SPDEs at Criticality

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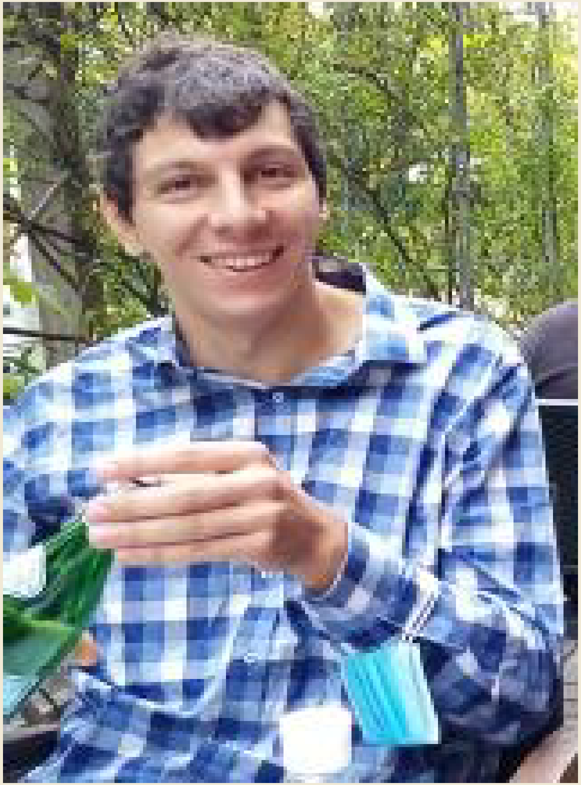


Tommaso Rosati

Stochastic Heat Equation

$$\partial_t u = \frac{1}{2}\Delta u + \xi\, u$$

Criticality of $d=2$

if $\quad u^\varepsilon(t,x) := u\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)$

then $\quad \partial_t u^\varepsilon = \Delta u^\varepsilon + \varepsilon^{\frac{d-2}{2}}\, \dot{W}\, u^\varepsilon$

SPDE's $\qquad$ Disordered Systems

criticality $\equiv$ marginality

Rigorous Approach
focus on $d=2$

Mollify

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + \beta_\varepsilon \xi_\varepsilon u_\varepsilon$$

with

$$\xi_\varepsilon(t,x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} j\left(\frac{x-y}{\varepsilon}\right) \xi(t,y)\, dy$$

&

$$\beta_\varepsilon = \hat{\beta} \sqrt{\frac{2\pi}{\log 1/\varepsilon}}$$

Goal: Establish (& describe) $\lim_{\varepsilon \downarrow 0} u_\varepsilon$

Thm (Caravenna-Sun-Z '17)

for any fixed x,t

$$u_\varepsilon(t,x) \longrightarrow \begin{cases} e^{\sigma_{\hat\beta} X - \frac{1}{2}\sigma_{\hat\beta}^2} & , \hat\beta < 1 \\ 0 & , \hat\beta \geq 1 \end{cases}$$

with $\sigma_{\hat\beta}^2 = \log \frac{1}{1-\hat\beta^2}$

Thm (Caravenna-Sun-Z '17)

$$\sqrt{\frac{\log 1/\varepsilon}{2\pi}} \int_{\mathbb{R}^2} \left(u_\varepsilon(t,x) - 1 \right) \phi(x)\, dx \longrightarrow \int v(t,x)\, \phi(x)\, dx$$

with

$$\partial_t v = \frac{1}{2}\Delta v + \sqrt{\frac{1}{1-\hat\beta^2}}\ \tilde\xi$$

Some other examples

2d KPZ

$$\partial_t h_\varepsilon = \frac{1}{2}\Delta h_\varepsilon + \frac{1}{2}|\nabla h_\varepsilon|^2 + \hat\beta\sqrt{\frac{2\pi}{\log 1/\varepsilon}}\,\xi_\varepsilon - C_\varepsilon$$

<u>Thm</u> (CSZ'18) $\forall\ \hat\beta < 1$ then

$\sqrt{\frac{\log 1/\varepsilon}{2\pi}}\left(h_\varepsilon(t,\cdot) - \mathbb{E}\,h_\varepsilon\right)$ has same EW limit as SHE with same variance !

Gu '18$^+$ (small $\hat\beta$) , Chatterjee - Dunlap '18$^-$ (tightness)

2d Anisotropic KPZ

$$\partial_t h_\varepsilon = \frac{1}{2}\Delta h_\varepsilon + \frac{\lambda}{\sqrt{\log 1/\varepsilon}}\left((\partial_x h_\varepsilon)^2 - (\partial_x h_\varepsilon)^2\right) + \xi_\varepsilon$$

Thm (Erhard-Cannizzaro-Toninelli '21)

Edwards-Wilkinson limit:

$$\partial_t h = \frac{1}{2}\nu_{\text{eff}}\Delta h + \sqrt{\nu_{\text{eff}}}\,\xi$$

$$\nu_{\text{eff}} = \sqrt{\frac{2\lambda^2}{\pi}+1} \quad \forall \; \lambda > 0$$

i.e. NO phase transition!

