Random Tensors, Loop Vertex Representation and Cumulants

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Introduction I

Random tensors, like matrix models, *originated in theoretical physics*.

In the 70's the hot stuff in theoretical physics was to quantize the elementary particles like quarks and gluons. In this period *matrix models* had some success in quantizing the strong interaction.

In the 90's the dominating theory in quantizing gravity was *string theory*. Random matrix models were seen at this time as a successful theory for quantizing gravity, but only in two dimensions.

The inventors of random tensor models, such as Ambjorn, Gross, Sasakura... wanted to replicate the success of matrix models for *dimensions three and four*. But they lacked an essential tool, *the 1/N expansion*.

Introduction II

Let's come to 2010's. The *tensor track* is an attempt to quantize gravity in dimensions greater than two, by combining *random tensor models*, *discrete geometry* and the *renormalisation group*.

The tensor track lies at the crossroad of several closely related approaches to quantize gravity, most notably *causal dynamical triangulations*, *quantum field theory on non-commutative spaces* and *group field theory*.

Random tensors share with random matrices the fact that they are a zero-dimensional world, and, as such, they are *background-independent*; they made no references whatsoever of any particular space-time.

Moreover, random tensors models, based on the quantum field theory of Feynman, are manageable by *renormalisation group techniques*. Simple models even share with non-Abelian gauge theories the property of *asymptotic freedom*.

Introduction III

Random tensors are expected to play a growing role in many areas of mathematics, physics, and computer science.

Communities using random tensors have grown apart, developing their own tools and results.

Nowadays there is an increasing circle of mathematicians and physicists working on random tensors. These people are recently inclined towards applications linked to *data analysis and artificial intelligence*.

The line which once separated them from computer scientists becomes a bit blurred...

Matrix Models

In the Wishart's complex matrix ensemble, the free partition function is defined by the integral over complex $N \times N$ matrices M,

$$Z_0(N) \quad := \quad \int dM \, e^{-\mathrm{Tr}[MM^\dagger]},$$

where dM is defined by the probability measure

$$dM \quad := \quad \pi^N \prod_{1 \le i,j \le N} d\mathrm{Re}(M_{ij}) d\mathrm{Im}(M_{ij}) \, .$$

and the expectation values of $U(N)$ invariants

$$< \mathrm{Tr}[MM^\dagger]^{p_1} \mathrm{Tr}[MM^\dagger]^{p_2} \ldots \mathrm{Tr}[MM^\dagger]^{p_k} >$$

is entirely determined by the propagator and by Wick's rule.

Matrix Models and the 1/N Expansion

t'Hooft made the fundamental observation that the $1/N$ expansion for matrix models is a *topological expansion*.

t'Hooft degree is *the number of holes of a surface* associated to the graph G. It is related to the Euler characteristic χ via $\chi = 2 - 2g$ and it is noted $g(G)$.

A sphere has a number of holes equal to 0, a torus has 1 hole, and so on...

Tensor Models, I

In 2010 a new kind of $1/N$ expansion was discovered for random tensor models. It relies on the *Gurău degree* of a colored graph G.

The Gurău degree ω is a positive number, with is *partly topological and partly combinatorial :*

$$\omega(G) \in \mathbb{N}.$$

To define it, we need a new notion, that of the *jackets*. For the moment, it is enough to say that a J_σ jacket is associated to a colored graph and to a cyclic permutation σ on the colors.

There are *three inequivalent jackets* for a tensor of rank 3, *twelve inequivalent jackets* for a tensor of rank 4, and so on.

See the work of J. Ben Geloun, V. Bonzom, R. Gurău and myself.

Tensor Models, II

Any scalar function of a tensor theory of quantum fields is a big functional integral on a Gaussian measure and an interactive part.

In the tensor case, this interactive part is a sum of tensor invariants, denoted $\sum_I S_I(T)$. For example, the partition function and the free energy are *scalar functions of N* and partition function and free energy are related by a *normalized logarithm* :

$$Z(N) = \int d\mu(T) e^{-\sum_I S_I(T)}, \qquad F(N) = \frac{1}{N^D} \log Z(N).$$

To each jacket is associated *a combinatorial map*, so a matrix model, so a 'tHooft degree of the corresponding matrix model. The Gurău degree is then proportional to the *sum of the 'tHooft degrees of the jackets* :

$$\omega(G) = \frac{1}{(D-1)!} \sum_\sigma g(J_\sigma).$$

Tensor Models, III

The invariants themselves can be classified in terms of graphs. Of course, these graphs depend crucially on the symmetries of the tensor.

