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Reduced basis methods

Success, limitations and future challenges

http://www.stephanrave.de/talks/algoritmy_2016.pdf

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Outline

1. Success:
   - *Introduction* to the *theory* of reduced basis methods for coercive, affinely decomposed problems.
   - Proof of (sub-)exponential convergence.
2. Limitations and future challenges:
   - Advection dominated problems and the need for nonlinear approximation.
   - The FrozenRB method.

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Introduction to Reduced Basis Methods

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Abstract Problem Formulation

Consider parametric problems

$$\Phi : \mathcal{P} \to V, \qquad s : V \to \mathbb{R}^S$$

where

- $\mathcal{P} \subset \mathbb{R}^P$ *compact* set (parameter domain).
- V Hilbert space (solution state space, $\dim V \gg 0$, possibly $\dim V = \infty$).
- Φ maps parameters to solutions (*hard* to compute).
- s maps state vectors to quantities of interest.

Objective

Compute

$$s \circ \Phi : \mathbb{R}^P \to V \to \mathbb{R}^S$$

for *many* $\mu \in \mathcal{P}$ or *quickly* for unknown single $\mu \in \mathcal{P}$.

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Abstract Problem Formulation

Objective

Compute

$$s \circ \Phi : \mathbb{R}^P \to V \to \mathbb{R}^S.$$

- When Φ, s sufficiently smooth, quickly computable low-dimensional approximation of $s \circ \Phi$ should exist.
- Could use interpolation scheme. However:
  - How to choose interpolation points?
  - Error control?!
- State space approximation:
  - Find $\Phi_N : \mathcal{P} \to V_N$ s.t. $\Phi \approx \Phi_N$ and $\dim V_N =: N \ll \dim V$.
  - W.l.g. can assume $V_N \subset V$ (orthogonal projection).
  - Approximate $s \circ \Phi \approx s \circ \Phi_N$.

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State Space Approximation

Main questions

1. Do good approximation spaces V_N exist?
2. How to find a good approximation space V_N?
3. How to construct a quickly-evaluable $\Phi_N : \mathcal{P} \to V_N$?
4. How to control the approximation errors $\Phi(\mu) - \Phi_N(\mu)$, $s(\Phi(\mu)) - s(\Phi_N(\mu))$?

- We answer these questions for the archetypical class of

linear, coercive, affinely decomposed problems.

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Problem Class

Linear, coercive problem

$\Phi(\mu) = u_\mu \in V$ is the solution of variational problem

$$a_\mu(u_\mu, v) = f(v) \qquad \forall v \in V,$$

where $a_\mu : V \times V \to \mathbb{R}$ is continuous, coercive bilinear form, $f \in V'$.
Moreover, $s : V \to \mathbb{R}^S$ is linear and continuous.

Linear, coercive, affinely decomposed problem

Additionally:

$$a_\mu = \sum_{q=1}^{Q} \theta_q(\mu) a_q \qquad \forall \mu \in \mathcal{P},$$

where $\theta_q : \mathcal{P} \to \mathbb{R}$ continuous, $a_q : V \times V \to \mathbb{R}$ continuous bilinear form, $(1 \leq q \leq Q)$.

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3. Definition of Φ_N

Full order problem

$\Phi(\mu) = u_\mu \in V$ is the solution of variational problem

$$a_\mu(u_\mu, v) = f(v) \qquad \forall v \in V,$$

where $a_\mu : V \times V \to \mathbb{R}$ is continuous, coercive bilinear form, $f \in V'$.

Reduced order problem

For given $V_N \subset V$, let $\Phi_N(\mu) := u_{\mu,N} \in V_N$ be the Galerkin projection of u_μ onto V_N, i.e.

$$a_\mu(u_{\mu,N}, v) = f(v) \qquad \forall v \in V_N.$$

- Since a_μ is coercive, $u_{\mu,N}$ is well-defined.

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3. Definition of Φ_N

Theorem (Céa)

Let c_μ denote the coercivity constant of a_μ. Then

$$\|u_\mu - u_{\mu,N}\| \leq \frac{\|a_\mu\|}{c_\mu} \inf_{v \in V_N} \|u_\mu - v\|.$$

- $u_{\mu,N}$ is quasi-optimal approximation of u_μ in V_N.
- For badly conditioned ($\|a_\mu\|/c_\mu \gg 0$) or non-coercive a_μ use Petrov-Galerkin projection!

