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Model Reduction for Transport Problems via Nonlinear State Space Transformation

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Outline

- Reduced Basis Methods for Nonlinear Evolution Equations:
  Trouble with Advection Dominated Problems.
- The FrozenRB scheme.
  (Joint work with Mario Ohlberger.)
- Nonlinear MOR via Lagrangian Formulation.
  (Joint work in progress with Christoph Lehrenfeld.)

Reduced Basis Approximation of Nonlinear Evolution Equations

Full order problem

Find $u_\mu(t) \in V_h$ such that

$$\partial_t u_\mu(t) + \mathcal{L}_\mu(u_\mu(t)) = 0, \quad u_\mu(0) = u_0,$$

where $\mathcal{L}_\mu : \mathcal{P} \times V_h \to V_h$ is a parametric (nonlinear) Finite Volume operator.

Reduced order problem

For given $V_N \subset V_h$, find $\tilde{u}_\mu(t) \in V_N$ such that

$$\partial_t \tilde{u}_\mu(t) + P_N(\mathcal{L}_\mu(\tilde{u}_{\mu,N}(t))) = 0, \quad \tilde{u}_\mu(0) = P_N(u_0),$$

where $P_N : V_h \to V_N$ is orthogonal projection onto V_N.

Empirical Operator Interpolation (a.k.a. DEIM, EIM)

Problem: Still expensive to evaluate

$$P_N \circ \mathcal{L}_\mu : V_N \longrightarrow V_h \longrightarrow V_N.$$

Solution:

- Use locality of finite volume operators:

  to evaluate M DOFs of $\mathcal{L}_\mu(u)$ we need $M' \leq C \cdot M$ DOFs of u.

- Approximate

$$\mathcal{L}_\mu \approx \mathcal{I}_M[\mathcal{L}_\mu] := I_M \circ \mathcal{L}_{M,\mu} \circ R_{M'},$$

  where

$R_{M'}: V_h \to \mathbb{R}^{M'}$	restriction to M' DOFs needed for evaluation
$\mathcal{L}_{M,\mu}: \mathbb{R}^{M'} \to \mathbb{R}^M$	$\mathcal{L}_\mu$ restricted to M interpolation DOFs
$I_M: \mathbb{R}^M \to V_h$	linear combination with interpolation basis

- Use greedy algorithm to determine DOFs and interpolation basis from operator evaluations on appropriate solution trajectories.

Full Reduction

Reduced order problem (with EI)

Find $\tilde{u}_\mu(t) \in V_N$ such that

$$\partial_t \tilde{u}_\mu(t) + \{(P_N \circ I_M) \circ \mathcal{L}_{M,\mu} \circ R_{M'}\}(\tilde{u}_{\mu,N}(t)) = 0, \quad \tilde{u}_\mu(0) = P_N(u_0).$$

Offline/Online decomposition

- Precompute the linear operators $P_N \circ I_M$ and $R_{M'}$ w.r.t. basis of V_N.
- Effort to evaluate $(P_N \circ I_M) \circ \mathcal{L}_{M,\mu} \circ R_{M'}$ w.r.t. this basis:

$$\mathcal{O}(MN) + \mathcal{O}(M) + \mathcal{O}(MN).$$

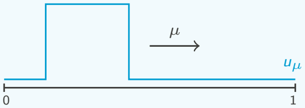
- Use POD-Greedy algorithm for the construction of V_N.

Trouble with Advection Dominated Problems

Typically slow decay of Kolmogorov N-widths d_N of the solution manifold, but RB will only work well for rapid decay!

$$d_N := \inf_{\substack{V_N \subseteq V_h \\ \dim V_N \leq N}} \sup_{\substack{\mu \in \mathcal{P} \\ t \in [0,T]}} \inf_{v \in V_N} \|u(t) - v\|.$$

Basic example



$$\partial_t u(t,x) + \mu \cdot \partial_x u_\mu(t,x) = 0$$
$$u_\mu(0,x) = u_0(x),\ u_\mu(0,t) = u_\mu(1,t)$$
$$\mu, x, t \in [0,1]$$

Here: $d_N \sim N^{-1/2}$ w.r.t. $L^2([0,1])$.