See also Erhard-Cannizzaro-Schönbauer '19

Cannizzaro-Gubinelli-Toninelli '23

on Burgers

Semilinear SHE

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \sqrt{\frac{1}{\log 1/\varepsilon}}\, \sigma(u_\varepsilon)\, \xi_\varepsilon$$

Thm (Dunlap–Gu '20)

if $\|\sigma\|_{Lip} < \sqrt{2\pi}$

then $u_\varepsilon(\varepsilon^{2-Q}, x) \xrightarrow{d} \Xi(Q)$ (pointwise fluctuation)

with

$$\left.\begin{aligned} d\Xi(q) &= J(Q-q, \Xi(q))\, dB(q) \\ J(q,b) &= \frac{1}{2\sqrt{\pi}} \sqrt{\mathbb{E}\left[\sigma^2(\Xi(q))\right]} \end{aligned}\right\}$$

Thm (Ran Tao '22)

Edwards-Wilkinson convergence to

$$\partial_t u = \frac{1}{2}\Delta u + \sqrt{\mathbb{E}\left[\sigma(\Xi(2))\right]}\, \xi$$

Allen-Cahn, $d=2$
critical noise scaling

$$\begin{cases} \partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + u_\varepsilon - u_\varepsilon^3 \\ u_\varepsilon(0,x) = \dfrac{\lambda}{\sqrt{\log 1/\varepsilon}}\, \xi_\varepsilon(x) \end{cases}$$

Thm (Gabriel-Rosati-Z '23+)

for $\lambda < \lambda_0$ & $\forall\, t,x$

$$\sqrt{\log 1/\varepsilon}\; u_\varepsilon(t,x) \xrightarrow{d} \sigma_\lambda\, P_t * \xi\,(x)$$

with
$$\begin{cases} \partial_\lambda \sigma = \frac{1}{\lambda}\left(\sigma - \frac{3}{\pi}\sigma^3\right) \\ \sigma_0 = 0 \end{cases}$$

$$\rightsquigarrow \quad \sigma_\lambda = \frac{\lambda}{\sqrt{1+\frac{3}{\pi}\lambda^2}}$$

also Hairer-Le-Rosati '22

: sub-critical noise scaling $\quad \varepsilon^{\frac{d}{2}-\alpha}\xi_\varepsilon\,,\ \alpha \in (0,1)$

Emergence of Strong Correlations

The critical 2d Stochastic Heat Flow

Thm (CSZ'22) if $\beta_N^2 = \frac{\pi}{\log N}\left(1 + \frac{\theta}{\log N}\right)$

then

$$\sum_{x,y \in \mathbb{Z}^2} \phi\left(\frac{x}{\sqrt{N}}\right)\left(Z^{\omega}_{N,\beta_N}(x,y) - 1\right)\varphi\left(\frac{y}{\sqrt{N}}\right) \longrightarrow \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \phi(x)\left(\mathcal{Z}^{SHF}_{\theta}(x,y) - 1\right)\varphi(y)\, dx\, dy$$

- log-correlated field
- not Gaussian or exp(Gaussian)
- singular wrt Lebesgue measure
- some hints of self-similarity

Structures

&

Methods

Directed Polymers

Feynman-Kac

$$u_\varepsilon(t,x;\varphi) = E_x\left[\varphi(B(t))\, e^{\beta\int_0^t \xi(s,B(s))ds - \frac{t\beta^2}{2}\langle \xi \rangle}\right]$$

Directed Polymer

$$Z_N^\beta(x,y) = E_x\left[e^{\sum_{n=1}^{N-1}\{\beta\omega(n,S_n) - \lambda(\beta)\}} \mathbb{1}_{S_N = y}\right]$$

disorder $\omega = \{\omega_{n,x}\}_{\substack{n\in\mathbb{N}\\ x\in\mathbb{Z}^2}}$ i.i.d.

