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Localized Reduced Basis Domain Decomposition Methods

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Outline

1. Reduced Basis Methods for Elliptic Problems

2. Localized Reduced Basis Methods for Elliptic Problems

3. Localized Reduced Basis Domain Decomposition Methods for Elliptic Problems

Reduced Basis Methods for Elliptic Problems

Reduced Basis Methods

Parametric linear elliptic problem (full order model)

For given parameter $\mu \in \mathcal{P}$, find $u_h(\mu) \in V_h$ s.t.

$$\begin{aligned} a(u_h(\mu), v_h; \mu) &= f(v_h) \qquad \forall v_h \in V_h \\ y_h(\mu) &= g(u_h(\mu)) \end{aligned}$$

Parametric linear elliptic problem (reduced order model)

For given $V_N \subset V_h$, let $u_N(\mu) \in V_N$ be given by Galerkin proj. onto V_N, i.e.

$$\begin{aligned} a(u_N(\mu), v_N; \mu) &= f(v_N) \qquad \forall v_N \in V_N \\ y_N(\mu) &= g(u_N(\mu)) \end{aligned}$$

RB Methods – Computing V_N

Weak greedy basis generation

function WEAK-GREEDY($\mathcal{S}_{train} \subset \mathcal{P}$, ε)
  $V_N \leftarrow \{0\}$
  while $\max_{\mu \in \mathcal{S}_{train}}$ ERR-EST(ROM-SOLVE(μ), μ) $> \varepsilon$ **do**
    $\mu^* \leftarrow$ arg-max$_{\mu \in \mathcal{S}_{train}}$ ERR-EST(ROM-SOLVE(μ), μ)
    $V_N \leftarrow$ span($V_N \cup$ {FOM-SOLVE(μ^*)})
  end while
  return V_N
end function

ERR-EST

Use residual-based error estimate w.r.t. FOM (finite dimensional $\rightsquigarrow$ can compute dual norms).

- Use dual weighted residual approach for improved convergence w.r.t to output $y_N(\mu)$.

RB Methods – Online Efficiency

Parametric linear elliptic problem (reduced order model)

For given $V_N \subset V_h$, let $u_N(\mu) \in V_N$ be given by Galerkin proj. onto V_N, i.e.

$$a(u_N(\mu), v_N; \mu) = f(v_N) \qquad \forall v_N \in V_N$$
$$y_N(\mu) = g(u_N(\mu))$$

Affine decomposition

Assume that a_μ can be written as

$$a(u, v; \mu) = \sum_{q=1}^{Q} \theta_q(\mu) a_q(u, v).$$

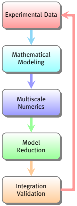
Offline/Online splitting

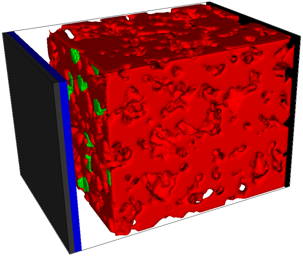
By pre-computing

$$a_q(\varphi_i, \varphi_j), \; f(\varphi_i), \; g(\varphi_i)$$

for a reduced basis $\varphi_1, \ldots, \varphi_N$ of V_N, solving ROM becomes independent of dim V_h.

Example: RB Approximation of Li-Ion Battery Models





MULTIBAT: Gain understanding of degradation processes in rechargeable Li-Ion Batteries through mathematical modeling and simulation at the pore scale.

FOM:

- 2.920.000 DOFs
- Simulation time: $\approx$ 15.5h

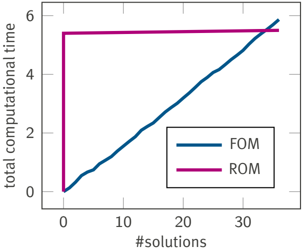
ROM:

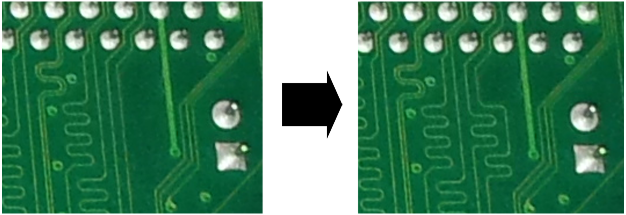
- Snapshots: 3
- $\dim V_N = 245$
- Rel. err.: $< 4.5 \cdot 10^{-3}$
- Reduction time: $\approx$ 14h
- Simulation time: $\approx$ 8m
- Speedup: 120

Localized Reduced Basis Methods for Elliptic Problems

Caveats

- Offline time too large in not-so-many-query scenarios?
- $\mathcal{P}$ too large?
- Only local influence of μ?
- Local geometry changes?