The FrozenRB Scheme

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Typically slow decay of Kolmogorov N-widths d_N of the solution manifold, but RB will only work well for rapid decay!

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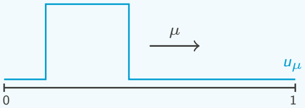

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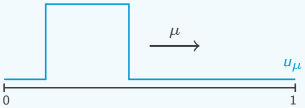

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Here: $d_N \sim N^{-1/2}$ w.r.t. $L^2([0,1])$.

However: We can describe solution easily as

$$u_\mu(t,x) = u_0(x - \mu \cdot t \mod 1).$$

Nonlinear Approximation

- Write $u_\mu(t, x)$ as

$$u_\mu(t, x) = u_0(x - \mu \cdot t \mod 1) =: ((\mu \cdot t) \,.\, u_0)(x)$$

Nonlinear Approximation

- Write $u_\mu(t,x)$ as

$$u_\mu(t,x) = u_0(x - \mu \cdot t \mod 1) =: ((\mu \cdot t) \,.\, u_0)(x)$$

- **General idea:** Write $u_\mu(t,x)$ as

$$u_\mu(t,x) = g_\mu(t).v_\mu(t,x)$$

dynamics of u_μ
large variation in time

shape of u_μ
small variation in time

where $\mathcal{V}$ function space, $v_\mu(t) \in \mathcal{V}$ and $g_\mu(t)$ is element of Lie group G acting on $\mathcal{V}$.

- $v_\mu(t,x)$ should be easier to approximate than $u_\mu(t,x)$!

Method of Freezing [Beyn, Thümmler, 2004], [Rowley et. al., 2000, 2003]

- Consider Lie group G acting on $\mathcal{V}$ and evolution equation of the form:

$$\partial_t u_\mu(t) + \mathcal{L}_\mu(u_\mu(t)) = 0, \quad u_\mu(0) = u_0, \quad u_\mu(t) \in \mathcal{V}$$

- Substituting the *ansatz* $u_\mu(t) = g_\mu(t).v_\mu(t)$ leads to:

$$\partial_t v_\mu(t) + g_\mu(t)^{-1}.\,\mathcal{L}_\mu(g_\mu(t).v_\mu(t)) + \mathfrak{g}_\mu(t).v_\mu(t) = 0$$
$$\mathfrak{g}_\mu(t) = g_\mu(t)^{-1}\partial_t g_\mu(t).$$

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- Have $\dim(G)$ additional degrees of freedom.
$\rightarrow$ Add additional algebraic constraint (phase condition):

$$\Phi(v_\mu(t), \mathfrak{g}_\mu(t)) = 0.$$

- Further assume invariance of $\mathcal{L}_\mu$ under action of G:

$$h^{-1}.\,\mathcal{L}_\mu(h.w) = \mathcal{L}_\mu(w) \quad \forall h \in G, w \in \mathcal{V}.$$

Method of Freezing [Beyn, Thümmler, 2004], [Rowley et. al., 2000, 2003]

Definition (Method of Freezing)

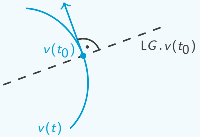
With initial conditions $v_\mu(0) = u(0),\ g_\mu(0) = e$, solve:

$$\begin{aligned} \partial_t v_\mu(t) + \mathcal{L}_\mu(v_\mu(t)) + \mathfrak{g}_\mu(t).v_\mu(t) &= 0 \\ \Phi(v_\mu(t), \mathfrak{g}_\mu(t)) &= 0 \end{aligned} \qquad \text{frozen PDAE}$$

$$\mathfrak{g}_\mu(t) = g(t)_\mu^{-1}\partial_t g_\mu(t) \qquad \text{reconstruction equation}$$

Orthogonality phase condition

$$\begin{aligned} \Phi(v, \mathfrak{g}) = 0 \ &:\Longleftrightarrow \partial_t v(t) \perp LG.v(t) \\ &\Longleftrightarrow (\mathcal{L}(v) + \mathfrak{g}.v,\ \mathfrak{h}.v) = 0 \quad \forall \mathfrak{h} \in LG \end{aligned}$$



