Westfälische Wilhelms-Universität Münster

Freezing Solutions of Time Evolution Problems for Reduced Basis Approximation

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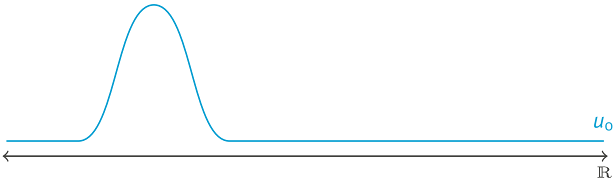
Stephan Rave

???

The Problem



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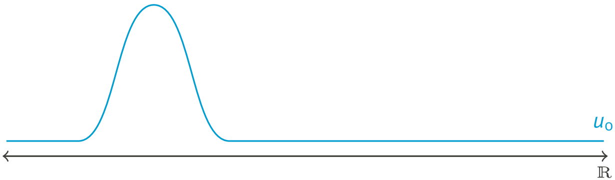

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$$u(0,x) = u_0(x)$$
$$x \in \mathbb{R}, t \in [0,T], u \in V$$

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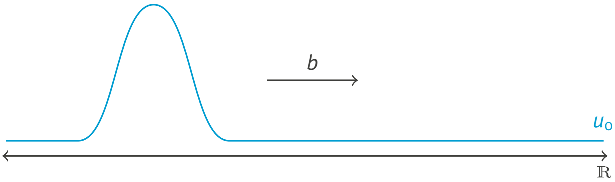

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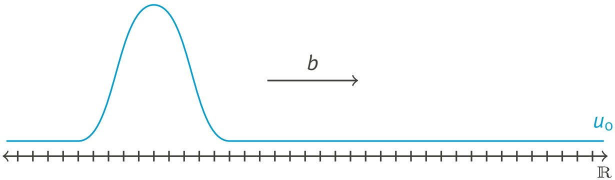

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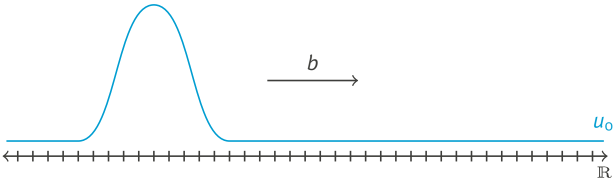

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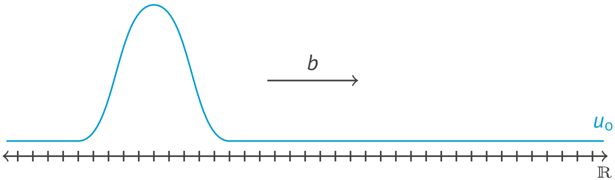

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- need $\sim C \cdot T$ basis functions to approximate solution by linear space
- **however**, we can describe solution easily by:

$$u(t,x) = u_0(x - bt)$$

Nonlinear Approximation

Using Groups of Transformations

- Rewrite $u(t,x)$ as
$$u(t,x) = u_0(x - bt) = \Phi_{bt}(u_0)(x)$$
with $\Phi_g(v)(x) := v(x-g)$ for $v \in V$.

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- **General idea:** Write $u(t,x)$ as
$$u(t,x) = g(t).v(t,x)$$
for general group G acting on function space V.

Nonlinear Approximation
using Groups of Transformations

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$g(t)$: dynamics of u, large variation in time

$v(t, x)$: shape of u, small variation in time

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$v(t,x)$: shape of u, small variation in time

- If this can be done, then $v(t,x)$ will be easier to approximate by a low-dimensional linear space than $u(t,x)$.

Lie Groups

- Problem: How to calculate/make sense of
$$\frac{d}{dt}\, g(t).v(t,x)\ ?$$
Have to derive the path $g(t) \in G$ and the action of G on V.

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Definition

A Lie group is a group G which is at the same time a smooth manifold such that group multiplication and inversion are smooth maps.

Lie Groups

- The Lie algebra LG of G is the tangent space $T_e(G)$ of G at its neutral element e. If the action of G on V is smooth (enough), it can be derived to an action of LG on V.

