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Nonlinear Reduced Basis Approximation of Parameterized Evolution Equations using the Method of Freezing

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The Problem

Model reduction for parameter dependent, convection dominated, nonlinear Cauchy problem

$$\partial_t u_\mu(t) + \mathcal{L}_\mu(u_\mu(t)) = 0, \quad u_\mu(0) = u_0$$

where

- $\mu \in \mathcal{P}$ (parameter space)
- $u_\mu(t) \in V$, $t \in [0, T]$ for appropriate function space V
- $\mathcal{L}_\mu$ partial differential operator

using reduced basis approach, i.e. ...

The Problem

$$\partial_t u_\mu(t) + \mathcal{L}_\mu(u_\mu(t)) = 0, \quad u_\mu(0) = u_0$$

Assume we have

- H-dimensional linear discrete space V_h $(0 \ll H)$
- Operator $\mathcal{L}_{\mu,h}$ on V_h approximating $\mathcal{L}_\mu$
- N-dimensional linear RB-space $V_N \subset V_h$ $(N \ll H)$

and solve

$$\partial_t u_{N,\mu}(t) + P_N(\mathcal{L}_{\mu,h}(u_{N,\mu}(t))) = 0, \quad u_{N,\mu}(0) = P_N(u_{h,0})$$

with appropriate projection $P_N : V_h \longrightarrow V_N$.

The Problem

$$\partial_t u_{N,\mu}(t) + P_N(\mathcal{L}_{\mu,h}(u_{N,\mu}(t))) = 0, \quad u_{N,\mu}(0) = P_N(u_{h,0})$$

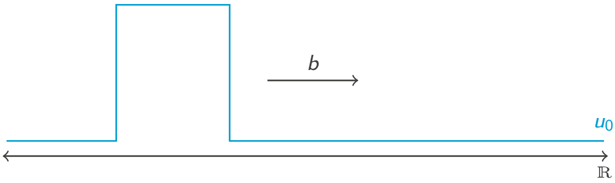
- Empirical operator interpolation to handle $\mathcal{L}_\mu$
- Greedy search to construct V_N and interpolation basis for $\mathcal{L}_\mu$, e.g. [Drohmann, Haasdonk, Ohlberger, 2012]
- Exponential convergence rates are preserved by greedy search [Haasdonk, 2011]
- However …

The Problem

$$\partial_t u_{N,\mu}(t) + P_N(\mathcal{L}_{\mu,h}(u_{N,\mu}(t))) = 0, \quad u_{N,\mu}(0) = P_N(u_{h,0})$$

- V_N has to approximate $u_{\mu,h}(t)$ for every t.
- If $u_{\mu,h}$ has low regularity and moves in space, this is really bad.
- Even for a single parameter!
- Worse: Velocity can depend on parameter!

The Problem
Basic example



$$\partial_t u(t,x) + b \cdot \partial_x u(t,x) = 0$$
$$u(0,x) = u_0(x)$$
$$x \in \mathbb{R},\ t \in [0,T]$$

- Need $\mathcal{O}(\varepsilon^{-2})$ basis functions for L^2-approximation error $< \varepsilon$
- **However,** we can describe solution easily by:

$$u(t,x) = u_0(x - bt)$$

Nonlinear Approximation

Using Groups of Transformations

- Rewrite $u(t, x)$ as
$$u(t, x) = u_0(x - bt) = \Phi_{bt}(u_0)(x)$$
with $\Phi_g(v)(x) := v(x - g)$.
- $g.v := \Phi_g(v)$ defines action of additive group $\mathbb{R}$:
$$(g + h).v = \Phi_{g+h}(v) = \Phi_g(\Phi_h(v)) = g.(h.v)$$
- **General idea:** Write $u(t, x)$ as
$$u(t, x) = g(t).v(t, x)$$
for group G acting on function space V.

Nonlinear Approximation
using Groups of Transformations

- **General idea:** Write $u(t, x)$ as

$$u(t, x) = g(t).v(t, x)$$

$g(t)$: dynamics of u, large variation in time

$v(t, x)$: shape of u, small variation in time

- If this can be done, then $v(t, x)$ will be easier to approximate by a low-dimensional linear space than $u(t, x)$.