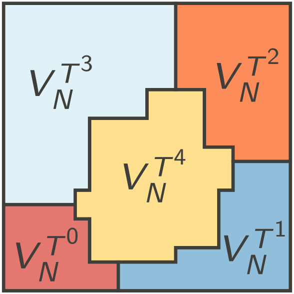


Localized RB Methods for Elliptic Problems

Idea of the **LRBMS**: given a finely-resolved grid τ_h [Albrecht et al., 2012]

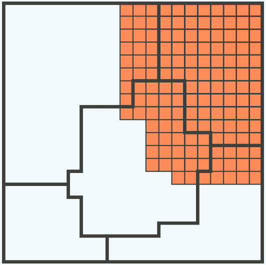
- decompose approximation space into *local* spaces $V_h = \bigoplus_{T \in \mathcal{T}_H} V_h^T$
- associated with subdomains $T \in \mathcal{T}_H$
  independent local discretizations and approximation spaces (CG or DG)
  and global SWIPDG coupling [Ern, Stephansen, Zunino, 2009]
- build local reduced spaces $V_N^T \subset V_h^T$
- reduced *broken* space $V_N = \bigoplus_{T \in \mathcal{T}_H} V_N^T$

- larger V_N, but sparse ROM system matrices
- initialization of V_N^T:
  - empty
  - global solution snapshots
  - **local training**



Offline Initialization of V_N

Training algorithm (adapted from [Buhr, Engwer, Ohlberger, R, 2017]) **for all $T \in \mathcal{T}_H$**



- For every $\mu \in \mathcal{S}_{train} \subset \mathcal{P}$:
  - Solve training problem on oversampling subdomain $T^\delta \supset T$:

$$a(\varphi_{h,0}(\mu), v_h; \mu) = f(v_h) \quad \text{in } T^\delta$$
$$\varphi_{h,0}(\mu) = 0 \quad \text{on } \partial T^\delta$$

  - For $1 \leq k \leq K$, solve training problem:

$$a(\varphi_{h,k}(\mu), v_h; \mu) = 0 \quad \text{in } T^\delta$$
$$\varphi_{h,k}(\mu) = g_k \quad \text{on } \partial T^\delta$$

    for K random Dirichlet data functions g_k on ∂T^δ.

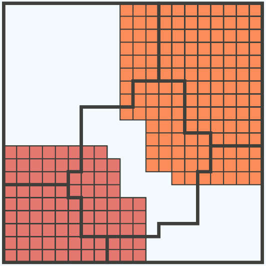
- Initialize local RB space on T as

$$V_N^T := \operatorname{span} \bigcup_{\mu \in \mathcal{S}_{train}} \{ \varphi_{h,0}(\mu)|_T, \ldots, \varphi_{h,K}(\mu)|_T \}.$$

- Use greedy algorithm for large $\mathcal{S}_{train}$.

Online-Adaptive Enrichment of V_N

Enrichment algorithm — for some $\mu \in \mathcal{P}$



- compute reduced solution $u_N(\mu)$
- estimate error $\eta_{h,N}(\mu)$
- if $\eta_{h,N}(\mu) > \Delta$, start intermediate local enrichment phase:
  - compute local error indicators
  - mark subdomains for enrichment: $\mathcal{X} = \mathtt{mark}(\mathcal{T}_H)$
  - solve corrector problem on oversampling subdomain $T^\delta \supset T$ for all $T \in \mathcal{X}$:

$$a(\varphi_h(\mu), v_h; \mu) = f(v_h) \qquad \text{in } T^\delta$$

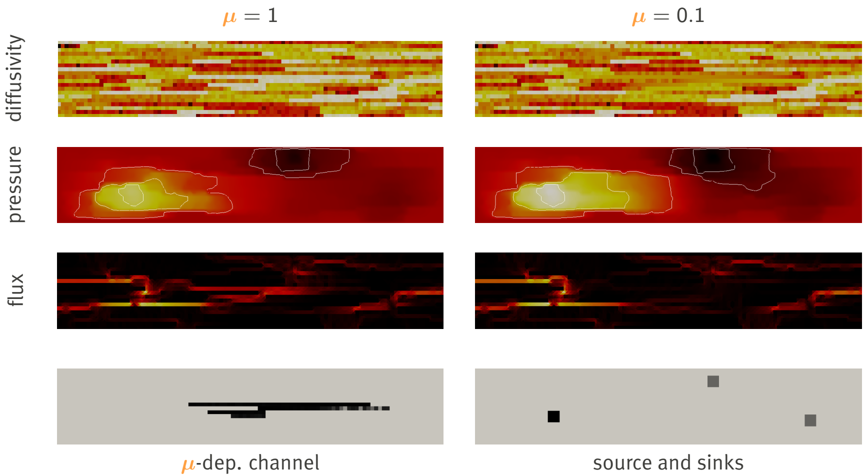
$$\varphi_h(\mu) = u_N(\mu) \qquad \text{on } \partial T^\delta$$