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3. Definition of Φ_N

Let $\varphi_1, \cdots, \varphi_N$ be a basis of V_N. Then $u_{\mu,N} = \sum_{l=1}^N \varphi_l \cdot \underline{u}_{\mu,N,l}$, where

$$\sum_{q=1}^{Q} \mu_q \cdot \left[a_q(\varphi_l, \varphi_k)\right]_{k,l} \cdot \underline{u}_{\mu,N,l} = \left[f(\varphi_k)\right]_k \qquad (1)$$

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Proposition

If $\left[a_q(\varphi_l, \varphi_k)\right]_{k,l}$ are pre-computed, (1) can be solved with effort $\mathcal{O}(QN^2 + N^3)$.

Warning

Using solution snapshots $u_{\mu_1}, \ldots, u_{\mu_N}$ as basis for V_N leads to (really!) badly conditioned reduced system matrices! Orthonormalize!

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4. Error Control

Define residual $\mathcal{R}_\mu(u) \in V'$ as

$$\mathcal{R}_\mu(u)[v] := f(v) - a_\mu(u, v).$$

Then

$$\begin{aligned}\|u_\mu - u_{\mu,N}\|^2 &\leq c_\mu^{-1} a_\mu(u_\mu - u_{\mu,N},\, u_\mu - u_{\mu,N}) \\ &= c_\mu^{-1}\mathcal{R}_\mu(u_{\mu,N})[u_\mu - u_{\mu,N}] \leq c_\mu^{-1}\|\mathcal{R}_\mu(u_{\mu,N})\|\,\|u_\mu - u_{\mu,N}\|.\end{aligned}$$

Proposition

The quantity $\Delta_\mu(u_{\mu,N}) := c_\mu^{-1} \cdot \|\mathcal{R}(u_{\mu,N})\|$ is a reliable and effective a posteriori estimate for the model reduction error:

$$\|u_\mu - u_{\mu,N}\| \leq \Delta_\mu(u_{\mu,N}) \leq \|a_\mu\| \cdot c_\mu^{-1} \cdot \|u_\mu - u_{\mu,N}\|.$$

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4. Error Control

We have

$$\|\mathcal{R}_\mu(u_{\mu,N})\|^2 = \left\| f + \sum_{q=1}^{Q}\sum_{n=1}^{N} \underline{u}_{\mu,N,n} a_q(\varphi_n, \cdot)\right\|^2.$$

Note that V' is a Hilbert space via the Riesz isomorphism.

Thus, we can pre-compute all $(1+QN)^2$ cross-terms in the scalar-product evaluation. Online effort: $\mathcal{O}((1+QN)^2) = \mathcal{O}(Q^2N^2)$.

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However, bad numerical stability (half machine precision). Better approach:

Stable estimator decomposition (Buhr, R, 2014)

Project $\mathcal{R}_\mu$ onto V_N and $\mathrm{span}\{f, a_q(\varphi_n, \cdot)\}$ w.r.t. orthonormal bases.

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4. Error Control

Simple output error bound

We have
$$|s \circ \Phi(\mu) - s \circ \Phi_N(\mu)| \leq \|s\| \cdot \Delta_\mu(u_{\mu,N}).$$

- Not very effective: Typically, error decays at faster rate than $\Delta_\mu(u_{\mu,N})$.
- When a_μ symmetric and $s = f$ ('compliant' case):
$$0 \leq s \circ \Phi(\mu) - s \circ \Phi_N(\mu) \leq c_\mu \cdot \Delta_\mu(u_{\mu,N})^2.$$
- For general a_μ, s: Improved estimates via dual weighted residual approach.
- If unknown, c_μ can be replaced by arbitrary lower bound $0 < \alpha_\mu \leq c_\mu$ ($\rightarrow$ sucessive constraint method).

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1. Existence of good V_N

Definition

The *Kolmogorov N-width* $d_N(\Phi(\mathcal{P}))$ of $\Phi(\mathcal{P})$ is given as

$$d_N(\Phi(\mathcal{P})) = \inf_{\substack{V_N \subseteq V \\ \text{lin subsp.} \\ \dim V_N \leq N}} \sup_{u \in \Phi(\mathcal{P})} \inf_{v \in V_N} \|u - v\|.$$

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- Cannot beat N-width with any V_N.
- For elliptic problems with fixed operator and arbitrary RHS in some unit ball: Polynomial decay of d_N.
- Hope for exponential decay of $d_N(\Phi(\mathcal{P}))$.