For matrix models the expectation values of the invariants can be classified by ribbon graphs, the two ribbons corresponding to the two indices defining the matrix.

The family of *melonic graphs*, i.e. the graphs that have Gurău degree 0, can perhaps be called too trivial from a topologist point of view ; it corresponds to some triangulations of the sphere S_D where D is the rank of the tenseur.

But, as the Gurău degree is not only purely topological, the interplay between combinatorics and topology *in sub-leading terms* can be amazingly complex !

Quantum Gravity I

In the field theory approach to quantum gravity, one has to perform a functional integral over random geometry *pondered by Einstein-Hilbert action* :

$$\text{Quantum} \quad \text{Gravity} <=> \text{Random} \quad \text{geometry} + \text{EH}.$$

One builds the geometry by gluing discrete blocks, or "space time quanta".

Fundamental interactions of *a few space time quanta* lead to effective behavior of *an huge ensemble* of *many space time quanta*.

Quantum Gravity II

Let us come to the Sachdev-Ye-Kitaev (SYK) model. It is a quartic model of N Majorana fermions coupled by a *disordered tensor*. It is a model of condensed matter, hence it depends on time though a Hamiltonian.

The disordered tensor is centered Gaussian independently and identically distributed (iid)

$$< J_{abcd} >= 0, \quad < J^2_{abcd} >= \frac{\lambda^2}{N^3},$$

and the Hamiltonian is simply $H = J_{abcd}\psi_a\psi_b\psi_c\psi_d$.

Quantum Gravity III

This model possesses three important properties :

- it is solvable at large N,

- at strong coupling there is a *conformal symmetry*, hence it can be *a fixed point of the renormalization group*,

- above all, from the point of view of quantum gravity, it is *maximally chaotic* in the sense of Maldacena, Shenker and Stanford.

Hence the SYK model, although very simple, offers a path to the *main theoretical concepts of quantum gravity*, such as *the Bekenstein-Hawking entropy and holography*.

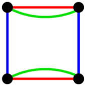
Quantum Gravity IV

SYK became a very active field. At large N the Schwinger-Dyson equation for the 2-point function is closed. The conformal symmetry can be broken and the corresponding subject goes under the name of near-AdS_2/near-CFT_1 correspondence.

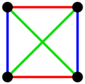
Witten has found a genuine field theory model (with no disorder), in which the tensors plays a *much more fundamental role*. In a nutshell, he discovered that his model has the same melonic limit as the tensors models pioneered by Gurău.

Klebanov and Tarnopolsky, when combined with an earlier work of Carrozza and Tanasa, allows a big simplification of the group symmetry of the main tensor, from $U(N)^{D(D-1)/2}$ to $O(N)^D$, but the graphs are not bipartite (see Figure).

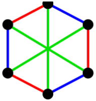
(a) Melon graph



(b) Pillow graph



(c) Tetra graph



(d) $K_{3,3}$ graph

Quantum Gravity V

The action of the KTCT model is

$$S = \frac{i}{2} \int dt \; \psi_{i_1,i_2,i_3} \partial_t \psi_{i_1,i_2,i_3} + \frac{\lambda}{4N^{3/2}} \psi_{i_1,i_2,i_3} \psi_{i_4,i_5,i_3} \psi_{i_4,i_2,i_6} \psi_{i_1,i_5,i_6}.$$

Unlike the initial SYK model, these tensor models fit in the framework of local quantum field theory with $D = 1$. Hence there is a possibility *to extend them in dimension greater than one.*

See the recent work of D. Benedetti, S. Carrozza, R. Gurău and collaborators.

A constructive field question

For a long time I was fascinated by the following question :

"how to make Feynman graphs more rigorous ?"

I mean "more rigorous for a mathematical perspective..."

This is typically a constructive field question,
that was bequeathed to me by my PhD director, Arthur Wightman.

Loop Vertex Expansion

In 2007 I was able to say something sensible about this question. I did discover the loop vertex expansion (in short LVE).

I hasten to say that the LVE only partly answers this question : for instance in the initial times, it was limited to matrix models with *quartic interactions* and, what's more, when *renormalization* is absent.

But still, I consider it to be a rather new technique of constructive field theory. For the first time it applies to Bosonic fields, and a main feature of the LVE is that it is written in terms of trees which are exponentially bounded.

It means that the outcome of the LVE is convergent, whereas the usual perturbative quantum field theory, that of Feynman, diverges.

Loop Vertex Expansion II

The LVE does indeed converge : the next question is "to which sum" ?

Answer : It converges to the Borel sum (named after the mathematician Emile Borel ; there are plenty of mathematicians) who were interested in divergent series, such as Borel and Nevanlinna and in more recent times Sokal, Ecalle...)