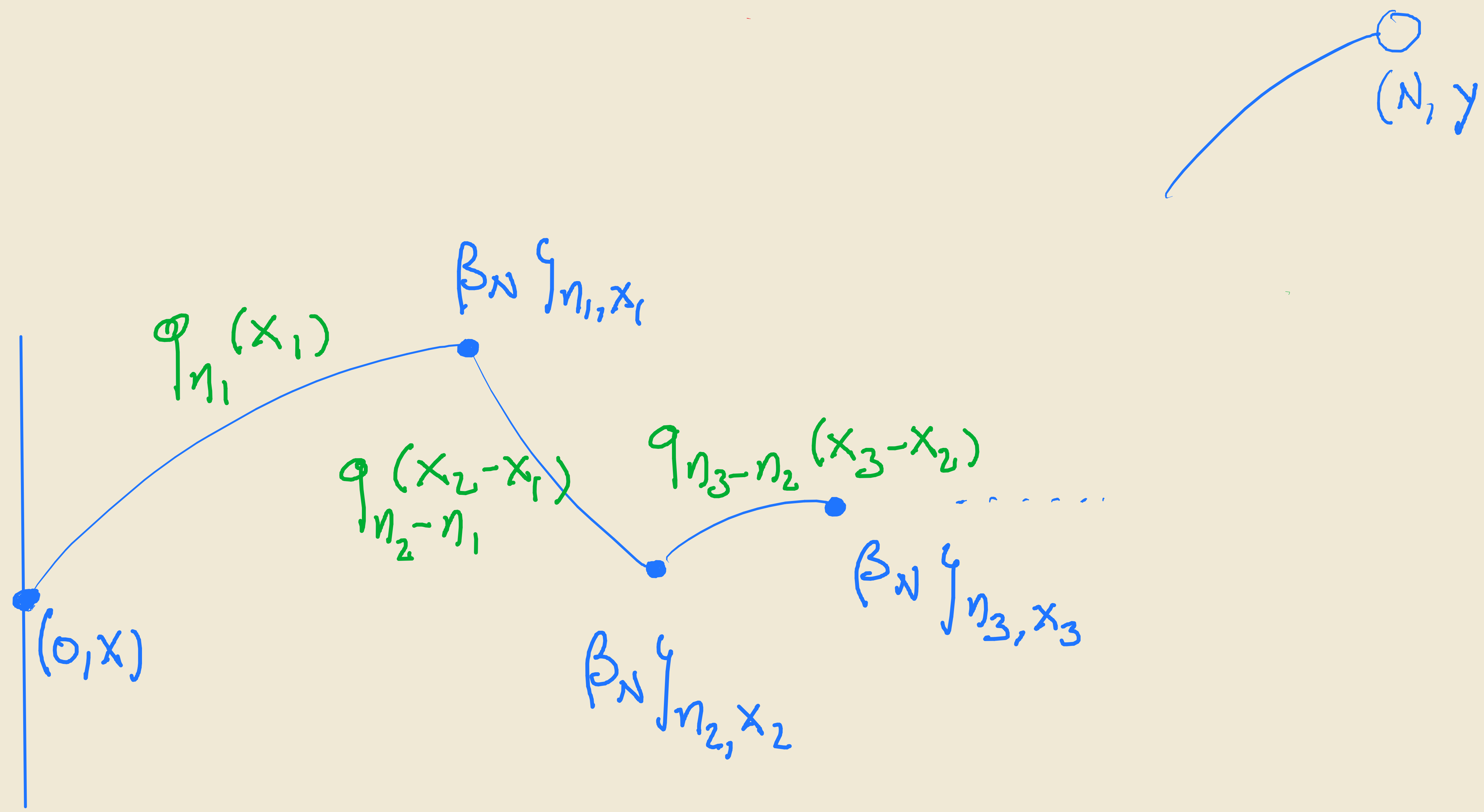
$$\mathbb{E}\omega = 0, \quad \mathrm{Var}(\omega) = 1, \quad \lambda(\beta) := \log \mathbb{E}e^{\beta\omega} < \infty$$

Averaged partition

$$Z_N^\beta(\phi,\varphi) := \frac{1}{N}\sum_{x,y\in\mathbb{Z}^2} \phi\left(\frac{x}{\sqrt{N}}\right) Z_N^\beta(x,y)\, \varphi\left(\frac{y}{\sqrt{N}}\right)$$

Chaos Expansion

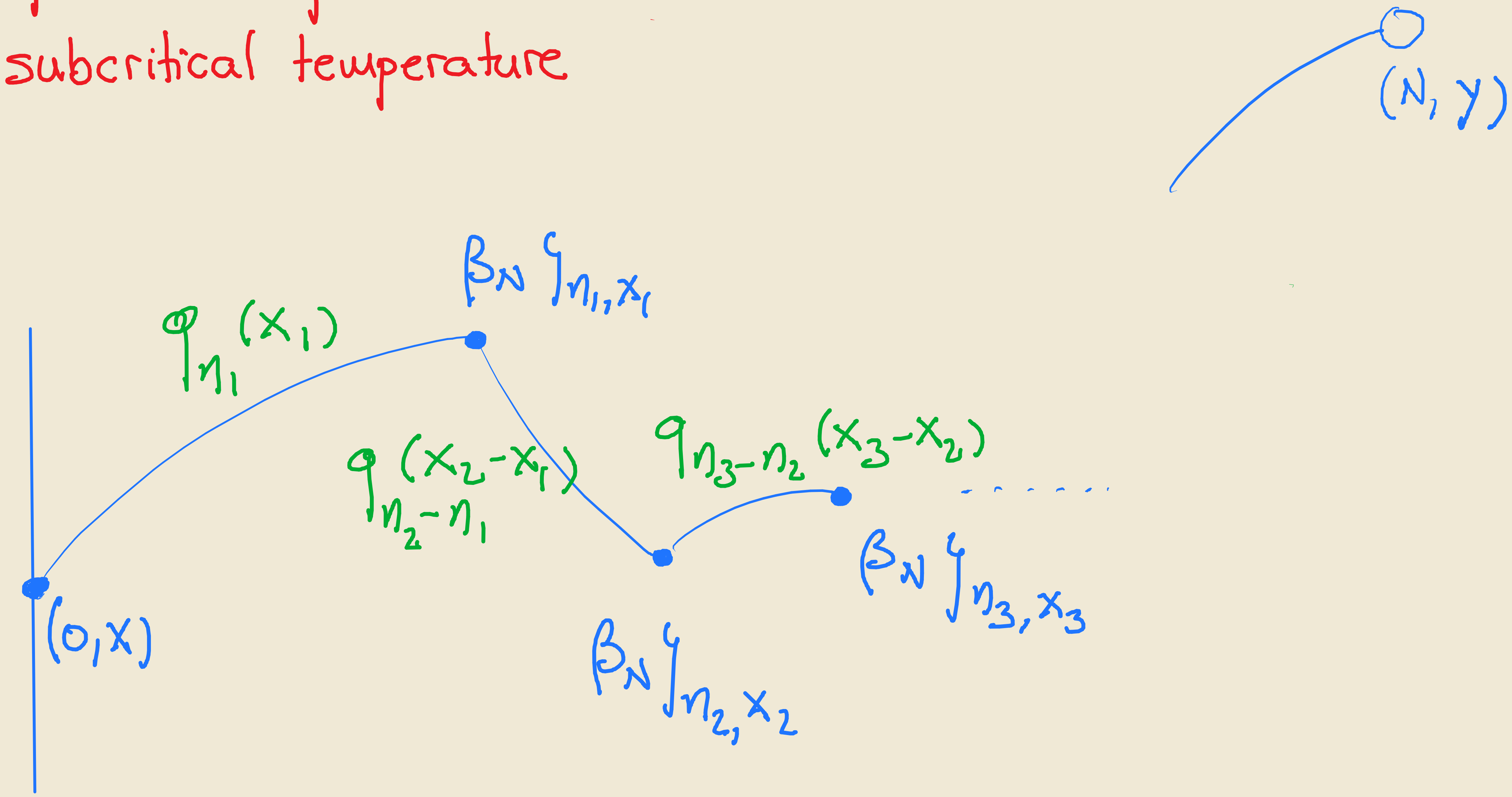
$$Z_{N,\beta_N}(x,y) = 1 + \sum_{k} \sum_{\substack{n_1 < \cdots < n_k \\ x_1, \dots, x_k}}$$



