposition see [20, Thm. 19.6.].

Theorem 7.10. (Taken from [20, Lemma 26.11]). Let (E, π, B, G) be a smooth G-principal bundle with principal connection $\Phi : TE \to VE$ and let $c : [0, 1] \to B$ be a geodesic. Then the following holds:

1. The length and energy of c are preserved, when lifting c horizontally:

$$L_0^t(c) = L_0^t(Pt^\Phi(c, \cdot, x)),$$
$$E_0^t(c) = E_0^t(Pt^\Phi(c, \cdot, x)),$$

   where $x \in E_{c(0)}$ denotes the starting point of the parallel transport.

2. The horizontal lift of a curve is orthogonal to its fibers:

$$Pt^\Phi(c, ., x) \perp E_{c(t)} \ \forall\, t \in [0, 1].$$

3. $t \mapsto Pt^\Phi(c, t, x)$ is a geodesic in E.

Proof: 1) Since π is a submersion by Lemma 7.6, we have that $g_E(x, y) = g_B(X, Y)$ for $\pi(x) = X$ and $\pi(y) = Y$. By the properties of the parallel transport we have $\Phi(\frac{d}{ds}Pt^\Phi(c, s, x)) = 0$, which implies that $\frac{d}{ds}Pt^\Phi(c, s, x) \in \ker \Phi$ and hence $\frac{d}{ds}Pt^\Phi(c, s, x)$ is a horizontal vector. Moreover, we know that $\pi(Pt^\Phi(c, s, x)) = c(s)$ and as $\frac{d}{ds}Pt^\Phi(c, s, x)$ is horizontal, we obtain by the horizontal lift C:

$$g_B\Big(c'(s), c'(s)\Big) = g_E\Big(\tfrac{d}{ds}Pt^\Phi(c, s, x), \tfrac{d}{ds}Pt^\Phi(c, s, x)\Big).$$

Now computing the length of c:

$$\begin{aligned} L_0^t(c) &= \int_0^t g_B\Big(c'(s), c'(s)\Big)^{\frac{1}{2}} ds \\ &= \int_0^t g_E\Big(\tfrac{d}{ds}Pt^\Phi(c, s, x), \tfrac{d}{ds}Pt^\Phi(c, s, x)\Big)^{\frac{1}{2}} ds = L_0^t(Pt^\Phi(c, ., x)). \end{aligned}$$

Similarly for the energy we have

$$\begin{aligned} E_0^t(c) &= \int_0^t g_B\Big(c'(s), c'(s)\Big) ds \\ &= \int_0^t g_E\Big(\tfrac{d}{ds}Pt^\Phi(c, s, x), \tfrac{d}{ds}Pt^\Phi(c, s, x)\Big) ds = E_0^t(Pt^\Phi(c, ., x)). \end{aligned}$$

2) This is due to our choice of Φ as the orthogonal projection onto the vertical bundle with respect to the Riemannian metric on E. The vertical bundle consists of all vectors that are tangential to any fiber the curve meets. Hence, this choice of Φ ensures that the parallel transport along c is

the only horizontal curve that covers c, and it remains orthogonal to any fiber $E_{c(t)}$.
3) Let $e : [0, 1] \to E$ be a (piecewise) smooth curve starting at $E_{c(0)}$ and ending at $E_{c(1)}$. Then the composition $\pi \circ e$ is a (piecewise) smooth curve that starts at $c(0)$ and ends at $c(1)$. Since c is a geodesic, we have $L_0^1(c) \leq L_0^1(\pi \circ e)$. Now we decompose the tangent vectors of e into their horizontal and vertical components. As the vertical vectors are not affected by $D\pi$, they can only increase the length. Moreover, the projection π is a submersion, hence $D\pi$ is an isometry on the horizontal vectors. Therefore, the length of the horizontal components of e is preserved under the projection π. All this together yields

$$L_0^1(e) = \int_0^1 |e'(t)^{\mathrm{ver}} + e'(t)^{\mathrm{hor}}|_{g_E}\, dt \geq \int_0^1 |e'(t)^{\mathrm{hor}}|_{g_E}\, dt = \int_0^1 |(\pi \circ e)'(t)|_{g_M}\, dt = L_0^1(\pi \circ e).$$

Now, by the previous observations we obtain

$$L_0^1(e) \geq L_0^1(\pi \circ e) \geq L_0^1(c) \overset{1)}{=} L_0^1(Pt^{\Phi}(c, ., x)).$$