The essential ingredients of LVE are the intermediate field and the BKAR formula, developed by D. Brydges, T. Kennedy and perfected by A. Abdesselam, me and my collaborator Z. Wang.

In 2014 T. Krajewski and R. Gurău solved *by an LVE method* the main *constructive problem of the cumulants*.

Loop Vertex Expansion, III

In 2022 another step using the Loop Vertex Expansion, was published by Benedetti, Gurău, Keppler and Lettera. It uses also the formalism of trans-series due to Ecalle. These authors study the small-N expansions of $Z(g,N)$ and $W(g,N)$.

For any $g = |g|e^{\imath\varphi}$ on the sector of the Riemann surface with $|\varphi| < 3\pi/2$, the small-N expansion of $Z(g,N)$ has infinite radius of convergence in N, while the expansion of $W(g,N)$ has a finite radius of convergence in N.

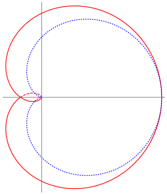


Figure – The cardioid domain (dotted blue line) and the extended cardioid (red line), for $\theta = \varphi/6$, in the complex g-plane.

Loop Vertex Expansion, IV

The Taylor coefficients of these expansions, $Z_n(g)$ and $W_n(g)$, exhibit analytic properties similar to $Z(g, N)$ and $W(g, N)$ and have transseries expansions.

The transseries expansion of $Z_n(g)$ is readily accessible : much like $Z(g, N)$, for any n, $Z_n(g)$ has *a zero- and a one-instanton contribution*.

The transseries of $W_n(g)$ is obtained using Möebius inversion and summing these transseries yields the transseries expansion of $W(g, N)$.

The transseries of $W_n(g)$ and $W(g, N)$ are markedly different : while $W(g, N)$ displays contributions from arbitrarily many multi-instantons, $W_n(g)$ exhibits contributions of *only up to n-instanton sectors*.

Loop Vertex Representation

In 2016 I discovered a refinement of the LVE, which I named LVR. The added ingredients of the LVR are essential combinatorial, based on *selective Gaussian integration* and the generating function of Fuss-Catalan numbers.

I think that the LVR has *more power* than the LVE, since the LVR can treat more models, with higher polynomial interactions.

It converges to the Borel-LeRoy sum (Le Roy was a PhD student of Borel).

It has been developed and perfected for approximately 6 years by T. Krajewski, V. Sazonov and me :

- T. Krajewski has added the *holomorphic matrix calculus*,
- V. Sazonov has recently and beautifully applied selective Gaussian integration in the quantum field theory formulation of the *Jacobian conjecture*.

Loop Vertex Representation II

LVR has an *additional parameter* p, the degree of the polynomial interaction. Hence the partition function for Wishart's complex matrix ensemble is

$$Z_p(\lambda, N) = \int dM e^{-\mathrm{Tr}[MM^\dagger + \frac{\lambda}{N^{p-1}}(MM^\dagger)^p]}$$

$p = 2$ is equivalent to quartic interaction ; but in this case LVE is sufficient, hence LVR is really necessary from $p = 3$ to infinity.

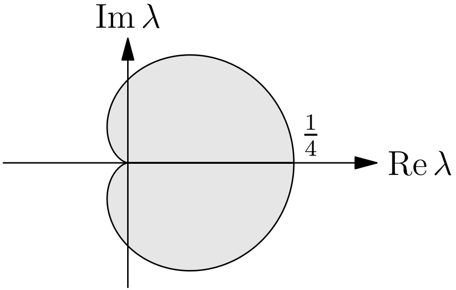
In the case $p = 3$ the LVR is somewhat simplified ; the *Fuss-Catalan equation for the generated function* is

$$zT_3^3(z) - T_3(z) + 1 = 0,$$

which is soluble by radicals.

Loop Vertex Representation for Cumulants, I

Last month, I constructed by the LVR method the *cumulants* in a cardioid-shaped domain $\mathcal{C}$ of the coupling constant (for instance for the Wishart's complex matrix model).



Cardioid domain $\mathcal{C}$ in the complex λ plane, in the case $p = 3$.