Example: 2D-Shifts

Consider $G = \mathbb{R}^2$, $LG = \mathbb{R}^2$ acting via

$$\begin{aligned} g.u(x) &:= u(x - g), \quad x \in \mathbb{R}^2 \\ \mathfrak{g}.u &= -\mathfrak{g} \cdot \nabla u \end{aligned}$$

The Method of Freezing for 2D-shifts

Solve

$$\begin{gathered} \partial_t v_\mu(t) + \mathcal{L}_\mu(v_\mu(t)) - \mathfrak{g}_\mu(t) \cdot \nabla v_\mu(t) = 0 \\ \left[(\partial_{x_i} v_\mu,\, \partial_{x_j} v_\mu)\right]_{i,j} \cdot \left[\mathfrak{g}_\mu\right]_j = \left[(\mathcal{L}_\mu(v_\mu),\, \partial_{x_i} v_\mu)\right]_i \end{gathered}$$

and

$$\partial_t g_\mu(t) = \mathfrak{g}_\mu(t)$$

with initial conditions $v_\mu(0) = u(0)$, $g_\mu(0) = (0, 0)^T$.

Test Problem

2D Burgers-type problem

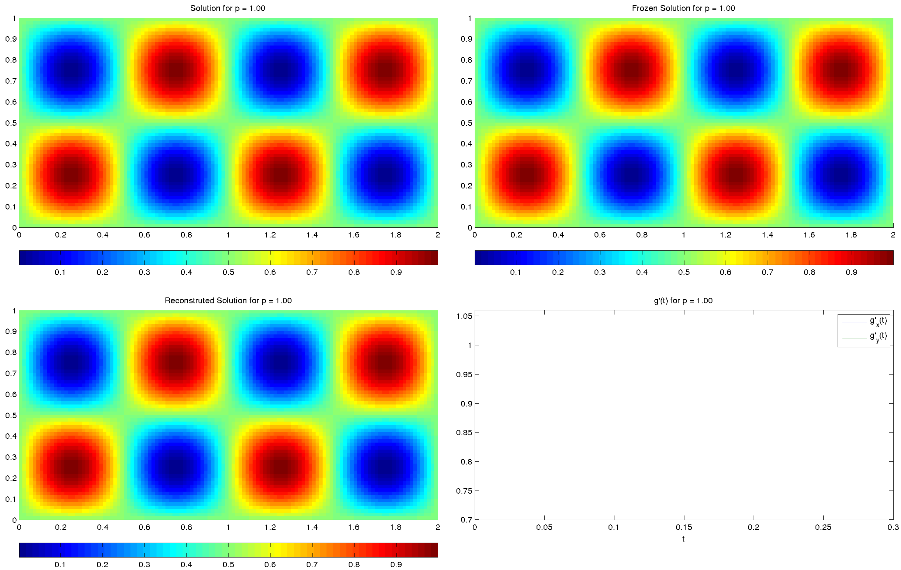
Solve on $\Omega = [0,2] \times [0,1]$:

$$\partial_t u + \nabla \cdot (\vec{v} \cdot u^\mu) = 0$$
$$u(0, x_1, x_2) = 1/2(1 + \sin(2\pi x_1)\sin(2\pi x_2))$$