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Lie Groups

- The Lie algebra LG of G is the tangent space $\mathrm{T}_e(G)$ of G at its neutral element e. If the action of G on V is smooth (enough), it can be derived to an action of LG on V.
- $g_t(t) \in \mathrm{T}_{g(t)}(G)$
- Left-multiplication by g induces map $(g \cdot \_) : \mathrm{L}G \longrightarrow \mathrm{T}_g(G)$.
  Left-multiplication by g^{-1} induces $(g^{-1} \cdot \_) : \mathrm{T}_g(G) \longrightarrow \mathrm{L}G$.
  Thus:
  $$g(t)^{-1} g_t(t) \in \mathrm{L}G \quad \text{for all } t.$$

The Method of Freezing

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or

$$v_t(t) = g^{-1}(t)F(g(t).v(t)) - \eta(t).v(t)$$
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  Add some (well-chosen) additional algebraic constraint

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- How to determine $g(t)$ and $v(t)$? (Have $\dim(G)$ additional DOFs!) Add some (well-chosen) additional algebraic constraint
$$\Phi(v(t), \eta(t)) = 0.$$
- Further assume that F is invariant under the action of G:
$$h^{-1}.F(h.w) = F(w) \quad \text{for all } h \in G, w \in V.$$

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Definition

The method of freezing for $u_t(t) = F(u(t))$ consists in solving

$$\begin{aligned} v_t(t) &= F(v(t)) - \eta(t).v(t) \\ 0 &= \Phi(v(t), \eta(t)) \end{aligned}$$ frozen PDAE

$$\eta(t) = g^{-1}g_t(t)$$ reconstruction equation

with initial conditions $v(0) = u(0), \; g(0) = e$.

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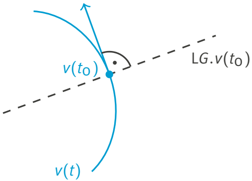
- Introduced for stability analysis of relative equilibria by Beyn, Thümmler, Rottman-Matthes (Bielefeld) and Rowley et. al.

Phase Conditions

- Possible choice:

$$\Phi(v, \eta) = 0 \;:\Longleftrightarrow\; v_t \perp LG.v$$
$$\Longleftrightarrow\; (F(v) - \eta.v \mid \xi.v) = 0 \quad \forall \xi \in LG$$



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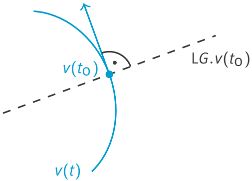

- Other choices: minimize $||v_t||$ or $||v - v_0||$ for some template function v_0

Example: 2D-Shifts

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$$(\eta.u)(x) = \frac{d}{dh}\left((h \cdot \eta).u\right)\Big|_{h=0}(x)$$
$$= \lim_{h \to 0} \frac{u(x - h \cdot \eta) - u(x)}{h} = -(\eta \cdot \nabla u)(x)$$

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- Writing $\eta(t) = (\eta^x(t), \eta^y(t))$, we have
$$v_t(t) = F(v(t)) + \eta^x(t) v_x(t) + \eta^y(t) v_y(t).$$

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$$\iff \begin{bmatrix} (v_x \mid v_x) & (v_y \mid v_x) \\ (v_x \mid v_y) & (v_y \mid v_y) \end{bmatrix} \cdot \begin{bmatrix} \eta^x \\ \eta^y \end{bmatrix} = - \begin{bmatrix} (F(v) \mid v_x) \\ (F(v) \mid v_y) \end{bmatrix}$$

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The Method of Freezing for 2D-Shifts

Solve

$$v_t(t) = F(v(t)) + \eta^x(t)v_x(t) + \eta^y(t)v_y(t)$$

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and

$$g_t(t) = \eta(t)$$

with initial conditions $v(0) = u(0)$, $g(0) = (0, 0)^T$.

