Bachelor thesis

The Douglas-Rachford Algorithm for Optimal Transport

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CHAPTER 1

Introduction

In this bachelor thesis the Douglas-Rachford Algorithm for optimal transport will be introduced and analysed. The idea of optimal transport and its relevant definitions will be given as well as the necessary analytical tools to apply the Douglas-Rachfrod Algorithm to the proposed optimal transport problem. The optimal transport problem, which is going to be the basis of our approach will be discretized and than transferred into the appropriate form demanded by the algorithm. Because the Douglas-Rachford Algorithm is a proximal splitting method, the proximal operator, a very important and crucial tool for proximal splitting methods is going to play a significant role in this work. Furthermore four constructed variations of the Douglas-Rachford Algorithm are given and evaluated by taking appropriate reference solutions f^* and m^* and looking at the $L1$ norm regarding the error $|f - f^*|$ and $|m - m^*|$. This can be achieved because the Douglas-Rachford Algorithm is implemented in Matlab for our discretized optimal transport problem. Comparing the four variations of the Douglas-Rachford Algorithm, analysing the amount of iterations one has to compute for a satisfying result and the accuracy of the approximations, is done at last to evaluate the approach and algorithms at hand.

CHAPTER 2

Optimal Transport

Optimal transport is a concept where one tries to find a cost efficient way of moving mass from one place to another. Mathematically this means one has a distribution of particles at a point x described by a density f and wants to find a map T which specifies the movement of the particles from x to their destination $T(x)$, given by a density g. The key is to try and find an optimal map T so the movement of the particles is minimal.

2.1 Monge formulation

Having two densities of mass $f, g \geq 0$ with

$$\int_{\mathbb{R}^d} f(x)\mathrm{dx} = \int_{\mathbb{R}^d} g(y)\mathrm{dy} = 1$$

the aim is to find a map $T : \mathbb{R}^d \mapsto \mathbb{R}^d$ minimizing

$$M(T) = \int_{\mathbb{R}^d} |T(x) - x| f(x)\mathrm{dx}$$

such that

$$\int_A g(y)dy = \int_{T^{-1}(A)} f(x)\mathrm{dx}$$

for every Borel subset $A \subset \mathbb{R}^d$. Reformulating this problem with measures instead of densities with the identity $(T_{\#\mu}) = \mu(T^{-1}(A))$ one gets:

$$\min \left\{ M(T) = \int c(x, T(x))\mathrm{d}\mu(x) : T_{\#\mu} = \nu \right\} \tag{1.1}$$

with $c : X \times Y \mapsto \mathbb{R}$ being a cost function for transferring the particles at x to its location $T(x)$. Furthermore one has a correlation between the densities and the map T when one considers the euclidean setting where f and g induce measures and the cost function is the euclidean norm. Then the identity

$$g(T(x))\mathrm{det}(\partial T(x)) = f(x)$$

can be found but with this setting there are certain limitations as to the way the mass is moved. The particles at x cannot be divided and moved to different destinations in this model, the splitting of mass cannot be described through a map T.[2] That is why some thought was done as to generalize this formula to make it applicable to more complex and general settings.

2.2 Kantorovich formulation

The Kantorovich formulation is concerned with the same optimal transport problem but uses more generalized measures and connects them with maps which can describe splitting of mass and are more general in their properties. One has $\mu \in \mathcal{P}(X), \nu \in \mathcal{P}(Y)$ and a cost function $c : X \times Y \mapsto [0, \infty]$. The Kantorovich formulation is

$$\inf \left\{ K(\gamma) := \int_{X \times Y} c(x, y) d(x, y) : \gamma \int \Pi(\mu, \nu) \right\} \tag{1.2}$$

where

$$\pi(\mu, \nu) = \{\gamma \in \mathcal{P}(X \times Y) : \pi_{x\#\gamma} = \mu, \pi_{y\#\gamma} = \nu\}$$

and π_x is a projection from $X \times Y$ to X, π_y is a projections form $X \times Y$ to Y. Instead of describing the destination $T(x)$ for the particles at x one specifies for (x, y) the amount of mass moving from x to y.[2][p.2]
Optimal transport always guaranties that the transported mass is preserved and the two general descriptions of the optimal transport problem given before, are the foundation on which optimal transport builds. In the following we are going to restrict our focus on the case where the cost c is given as $c(x, y) = |x - y|^p$. Moreover due to interest in the development over time we are going to consider only those settings where the continuity equation in time, is solved.

2.3 Optimal transport for time dependent arguments

Regarding optimal transport with the relevant arguments being dependent on time the continuity equation $\partial_t f + \nabla v = 0$ guaranties that existing mass is preserved after applying the optimal map.
Following[2][p.169,170] a vector field $v_t(x)$ solving the continuity equation and a density $\mu(t, x)$ describing the movement of the particles, can be interpreted as the ODE

$$\dot{Y}_x(t) = v_t(Y_x(t)) \text{ with } Y_x(0) = x$$

where $Y_t(x) = y_x(t)$ and one tries to find a measure μ_t which fulfils

$$\mu_t := (Y_t \# \mu_0)$$

with μ_0 being the initial distribution of the particles we are interested in.
This means, that the continuity equation in relation to our arguments reads

$$\partial_t \mu_t + \nabla(\mu_t v_t) = 0.$$

For further restriction to simpler settings we give some relevant definitions as defined in chapter 5 of[2].

Definition 2.3.1. *The Wassersteinspace $\mathbb{W}_p$*
For $X = \mathbb{R}^d$, $p \in [1, \infty)$ define the Wassersteinspace $\mathbb{W}_p(X)$, as the space $\mathcal{P}_p(X)$, with the metric W_p.

Definition 2.3.2. *The Wassersteindistance W_p*
For $\mu, \nu \in \mathcal{P}_p(\Omega)$ and $p \in [1, \infty)$ we define

$$W_p(\mu, \nu) := \min \left\{ \int_{\Omega \times \Omega} |x - y|^p d\gamma : \gamma \in \Pi(\mu, \nu) \right\}^{\frac{1}{p}}$$

as the Wassersteindistance. This is the p-root of the minimal transport costs $c(x, y) = |x - y|^p$. This distance is also a metric.

Definition 2.3.3. *constant speed geodesic*
A curve $\omega : [0, 1] \mapsto X$ is said to be a geodesic between x_0 and $x_1 \in X$ if it minimizes the length of all curves such that $\omega_0 = x_0$ and $\omega_1 = x_1$.
ω is said to be a constant speed geodesic between ω_0 and $\omega_1 \in X$ if it satisfies

$$d(w(t), w(s)) = |t - s| d(w(0), w(1)) \ \forall t, s \in [0, 1]$$

Definition 2.3.4. *geodesic space*
A space (X, d) is a geodesic space if

$$d(x, y) = \min\{\textit{Length}(\omega) : \omega \in AC(X) \text{ with } \omega(0) = x \text{ and } \omega(1) = y\}.$$

AC(X)is the set of all absolutely continuous curves in X and

$$\textit{Length}(\omega) := \int_0^1 |\omega'|(t)\, dt.$$

Benamou and Brenier found a way to rewrite and improve the optimal transport problem described with a velocity field. With the Benamou-Brenier formula the non-convex optimal transport problem is now a convex optimization problem and thus admits a solution. The different functional's involved are defined as

$$\mathcal{B}_p(\rho, E) := \int_\Omega f_p(\rho, E) \text{ and } f_p : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R} \cup \{\infty\}$$

with

$$f_p(t, x) := \begin{cases} \frac{1}{p} \frac{|x|^p}{t^{p-1}}, & \text{if } t > 0 \\ 0, & \text{if } t = 0, x = 0 \\ \infty, & \text{otherwise} \end{cases}$$

and the resulting optimisation problem becomes

$$(B_p P) \ \min\{\mathcal{B}(\rho, E) : \partial_t p_t + \nabla \cdot E_t = 0, \rho_0 = \mu, \rho_1 = \nu\}.$$