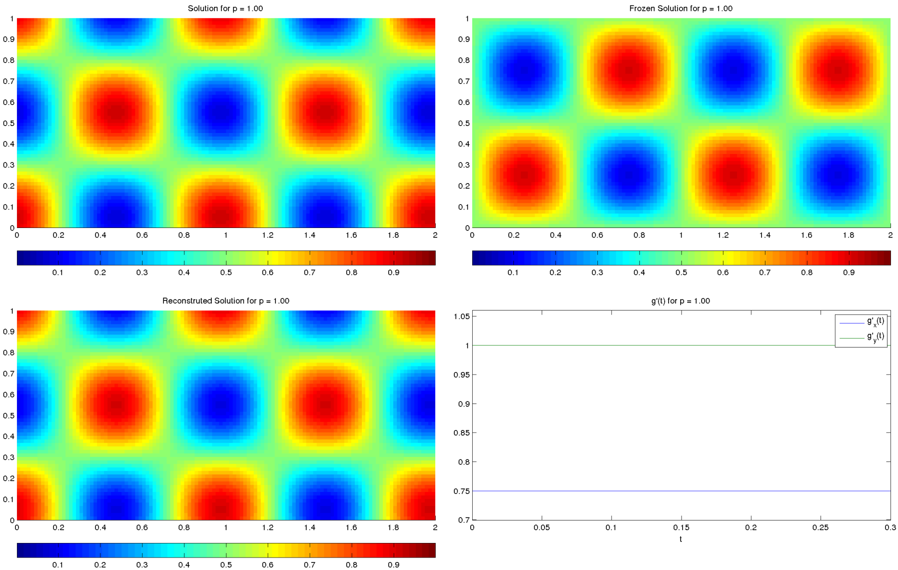
for $t \in [0, 0.3]$, $\vec{v} \in \mathbb{R}$ with periodic boundary conditions and $\mu \in \mathcal{P} = [1, 2]$.

- Finite volume (Lax-Friedrichs) space discretization on 240 x 120 grid.
- Explicit Euler time-stepping (200 time steps).
- Same problem as in [Drohmann, Haasdonk, Ohlberger, 2012].
- (The following videos are actually computed on a 120 x 60 grid.)

Frozen vs. Non-frozen Solution $(\mu = 1, \vec{v} = (0.75, 1)^T)$



Frozen vs. Non-frozen Solution ($\mu = 1, \vec{v} = (0.75, 1)^T$)



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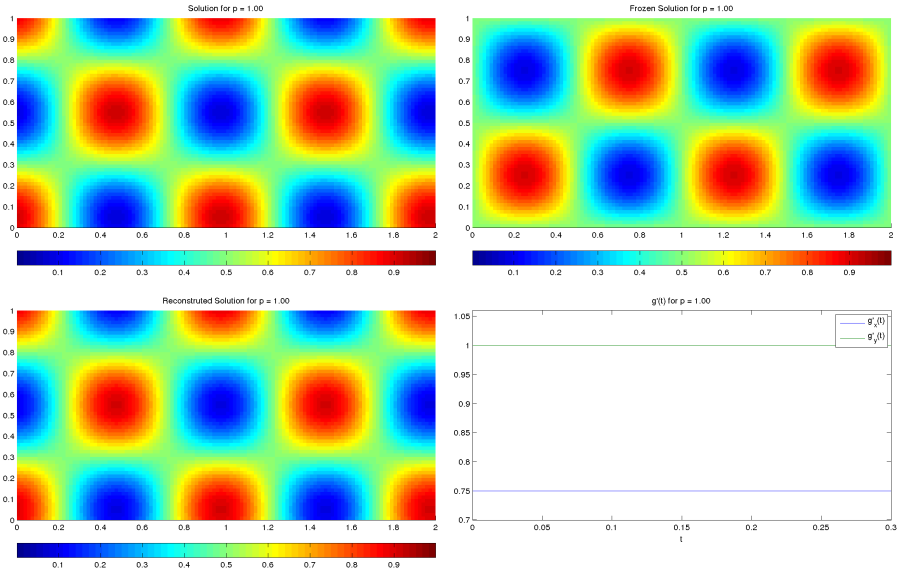
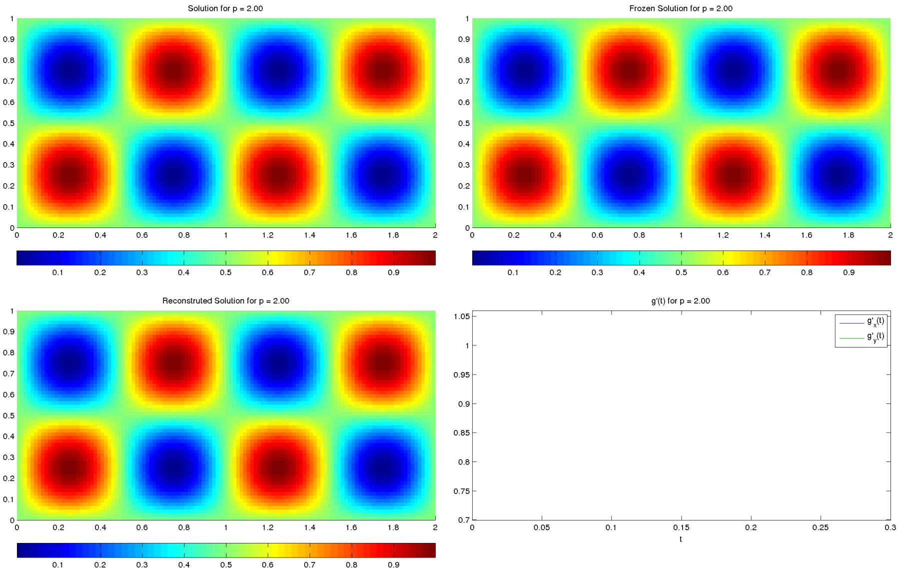