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1. Existence of good V_N

Proposition (Cohen, DeVore, 2014)

Let $F: V \times X \longrightarrow W$ holomorphic map between Banach spaces and $\mathcal{P} \subseteq X$.
If for all $\mu \in \mathcal{P}$

- $\Phi(\mu) := u_\mu$ is the unique solution of $F(u_\mu, \mu) = 0$
- $\partial_u F(u_\mu, \mu) : V \longrightarrow W$ is invertible,

then there is holomorphic extension $\Phi : \mathcal{O} \longrightarrow V$ with $\mathcal{P} \subseteq \mathcal{O}$ open.

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Proof

Implicit function theorem (for complex Banach spaces).

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Proof

Implicit function theorem (for complex Banach spaces).

- For affinely decomposed, linear coercive problems:

$$F : V \times \mathbb{C}^Q \to V', \quad F(u, \underline{z})[v] := \sum_{q=1}^{Q} z_q \cdot a_q(u, v) - f$$

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1. Existence of good V_N

Corollary

There are $C, c > 0$ s.t.

$$d_N(\Phi(\mathcal{P})) \leq Ce^{-cN^{1/Q}}$$

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Corollary

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$$d_N(\Phi(\mathcal{P})) \leq Ce^{-cN^{1/Q}}$$

Proof

- $\hat{\mathcal{P}} := \{(\theta_1(\mu), \ldots, \theta_Q(\mu) \mid \mu \in \mathcal{P}\} \subset \mathbb{C}^Q$ is compact ($\mathcal{P}$ cpct., θ_q cont.)
- $\hat{\Phi} : \hat{\mathcal{P}} \to V$, $\hat{\Phi}[\theta_1(\mu), \ldots, \theta_Q(\mu)] := \Phi(\mu)$ has holom. ext. to $\hat{\mathcal{P}} \subset \mathcal{O}$.

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1. Existence of good V_N

Corollary

There are $C, c > 0$ s.t.

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- Thus, $\hat{\Phi}$ can be extended as multivariate power series for any $z \in \hat{\mathcal{P}}$.
- By compactness of $\hat{\mathcal{P}}$, finitely many power series expansions suffice to represent any $\hat{\mathcal{P}}(z)$, $z \in \hat{\mathcal{P}}$.

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- Thus, $\hat{\Phi}$ can be extended as multivariate power series for any $z \in \hat{\mathcal{P}}$.
- By compactness of $\hat{\mathcal{P}}$, finitely many power series expansions suffice to represent any $\hat{\mathcal{P}}(z)$, $z \in \hat{\mathcal{P}}$.
- $V_N :=$ span{first $k(N)$ coeffs. in expansions}.

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2. Construction of V_N

Definition (weak greedy sequence)

Let $0 < \gamma \leq 1$ and $s_1, s_2, \ldots \in \Phi(\mathcal{P})$ be such that

$$\inf_{v \in V_{N-1}} \|s_N - v\| \geq \gamma \cdot \sup_{u \in \Phi(\mathcal{P})} \inf_{v \in V_{N-1}} \|u - v\| \qquad V_N := \operatorname{span}\{s_1, \ldots s_N\}$$

Then (s_n) is called weak greedy sequence for $\Phi(\mathcal{P})$ with parameter γ.

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Then (s_n) is called weak greedy sequence for $\Phi(\mathcal{P})$ with parameter γ.

Theorem (DeVore, Petrova, Wojtaszczyk, 2013)

Let (s_n) be a weak greedy series for $\Phi(\mathcal{P})$ with param. γ. Assume there are $C, c, \alpha > 0$ such that

$$d_N(\Phi(\mathcal{P})) \leq Ce^{-cN^\alpha}.$$

Then with $V_N := \operatorname{span}\{s_1, \ldots s_N\}$ we have

$$\sup_{u \in \Phi(\mathcal{P})} \inf_{v \in V_N} \|u - v\| \leq \sqrt{2C}\gamma^{-1} e^{-c' N^\alpha}, \qquad c' = 2^{-1-2\alpha} c.$$

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2. Construction of V_N

Greedy algorithm with error estimator

Choose snapshots $s_N := u_{\mu_N}$ where μ_N is such that

$$\mu_N = \operatorname*{arg\,max}_{\mu \in \mathcal{P}} \Delta_\mu(u_{\mu,N-1})$$