Inserting the definitions of f and B into the problem $(B_p P)$ we get

$$B_p P = \min_{(\rho_t, E_t) \in \mathcal{C}} \int_0^1 \mathcal{B}_p(\rho_t, E_t) \mathrm{dt} = \int_0^1 \int_\Omega f_p(\rho_t(x), E_t(x)) \mathrm{dxdt} \tag{1.3}$$

as our optimal transport problem with the set of constraints

$$\mathcal{C}_p = \{(\rho_t, Et) : \partial_t p_t + \nabla \cdot E_t = 0, \rho_0 = \mu, \rho_1 = \nu\}.$$

Theorem 2.3.5. *Let Ω be compact and convex, with $\mu, \nu \in \mathcal{P}(\Omega)$*

$$W_p^\rho(\mu,\nu) = \min\{\mathcal{B}(\rho,E) : \partial t\rho + \nabla\cdot E = 0, \rho_0 = \mu, \rho_1 = \nu\}.$$

ρ and E are interpreted as measures on $\Omega \times [0,1]$, with $[0,1]$ representing time.
***idea for proof**:*
Because of W_p^ρ being a geodesic space we can write

$$W_p^\rho(\mu,\nu) = \left(\min\left\{\int_0^1 |\rho|\,dt, \rho_0 = \mu, \rho_1 = \nu\right\}\right)^p.$$

Restricting the minimum to constant speed geodesic it follows

$$W_p^\rho(\mu,\nu) = \min\left\{\int_0^1 |\rho|^p\,dt, \rho_0 = \mu, \rho_1 = \nu\right\}.$$

Now, due to the existence of a vector field

$$W_p^\rho(\mu,\nu) = \min\left\{\int_0^1 ||v_t||^p\,dt, \partial_t\rho_t \nabla\cdot(\rho_t v_t) = 0,\ \rho_0 = \mu, \rho_1 = \nu\right\}.$$

Changing the variables by setting $E_t = \rho_t v_t$ we get the proposed result.

For more details we refer to p.184 of[2].
Because of this theorem we know, that the Benamou-Brenier formula and the Monge and Kantorovich formulation are all equivalent optimal transport problems. With the restriction to constant speed geodesics in W_p and $\gamma = \gamma_t := (id, T)\#\mu$, which essentially means there exists a transport map in the sense of the Monge formulation, one gets the following optimal transport problem.
With the geodesic path between f^0 and f^1 being the density $t \mapsto f(x,t)$ and t parametrizing:

$$f(x,t) = f^0(T_t(x))| * \det(\partial T_t(x))|$$

and

$$T_t = (1-t)\mathrm{Id}_d + tT,$$

the densities $f(x,t)$ and the velocity field $v(x,t)$ solve the non-convex optimal transport problem,

$$\min_{(v,f)\in C^0} \frac{1}{2}\int_{[0,1]}^d \int_0^1 f(x,t)\|v(x,t)\|^2 \mathrm{dtdx} \tag{1.4}$$

under the constraints

$$C^0 = \{(v,f) \in: \partial_t f + \mathrm{div}_x(fv) = 0, v(0,*) = v(1,*) = 0, f(*,0) = f^0, f(*,1) = f^1\}.$$

Because of this context the transport map T_t with the velocity field v_t solve the ODE and we get

$$\dot{T}_x(t) = v_t(T_x(t)) \text{ with } T_x(0) = x$$

where $T_t(x) = T_x(t)$.[1]
The value $v_t(x)$ of the velocity field in this circumstances can be interpreted as the speed of the particles passing x at time t.[2]

In the following we will only be concerned with the optimal transport problem in Benamou-Brenier form for $p = 2$ and $\Omega = [0, 1]^d$.
Based on everything we looked at in this chapter from now on we will regard the convex optimal transport problem defined in [1]as

$$\min_{(m,f)\in\mathcal{C}} \mathcal{J}(m, f) = \int_{[0,1]^d} \int_0^1 J(m(x, t), f(x, t))\mathrm{dtdx} \tag{1.5}$$

where $\forall(m, f) \in \mathbb{R}^d \times \mathbb{R}$, $J(m, f) = \begin{cases} \frac{\|m\|^2}{2f}, & \text{if } f > 0 \\ 0, & \text{if } (m, f) = (0, 0) \\ \infty & \text{otherwise} \end{cases}$

and

$$\mathcal{C} = \{(m, f) : \partial_t f + \mathrm{div}_x(m) = 0, m(0, \cdot) = m(1, \cdot) = 0, f(\cdot, 0) = f^0, f(\cdot, 1) = f^1\}.$$

It is important to stress the fact, that our constraint only allows solutions for the continuity equation and sets our arguments to have Dirichlet boundary conditions. To simplify our problem we are going to focus on the one dimensional case and begin discretizing every component of our proposed problem.

CHAPTER 3

Discretization

In this chapter we are going to discretize the relevant functions and constraints of our original problem (1.5). For that we need to consider a discretization grid, reformulate the relevant conditions and approximate every involved function.

3.1 Grids

The Grid on which we want to approximate the arguments m, f is given by dividing $[0,1]^2$ into $(N+1) \times (P+1)$ points. This is going to be the centered grid in our context

$$G_c = \left\{ \left(x_i = \frac{i}{N}, t_j = \frac{j}{P} \right) \in [0,1]^2, 0 \leq i \leq N \ , \ 0 \leq j \leq P \right\}.$$

$\mathcal{E}_c = (\mathbb{R}^2)^{G_c}$ is the finite dimensional space of centered variables and our arguments are going to be defined as

$$V = (m, f) \in \mathcal{E}_c = (m_{i,j}, f_{i,j})_{0 \leq i \leq N}^{0 \leq j \leq P}.$$

Following [1]we use a staggered grid to approximate the relevant arguments m,f by $\tilde{m}$, $\tilde{f}$ with the linear operators explained at the end of this chapter. The introduction of those auxiliary variables is said to be natural but why there is an advantage is not explicitly explained. On this introduced staggered grid we can use a symmetric difference scheme and therefore accomplish second order consistency. The grid, which refines the original centered grid in space is defined as

$$G_s^x = \left\{ x_i = \frac{(i + \frac{1}{2})}{N}, t_j = \frac{j}{P} \in \frac{[-1, 2N+1]}{2N} \times [0,1] \ , \ -1 \leq i \leq N \ , \ 0 \leq j \leq P \right\}.$$

This means, that in space our grid is redefined with smaler steps taken but in time we take steps as before. This is also done in time so the according grid is

$$G_s^t = \left\{ x_i = \frac{i}{N}, t_j = \frac{(j + \frac{1}{2})}{P} \in [0,1] \times \frac{[-1, 2P+1]}{2P}, 0 \leq i \leq N; -1 \leq j \leq P \right\}.$$

Consequently we have two grids, which enables the use of symmetric differences and is very differently to the centered grid, with which it has no overlap. The arguments we define on those grids are

$$U = (\tilde{m}, \tilde{f}) \in \mathcal{E}_s = \left((\tilde{m}_{i,j})_{-1 \leq i \leq N}^{0 \leq j \leq P}, (\tilde{f}_{i,j})_{0 \leq i \leq N}^{-1 \leq j \leq P} \right)$$

and $\mathcal{E}_s = \mathbb{R}^{G_s^x} \times \mathbb{R}^{G_s^t}$ is the finite dimensional space of staggered variables.