  - extend local reduced basis for all $T \in \mathcal{X}$:

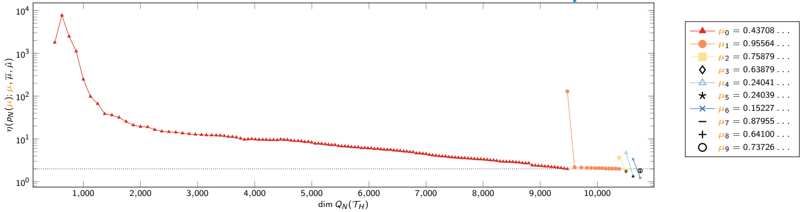
$$V_N^T := \operatorname{span} V_N^T \cup \{\varphi_h(\mu)|_T\}$$

  - update reduced quantities
  - compute updated reduced solution $u_N(\mu)$ and $\eta_{h,N}(\mu)$
- iterate until $\eta_{h,N}(u_{\mu,N}) \leq \Delta$, return $u_N(\mu)$

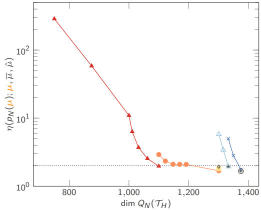
LRBMS with online enrichment: Example SPE10



LRBMS with online enrichment: Example SPE10



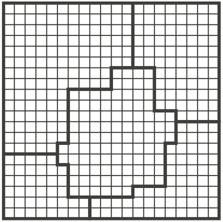
Convergence history of LRBMS with initially empty V_N



LRBMS initialized with 2 solution snapshots

Distribution of local basis size after online enrichment.

Related Approaches



- Reduced basis element Method
  [Maday, Ronquist, 2002]

- Reduced basis hybrid Method
  [Iapichino, Quarteroni, Rozza, Volkwein, 2014]

- Port-reduced static condensation Reduced basis element Method
  [Eftang, Patera, 2013]

- ArbiLoMod, a Simulation Technique Designed for Arbitrary Local Modifications
  [Buhr, Engwer, Ohlberger, R, 2017]

Localized Reduced Basis Domain Decomposition Methods for Elliptic Problems

Questions

- How fast does enrichment converge?
- How to balance training and enrichment?
- Which training method to combine with enrichment?

Connections with Domain Decomposition Methods

- Local enrichment function $\varphi_h(\mu)|_T$

$$a(\varphi_h(\mu), v_h; \mu) = f(v_h) \qquad \text{in } T^\delta$$
$$\varphi_h(\mu) = u_N(\mu) \qquad \text{on } \partial T^\delta$$

corresponds to subdomain solution in Restricted Additive Schwarz (RAS) method.

- In particular (for minimal overlap):

enrichment + Galerkin projection onto V_N

$\cong$

adaptive [Spillane, 2016] RAS multi-preconditioned CG [Bridson, Greif, 2006]

- Moreover:

offline training of V_N

$\cong$

construction of coarse space

e.g. DtN [Nataf et al., 2011], GenEO [Spillane et al., 2014], SHEM [Gander, Loneland, Rahman, 2015]

A Localized RB Additive Schwarz Method

1. Choose overlapping DD $T \in \mathcal{T}_H$ and define local FEM spaces $V_h^T \subset V_h$ as usual.
2. Use RB methods to construct coarse space V_N^0 for which abstract Schwarz framework guarantees robustness of AS+CG iterations for every μ.
3. Build local RB spaces V_N^T from AS+CG solutions.
4. Use RB estimator $\eta_{h,N}(u_N(\mu))$ to only enrich V_N when needed:

$$\eta_{h,N}(\mu)^2 := C(\mu)^2 \sum_{T \in \mathcal{T}_H} \left(\sup_{v_h \in V_h^T} \frac{f(v_h) - a(u_N(\mu), v_h; \mu)}{\|v_h\|} \right)^2$$

where, with C_{stab} the stability constant of decomposition $V_h = V_N^0 + \sum_{T \in \mathcal{T}_H} V_h^T$:

$$C(\mu) \leq C_{inf\text{-}sup}(\mu) \cdot C_{stab}$$

5. Use local error indicators to only compute AS corrections in $T \in \mathcal{T}_H$ with high residual.

Thank you for your attention!

Ohlberger, Schindler, *Error Control for the Localized Reduced Basis Multiscale Method with Adaptive On-Line Enrichment*, SISC, 37(6) (2015).

Ohlberger, R, Schindler, *True Error Control for the Localized Reduced Basis Method for Parabolic Problems*, Model Reduction of Parametrized Systems, Springer (2017).

Buhr, Engwer, Ohlberger, R, *ArbiLoMod, a Simulation Technique Designed for Arbitrary Local Modifications*, SISC, 39(4) (2017).

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