The Method of Freezing

$$\partial_t u(t) + \mathcal{L}(u(t)) = 0, \quad u(0) = u_0$$

- Substitue the *ansatz* $u(t) = g(t).v(t)$:
$$\partial_t g(t).v(t) + g(t).\partial_t v(t) + \mathcal{L}(g(t).v(t)) = 0$$
(G Lie group, action smooth)

- Multiply by $g(t)^{-1}$:
$$\partial_t v(t) + g(t)^{-1}.\mathcal{L}(g(t).v(t)) + \mathfrak{g}(t).v(t) = 0$$
$$\mathfrak{g}(t) = g(t)^{-1}\partial_t g(t).$$

The Method of Freezing

$$\partial_t v(t) + g(t)^{-1}.\mathcal{L}(g(t).v(t)) + \mathfrak{g}(t).v(t) = 0$$
$$\mathfrak{g}(t) = g(t)^{-1}\partial_t g(t)$$

- Have $\dim(G)$ additional degrees of freedom!
- Add additional algebraic constraint (phase condition)
$$\Phi(v(t), \mathfrak{g}(t)) = 0.$$
- Further assume invariance of $\mathcal{L}$ under action of G:
$$h^{-1}.\mathcal{L}(h.w) = \mathcal{L}(w) \quad \text{for all } h \in G, w \in V.$$

The Method of Freezing

Definition

The method of freezing for $\partial_t u(t) + \mathcal{L}(u(t)) = 0$ consists in solving

$$\begin{aligned} \partial_t v(t) + \mathcal{L}(v(t)) + \mathfrak{g}(t).v(t) &= 0 \\ \Phi(v(t), \mathfrak{g}(t)) &= 0 \end{aligned}$$ frozen PDAE

$$\mathfrak{g}(t) = g(t)^{-1}\partial_t g(t)$$ reconstruction equation

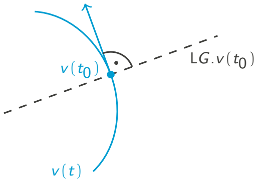
with initial conditions $v(0) = u(0),\ g(0) = e$.

- Introduced for stability analysis of relative equilibria [Beyn, Thümmler, 2004] and [Rowley et. al., 2003]

Phase Conditions

- Possible choice:

$$\Phi(v, \mathfrak{g}) = 0 \;:\Longleftrightarrow\; \partial_t v(t) \perp LG.v(t)$$
$$\Longleftrightarrow\; (\mathcal{L}(v) + \mathfrak{g}.v,\ \mathfrak{h}.v) = 0 \quad \forall \mathfrak{h} \in LG$$



- Other choices: minimize $\|\partial_t v\|$ or $\|v - v_0\|$ for some template function v_0

Example: 2D-Shifts

- $G = \mathbb{R}^2$, $\mathrm{L}G = \mathbb{R}^2$,

$$g.u(x) := u(x-g), \quad x \in \mathbb{R}^2$$

$$\mathfrak{g}.u = -\mathfrak{g} \cdot \nabla u$$

- Phase Condition:

$$\Phi(v,\mathfrak{g}) = 0 \iff (\mathcal{L}(v) + \mathfrak{g}.v,\ \mathfrak{h}.v) = 0 \quad \forall \mathfrak{h} \in \mathrm{L}G$$

$$\iff \left[(\partial_{x_i} v,\ \partial_{x_j} v)\right]_{i,j} \cdot \left[\mathfrak{g}_j\right]_j = \left[(\mathcal{L}(v),\ v_{x_r})\right]_i$$

$$1 \leqslant i,j \leqslant 2$$

Example: 2D-Shifts

The Method of Freezing for 2D-Shifts

Solve

$$\partial_t v(t) + \mathcal{L}(v(t)) - \mathfrak{g}(t) \cdot \nabla v(t) = 0$$
$$\left[(\partial_{x_i} v,\ \partial_{x_j} v)\right]_{i,j} \cdot \left[\mathfrak{g}_j\right]_j = \left[(\mathcal{L}(v),\ \partial_{x_i} v)\right]_i$$

and

$$\partial_t g(t) = \mathfrak{g}(t)$$

with initial conditions $v(0) = u(0)$, $g(0) = (0, 0)^T$.

Example

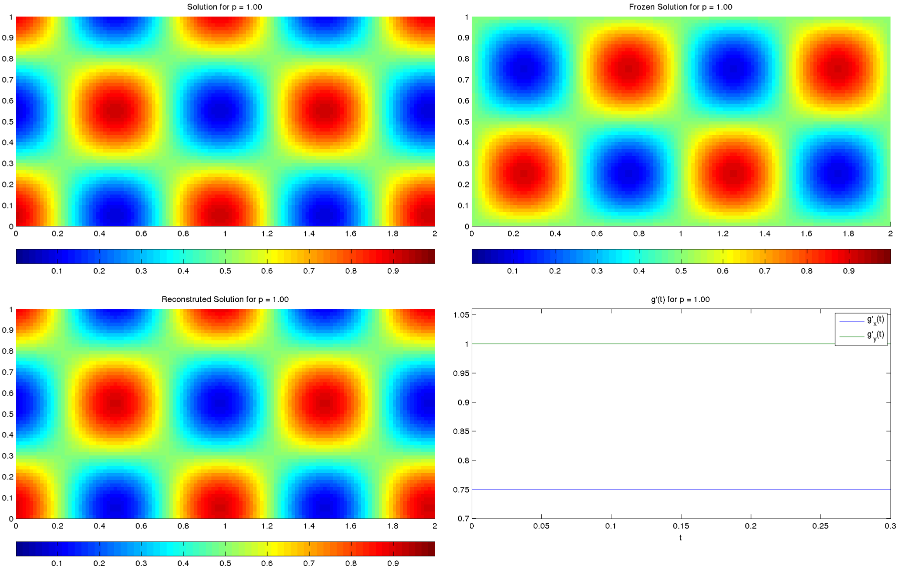
Consider on $\Omega = [0,2] \times [0,1]$ the two-dimensional Burgers-type problem