Frozen vs. Non-frozen Solution $(\mu = 2, \vec{v} = (0.75, 1)^T)$



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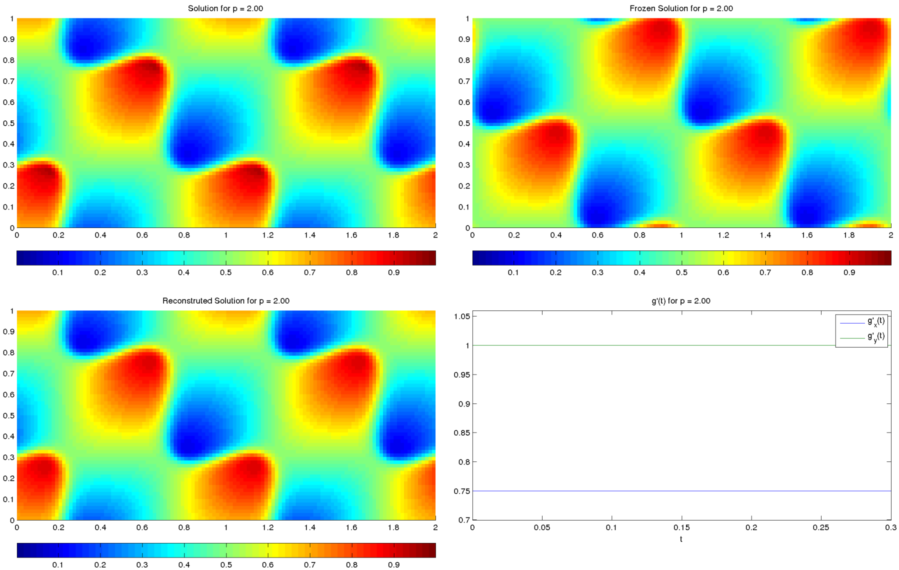

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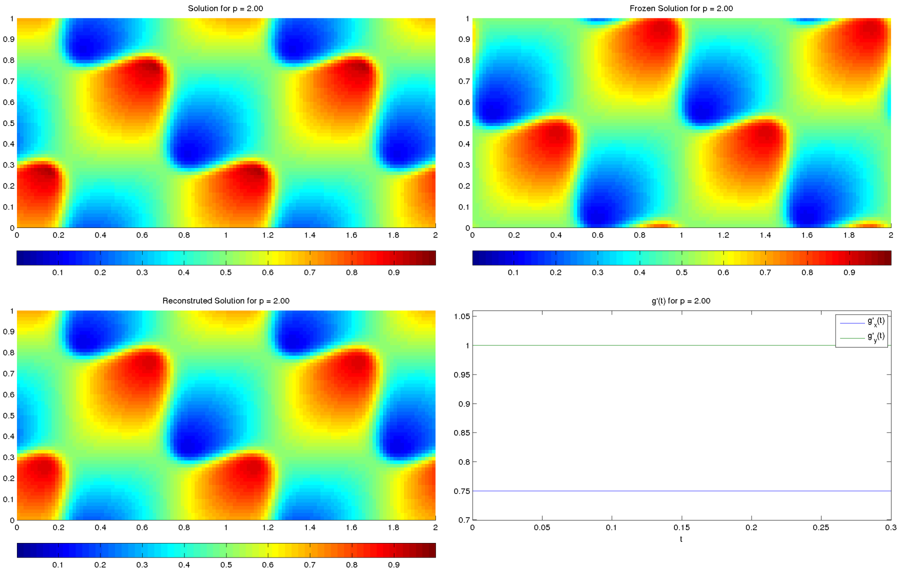