Separation of Scales

at subcritical temperature

$$Z_{N,\beta_N}(x,y) = 1 + \sum_{k} \sum_{\substack{n_1 < \cdots < n_k \\ x_1, \ldots, x_k}}$$



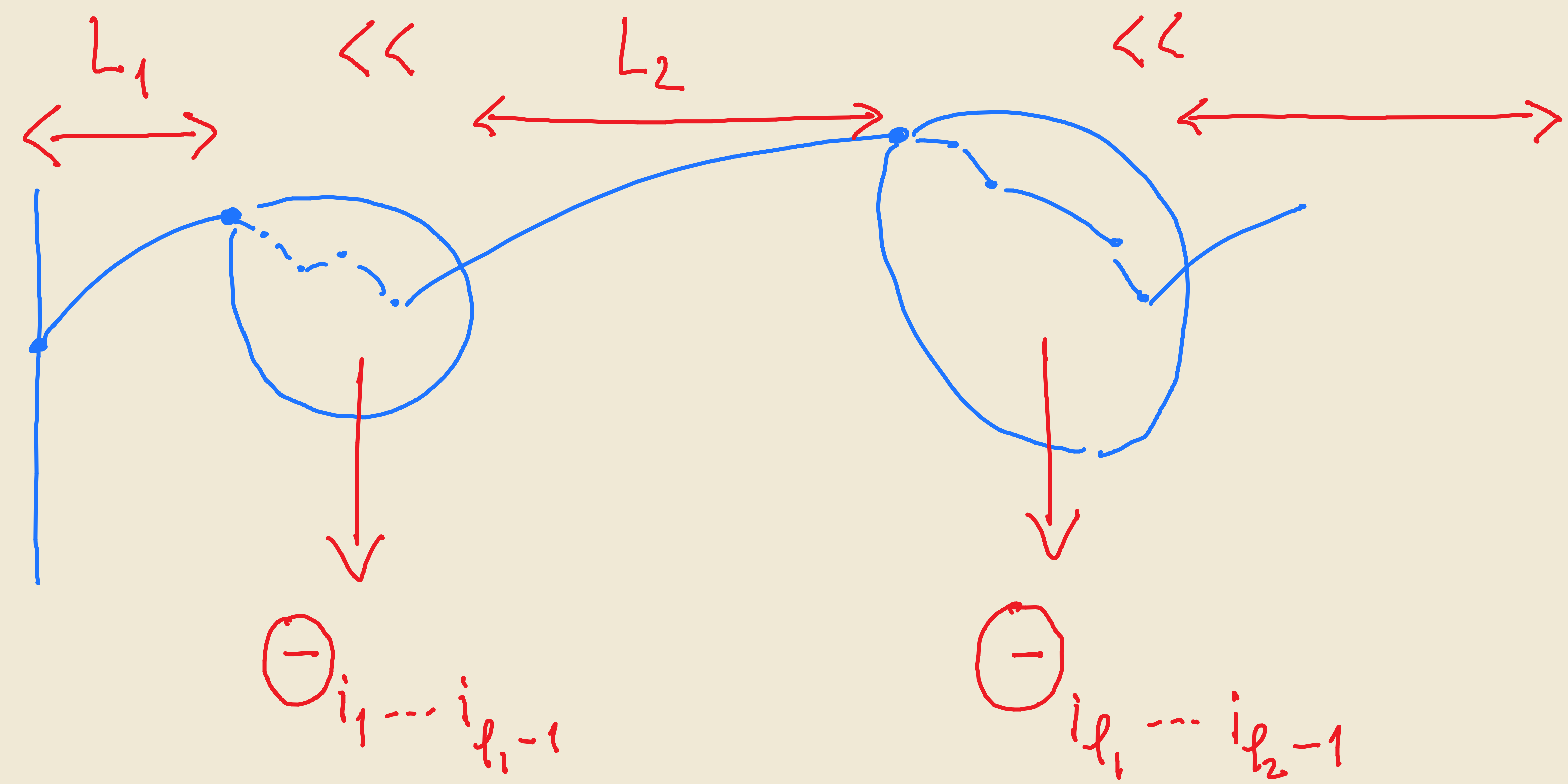
Main Contribution $\quad n_j - n_{j-1} \simeq N^{\alpha} \quad$ with $\alpha < 1$