3.2 Midpoint Interpolation

The midpoint interpolation $\mathcal{I}$ is a linear operator defined as

$$\mathcal{I}\colon \mathcal{E}_s \mapsto \mathcal{E}_c(\tilde{m}, \tilde{f}) \mapsto (m, f)$$

for all $0 \leq i \leq N$ and for all $0 \leq j \leq P$ the operator calculates

$$m_{i,j} = \frac{(\tilde{m}_{i-1,j} + \tilde{m}_{i,j})}{2} \text{ and } f_{i,j} = \frac{(\tilde{f}_{i,j-1} + \tilde{f}_{i,j})}{2}.$$

3.3 Divergence Operator

The divergence operator div is a linear operator defined as

$$\operatorname{div}\colon \mathcal{E}_s \mapsto \mathbb{R}^{G_c}(\tilde{m}, \tilde{f}) \mapsto \begin{pmatrix} \operatorname{div}(U)_{1,1} & \dots & \operatorname{div}(U)_{1,P+1} \\ \vdots & & \vdots \\ \operatorname{div}(U)_{N+1,1} & \dots & \operatorname{div}(U)_{N+1,P+1} \end{pmatrix}.$$

for all $0 \leq i \leq N$and for all $0 \leq j \leq P$ the operator estimates the divergence by

$$\operatorname{div}(U)_{i,j} = N(\tilde{m}_{i,j} - \tilde{m}_{i-1,j}) + P(\tilde{f}_{i,j} - \tilde{f}_{i,j-1}).$$

It is important to point out that the linear operators map variables from the staggered grid into the centered grid and especially when implementing the algorithm one must have the different dimensions of the domains in mind.

3.4 Boundary Condition

The proposed problem on our set C has Dirichlet boundary conditions and therefore we need to define a linear operator b as

$$b\colon \mathcal{E}_s \mapsto \mathbb{R}^{P+1} \times \mathbb{R}^{P+1} \times \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$$

with

$$b(U) = ((\tilde{m}_{-1,j}, \tilde{m}_{N,j})_{j=0}, (\tilde{m}_{-1,j}.\tilde{m}_{N,j})_{j=P}, (\tilde{f}_{i,-1}, \tilde{f}_{i,P})_{i=0}, (\tilde{f}_{i,-1}, \tilde{f}_{i,P})_{i=N}).$$

To ensure the boundary conditions are fulfilled we set

$$b_0 = (0, 0, f^0, f^1) \in \mathbb{R}^{P+1} \times \mathbb{R}^{P+1} \times \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}.$$

Here f^0 and f^1 are densities related to our original optimal transport problem.
The formula $b(U) = b_0$ needs to be set for our problem to ensure the boundary conditions are met.
By applying all our operators and the definition of our arguments U, V the original problem can now be written as the discrete convex problem

$$\min_{U \in \mathcal{E}_s} \mathcal{J}(\mathcal{I}(U)) + \iota_C(U), \tag{2.1}$$

with

$$\iota_C := \begin{cases} 0, & U \in C \\ \infty, & \text{otherwise} \end{cases},$$

where

$$\mathcal{J}(V) = \sum_{k \in G_c} J(m_k, f_k) \text{ with } k = (i,j) \in G_c.$$

Our set C now becomes

$$C = \{U \in \mathcal{E}_s : \operatorname{div}(U) = 0 \text{ and } b(U) = b_0\}.$$

Because we defined our argument V with the interpolation operator using U, the identity $\mathcal{I}(U) = V$ must be maintained. This is realized by the indicator function

$$\iota_{C_{s,c}}(U, V)$$

with

$$C_{c,s} = \{(U, V) \in \mathcal{E}_s \times \mathcal{E}_c V = \mathcal{I}(U)\}.$$

Hence our original problem with all the relevant conditions and functions now reads

$$\min_{U,V \in \mathcal{E}_s \times \mathcal{E}_c} \mathcal{J}(V) + \iota_c(U) + \iota_{c,s}(U, V). \tag{2.2}$$

The problem being broken down into something discrete, which can be implemented, gives rise to outlining four possible variations of the Douglas-Rachford proximal splitting Algorithm.

CHAPTER 4

The Douglas-Rachford Algorithm

The main idea behind proximal splitting methods is to divide the relevant functions of a problem into a sum of functions, which can easily be computed. The proximal operator is an approximative convex functional acting as the substitute for those functions because its properties are supporting simpler handling and efficiency. The Douglas-Rachford Algorithm is one of the algorithms using this functional to approximate a solution to an optimization problem. We first want to give an overview of what the algorithm is going to be like before defining and calculating the necessary proximal operators. That definition and the actual proximal operators in our problem will be dealt with in chapter four.
The Douglas-Rachford Algorithm is a Proximal splitting method to approximate the solution of

$$\min_{z\in\mathcal{H}} G_1(z) + G_2(z). \tag{3}$$

The algorithm uses a sequence $(z^{(l)}, w^{(l)}) \in \mathcal{H}^2$ with the initial values $(z^{(0)}, w^{(0)}) \in \mathcal{H}$ and computes

$$\begin{aligned} w^{(l+1)} &= w^{(l)} + \alpha(\mathrm{Prox}_\gamma G_1(2z^{(l)} - w^{(l)}) - z^{(l)}) \\ z^{(l+1)} &= \mathrm{Prox}_\gamma G_2(w^{(l+1)}). \end{aligned}$$

In each iteration the algorithm applies the proximal operator for the minimization of G_1 and then applies the proximal operator for minimizing G_2 based on the step before.

The following four algorithms are tailored to our discrete convex problem.
Asymmetric DR
For every argument z, we are going to define our two functions G_1 and G_2 as

$$\begin{aligned} G_1(z) &:= \mathcal{J}(V) + \iota_c(U) \\ G_2(z) &:= \iota_{c,s}(z) \end{aligned} \tag{3.1}$$

with $z = (U, V) \in \mathcal{H} = \mathcal{E}_s \times \mathcal{E}_c$. By swapping the definitions of G_1 and G_2 we receive a different algorithm due to G_1 and G_2 having different roles in the algorithm.
Asymmetric DR'
With the argument being the same $z = (U, V) \in \mathcal{H} = \mathcal{E}_s \times \mathcal{E}_c$ and now G_1 and G_2 defined as

$$\begin{aligned} G_1(z) &:= \iota_{c,s}(z) \\ G_2(z) &:= \mathcal{J}(V) + \iota_c(U) \end{aligned} \tag{3.2}$$

the second variation of the Douglas-Rachford algorithm is found. To have symmetric roles for all the relevant functions in our discrete problem one can consider splitting the problem into

Symmetric DR

$$\begin{aligned} G_1(z) &:= \mathcal{J}(V) + \iota_c(U) + \iota_{c,s}(\tilde{U}, \tilde{V}) \\ G_2(z) &:= \iota_D(z) \end{aligned} \tag{3.3}$$

with $z = (U, V, \tilde{U}, \tilde{V}) \in \mathcal{H} = (\mathcal{E}_s \times \mathcal{E}_c)^2$ whereas $D = \{z = (U, V, \tilde{U}, \tilde{V}) \in \mathcal{H} = (\mathcal{E}_s \times \mathcal{E}_c)^2; U = \tilde{U}, V = \tilde{V}\}$ thus adding a function ι_D to have all the necessary functions of our original problem combined in G_1. Again by swapping the definitions of G_1 and G_2

Symmetric DR'

with $z = (U, V, \tilde{U}, \tilde{V}) \in \mathcal{H} = (\mathcal{E}_s \times \mathcal{E}_c)^2$ we define

$$\begin{aligned} G_1(z) &:= \iota_D(z) \\ G_2(z) &:= \mathcal{J}(V) + \iota_c(U) + \iota_{c,s}(\tilde{U}, \tilde{V}). \end{aligned} \tag{3.4}$$

These four variations of the Douglas-Rachford Algorithm are approximates of the optimal transport problem (2.2) and will later be evaluated and compared. But first we need to calculate the relevant proximal operators to apply the proposed algorithm.