Combining RB with the Method of Freezing

FrozenRB-Scheme for 2D-shifts [Ohlberger, R, 2013]

Solve

$$\partial_t \tilde{v}_{\mu(t)} + P_N \circ \mathcal{I}_M[\mathcal{L}_\mu](\tilde{v}_\mu(t)) - \tilde{\mathfrak{g}}_{\mu(t)} \cdot (P_N \circ \nabla)(\tilde{v}_\mu(t)) = 0$$
$$\left[(\partial_{x_i}\tilde{v}_\mu, \partial_{x_j}\tilde{v}_\mu)\right]_{i,j} \cdot \left[\tilde{\mathfrak{g}}_\mu\right]_j = \left[(\mathcal{I}_M[\mathcal{L}_\mu](\tilde{v}_\mu), \partial_{x_i}\tilde{v}_\mu)\right]_i$$

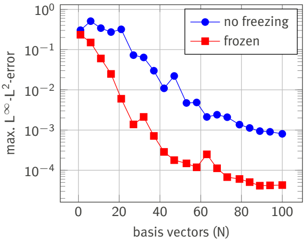
and

$$\partial_t \tilde{g}_\mu(t) = \tilde{\mathfrak{g}}_\mu(t)$$

with initial conditions $\tilde{v}_\mu(0) = u(0)$, $\tilde{g}_\mu(0) = (0,0)^T$.

- EI-Greedy, POD-Greedy algorithms for basis generation.
- Full offline/online decomposition.
- No additional evaluations of nonlinearity (small overhead).

Results for the Burgers Problem $(\vec{v} = (1,1)^T)$

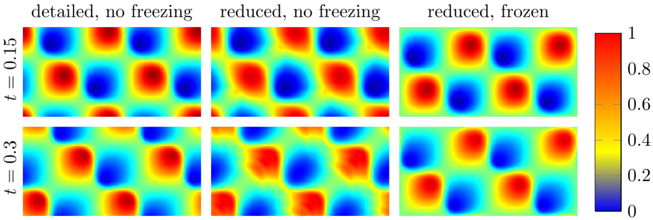


Left:

- $1.9 \cdot N$ interpolation points.
- Test set: 100 random μ.

Bottom:

- $\dim V_N = 20$, 38 interpolation points.
- $\mu = 1.5$.



Nonlinear MOR via Lagrangian Formulation

A Free Boundary Problem

Osmotic cell swelling model

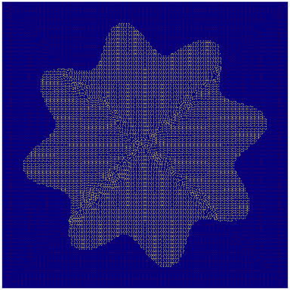
$$\begin{aligned} \partial_t u - \alpha \Delta u &= 0 && \text{in } \Omega(t) \\ \mathcal{V}_n u + \alpha \partial_n u &= 0 && \text{on } \partial\Omega(t) \\ -\beta\kappa + \gamma(u - u_0) &= \mathcal{V}_n && \text{on } \partial\Omega(t) \end{aligned}$$

- u: concentration field
- u_0: concentration in outside
- $\mathcal{V}_n$: normal velocity of $\partial\Omega(t)$
- κ: curvature of $\partial\Omega(t)$

osmosis

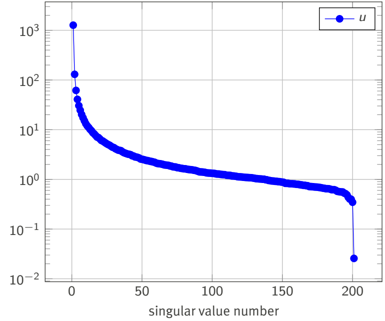
Eulerian Approximation in $L^2(\mathbb{R}^2)$

- Could consider $u(t) \in L^2(\Omega(t)) \hookrightarrow L^2(\mathbb{R}^2)$ to define joint approximation space.