Split over scales

$$Z_{N,\beta_N} \cong 1 + \sum_{k} \sum_{i_1 \dots i_k} \sum_{n_j - n_{j-1} \simeq N^{i_j/M}} \text{Weight}$$

Running Maxima

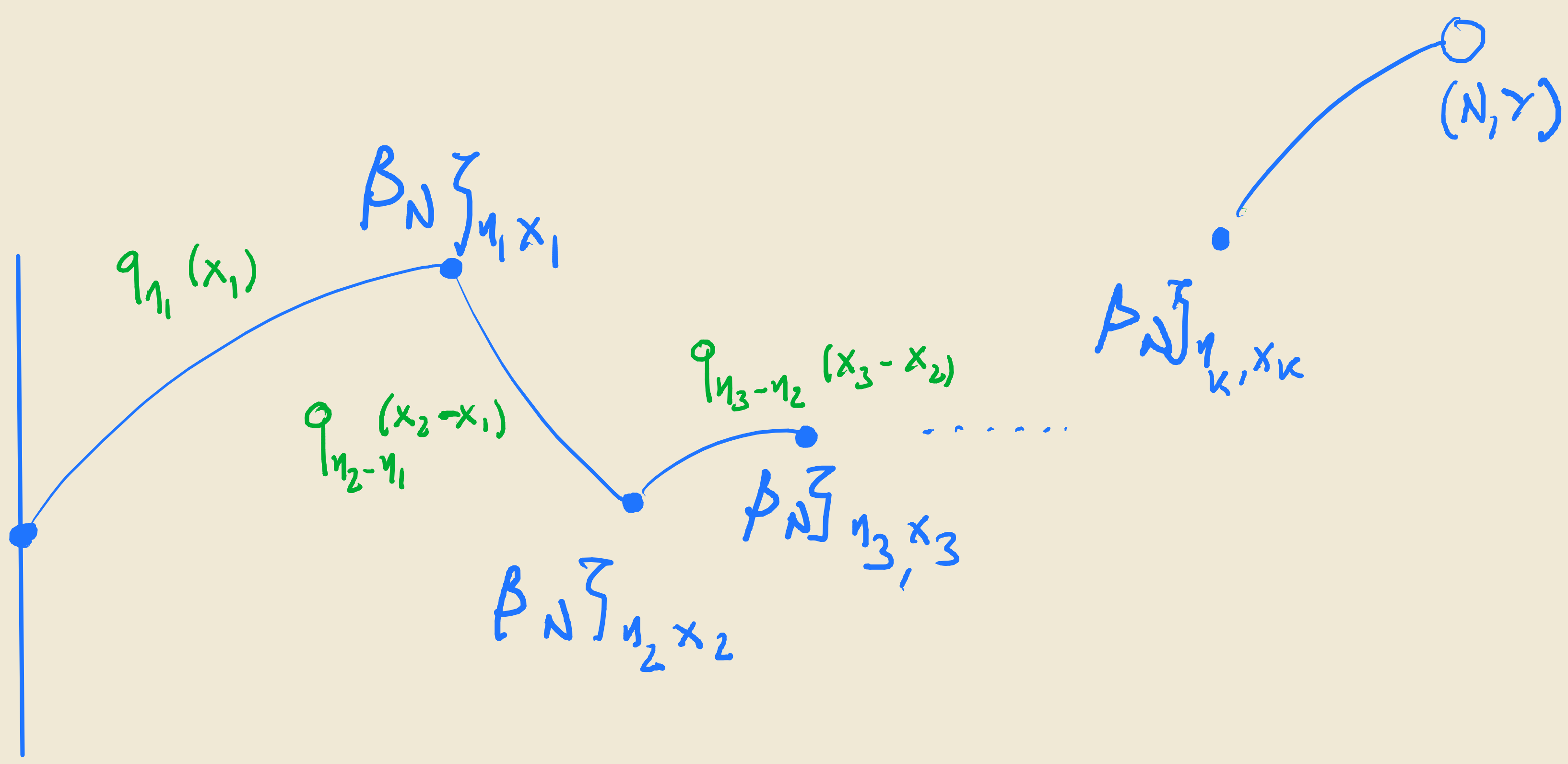
$$Z_{N,\beta_N} = 1 + \sum_{k} \sum_{i_1 \dots i_k}$$