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2. Construction of V_N

Greedy algorithm with error estimator

Choose snapshots $s_N := u_{\mu_N}$ where μ_N is such that

$$\mu_N = \underset{\mu \in \mathcal{P}}{\arg\max}\, \Delta_\mu(u_{\mu,N-1})$$

Then

$$\begin{aligned}\inf_{v \in V_{N-1}} \|s_N - v\| &\geq \|a_\mu\|^{-1} \cdot c_\mu \cdot \|u_{\mu_N} - u_{\mu_N,N-1}\| \\ &\geq \|a_\mu\|^{-2} \cdot c_\mu^2 \cdot \Delta_\mu(u_{\mu_N,N-1}) \\ &\geq \|a_\mu\|^{-2} \cdot c_\mu^2 \cdot \Delta_\mu(u_{\mu,N-1}) \geq \|a_\mu\|^{-2} \cdot c_\mu^2 \inf_{v \in V_{N-1}} \|u_\mu - v\|\end{aligned}$$

Proposition

The greedy algorithm with error estimator generates a weak greedy sequence with parameter $\inf_{\mu \in \mathcal{P}} \|a_\mu\|^{-2} \cdot c_\mu^2$.

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Summary

1. Do good approximation spaces V_N exist?

   $d_N(\Phi(\mathcal{P})) \leq Ce^{-cN^{1/Q}}$

2. How to find a good approximation space V_N?

   Greedy algorithm with error estimator

3. How to construct a quickly-evaluable $\Phi_N : \mathcal{P} \to V_N$?

   Galerkin projection

4. How to control the approximation errors $\Phi(\mu) - \Phi_N(\mu)$, $s(\Phi(\mu)) - s(\Phi_N(\mu))$?

   Residual-based error estimator

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Advertisement Break

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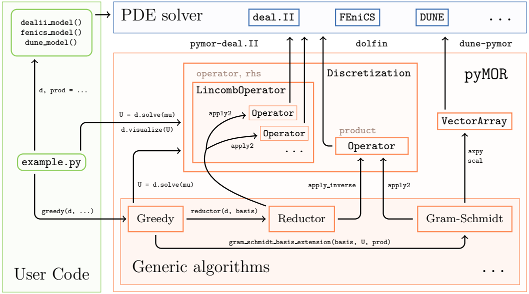
pyMOR – Model Reduction with Python



- Quick prototyping with Python.
- Seamless integration with high-performance PDE solvers.
- Out of box MPI support for reduction algs. and PDE solvers.
- BSD-licensed, fork us on Github!

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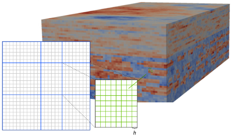
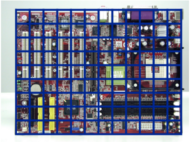
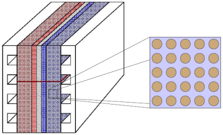
Some Projects using pyMOR



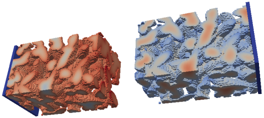
Localized Reduced Basis MultiScale method



Reduction of Maxwell's equations allowing Arbitrary Local Modifications



Reduced basis approximation for multiscale optimization problems



Reduction of microscale Li-ion battery models

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Advection Dominated Problems and the Method of Freezing

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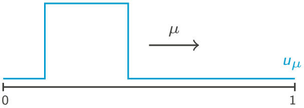
The Problem

- Typically slow decay of $d_N(\Phi(\mathcal{P}))$.
- Even for very simple examples:



$$\partial_t u(t,x) + \mu \cdot \partial_x u_\mu(t,x) = 0$$
$$u_\mu(0,x) = u_0(x), \;\; u_\mu(0,t) = u_\mu(1,t)$$
$$\mu, x, t \in [0,1]$$

Here: $d_N(\Phi(\mathcal{P}) \subset L^2) \sim N^{-1/2}$.

- Note that $\Phi : \mathcal{P} \to L^2$ is not differentiable.
- **However:** Can describe solution easily by

$$u_\mu(t,x) = u_0(x - \mu \cdot t \mod 1).$$

Nonlinear Approximation

Using Groups of Transformations

- Write $u_\mu(t, x)$ as

$$u_\mu(t, x) = u_0(x - \mu \cdot t \mod 1) =: ((\mu \cdot t) \,.\, u_0)(x)$$

- Shifts mod 1 define action of additive group $\mathbb{R}$, i.e.

$$(a + b).v = a.(b.v) \qquad \forall a, b \in \mathbb{R},\ v \in V$$

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- Shifts mod 1 define action of additive group $\mathbb{R}$, i.e.

$$(a + b).v = a.(b.v) \qquad \forall a, b \in \mathbb{R}, \ v \in V$$

- **General idea:** Write $u_\mu(t, x)$ as

$$u_\mu(t, x) = g_\mu(t).v_\mu(t, x)$$

where $v_\mu(t) \in V$ and $g_\mu(t)$ is element of Lie group G acting on V.