Example: 2D-Shifts

Detailed Scheme

Let $\tilde{u}(t), \tilde{v}(t) \in \tilde{V}$ be discrete functions, $\tilde{F}, \tilde{v}_x, \tilde{v}_y$ discrete operators. Solve

$$\tilde{v}_t(t) = \tilde{F}(\tilde{v}(t)) + \eta^x(t)\tilde{v}_x(t) + \eta^y(t)\tilde{v}_y(t)$$

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Basic time-stepping:

1. Evaluate $\tilde{F}(\tilde{v}(t^n))$
2. Solve phase condition for $\eta^x(t^n), \eta^y(t^n)$
3. $\tilde{v}(t^{n+1}) := \tilde{v}(t^n) + dt \cdot \left(\tilde{F}(\tilde{v}(t^n)) + \eta^x(t^n)\tilde{v}_x(t^n) + \eta^y(t^n)\tilde{v}_y(t^n)\right)$

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RB Scheme

Let F depend on some parameter. Generate reduced basis with standard methods and use same time stepping method.

- Replace $\tilde{F}$ by some appropriate operator on RB-space (empirical operator interpolation, etc.)
- Scalar products with $\tilde{v}_x, \tilde{v}_y$ are linear in v, thus can easily be precomputed for an RB-space basis.

Burgers' equation

Consider on $\Omega = [0, 2] \times [0, 1]$ the two-dimensional Burgers equation

$$\begin{aligned} u_t &= -\nabla \cdot (bu^p) \\ u(0,x,y) &= 1/2(1 + \sin(2\pi x)\sin(2\pi y)) \end{aligned}$$

for $t \in [0, 0.3]$, $b = (1, 1)^T$ with periodic boundary conditions and $p \in [0, 2]$ as parameter.

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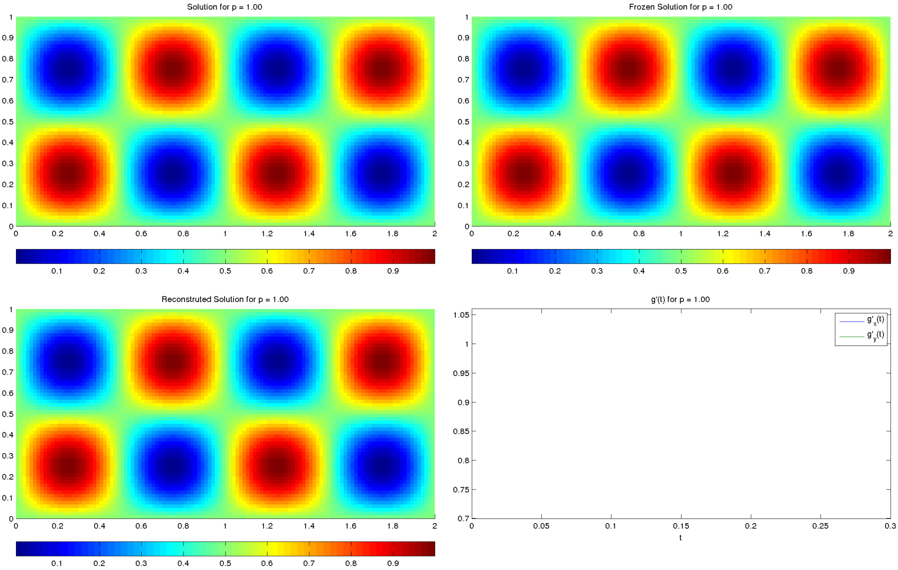
- Finite volume discretization on quadratic grid ($h = 1/60$), Euler time-stepping with 100 time steps
- Use empirical operator interpolation for nonlinearity
- Same setting as in Drohmann, Haasdonk, Ohlberger (2012)

Frozen vs. Nonfrozen Solution (p=1, b=(0.75,1))