Eulerian Approximation in $L^2(\mathbb{R}^2)$

- Could consider $u(t) \in L^2(\Omega(t)) \hookrightarrow L^2(\mathbb{R}^2)$ to define joint approximation space.
- However, moving domain boundary leads to slow singular value decay of solution trajectory:



osmosis

Lagrangian Formulation

- Fix reference domain $\widehat{\Omega}$ and introduce deformation field $\Psi(t)$ s.t. $\Psi(t)(\widehat{\Omega}) = \Omega(t)$.
- Time-discrete concentration equation on $\widehat{\Omega}$,

$$\int_{\hat{\Omega}} J_{n+1}\hat{u}_{n+1}\hat{v}\,dx + \Delta t \int_{\hat{\Omega}} J_{n+1}\partial_t \Psi_{n+1} \cdot (\partial_x \Psi_{n+1}^{-T} \cdot \nabla_{\hat{x}}\hat{v})\hat{u}_{n+1} dx + \Delta t \int_{\hat{\Omega}} \alpha J_{n+1}(\partial_x \Psi_{n+1}^{-T} \nabla_{\hat{x}} u) \cdot (\partial_x \Psi_{n+1}^{-T} \nabla_{\hat{x}} \hat{v})\,dx = \int_{\hat{\Omega}} J_n \hat{u}_n \hat{v}\,dx,$$

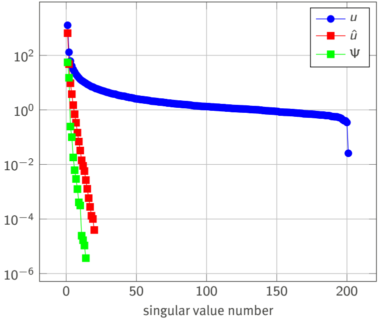
where $J_n := \left|\det(\partial_x \Psi_n)\right|$.

- Compute updated Ψ_{n+1} on $\partial\widehat{\Omega}$, and extend to $\widehat{\Omega}$ via harmonic extension.

osmosis

- After space discretization this corresponds to moving-mesh approach ($\rightarrow$ ALE), where $\Psi(t)(v)$ is the trajectory of the vertex v.

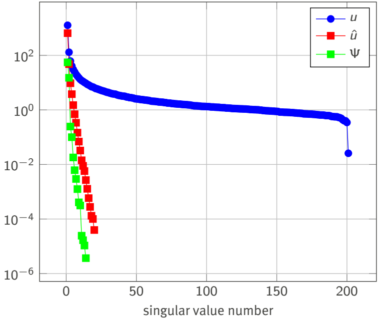
Nonlinear MOR via Lagrangian Formulation



Lagrangian ROM construction:

- Both trajectories $\hat{u}(t)$, $\Psi(t)$ are smooth and exhibit fast singular value decay.
- Compute low-rank approximation spaces $V_{\hat{u}}$, V_{Ψ} via POD.
- Note: V_{Ψ} acts nonlinearly on $V_{\hat{u}}$.
- Use Matrix-DEIM to approximate system matrices.

Nonlinear MOR via Lagrangian Formulation



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Preliminary MOR results:

- FOM: 2796 / 5592 DOFs
- ROM: 12 / 11 DOFs
- 10 / 14 / 12 / 16 interpolation points
- rel. space-time error: 10^{-4}

Thank you for your attention!

My homepage
`http://stephanrave.de/`

Ohlberger, R, *Nonlinear reduced basis approximation of parameterized evolution equations via the method of freezing*, C. R. Math. Acad. Sci. Paris, 351 (2013).

Ohlberger, R, *Reduced Basis Methods: Success, Limitations and Future Challenges*, Proceedings of ALGORITMY 2016.