CHAPTER 5

The Proximal Operator

The Proximal operator is a crucial tool in being able to minimize a non-smooth functional. It is a single valued map that minimizes a strongly convex functional dependent on the original function. By compromising between finding the minimum of the non-smooth function F and being close to the argument z, the proximal operator approximates the minimum of that function F. The scalar γ plays a role as to what extent steps toward the minimum are taken. But most importantly the proximal operator has a unique minimizer, which is why it plays a significant role in optimization. [3]

5.1 Definition proximal operator

Definition 5.1.1. *Proximal operator*
Let $\mathcal{H}$ be a Hilbert space we define

$$Prox_{\gamma F}(z) = \underset{\tilde{z}\in\mathcal{H}}{\operatorname{argmin}} \frac{1}{2}\|z-\tilde{z}\|^2 + \gamma F(\tilde{z}).$$

The proximal operator has several useful properties which make computing easier for our context. We will only give the ones relevant for our approach but we refer the reader to[3]. The proximal operator is separable in the sense that if f is separable, meaning

$$f(x,y) = \varphi(x) + \psi(y)$$

the resulting proximal operator then becomes

$$\operatorname{Prox}_{\gamma f}(u,v) = (\operatorname{Prox}_{\gamma\varphi}(u), \operatorname{Prox}_{\gamma\psi}(v)).$$

When F(z) is fully separable, given as

$F(z) = \sum_{i=1}^{n} F_i(z_i)$ then the proximal operator can be written as $(\operatorname{Prox}_{\gamma}F(v))_i = \operatorname{Prox}_{\gamma}F_i(v_i)$.

The proximal operator is often considered to be a generalization of projections, because of their strong correlation. Essentially when F is an indicator function one also tries to minimize that function, but in a projection restricting the domain has, roughly speaking, the same effect. This relation is going to become very apparent, when calculating the relevant proximal operators for our algorithms later on.

Additionally we will need two important analytical tools for working with proximal operators.

5.2 Legendre-Fenchel transform

Definition 5.2.1. *Legendre-Fenchel transform*

$$F^*(w) = \max_{z \in \mathcal{H}} \langle z, w \rangle - F(z).$$

The proximal operator of the Legendre-Fenchel functional can be computed as efficiently and easily as the proximal operator of the original function F and plays an important role in the next theorem.

5.3 Fenchel-Rockafellar

Theorem 5.3.1. *Fenchel-Rockafellar*
let $X \subseteq \mathbb{R}^d$, $A \in \mathbb{R}^{d \times d}$ and X^* dual space of X. The theorem states

$$\inf\{f(x) + g(Ax) : x \in X\} = \max\{-f^*(A^*y) - g^*(-y) | y \in X^*\}$$

for f, g convex $\exists$ x subject to $f(x) < +\infty$ $g(Ax) < +\infty$ g continuous at Ax.

This Theorem is very important for calculating the proximal operators because of its strong and useful statement. We will not be giving the proof of this theorem but refer the reader to[4].
The proposed problem in (2.2) consists of several functions, which need to be 'translated' into their proximal operator for our proximal splitting approach to work. As outlined in the chapter before, the different Douglas-Rachford Algorithms also consist of another function $\iota_D(z)$ not explicitly defined in (2.2) but crucial to our setting for the symmetric DR algorithms. Therefore we need to calculate several proximal operators most of which are indicator functions thus having projections as their proximal operator.

5.4 Proximal operator for the functional $\mathcal{J}$

Calculating this proximal operator we use the separability of the proximal operator mentioned at the beginning of the chapter because $\mathcal{J}$ is a fully separable function.

$$\text{Prox}_{\gamma \mathcal{J}}(V) = \text{Prox}_{\gamma J}(V_k)_{k \in G_c}$$

To get the proximal operator for $J(V_k)$ we set $\tilde{m}, \tilde{f}$ as arbitrary but fixed arguments and get

$$\begin{aligned} \text{Prox}_{\gamma J}(\tilde{m}, \tilde{f}) &= \underset{(m,f)}{\text{argmin}} \frac{1}{2} \|(m, f) - (\tilde{m}, \tilde{f})\|^2 + J(m, f) \\ &= \underset{(m,f)}{\text{argmin}} \frac{1}{2}((m - \tilde{m})^2 + (f - \tilde{f})^2) + \gamma \frac{m^2}{2f}. \end{aligned}$$

To solve this minimization problem we define $F(m,f) = \frac{1}{2}(m-\tilde{m})^2 + (f-\tilde{f})^2 + \gamma\frac{m^2}{2f}$ and now set $F'(m,f) = 0$ thus get $0 = \frac{\partial F}{\partial f}$ and $0 = \frac{\partial F}{\partial m}$. Differentiating the function F we have the equations

$$0 = f - \tilde{f} - \gamma\frac{m^2}{2f^2}$$

and

$$0 = m - \tilde{m} + \frac{\gamma m}{f}.$$

This means for the value m we get the formula

$$m = \frac{f\tilde{m}}{f+\gamma}.$$

For f we have to solve the third order polynomial equation

$$0 = f^3 + 2f^2\gamma + f\gamma^2 - \tilde{f}f^2 - 2f\tilde{f}\gamma - \tilde{f}\gamma^2 - \frac{\gamma}{2}\tilde{m}^2$$

and set f^* as the solution. We can therefore define, for all $f^* \geq 0$ and

$$\mu(f^*) = \frac{f^*\tilde{m}}{f^*+\gamma}$$

the proximal operator as

$$\operatorname{Prox}_{\gamma J}(m,f) = \begin{cases} (\mu(f^*), f^*) & \text{if} f^* > 0 \\ (0,0) & \text{otherwise} \end{cases}.$$

Because of

$$\operatorname{Prox}_{\gamma\mathcal{J}}(V) = \operatorname{Prox}_{\gamma J}(V_k)_{k\in G_c} = \operatorname{Prox}_{\gamma J}(m_k, f_k)_{k\in G_c}$$

our final proximal operator is

$$\operatorname{Prox}_{\gamma\mathcal{J}}(V) = \begin{cases} (\mu(f_k^*), f_k^*) \text{ if} f_k^* > 0 \\ (0,0) \end{cases} \tag{4.1}$$

for all $k \in G_c$ and for all $f_k^* \geq 0$ with $\mu(f_k^*) = \frac{f_k^*\tilde{m}_k}{f_k^*+\gamma}$.

5.5 Proximal operator of ι_C

The Proximal operator of $\iota_C(U)$ can be found by redrafting the set C with the linear equation $Ax = y$. To do so one must consider the constraints in the original set C

$$C = \{U \in \mathcal{E}_s,\ div(U) = 0 \text{ and } b(U) = b_0\}$$

following this we define

$$A = \begin{pmatrix} div \\ b \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 \\ b_0 \end{pmatrix}.$$

This means we can write our proximal operator as follows

$$\mathrm{Prox}_{\gamma\iota_c}(U) = \underset{\tilde{U}\in C}{\operatorname{argmin}} \frac{1}{2}\|U-\tilde{U}\|^2 + \gamma\iota_c(\tilde{U}).$$

For simplification we now only concern ourselves with the minimum of this function but we will later give the argument minimum U.

$$\begin{aligned}
&\min_{\tilde{U}\in C} \frac{1}{2}\|U-\tilde{U}\|^2 + \gamma\iota_c(\tilde{U}) \\
&= \min_{\substack{\tilde{U}\in C \\ A\tilde{U}=y}} \frac{1}{2}\|U-\tilde{U}\|^2 \\
&= \min_{\tilde{U}\in C} \frac{1}{2}\|U-\tilde{U}\|^2 + \sup_{\lambda}\langle\lambda, AU-y\rangle \qquad (1)\\
&= \sup_{\lambda}\min_{U} \frac{1}{2}\|U-\tilde{U}\|^2 + \langle\lambda, AU\rangle - \langle\lambda, y\rangle \\
&= \sup_{\lambda}\min_{U} \frac{1}{2}\|U-\tilde{U}\|^2 + \langle A^*\lambda, U\rangle - \langle\lambda, y\rangle \\
&= \sup_{\lambda}\min_{U} \frac{1}{2}(\langle U-\tilde{U}, U-\tilde{U}\rangle) + \langle A^*\lambda, U\rangle - \langle\lambda, y\rangle \\
&= \sup_{\lambda}\min_{U} \frac{1}{2}(\langle U-\tilde{U}, U\rangle - \langle U-\tilde{U}, \tilde{U}\rangle) + \langle A^*\lambda, U\rangle - \langle\lambda, y\rangle \\
&= \sup_{\lambda}\min_{U} \frac{1}{2}(\langle U, U\rangle - \langle\tilde{U}, U\rangle - \langle U, \tilde{U}\rangle + \langle\tilde{U}, \tilde{U}\rangle) + \langle A^*\lambda, U\rangle - \langle\lambda, y\rangle \\
&= \sup_{\lambda}\min_{U} \frac{1}{2}(\langle U, U\rangle - 2\langle\tilde{U}, U\rangle + \langle\tilde{U}, \tilde{U}\rangle) + \langle A^*\lambda, U\rangle - \langle\lambda, y\rangle
\end{aligned}$$