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Nonlinear Approximation

using Groups of Transformations

- **General idea:** Write $u_\mu(t,x)$ as

$$u_\mu(t,x) = g_\mu(t).v_\mu(t,x)$$

dynamics of u_μ
large variation in time

shape of u_μ
small variation in time

- $v_\mu(t,x)$ should be easier to approximate than $u(t,x)$!

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The Method of Freezing

- Consider Lie group G acting on V and evolution equation of the form:

$$\partial_t u_\mu(t) + \mathcal{L}_\mu(u_\mu(t)) = 0, \quad u_\mu(0) = u_0, \quad u_\mu(t) \in V$$

- Substituting the *ansatz* $u_\mu(t) = g_\mu(t).v_\mu(t)$ leads to:

$$\partial_t v_\mu(t) + g_\mu(t)^{-1}.\,\mathcal{L}_\mu(g_\mu(t).v_\mu(t)) + \mathfrak{g}_\mu(t).v_\mu(t) = 0$$
$$\mathfrak{g}_\mu(t) = g_\mu(t)^{-1}\partial_t g_\mu(t).$$

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$$\mathfrak{g}_\mu(t) = g_\mu(t)^{-1}\partial_t g_\mu(t).$$

- Have $\dim(G)$ additional degrees of freedom.
  $\rightarrow$ Add additional algebraic constraint (phase condition):

$$\Phi(v_\mu(t), \mathfrak{g}_\mu(t)) = 0.$$

- Further assume invariance of $\mathcal{L}_\mu$ under action of G:

$$h^{-1}.\mathcal{L}_\mu(h.w) = \mathcal{L}_\mu(w) \quad \forall h \in G, w \in V.$$

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The Method of Freezing

Definition (Method of Freezing)

With initial conditions $v_\mu(0) = u(0),\ g_\mu(0) = e$, solve:

$$\begin{aligned} \partial_t v_\mu(t) + \mathcal{L}_\mu(v_\mu(t)) + \mathfrak{g}_\mu(t).v_\mu(t) = 0 \\ \Phi(v_\mu(t), \mathfrak{g}_\mu(t)) = 0 \end{aligned}$$

frozen PDAE

$$\mathfrak{g}_\mu(t) = g(t)_\mu^{-1}\partial_t g_\mu(t)$$

reconstruction equation

- Introduced for stability analysis of relative equilibria [Beyn, Thümmler, 2004] and [Rowley et. al., 2003]

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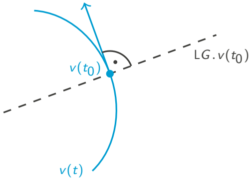
Phase Condition

- Orthogonality condition:

$$\begin{aligned}\Phi(v, \mathfrak{g}) = 0 \;&:\Longleftrightarrow \partial_t v(t) \perp LG.v(t) \\ &\Longleftrightarrow (\mathcal{L}(v) + \mathfrak{g}.v,\, \mathfrak{h}.v) = 0 \quad \forall \mathfrak{h} \in LG\end{aligned}$$



- Other choices possible.

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Example: 2D-Shifts

- $G = \mathbb{R}^2, \mathrm{L}G = \mathbb{R}^2,$

$$g.u(x) := u(x - g), \quad x \in \mathbb{R}^2$$
$$\mathfrak{g}.u = -\mathfrak{g} \cdot \nabla u$$

- Phase Condition:

$$\Phi(v, \mathfrak{g}) = 0 \iff (\mathcal{L}(v) + \mathfrak{g}.v,\ \mathfrak{h}.v) = 0 \quad \forall \mathfrak{h} \in \mathrm{L}G$$
$$\iff \left[(\partial_{x_i} v,\ \partial_{x_j} v)\right]_{i,j} \cdot \left[\mathfrak{g}_j\right]_j = \left[(\mathcal{L}(v),\ v_{x_r})\right]_i$$
$$1 \le i, j \le 2$$

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Example

Consider on $\Omega = [0, 2] \times [0, 1]$ the two-dimensional Burgers-type problem

$$\partial_t u = -\nabla \cdot (\vec{v} \cdot u^\mu)$$
$$u(0, x_1, x_2) = 1/2(1 + \sin(2\pi x_1) \sin(2\pi x_2))$$

for $t \in [0, 0.3]$, $\vec{v} = (1, 1)^T$ with periodic boundary conditions and $\mu \in \mathcal{P} = [1, 2]$.