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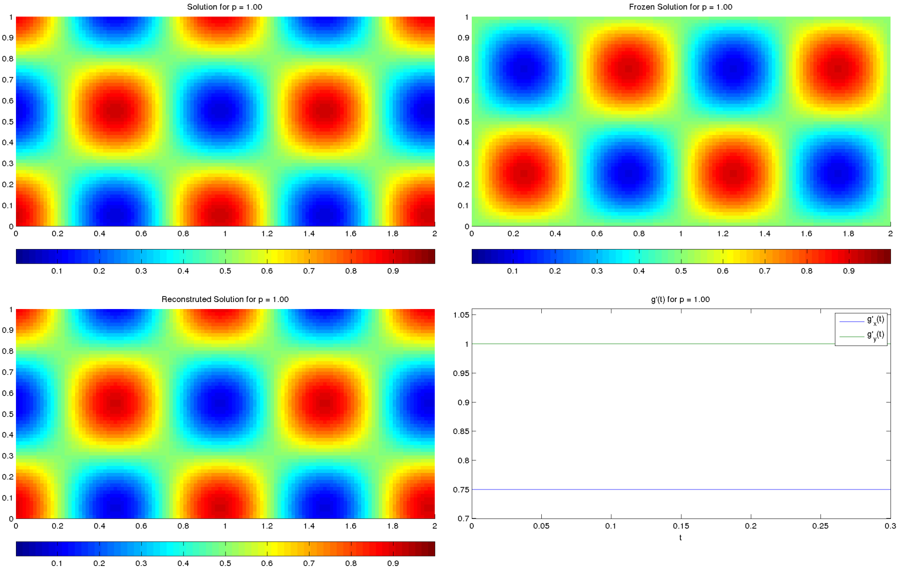
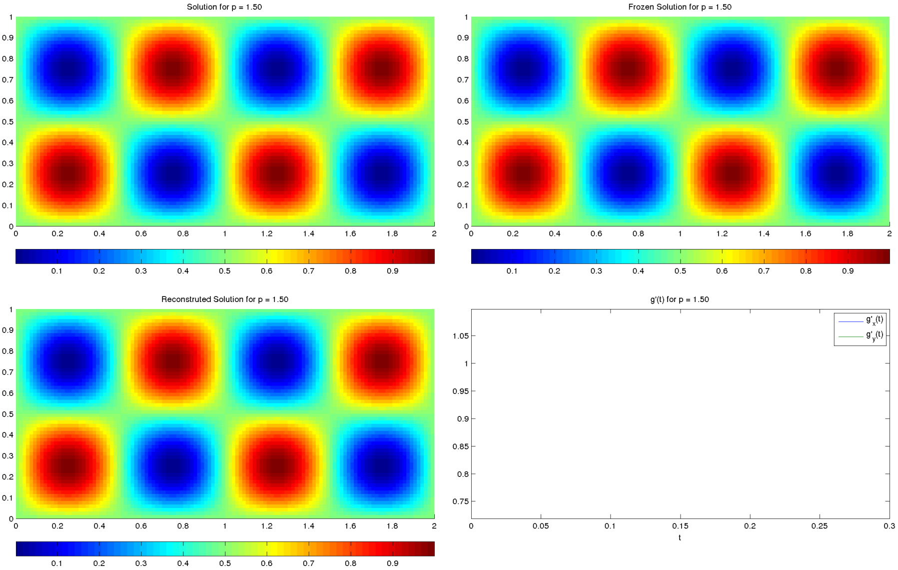


Frozen vs. Nonfrozen Solution (p=1.5, b=(0.75,1))