Critical Structure

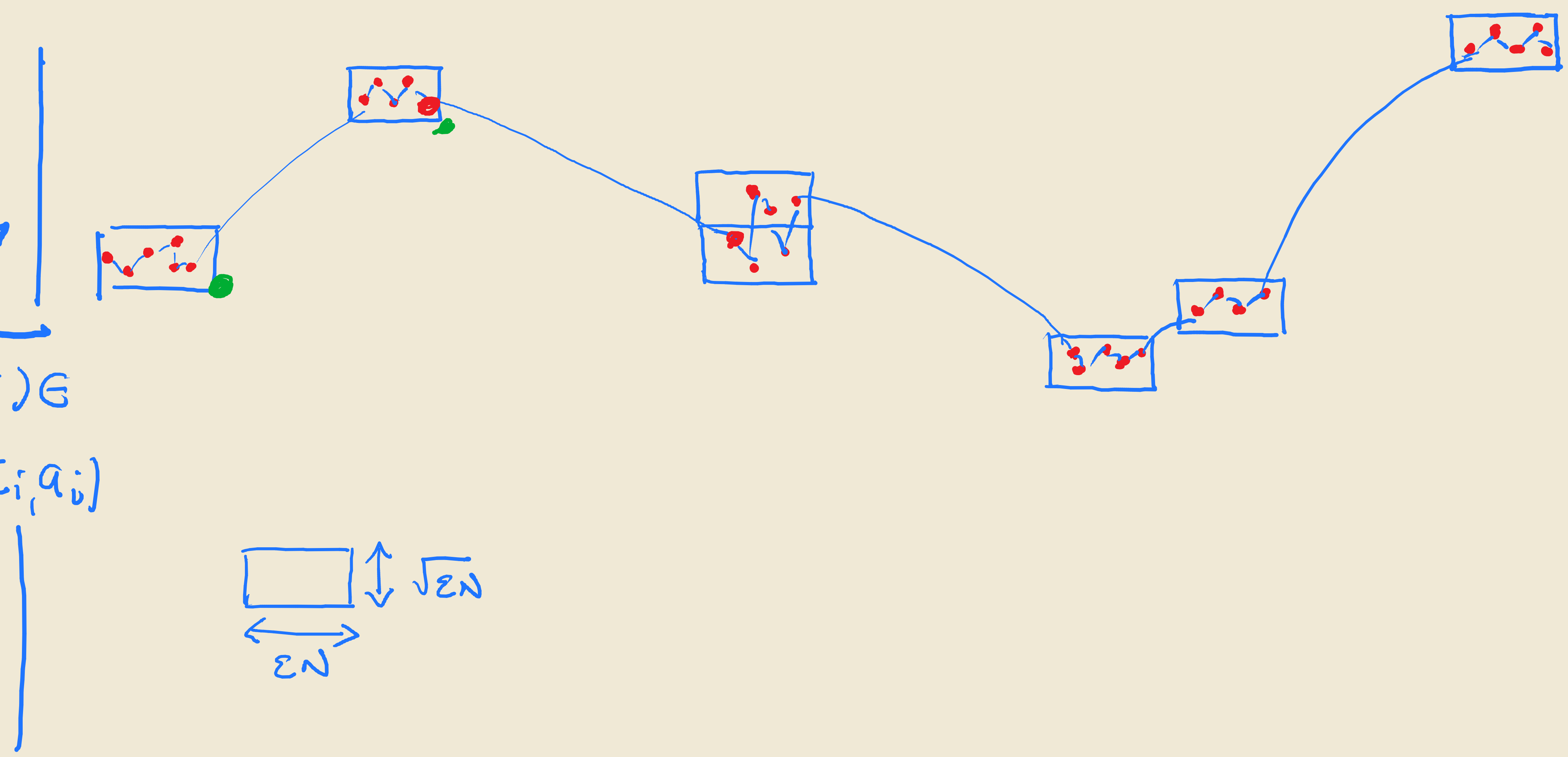
Microscopic Structure

$$Z_{N,\beta_N} = 1 + \sum_{k} \sum_{\substack{n_1 < \cdots < n_k \\ x_1 \ldots x_k}}$$

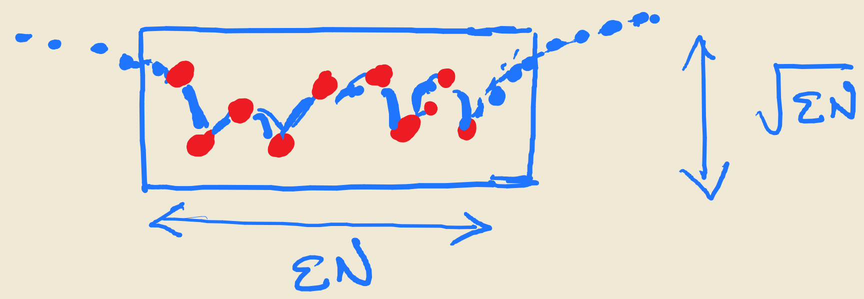


Coarse-Grained Structure

$$\sum_{k} \sum_{\substack{\tau_1, \ldots \tau_k \in \mathbb{N} \\ a_1 \ldots a_k \in \mathbb{Z}^2}} \sum_{(n_i, x_i) \in B(\tau_i, a_i)}$$