- Finite volume discretization on 120 x 60 grid, explict Euler time-stepping
- Same problem as in [Drohmann, Haasdonk, Ohlberger, 2012]

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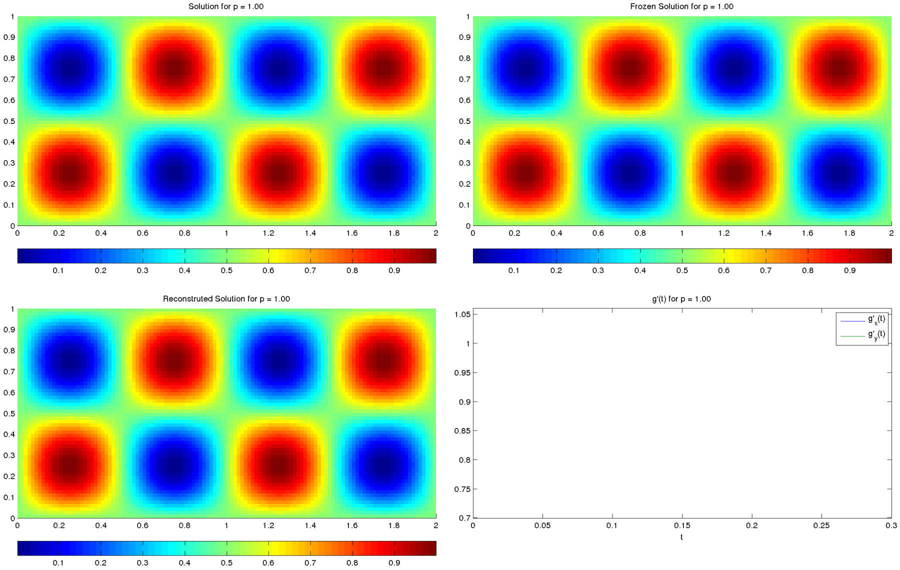
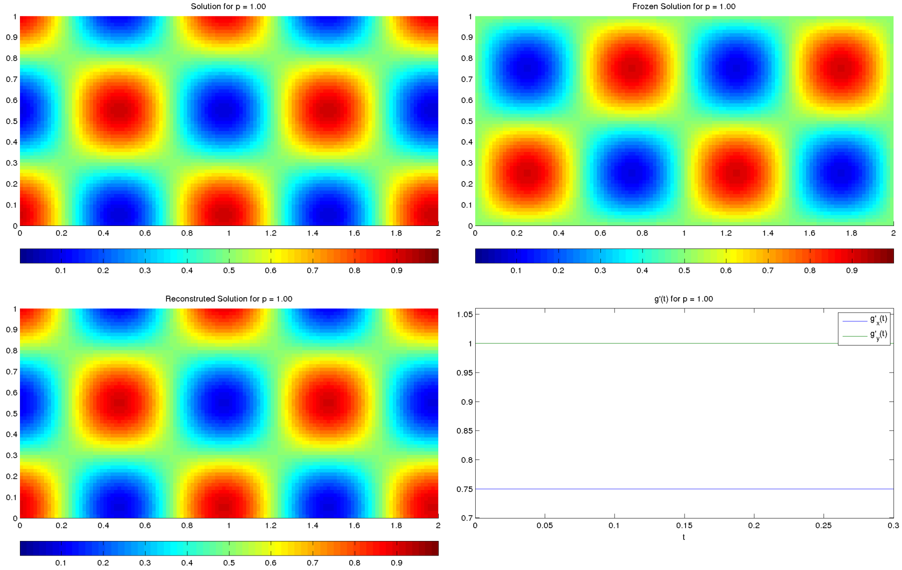
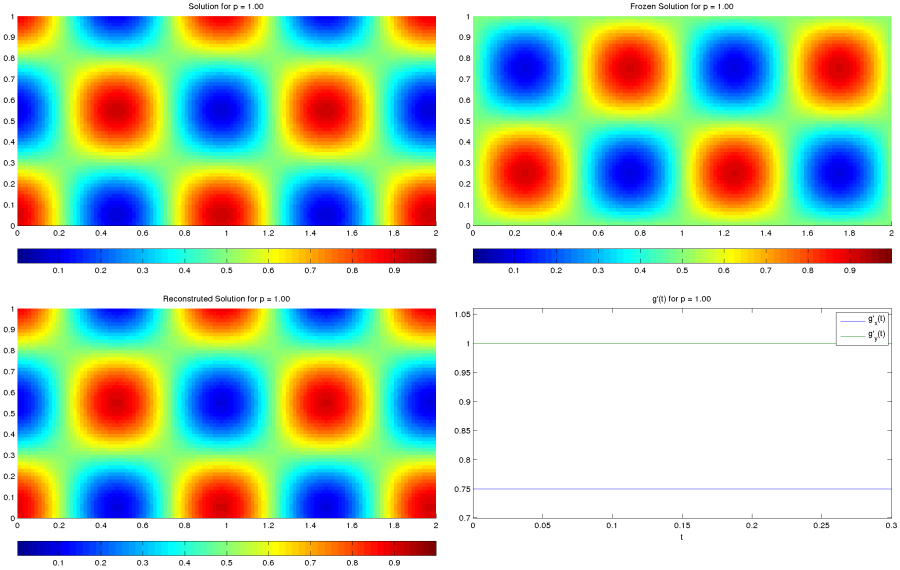
Frozen vs. Non-frozen Solution (μ=1, b=(0.75,1))