Solving the minimization problem dependent on U we get a formula for U.

$$\begin{aligned}
&\frac{1}{2}(U^{tr}U - 2(\tilde{U}^{tr}U) + \tilde{U}^{tr}\tilde{U}) + (A^*\lambda)^{tr}U - (\lambda^{tr}y) \\
&\qquad \Rightarrow \partial U \quad \frac{1}{2}(2U^{tr} - 2\tilde{U}^{tr}) + (A^*\lambda)^{tr} = U^{tr} - \tilde{U}^{tr} + (A^*\lambda)^{tr}
\end{aligned}$$

To get the minimum we must now set this derivative to 0

$$U^{tr} - \tilde{U}^{tr} + (A^*\lambda)^{tr} = 0 \Leftrightarrow U = (\tilde{U}^{tr} - (A^*\lambda)^{tr})^{tr} = \tilde{U} - A^*\lambda$$

and by inserting this for U we get

$$\sup_{\lambda}\frac{1}{2}\|\tilde{U} - A^*\lambda - \tilde{U}\|^2 + \langle A^*\lambda, \tilde{U} - A^*\lambda\rangle - \langle\lambda, y\rangle.$$

By reorganizing and using definitions of the norm and the scalar product we can calculate

$$\begin{aligned}
&= \sup \frac{1}{2}\| - A^*\lambda\|^2 + \langle A^*\lambda, \tilde{U}\rangle - \langle A^*\lambda, A^*\lambda\rangle - \langle \lambda, y\rangle \\
&= \sup \frac{1}{2}\|A^*\lambda\|^2 + \langle A^*\lambda, \tilde{U}\rangle - \|A^*\lambda\|^2 - \langle \lambda, y\rangle \\
&= \sup -\frac{1}{2}\|A^*\lambda\|^2 + \langle A^*\lambda, \tilde{U}\rangle - \langle \lambda, y\rangle \\
&= \sup -\frac{1}{2}\|A^*\lambda\|^2 + \langle \lambda, A\tilde{U}\rangle - \langle \lambda, y\rangle \\
&= \sup -\frac{1}{2}\|A^*\lambda\|^2 + \langle \lambda, A\tilde{U} - y\rangle \\
&= \sup -\frac{1}{2}\langle A^*\lambda, A^*\lambda\rangle + \langle \lambda, A\tilde{U} - y\rangle \qquad (2) \\
&= \sup -\frac{1}{2}\langle \lambda, AA^*\lambda\rangle + \langle \lambda, A\tilde{U} - y\rangle \\
&= \sup -\frac{1}{2}(\lambda^{tr}AA^*\lambda) + (\lambda^{tr}(A\tilde{U} - y))
\end{aligned}$$

. deviating the gotten term for λ we have

$$\Rightarrow \partial\lambda \qquad -\frac{1}{2}(\lambda^{tr}AA^* + \lambda^{tr}(AA^*)^{tr}) + (A\tilde{U} - y)^{tr} = \lambda^{tr}(AA^*) - (A\tilde{U} - y)^{tr}).$$

Again setting that to 0 we get

$$\begin{aligned}
\lambda^{tr}(AA^*) - (A\tilde{U} - y)^{tr}) = 0 &\Leftrightarrow \lambda^{tr} = (A\tilde{U} - y)^{tr}(AA^*)^{-1} \\
&\Leftrightarrow \lambda = ((A\tilde{U} - y)^{tr}(AA^*)^{-1})^{tr} \\
&\Leftrightarrow \lambda = (AA^*)^{-1}(A\tilde{U} - y).
\end{aligned}$$

Inserting the found formula for λ into the one we have for U

$$\begin{aligned}
U &= \tilde{U} - A^*(AA^*)^{-1}(A\tilde{U} - y) \\
&= \tilde{U} - A^*((AA^*)^{-1}A\tilde{U} - (AA^*)^{-1}y) \\
&= \tilde{U} - A^*(AA^*)^{-1}A\tilde{U} + A^*(AA^*)^{-1}y.
\end{aligned}$$

U is now the solution of

$$\underset{\tilde{U}}{\operatorname{argmin}} \frac{1}{2}\|U - \tilde{U}\|^2 + \sup_{\lambda}\langle \lambda, AU - y\rangle$$

and considering every step of our calculation is allowed, the resulting proximal operator is the orthogonal projector onto C

$$\mathrm{Proj}_c = Id - A^*(AA^*)^{-1}A + A^*(AA^*)^{-1}y.$$

Now the only thing left to proof is that (1) and (2) are equivalent, meaning every step from (1) to (2) is indeed allowed and the swapping of *min* and *sup* is a legitimate step in our calculation. The Fenchel-Rockafellar theorem introduced at the beginning of this chapter allows us to proof the necessary equivalence of (1) and (2). The theorem was

$$\inf\{f(x) + g(Ax)|x \in X\} = \max\{-f^*(A^*y) - g^*(-y)|y \in X^*\}$$

with f, g convex and $\exists x$ subject to $f(x) < +\infty; g(Ax) < +\infty$ with g continous at Ax. To use it in our context we need to set f and g appropriately. We define

$$f(x) = -\langle x, A\tilde{U} - y\rangle = -\langle x, b\rangle$$

and calculate the Legendre-Fenchel transform

$$f^*(w) = \sup_{z\in\mathcal{H}} \langle z, w\rangle + \langle z, b\rangle = \sup_{z\in\mathcal{H}} \langle z, w + b\rangle = \begin{cases} 0, & w = -b \\ \infty, & \text{otherwise} \end{cases} =: \iota_{\{b\}}$$

while obviously our f satisfies the Fenchel-Rockafellar assumptions. Furthermore we define

$$g(x) = \frac{1}{2}\|x\|^2$$

and

$$g^*(w) = \sup_{z\in\mathcal{H}} \langle z, w\rangle - \frac{1}{2}\|z\|^2 = w^{tr}w - \frac{1}{2}\|w\|^2 = \|w\|^2 - \frac{1}{2}\|w\|^2 = \frac{1}{2}\|w\|^2$$

where g also satisfies the Fenchel-Rockafellar assumptions. Now the actual proof is quite short and easy,

Proof.

$$\begin{aligned}
(2) &= \sup -\frac{1}{2}\|A^*\lambda\|^2 - \langle \lambda, A\tilde{U} - y\rangle \\
&= -\inf\left(\frac{1}{2}\|A^*\lambda\|^2 + \langle \lambda, A\tilde{U} - y\rangle\right) \\
&= -\max\left\{\iota_{\{b\}}(Az) - \frac{1}{2}\|z\|^2 | z \in X^*\right\} \\
&= \min\left\{\frac{1}{2}\|z\|^2 z \in X^* s.t\ Az = -b = -A\tilde{U} + y\right\}
\end{aligned}$$

with $\tilde{U} = U + z$ and $z = \tilde{U} - U$

$$= \min\left\{\frac{1}{2}\|z\|^2 : AU = y\right\} = (1)$$

. □

5.6 Proximal operator of $\iota_{C_{c,s}}$

The Proximal operator of $\iota_{C_{c,s}}(U, V)$ can be found by now redrafting the set $C_{c,s}$ with the linear equation $Ax = y$. The original set $C_{c,s}$ was defined as

$$C_{c,s} = \{Z = (U, V) \in \mathcal{E}_s \times \mathcal{E}_c; \mathcal{I}(U) = V\}$$

following this we set

$$A = \begin{pmatrix} \mathcal{I} & -Id \end{pmatrix} \text{ and } y = 0.$$

This means we can calculate the proximal operator exactly like we did with the indicator function of the set C. We then have

$$\text{Prox}_{\gamma\iota_{C_{c,s}}}(Z) = \underset{\tilde{Z}\in C_{c,s}}{\text{argmin}}\, \frac{1}{2}\|Z-\tilde{Z}\|^2 + \gamma\iota_{C_{c,s}}(\tilde{Z})$$