Loop Vertex Representation for Cumulants, II

Let the partition function be $Z_p(\lambda, N, J)$; the cumulants are defined by

$$\mathfrak{K}_p^k(\lambda, N, J) \quad := \quad \Big[\frac{\partial^2}{J^\dagger_{a_1 b_1} J_{c_1 d_1}} \cdots \frac{\partial^2}{J^\dagger_{a_k b_k} J_{c_k d_k}} \log Z_p(\lambda, N, J) \Big]_{\{J\}=0} . \qquad (1)$$

Let me fix $k_{\max}$. My last month's main result extend to higher interactions $p = 3, ..., \infty$ those of Krajewski and Gurău in 2014 :

Theorem

Let $k \leq k_{\max}$. The series (1) is absolutely convergent in the cardioid domain $\mathcal{C}$. This expansion is analytic for any $\lambda \in \mathcal{C}$ and Borel-LeRoy summable. The proof holds uniformly in the external variables.

Sketch of proof of the main result, I

The *crux of the LVR* lies in the matrix-valued function A and the functional change of variables from M to X such that

$$A(\lambda, X) \quad := \quad X T_p(-\lambda X^{p-1}) \,,$$

so that Equation (1) can now be rewritten as

$$X = A(\lambda, X) + \lambda A^p(\lambda, X) \,.$$

Sketch of proof of the main result, II

We define

$$\begin{aligned} d\mu(M) &:= dM e^{-\mathrm{Tr} MM^\dagger}, \\ \Sigma(\lambda, X) &:= \sum_{k=0}^{p-1} A^k(X) \otimes A^{p-1-k}(X), \\ S(\lambda, X) &:= -\mathrm{Tr}_\otimes \log \left[\mathbf{1}_\otimes + \lambda\Sigma(\lambda, X)\right], \\ K_{J,N}(\lambda, X) &:= N(J^\dagger, \left[\mathbf{1}_\otimes + \lambda\Sigma(\lambda, X)\right]^{-1} J) , \end{aligned}$$

so that the formula for the *moments* of the partition function is

$$\mathfrak{S}^k(\lambda, N) \quad := \quad \left\{ \frac{\partial^2}{J^\dagger_{a_1 b_1} J_{c_1 d_1}} \cdots \frac{\partial^2}{J^\dagger_{a_k b_k} J_{c_k d_k}} \int d\mu(M) e^{K_{J,N}(\lambda,X) - S(\lambda,X)} \right\}_{J=0}$$

Our goal is to deduce the formula for the *cumulants* $\mathfrak{K}^k(\lambda, N)$, which are the *connected moments*.

Sketch of proof of the main result, III

We want to *factorize*

$$Z_p(\lambda, N, J) \quad = \quad \int d\mu(M) e^{K_{J,N}(\lambda, X) - S(\lambda, X)}.$$

In the *first step* we define $W := W_1 + W_2$, $W_1 := e^{K_{J,N}(\lambda,X)}$, $W_2 := e^{-S(\lambda,X)}$, and we *expand to infinity* the exponential of the interaction

$$Z(\lambda, N, J) \quad = \quad \sum_{n=0}^{\infty} \frac{1}{n!} \int d\mu(M) \; W^n = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\mu(M) \; \prod_{i=1}^{n} W^i,$$

provided $\forall i \; , W^i = W$.

The *second step* is to introduce *replicas* and to *subtly replace* the integral over the single $N \times N$ complex matrix M by an integral over an n-tuple of such $N \times N$ matrices $M_i, 1 \le i \le n$.

In the *third step* we perform the *BKAR expansion*.

Sketch of proof of the main result, IV

After the BKAR expansion we perform the *Cauchy holomorphic matrix calculus* introduced by Krajewski, and the partition function *with sources* can be rewritten as :

$$\mathfrak{S}^k(\lambda, N, J) = N^k \sum_{n=k}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}} \int dw_{\mathcal{F}} \partial_{\mathcal{F}} \int d\mu_{C\{x^{\mathcal{F}}\}}(M) \prod_{i=1}^{k} (J^{\dagger}_{a_i b_i}, \left[\mathbf{1}_{\otimes} + \lambda\Sigma(\lambda, X^i)\right]^{-1} J_{c_i,d_i}) \prod_{i=k+1}^{n} e^{-S(\lambda, X^i)}.$$

Remark that this formula *does not require the intermediate field*, because it is defined as a *Gaussian integral over* $X^i = \frac{1}{N} M_i M_i^{\dagger}$.

Sketch of proof of the main result, V

Now our result can be summarized as a convergent expansion of $\mathfrak{K}^k(\lambda, N, J)$ as a sum over LVR trees with k cilia [L. Ferdinand & co, 09/22] : since the fields, the measure and the integrand are now factorized over the connected components of $\mathcal{F}$, its logarithm is computed as exactly the same sum *but restricted to the spanning trees*

$$\mathfrak{K}^k(\lambda, N, J) = N^k \sum_{n=k}^{\infty} \frac{1}{n!} \sum_{T} \int dw_T \partial_T \int d\mu_{C\{x^T\}}(M) \prod_{i=1}^{k} (J^\dagger_{a_i b_i}, \left[\mathbf{1}_\otimes + \lambda \Sigma(\lambda, X^i)\right]^{-1} J_{c_i, d_i}) \prod_{i=k+1}^{n} e^{-S(\lambda, X^i)}.$$

After some combinatoric tricks (counting the cilia, trees, etc...), this concludes the proof.