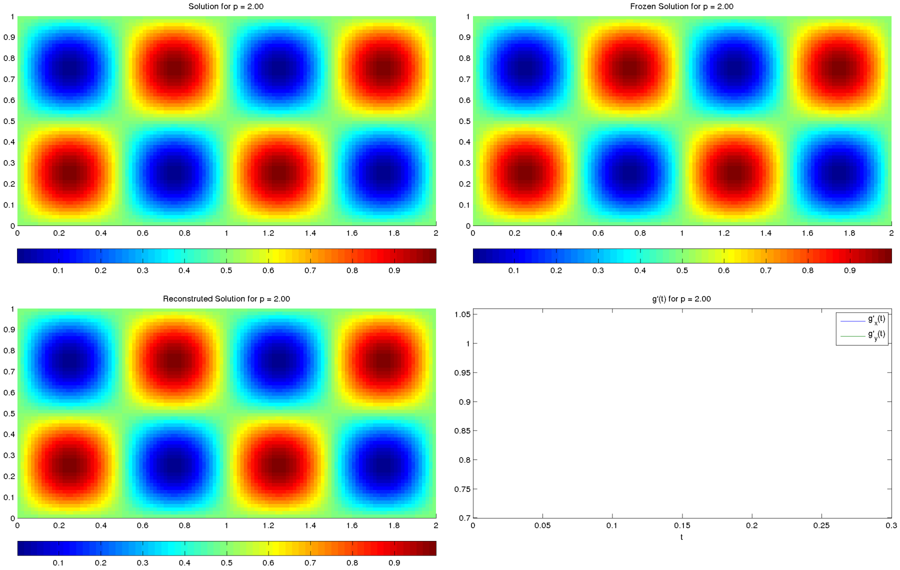
Frozen vs. Non-frozen Solution (μ=1, b=(0.75,1))



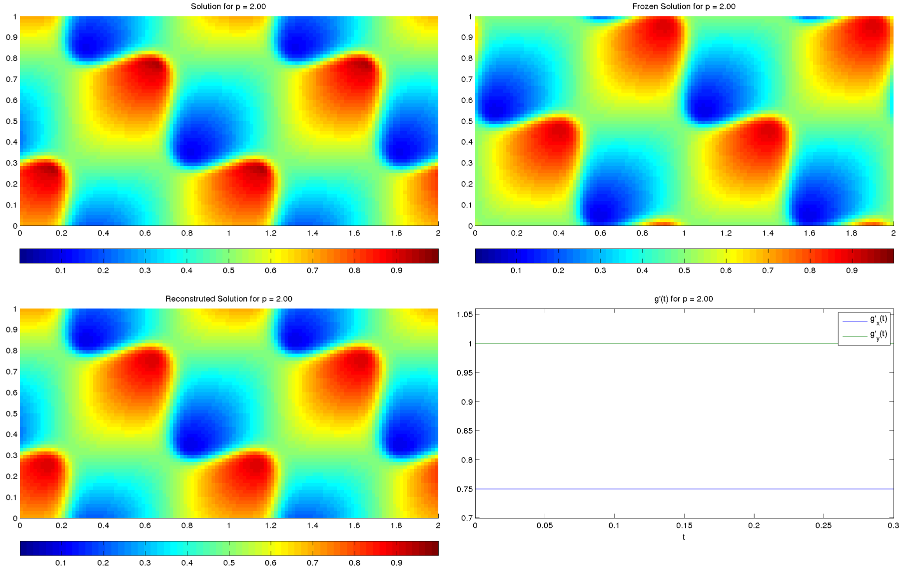
Frozen vs. Non-frozen Solution (μ=1, b=(0.75,1))