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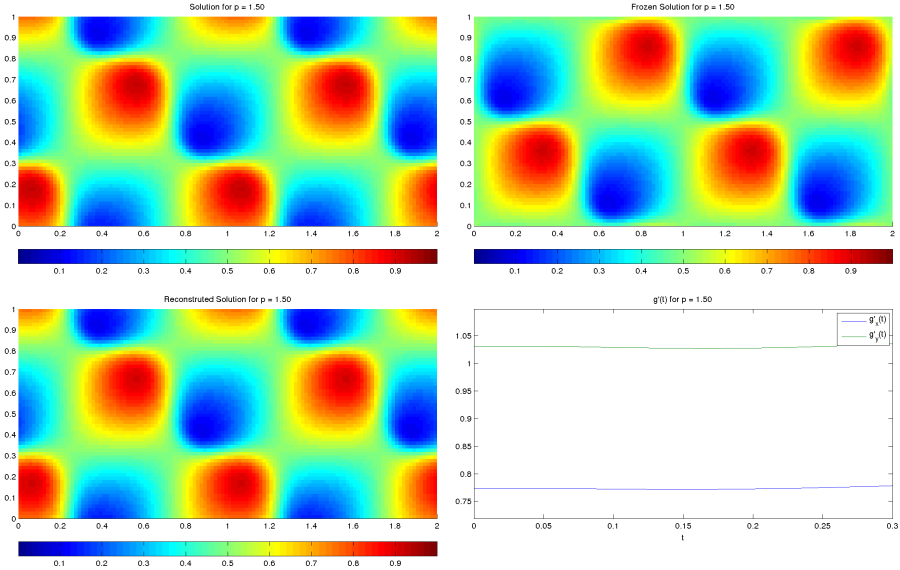
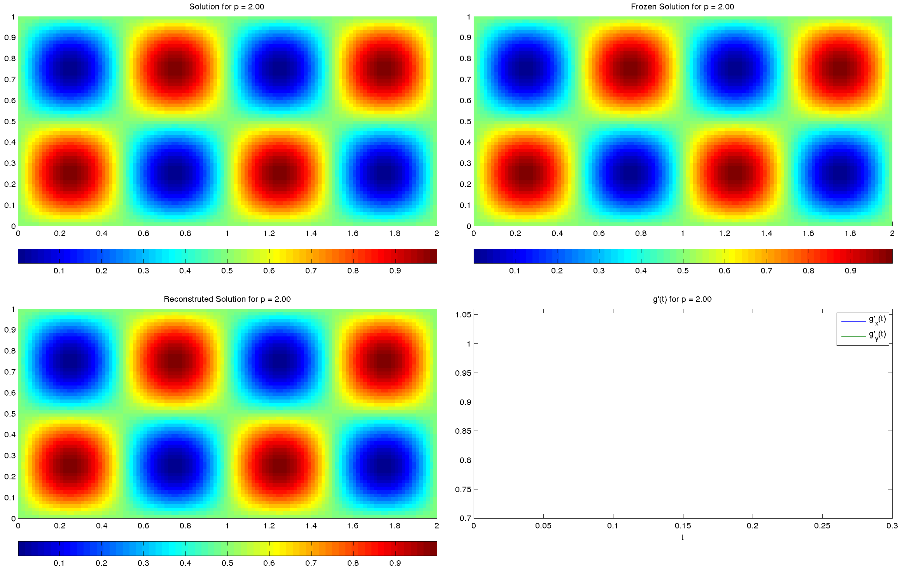


Frozen vs. Nonfrozen Solution (p=2, b=(0.75,1))



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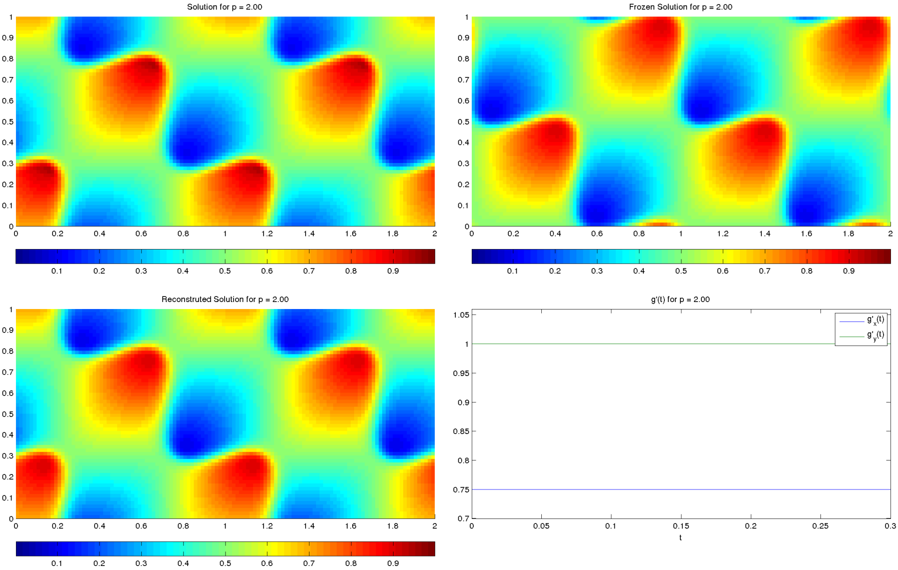
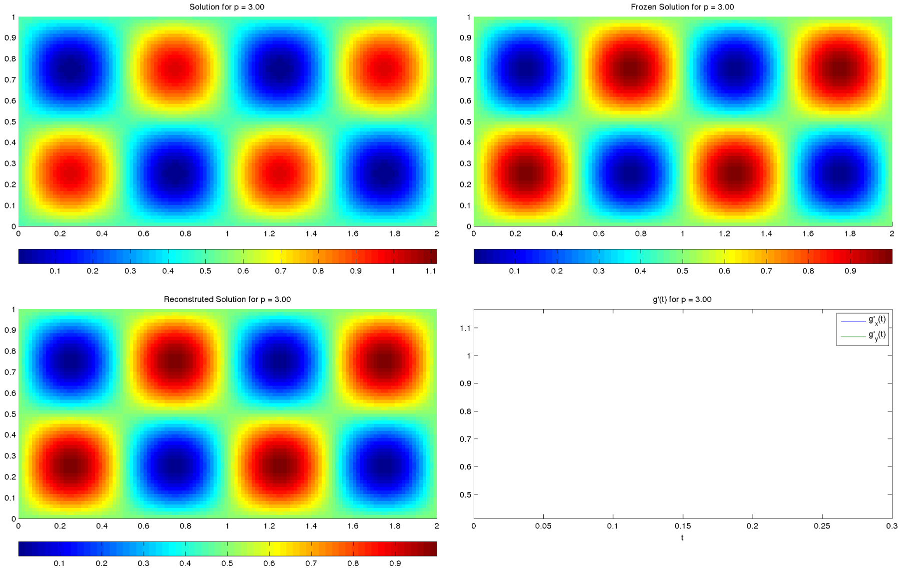