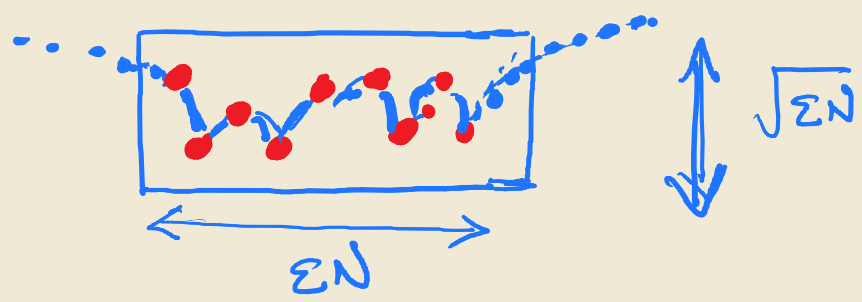


Critical Coarse-Grained Disorder

$\mathrm{Var}\Big(\frac{1}{N^2}$  $\Big) \underset{N\to\infty}{\cong} \frac{\beta_{crit}}{\log 1/\varepsilon}$

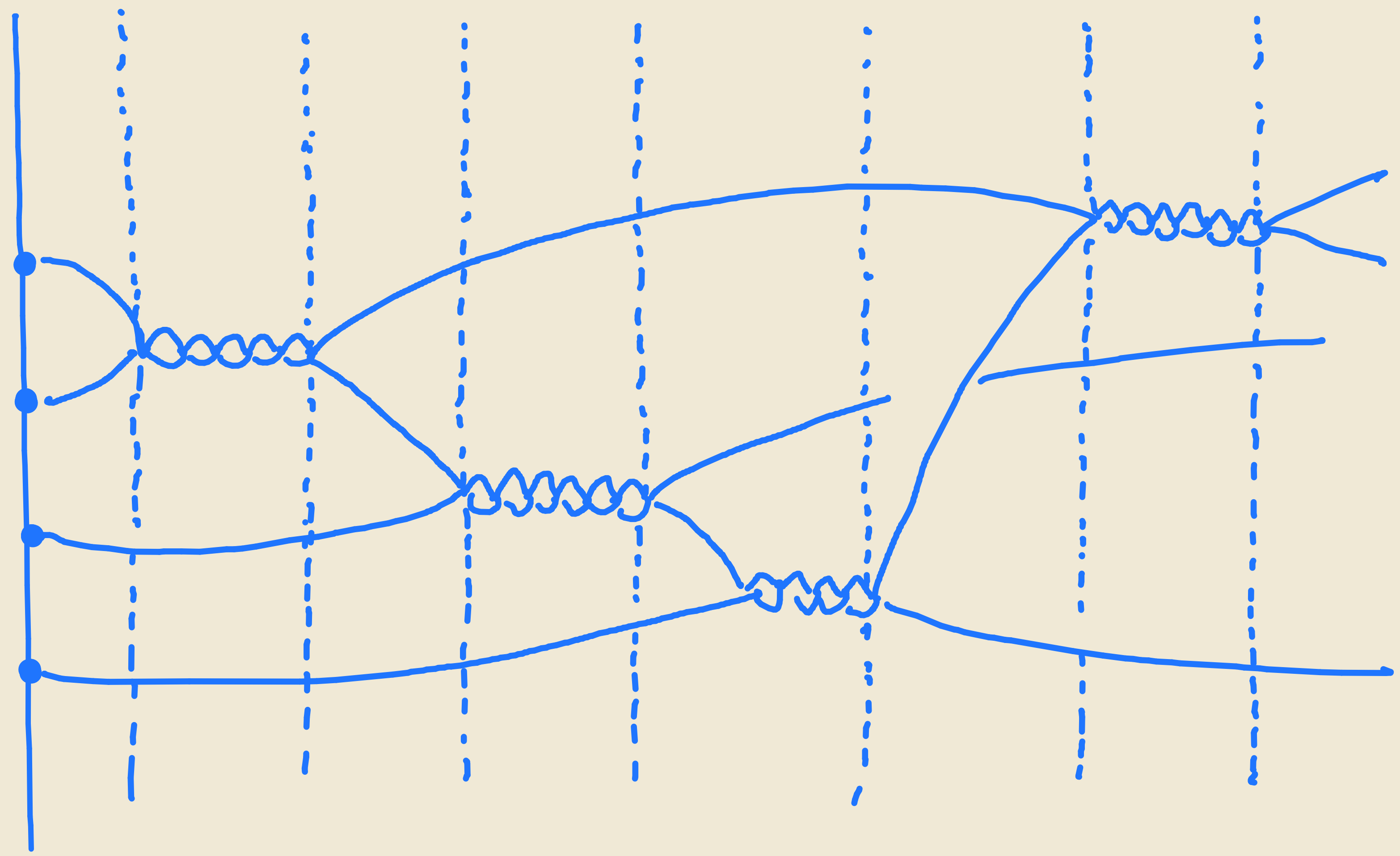
$\Theta_\varepsilon(i,a)$

we obtain these precise estimates through the renewal structure

$\mathrm{Var}\Big(\frac{1}{N^2}$  $\Big) \approx \int_0^\infty e^{\vartheta s}\, \mathrm{P}(Y_s \leq \varepsilon)\, ds$