$$\partial_t u = -\nabla \cdot (b u^\mu)$$
$$u(0, x_1, x_2) = 1/2(1 + \sin(2\pi x_1)\sin(2\pi x_2))$$

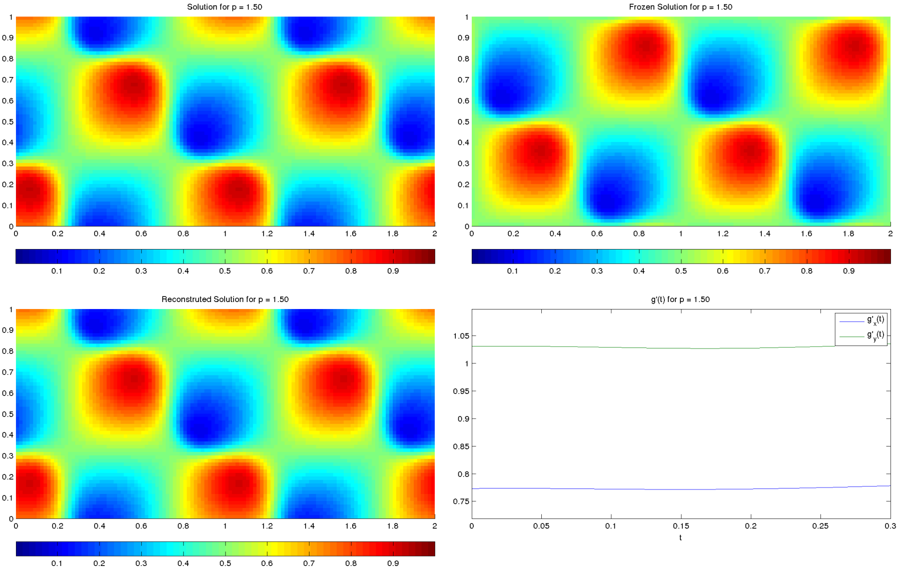
for $t \in [0, 0.3]$, $b = (1,1)^T$ with periodic boundary conditions and $\mu \in \mathcal{P} = [1,2]$.

- Finite volume discretization on 120 x 60 grid, explict Euler time-stepping
- Same problem as in [Drohmann, Haasdonk, Ohlberger, 2012]

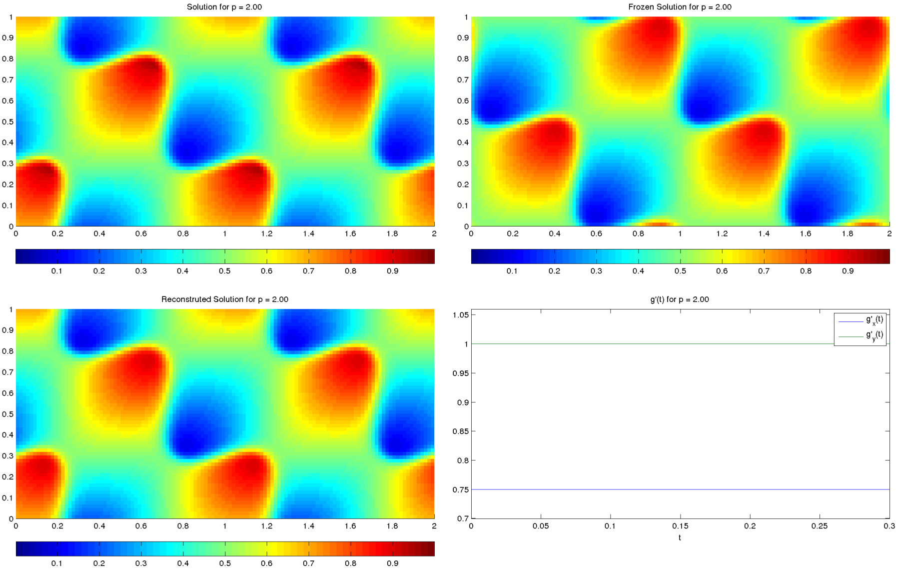
Frozen vs. Non-frozen Solution (μ=1, b=(0.75,1))