Frozen vs. Nonfrozen Solution (p=3, b=(0.75,1))



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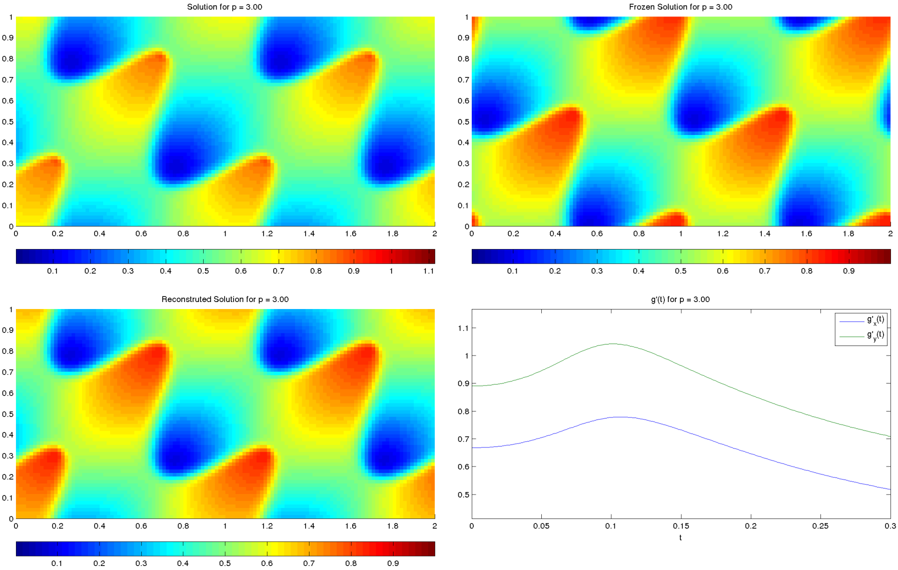
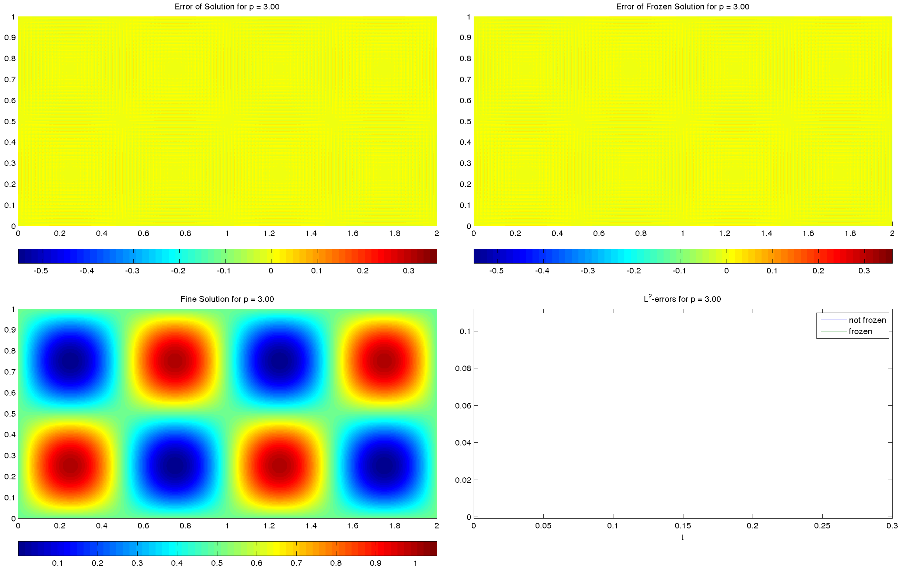