Since e was an arbitrary curve connecting $E_{c(0)}$ and $E_{c(1)}$, we see that the infimum of the length functional over all curves from $E_{c(0)}$ and $E_{c(1)}$ gets attained in $Pt^{\Phi}(c, ., x)$, thus

$$L_0^1(Pt^{\Phi}(c, ., x)) = \mathrm{dist}\,(E_{c(0)}, E_{c(1)}).$$

Hence, $t \mapsto Pt^{\Phi}(c, t, x)$ is a geodesic in E, that connects $E_{c(0)}$ and $E_{c(1)}$. □

Now we come back to the case where $E = \mathrm{Imm}_f$ and $B = \mathrm{Imm}_f/\mathrm{Diff}$. In Section 7.2 we have seen that $(\mathrm{Imm}_f, \pi, \mathrm{Imm}_f/\mathrm{Diff}, \mathrm{Diff})$ is a G-principal fiber bundle. Hence, we may apply the above theorem, which shows that geodesics in $\mathrm{Imm}_f/\mathrm{Diff}$ can be lifted to horizontal geodesics on Imm_f, which was the main object of this section.

8 Conclusion and Outlook

This master's thesis focused on investigating the existence of horizontal geodesics in the shape space of unparameterized Sobolev immersions. To do so, we analysed properties of the Sobolev norms and of the induced geodesic distance. Our findings revealed that the $H^n(d\theta)$- and $H^n(ds)$-norms are equivalent on metric balls in Imm. However, the equivalence between the induced geodesic distance and the $H^n(d\theta)$-norm could only be shown on metric balls in Imm^n. Nevertheless, we concluded that the space Imm^n is metrically complete. Building upon these results, we deduced that the quotient space $\mathrm{Imm}^n/\mathrm{Diff}^n$ equipped with the metric $d_{I/D}$ respectively $\mathrm{dist}_{I/D}$ is also metrically complete. This conclusion relied on the fact that $d_{I/D}$ is intrinsic and hence coincides with the induced distance on the quotient space. Afterwards, we used a lemma which provided the existence of geodesics, assuming the existence of midpoints and the metric completeness of the space. The latter one was already shown and the first one followed from the existence of geodesic in the original space. Furthermore, we have shown that a geodesic between two free immersions is itself a free immersion.

We showed that the dense open subset $\mathrm{Imm}_f/\mathrm{Diff}$ of $\mathrm{Imm}^n/\mathrm{Diff}^n$ serves as a suitable space for proving the horizontality of geodesics, since this space admits a manifold structure. Leveraging this structure, we easily established that $(\mathrm{Imm}_f, \pi, \mathrm{Imm}_f/\mathrm{Diff}, \mathrm{Diff})$ forms a G-principal fiber bundle, thereby ensuring that the canonical projection π is always a submersion. This was an important prerequisite for defining horizontal and vertical vectors. Subsequently, we employed parallel transport to demonstrate that we can lift the geodesics from $\mathrm{Imm}_f/\mathrm{Diff}$ horizontally to Imm_f. The one-to-one correspondence between geodesics on the quotient space and horizontal geodesics in the total space provided an efficient way to compute geodesics in the quotient space, on the assumption that the horizontal bundle is not too complicated.

In further work, one could analyse the existence of geodesics in the space of smooth immersions. As previously noted, we were unable to establish all estimates presented in Section 4 for this space and therefore, we cannot employ them in the same manner as in the proof for the existence of geodesics in Imm^n. Also, it is not possible to prove metric completion and therefore we cannot conclude that the quotient space is complete either. In fact, it was shown in [3] that both spaces are not complete. Since the space of smooth immersions is contained in the space of Sobolev immersions, we know that there exists a geodesic between two smooth curves in Imm. However, the question of whether this geodesic itself qualifies as a smooth immersion remains open. Nevertheless, the results for the space Imm^n are still valuable since we use geodesics in numerical applications, where we often work with approximations of smooth curves.

Furthermore, one could study the concept of geodesic completeness, which explores the existence of geodesics for all time. Geodesic completeness ensures that certain dynamical systems or evolution equations defined on the space have well-defined solutions for all time. In this work we focused on geodesics between two curves, and we did not delve into the behavior of geodesics beyond the curves themselves. It is shown in [17, Ch. VIII, Prop. 6.5] that on a strong Riemannian manifold the metric completeness implies geodesic completeness and hence the space Imm^n is geodesically complete. In this context the term "strong" refers to the regularity of the Riemannian metric. However, it raises the question of the long-time existence of geodesics when considering a weak Riemannian manifold or geodesics in the quotient space $\mathrm{Imm}^n/\mathrm{Diff}^n$ which is not a manifold .

Additionally, it would be interesting to have a look at the stability and sensitivity of geodesics with respect to perturbations in the initial curves or variations in the underlying metric. Understanding how small changes in the curves or metric affect the corresponding geodesics can provide insights into the stability of the metric space.