Here we will also restrict ourselves to looking at the minimum but later on will have calculated the argument minimum Z.

$$\begin{aligned}
\min_{\tilde{Z}\in C_{c,s}} \frac{1}{2}\|Z-\tilde{Z}\|^2 + \gamma\iota_{C_{c,s}}(\tilde{Z}) &= \min_{\substack{\tilde{Z}\in C_{c,s}\\ \tilde{Z}:AZ=0}} \frac{1}{2}\|Z-\tilde{Z}\|^2 \\
&(3) = \min_{\tilde{Z}} \frac{1}{2}\|Z-\tilde{Z}\|^2 + \sup_{\lambda}\langle\lambda, AZ\rangle \\
&= \sup_{\lambda}\min_{Z} \frac{1}{2}\|Z-\tilde{Z}\|^2 + \langle\lambda, AZ\rangle \\
&= \sup_{\lambda}\min_{Z} \frac{1}{2}\|Z-\tilde{Z}\|^2 + \langle A^*\lambda, Z\rangle \\
&= \sup_{\lambda}\min_{Z} \frac{1}{2}(\langle Z-\tilde{Z}, Z-\tilde{Z}\rangle) + \langle A^*\lambda, Z\rangle \\
&= \sup_{\lambda}\min_{Z} \frac{1}{2}(\langle Z-\tilde{Z}, Z\rangle - \langle Z-\tilde{Z}, \tilde{Z}\rangle) + \langle A^*\lambda, Z\rangle \\
&= \sup_{\lambda}\min_{Z} \frac{1}{2}(\langle Z, Z\rangle - \langle\tilde{Z}, Z\rangle - \langle Z, \tilde{Z}\rangle + \langle\tilde{Z}, \tilde{Z}\rangle) + \langle A^*\lambda, Z\rangle \\
&= \sup_{\lambda}\min_{Z} \frac{1}{2}(\langle Z, Z\rangle - 2\langle\tilde{Z}, Z\rangle + \langle\tilde{Z}, \tilde{Z}\rangle) + \langle A^*\lambda, Z\rangle
\end{aligned}$$

Now we can solve the minimization problem dependent on Z.

$$\begin{aligned}
&\frac{1}{2}(Z^{tr}Z - 2(\tilde{Z}^{tr}Z) + \tilde{Z}^{tr}\tilde{Z}) + A^*\lambda^{tr} \\
&\qquad \Rightarrow \partial Z \quad \frac{1}{2}(2Z^{tr} - 2\tilde{Z}^{tr}) + (A^*\lambda)^{tr} = Z^{tr} - \tilde{Z}^{tr} + (A^*\lambda)^{tr}
\end{aligned}$$

as before we set the derivative to 0 and then have a formula

$$Z^{tr} - \tilde{Z}^{tr} + (A^*\lambda)^{tr} = 0 \Leftrightarrow Z = (\tilde{Z}^{tr} - (A^*\lambda)^{tr})^{tr} = \tilde{Z} - A^*\lambda$$

inserting this for Z we get

$$\sup \frac{1}{2}\|\tilde{Z} - A^*\lambda - \tilde{Z}\|^2 + \langle A^*\lambda, \tilde{Z} - A^*\lambda\rangle.$$

By reorganizing and using definitions of the norm and the scalar product we can calculate

$$\begin{aligned}
&= \sup \frac{1}{2}\| - A^*\lambda\|^2 + \langle A^*\lambda, \tilde{Z}\rangle - \langle A^*\lambda, A^*\lambda\rangle \\
&= \sup \frac{1}{2}\|A^*\lambda\|^2 + \langle A^*\lambda, \tilde{Z}\rangle - \|A^*\lambda\|^2 \\
&= \sup -\frac{1}{2}\|A^*\lambda\|^2 + \langle A^*\lambda, \tilde{Z}\rangle \\
&= \sup -\frac{1}{2}\|A^*\lambda\|^2 + \langle \lambda, A\tilde{Z}\rangle \\
(4) &= \sup -\frac{1}{2}\langle A^*\lambda, A^*\lambda\rangle + \langle \lambda, A\tilde{Z} - y\rangle \\
&= \sup -\frac{1}{2}\langle AA^*\lambda, A^*\lambda\rangle + \langle \lambda, A\tilde{Z} - y\rangle \\
&= \sup -\frac{1}{2}\langle \lambda, AA^*\lambda\rangle + \langle \lambda, A\tilde{Z} - y\rangle \\
&= \sup -\frac{1}{2}(\lambda^{tr}AA^*\lambda) + (\lambda^{tr}(A\tilde{Z})).
\end{aligned}$$

Solving this we deviate for λ and receive

$$\Rightarrow \partial\lambda - \quad \frac{1}{2}(\lambda^{tr}AA^* + \lambda^{tr}(AA^*)^{tr}) + (A\tilde{Z})^{tr} = \lambda^{tr}(AA^*) - (A\tilde{Z})^{tr}).$$

Setting $\partial\lambda = 0$ it follows

$$\begin{aligned}
\lambda^{tr}(AA^*) - (A\tilde{Z})^{tr}) = 0 &\Leftrightarrow \lambda^{tr} = (A\tilde{Z})^{tr}(AA^*)^{-1} \\
&\Leftrightarrow \lambda = ((A\tilde{Z})^{tr}(AA^*)^{-1})^{tr} \\
&\Leftrightarrow \lambda = (AA^*)^{-1}(A\tilde{Z}).
\end{aligned}$$

Inserting this formula into the one we found for Z, we now have

$$Z = \tilde{Z} - A^*(AA^*)^{-1}(A\tilde{Z})$$

where Z is the minimizer we wanted to get. The resulting proximal operator is the orthogonal projector onto $C_{c,s}$ and can be written as

$$\mathrm{Proj}_{c_{c,s}} = Id - A^*(AA^*)^{-1}A.$$

Now we also have to proof the equivalence of (3) and (4) with the Fenchel-Rockafellar theorem as before. For f we now set

$$f(x) = -\langle x, A\tilde{Z}\rangle = -\langle x, b\rangle.$$

The Legendre-Fenchel transform of this is

$$f^*(x) = \sup_{z\in\mathcal{H}}\langle z, w\rangle - \langle z, AZ\rangle = \sup_{z\in\mathcal{H}}\langle z, w - b\rangle = \begin{cases} 0, & w = -b \\ \infty, & \text{otherwise} \end{cases} =: \iota_{\{b2\}}$$

while f satisfies the Fenchel-Rockafellar assumptions. Setting g as

$$g(x) = \frac{1}{2}\|x\|^2$$

and the Legendre-Fenchel transform being

$$g^*(x) = \sup_{z\in\mathcal{H}} \langle z, w\rangle - \frac{1}{2}\|z\|^2 = w^{tr}w - \frac{1}{2}\|w\|^2 = \|w\|^2 - \frac{1}{2}\|w\|^2 = \frac{1}{2}\|w\|^2$$

with g satisfying the Fenchel-Rockafellar assumptions, we can now proof the wanted equivalence of (3) and (4).

Proof.

$$\begin{aligned}(4) &= \sup -\frac{1}{2}\|A^*\lambda\|^2 - \langle\lambda, A\tilde{Z}\rangle = -\inf\left(\frac{1}{2}\|A^*\lambda\|^2 + \langle\lambda, A\tilde{Z}\rangle\right)\\ &= -\max\left\{\iota_{\{b\}}(Az) - \frac{1}{2}\|z\|^2 | z\in X^*\right\}\\ &= \min\left\{\frac{1}{2}\|z\|^2 z\in X^* s.t\ Az = -b = -A\tilde{Z}\right\}\end{aligned}$$

with $\tilde{Z} = Z + z$ and $z = \tilde{Z} - Z$

$$= \min\left\{\frac{1}{2}\|z\|^2 : AZ = 0\right\} = (3)$$

□

The last proximal operator we are going to calculate is the one, which was constructed, so the relevant functions could be in the firs argument of the Douglas-Rachford algorithm.