Introduction

The Jacobian conjecture can be formulated as follows. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map written as

$$F(x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n)),$$

where $x_1, \ldots, x_n$ are coordinates and the functions $F_i : \mathbb{C}^n \to \mathbb{C}$ are polynomials. Hereafter we write just x for $(x_1, \ldots, x_n)$. Suppose that the Jacobian determinant

$$JF(x) := \det \left(\frac{\partial F_i(x)}{\partial x_j} \right)$$

is identically a nonzero constant – without loss of generality it can be fixed to the value $JF(x) = 1$. *Show then that F is globally invertible in sense of composition and its inverse $G := F^{-1}$ is also a polynomial map.*

Abdesselam-Rivasseau (AR) model, I

The AR model is defined by the action

$$S := \sum_{i=1}^{n} \psi_i F_i(\phi_1, \dots, \phi_n) - \sum_{i=1}^{n} \psi_i y_i,$$

where $F_i(\phi) = \phi_i - H_i(\phi)$, and where H is a polynomial with $d := \deg H$. In tensorial notations the functions $H_i(\phi)$ can be written as

$$H_i(\phi) = \frac{1}{d!} \sum_{j_1, \dots j_d = 1}^{n} w_{i, j_1 \dots j_d} \phi_{j_1} \dots \phi_{j_d} \ ,$$

with the tensor $w_{i,j_1 \dots j_d}$ being completely symmetric in j indices.

AR model, II

The inverse map $G_k(y)$ is equal to the power series of the first cumulant of the model

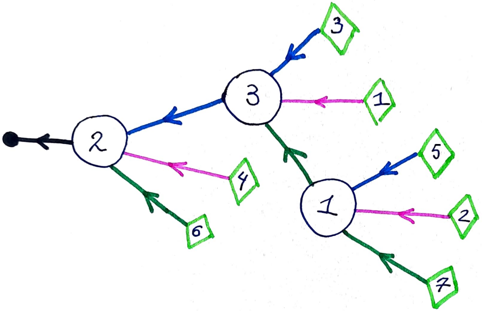
$$G_k(y) = \frac{1}{Z} \int d\psi d\phi \; \phi_k e^{-S} ,$$

and is *convergent for the small* $|y|$ (Abdesselam, 2002).

This *reduces* the proof of Jacobian Conjecture to prove the vanishing of all the coefficients *after a certain order* of this perturbative series for $G_k(y)$.

Colored AR model

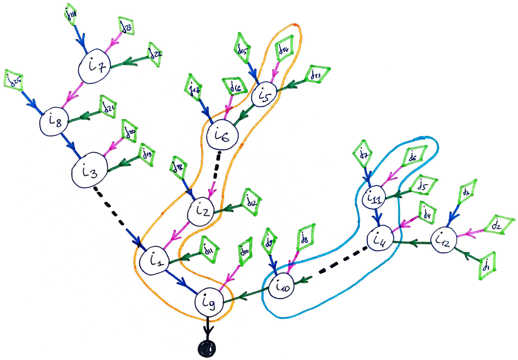
An example of the tree with $d = 3$ and $V = 3$.



Is it possible that colored trees can contain some chains of vertices of a length greater than n?

Colored AR model, II

An example of the tree of a not specified order.



Can the chains circled by the sky-blue and orange lines be longer than n? This question is equivalent to the question about the maximal possible depth of the trees and is directly related to the proof of the Jacobian Conjecture.

Termination of the series !

For trees with maximal depth $(n-1)$, the maximal number of leaves is d^{n-1}.

In 1982 Bass, Connell and Wright already reduced the Jacobian Conjecture to the case $d=3, \ \forall n$.

I think the argument of V. Sazonov is *correct and optimal*, in the sense that it works for *any homogeneous polynomial*

$$H_i(\phi) = \sum_{j_1, \dots j_d = 1}^{n} w_{i, j_1 \dots j_d} \phi_{j_1} \dots \phi_{j_d}$$

and make no use of the reduction of Bass, Connell and Wright.

I could be wrong, but I think that the Jacobian Conjecture has now become the *Jacobian Theorem*.

- Stay tune !