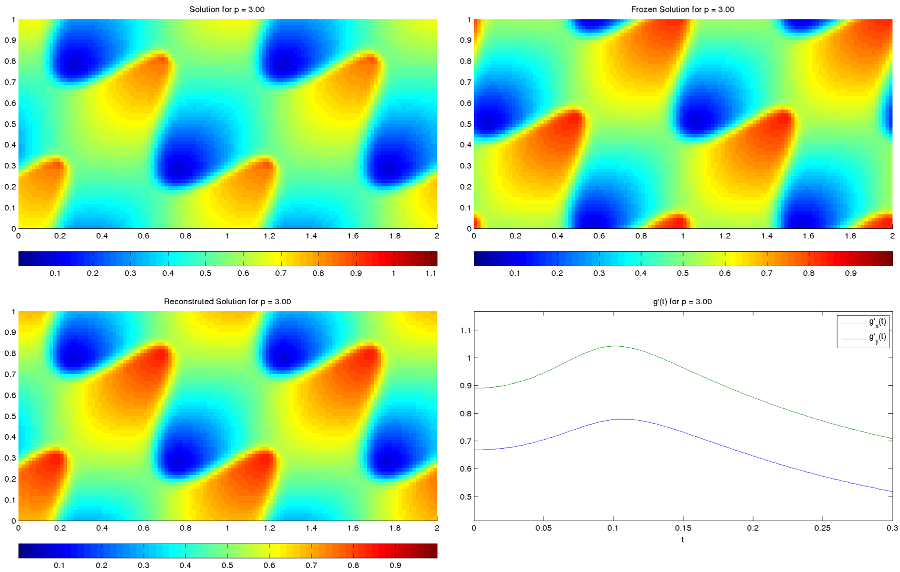
Frozen vs. Non-frozen Solution (μ=1.5, b=(0.75,1))



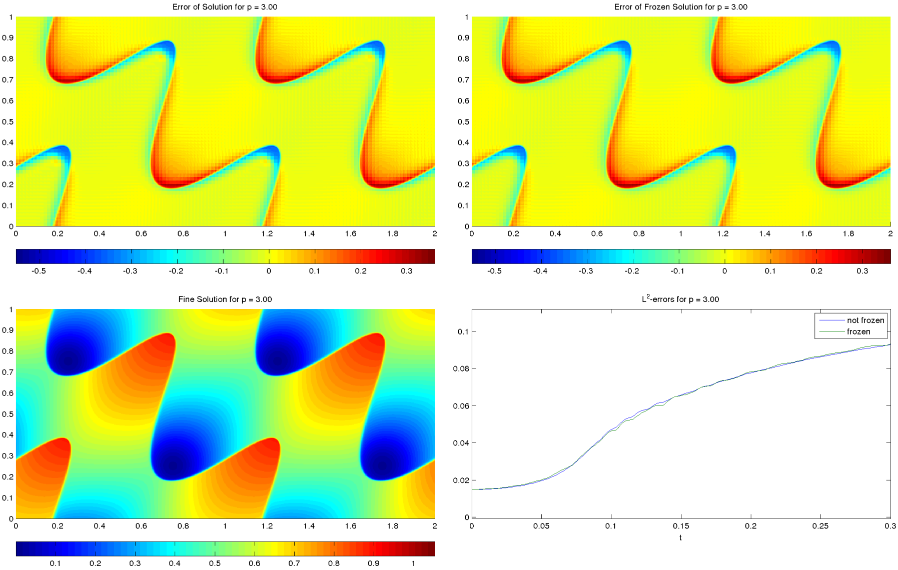
Frozen vs. Non-frozen Solution (μ=2, b=(0.75,1))



Frozen vs. Non-frozen Solution (μ=3, b=(0.75,1))



Errors of Frozen and Non-frozen Solution (μ=3)



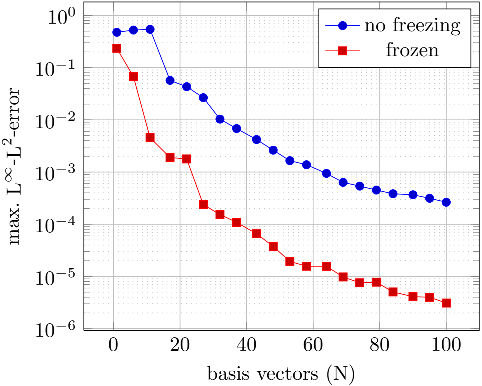
RB-Approximation

FrozenRB-Scheme [Ohlberger, R., 2013]

1. Replace original parameterized PDE with frozen parameterized PDAE and reconstruction ODE
2. Discretize
3. Use EI and greedy basis generation

- Offline/online decomposition possible
- No additional evaluations of nonlinearity (small overhead)

RB-Approximation Error for Burgers Problem



Thank you for your attention!

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