5.7 Proximal operator of ι_D

The constructed function ι_D is the simplest one yet and the minimizer can be calculated like this

$$\begin{aligned}\mathrm{Prox}_{\gamma\iota_D}(U,V,W,X) &= \operatorname*{argmin}_{(\tilde{U}\tilde{V}\tilde{W}\tilde{X})\in\mathcal{H}} \frac{1}{2}\left(\|(U,V,W,X) - (\tilde{U}\tilde{V}\tilde{W}\tilde{X})\|^2\right) + \gamma\iota_D(\tilde{U}\tilde{V}\tilde{W}\tilde{X})\\ &= \operatorname*{argmin}_{\substack{(\tilde{U}\tilde{V}\tilde{W}\tilde{X})\in\mathcal{H}\\ \tilde{U}=\tilde{W},\tilde{V}=\tilde{X}}} \frac{1}{2}\|(U,V,W,X) - (\tilde{U}\tilde{V}\tilde{W}\tilde{X})\|^2\\ &= \operatorname*{argmin}_{\substack{(\tilde{U}\tilde{V}\tilde{W}\tilde{X})\in\mathcal{H}\\ \tilde{U}=\tilde{W},\tilde{V}=\tilde{X}}} \frac{1}{2}\|(U-\tilde{U}) + (V-\tilde{V}) + (W-\tilde{W}) + (X-\tilde{X})\|^2\\ &= \operatorname*{argmin}_{\substack{(\tilde{U}\tilde{V}\tilde{W}\tilde{X})\in\mathcal{H}\\ \tilde{U}=\tilde{W},\tilde{V}=\tilde{X}}} \frac{1}{2}\left(\|U-\tilde{U}\|^2 + \|V-\tilde{V}\|^2 + \|W-\tilde{W}\|^2 + \|X-\tilde{X}\|^2\right)\\ &= \operatorname*{argmin}_{(\tilde{U}\tilde{V}\tilde{W}\tilde{X})\in\mathcal{H}} \frac{1}{2}\left(\|U-\tilde{U}\|^2 + \|V-\tilde{V}\|^2 + \|W-\tilde{U}\|^2 + \|X-\tilde{V}\|^2\right)\\ &= \frac{1}{2}((\langle U,U\rangle - 2\langle\tilde{U},U\rangle + \langle\tilde{U},\tilde{U}\rangle) + (\langle W,W\rangle - 2\langle\tilde{U},W\rangle + \langle\tilde{U},\tilde{U}\rangle)\\ &+ (\langle V,V\rangle - 2\langle\tilde{V},V\rangle + \langle\tilde{V},\tilde{V}\rangle) + (\langle X,X\rangle - 2\langle\tilde{V},X\rangle + \langle\tilde{V},\tilde{V}\rangle))\end{aligned}$$

Solving this minimization problem we differentiate for component $\tilde{U}$ and $\tilde{V}$ and set the derivative$= 0$

$$\Rightarrow^{\partial \tilde{U}} 4\tilde{U} - 2(U + W) = 0 \Leftrightarrow \frac{U + W}{2} = \tilde{U}$$
$$\Rightarrow^{\partial \tilde{V}} 4\tilde{V} - 2(V + X) = 0 \Leftrightarrow \frac{V + X}{2} = \tilde{V}$$

Because $\tilde{U} = \tilde{W}$ and $\tilde{V} = \tilde{X}$ our Proximal operator is defined as

$$\text{Prox}_{\gamma\iota D}(U, V, \tilde{U}, \tilde{V}) = \frac{1}{2}(U + \tilde{U}, V + \tilde{V}, U + \tilde{U}, V + \tilde{V})$$

Having calculated all proximal operators necessary for the Douglas-Rachford-Algorithm we can now implement every relevant component and apply this algorithm to our optimal transport problem.

CHAPTER 6

Numerical Results

6.1 Implementation

As shown in chapter three the algorithm consist of two functions $G1$ and $G2$, which are composed of the relevant functions calculated in the chapter before. To implement those functions we transform every component of the necessary proximal operators into Matlab code. For the proximal operator of $\mathcal{J}$ we use the newton method [5] to solve the third order polynomial equation thus receiving f^*. The additional calculations concerning this proximal operator as defined in the relevant section are easily transformed into Matlab functions.
The proximal operators of the two important indicator functions are implemented by constructing the matrices representing the linear functions of our disretization and solving the linear system of equations given by the projections of each function $\iota_C(U)$ and $\iota_{C_{c,s}}(U,V)$. The code for the constructed indicator function ι_D is implemented by transforming $\text{Prox}_{\gamma\iota D}(U,V,\tilde{U},\tilde{V}) = \frac{1}{2}(U+\tilde{U}, V+\tilde{V}, U+\tilde{U}, V+\tilde{V})$ into code.
With all these functions we are able to implement the four different variations of the Douglas-Rachford Algorithm by seperating the argument to calculate each step and regarding appropriate parameters. In addition to those essential function we also implement functions to calculate the discrete $L1$ norm for our approximations and reference solutions as to be able to evaluate the proposed algorithms.

6.2 Evaluation

To evaluate the different algorithms given in chapter three we set the boundary conditions for f and consequently our reference solution f^*, as the Gaussian curves to approximate delta peaks. For delta peaks it is known that the optimal mass transport is linear movement of that mass in time. That means the mass distribution after a set amount of time p will be the same delta peak at $t = p$. Furthermore we will set the reference solution m^* for our setting using the identity $m = fv$ with v being the velocity field talked about in the first chapter.
In our setting this means the velocity v is the distance between c_0 and c_1, where c_0 represents the x position of the Gaussian peak of the boundary condition f^0 and c_1 of f^1 accordingly. We set those Gaussian curves with parameters $c_0 = 0.25$, $c_1 = 0.75$ and $\sigma = 0.1$. When evaluating the error of m and f for simplification we calculate their distance to the reference solutions using the $L1$ norm rather than the more appropriate wassersteinmetric.
First we are looking at the distance between f, m and their reference solutions f^*, m^* by

calculating their discrete $L1$ norm in time and space. We plot the error for various time and space steps, whereas we set $P = 2N$ and observe the changes for each algorithm, $DR1$ being the asymmetric Douglas-Rachford Algorithm, $DR2$ the asymmetric' DR-Algorithm and $DR2$ and $DR3$ the symmetric ones.

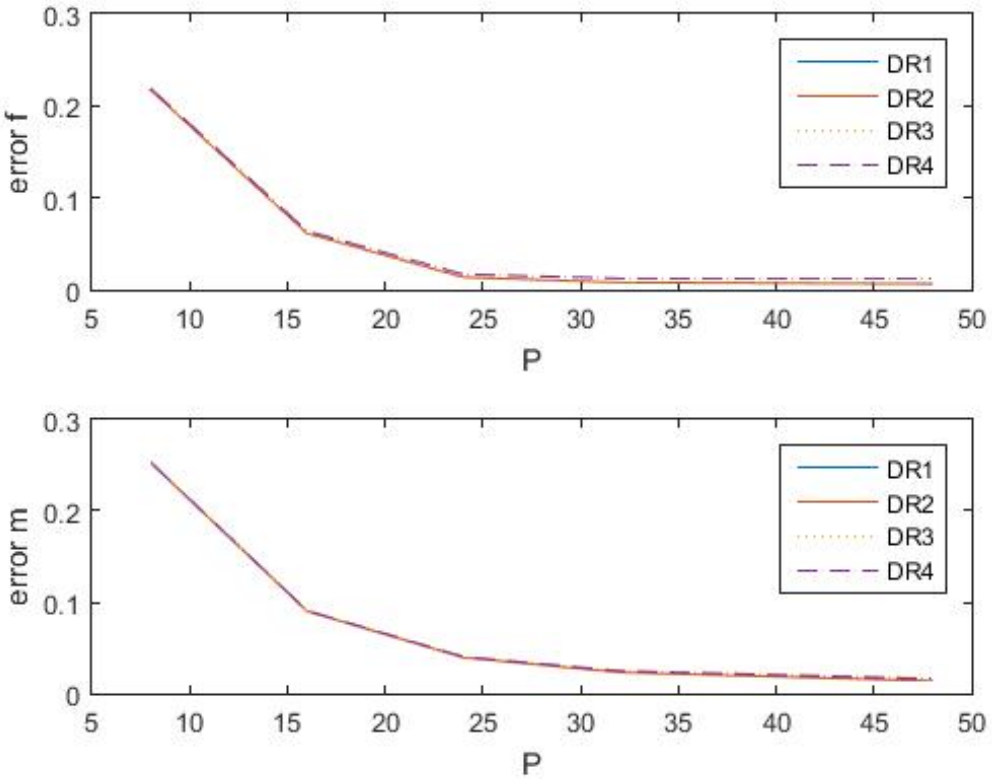

Figure 6.1: Comparison of the four DR-Algorithms with respect to the time step P after 1000 Iterations