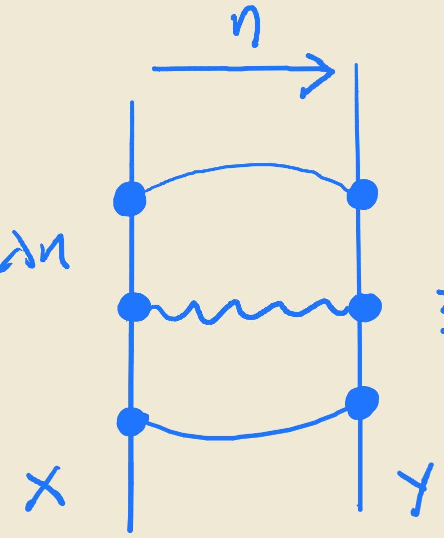
Dickman subordinator

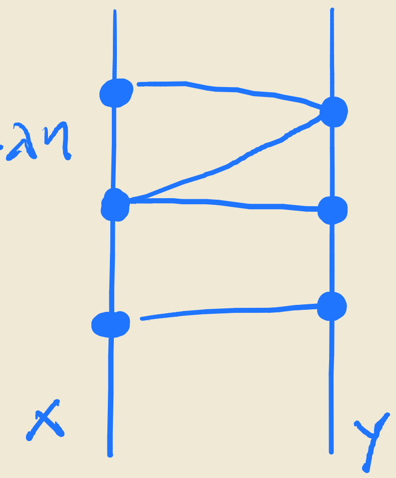
Higher Moments

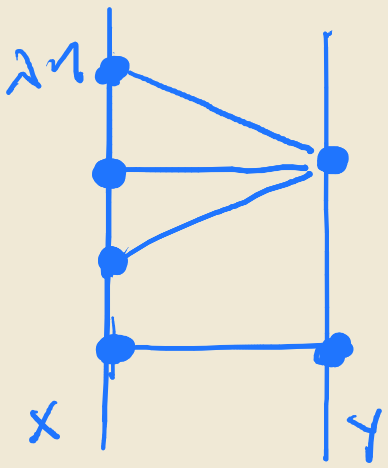
$$\mathbb{E}\, Z_N^4(\phi) =$$ 

OPERATORS

Need to control the norm of the Laplace transform of

Collision evolution operator $\sum_n e^{-\lambda n}$  $\equiv U_{\lambda,N}(x,y)$

The "free" evolution operator $\sum_n e^{-\lambda n}$ 

or $\sum_n e^{-\lambda n}$ 

$\equiv Q^{I,J}_{\lambda,N}(x,y)$

etc.

Critical Hardy-Littlewood-Sobolev Inequality

Prop (CSZ'21)

$\vec{x}, \vec{y} \in (\mathbb{Z}^2)^h$ & $f \in \ell^p((\mathbb{Z}^2)^{h-1})$,

$g \in \ell^q((\mathbb{Z}^2)^{h-1})$,

$$\sum_{\vec{x},\vec{y}} f(\vec{x}) \frac{\delta(x_i - x_j)\,\delta(y_k - y_l)}{|\vec{x} - \vec{y}|^{2h-2}} g(\vec{y}) \le C_{p,q} \|f\|_{\ell^p} \|g\|_{\ell^q}$$

Progeny **Prop** (Dell'Antonio-Figari-Teta '94)

for $L^2((\mathbb{R}^2)^{h-1})$ & Green's function of Δ

Motivation: self-adjoint extension of

$$-\Delta + \sum \delta(x_i - x_j)$$

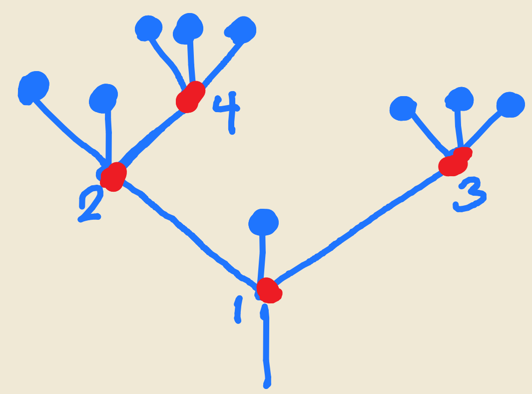
On the Allen-Cahn

$$\partial_t u = \Delta u + u - u^3$$

$$u(0,x) = \frac{\lambda}{\sqrt{\log 1/\varepsilon}}\, \xi(x)$$

$$\Longrightarrow \quad u(t,x) = P_t * \xi_\varepsilon(x) + \int_0^t P_{t-s} * (u - u^3)\, ds$$

$$\Longrightarrow \quad u(t,x) = \sum_{\text{ternary trees}}$$



Contraction Structure