Bibliography

[1] Atkin, C. J. (1975). The Hopf-Rinow theorem is false in infinte dimensions. *Bulletin of the London Mathematical Society*, Vol.7, 261-266.

[2] Aubin, J. P. (1963). Un théorème de compacité. *C.R. Acad. Sc. Paris*, 256, 5042-5044.

[3] Bruveris, M. et al. (2015). Completeness properties of Sobolev metrics on the space of curves. *J. Geom. Mech.,* 7(2), 125-150.

[4] Bruveris, M. et al. (2014). Geodesic completeness for Sobolev metrics on the space of immersed plane curves. *Forum Math. Sigma*, 2, e19

[5] Bruveris, M. (2020). Riemannian geometry for shape analysis and computational anatomy. *Mathematics Of Shapes And Applications*, 15-44.

[6] Burago, D. (2001). *A course in metric geometry*. Vol. 33 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI.

[7] Caselles, V. et. al. (1995). Geodesic active contours. *Int. J. Comput. Vis.*, 22, 61-79

[8] Cervera, V. et al. (1991). The action of the diffeomorphism group on the space of immersions. *Differential Geo. Appl.,*1, 391-401.

[9] Elíasson, H. I. (1967).*Geometry of manifolds of maps*. J. Differential Geom. 1 (1-2), 169 - 194.

[10] Eslitzbichler, M. (2015). Modelling character motions on infinite-dimensional manifolds. *The Visual Computer*, 31, 1179–1190.

[11] Freire, A. (2020). Winding number and applications. University of Tennessee, URL https://web.math.utk.edu/ freire/teaching/m462s20/WindingNumber.pdf

[12] Hopf, H. and Rinow, W. (1931). Ueber den Begriff der vollständigen differential-geometrischen Fläche. *Commentarii Mathematici Helvetici*, 3, 209-225.

[13] Huckemann, S. et. al. (2010). Intrinsic Shape Analysis: Geodesic PCA for Riemannian Manifolds modulo isometric Lie Group Actions. *Statistica Sinica*, Vol. 20, 1-58.

[14] Kahn, D. (1980). *Introduction to Global Analysis.* Vol. 91 of Pure and Applied Mathematics, Academic Press, Camebridge, MA

[15] Klingenberg, W. P. A. (1995). *Riemannian Geometry*. Vol. 1 of De Gruyter Studies in Mathematics, 2nd edition, De Gruyter, Berlin.

[16] Kielhöfer, H. (2018). *Calculus of Variations: An Introduction to the One-Dimensional Theory with Examples and Exercises,* Vol. 67 of Texts in Applied Mathematics, Springer, Cham.

[17] Lang, S. (1999). *Fundamentals of differential geometry.* Vol. 191 of Graduate Texts in Mathematics, Springer-Verlag, New York, NY.

[18] Mcintyre, M. and Cairns, G. (1993). A new formula for winding number. *Geom Dedicata*, 46, 149–159.

[19] Mennucci, A. C. G. (2021). Neighborhoods and Manifolds of Immersed Curves. *International Journal of Mathematics and Mathematical Sciences*, Vol. 2021, 1-22.

[20] Michor, P. (2008). *Topics in differantial geometry.* Vol. 93 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI.

[21] Michor, P. and Mumford, D. (2007). An overview of the riemannian metrics on spaces of curves using the hamiltonian approach. *Appl. Comput. Harmonic Anal.*, 23, 64-113.

[22] Michor, P. and Mumford, D. (2006). Riemannian geometries on spaces of plane curves. *J. Eur. Math. Soc. (JEMS),* 8, 1–48.

[23] Neeb, K.-H. (2005).*Infinite-Dimensional Lie Groups*. 3rd cycle. Monastir (Tunisie), pp.76.

[24] O'Neill, B. (1966). *Elementary differential geometry.* Academic Press, New York, NY.

[25] Rumpf, M. and Wirth, B. (To be published). *Riemannian Calculus on Shape Spaces and its Discretization*.

[26] Srivastava, A. et. al. (2011). Shape analysis of elastic curves in euclidean spaces. *Transactions on Pattern Analysis and Machine Intelligence*. 33(7), 1415-1428.

[27] Whitney, H. (1937). On regular closed curves in the plane. *Compositio Mathematica*, tome 4, 276-284.

[28] Wu, K-L. et al. (2016). Path Planning and Replanning for Mobile Robot Navigation on 3D Terrain: An Approach Based on Geodesic. *Mathematical Problems in Engineering*, Vol. 2016, 1-12.

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