The graphic 6.1 shows, selecting larger values for P improves the accuracy of the approximated solution to our optimal transport problem f and m. One can also detect the differences between the symmetric and asymmetric DR-Algorithms, which suggests the constructed functions $G1$ and $G2$ for the symmetric Douglas-Rachford Algorithms, where we wanted to have the influence of all the relevant functions independent of each other is in fact less accurate than the asymmetric algorithms, even though the discrepancy is very small. Furthermore we can easily determine, that a certain value of P, namely more than 25 needs to be set to have good results in our example. The last aspect, which is illustrated by the above figure is that the variations where one simply switched the role of the two functions $G1$ and $G2$, seems to have no notable effect of improvement either way although the roles of those functions in the Douglas-Rachford Algorithm is quite different.
Now we would like to take a look at the importance of the amount of Iterations one has to compute to get a satisfing approximation.

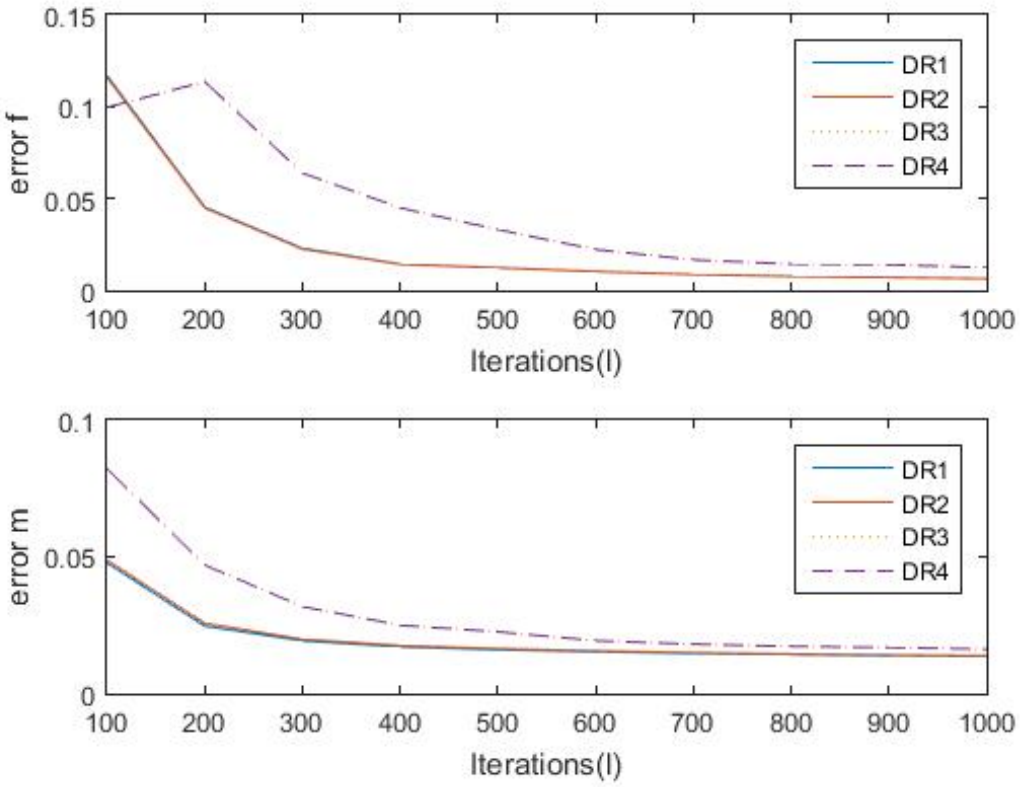

Figure 6.2: Comparison of the four DR-Algorithms for N=24,P=48 with respect to the amount of iterations

Here the argument form before, that the error of the symmetric variations is greater than the error of the asymmetric variations of the Douglas-Rachford Algorithm is more obvious. Especially after a small number of iterations $DR3$ and $DR4$ have an apparent disadvantage to the asymmetric variations. After 500 iterations the approximation are quite accurate and iterating further does not significantly improve the accuracy.
Consequently the asymmetric Douglas-Rachford variations are sufficient for finding an approximative solution of an optimal transport problem, such as or similar as the one we looked at throughout this work.

CHAPTER 7

Conclusion

The Douglas-Rachford Algorithm for the Benamou-Brenier approach to an optimal transport problem was handled throughout this work by transforming the proposed problem into a discrete one the Douglas-Rachford Algorithm can be applied to. The relevant optimal transport theory background and necessary assumptions for simpler handling have been taken into account and have been addressed. Furthermore the algorithm was evaluated and certain aspects have been criticised. The important aspect of optimal transport one should have in mind is the fact, that as outlined in *theorem*2.3.5 the Monge, Kantorovich and Benamou-Brenier formulations concerning optimal transport are equivalent, which allows one to choose the best or better yet easiest approach according to a given problem. Discretizing given problems is a crucial aspect of numerical mathematics and therefore plays a significant role in this paper, especially the use of appropriate grids and analytical tools for approximating the relevant functions. Most importantly the proximal operator introduced in this work is essential not only for this specific algorithm but for many more numeric approaches. Having evaluated the four variations of the Douglas-Rachford Algorithm one can say the need for constructing the symmetry between the functions is not necessary and offers no improvement to our setting.
But one must have in mind the considerable simplifications, which were made and therefore keep in mind the restricted purpose and usage of this specific approach. This thesis only considered the one dimensional case with regard only to basic settings, therefore it presents many opportunities for further in depth analysis of the Douglas-Rachford Algorithm.
The four variations dealt with in this work are just some of many possible definitions to transform the optimal transport problem into the algorithm setting. The settings such as the cost function could be formulated more generally thus adapting the optimal transport problem accordingly. Furthermore other proximal splitting methods could be implemented and evaluated regarding the specific problem. Additionally one could consider not using the staggered variables,thus discretizing the proposed problem with regard to only the centered grid. Comparing the outcome of both versions and therefore evaluating each approach could shed some light onto the usefulness of the staggered grid for the optimal transport problem.

Due to the code being implemented with very basic tools and approaches,one other issue would be to take a look at the efficiency and performance of the implementation, especially with regard to the inversion during the calculation of the projections onto the constraint sets.
All in all this work dealt with many important aspects of optimal transport and gave an introduction to the numeric handling of a given problem and the several considerations, steps necessary to use an algorithm for receiving a solution.

Bibliography

[1] Papadakis, Peyre, Oudet(2013):Optimal Transport With Proximal Splitting

[2] Santambrogio, Filippo (2015): Optimal Transport for Applied Mathematicians. Calculus of Variations, PDEs and Modeling, 159-166.

[3] Parikh,Boyd(2013):Proximal Algorithm.Foundation and Trends in Optimization,123-231

[4] R.T Rockafellar:Extension Of Fenchel's Duality Theorem For Convex Functions

[5] Anderson(2009):Computational Fluid Dynamics,223-232

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Münster, August 3, 2017

(Katharina Krabel)