Errors of Frozen and Nonfrozen Solution (p=3)



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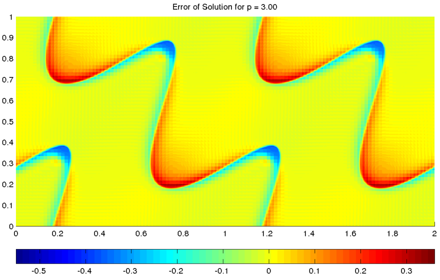
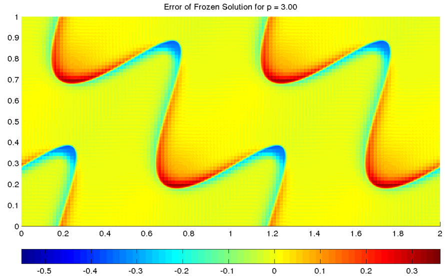
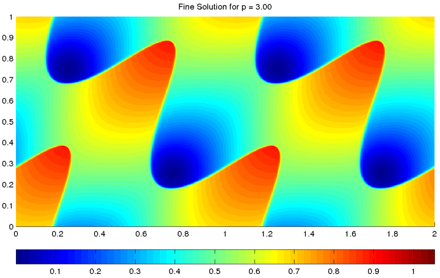
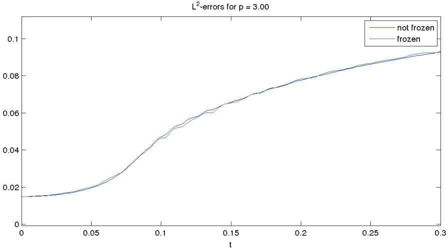
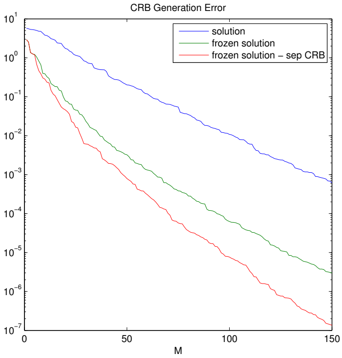


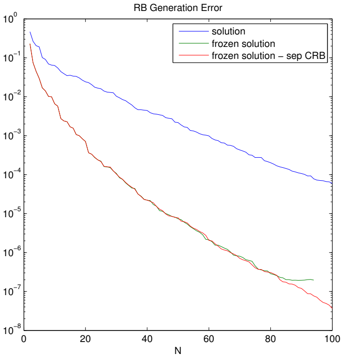




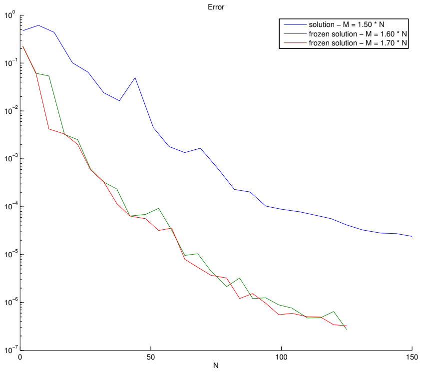


Reduced Basis Generation





Reduced Basis Approximation Error



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Thanks for your attention!

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