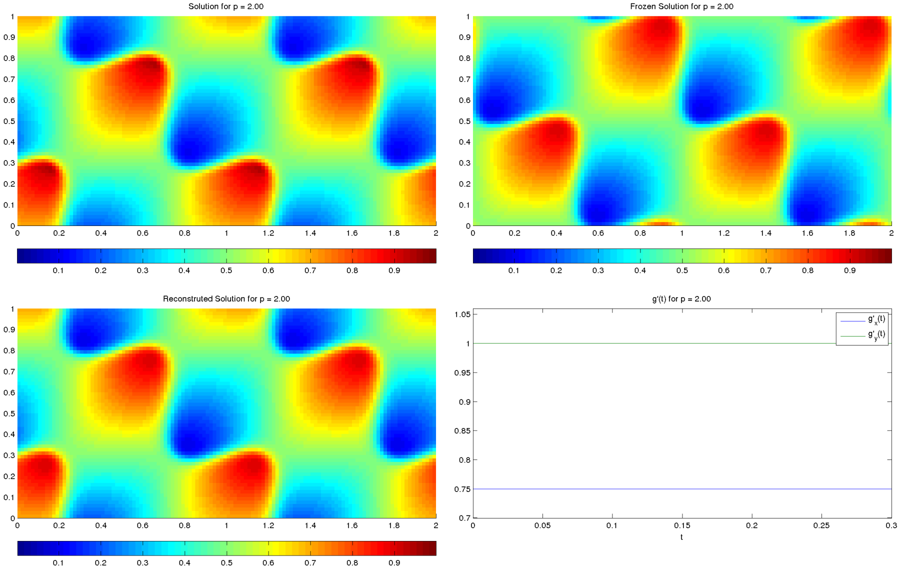
Frozen vs. Non-frozen Solution (μ=2, b=(0.75,1))



Frozen vs. Non-frozen Solution (μ=2, b=(0.75,1))



Frozen vs. Non-frozen Solution (μ=2, b=(0.75,1))



RB-Approximation

FrozenRB-Scheme [Ohlberger, R., 2013]

1. Replace PDE by frozen PDAE and reconstruction ODE.
2. Apply RB methods to frozen PDAE.

- $V_N :=$ span{POD modes of solution trajectores}. (POD-Greedy)
- Empirical operator interpolation to treat nonlinearity. (EI-Greedy)
- Offline/online decomposition possible
- No additional evaluations of nonlinearity (small overhead)

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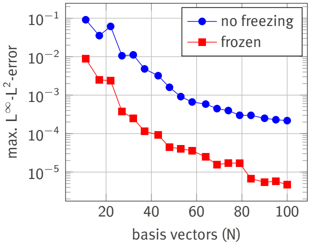
Results for Burgers Problem

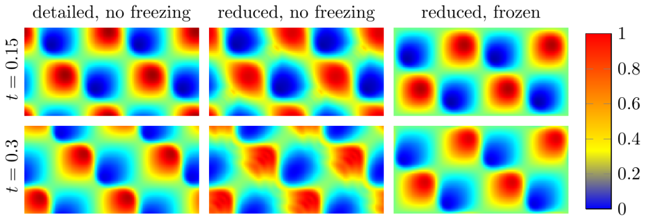


Left:

- $1.8 \cdot N$ interpolation points.
- Test set: 100 random μ.

Bottom:

- $\dim V_N = 20$, 38 interpolation points.



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Future Challenges

- Handling of (non-periodic) boundaries?
- More complicated group actions / local effects?
- Non-equivariant group actions?

Thank you for your attention!

My homepage:
`http://stephanrave.de/`

Ohlberger, R., *Reduced Basis Methods: Success, Limitations and Future Challenges*, Proceedings of ALGORITMY 2016.

Ohlberger, R., *Nonlinear reduced basis approximation of parameterized evolution equations via the method of freezing*, C. R. Math. Acad. Sci. Paris, 351 (2013).

pyMOR – Model Order Reduction with Python
`http://www.pymor.org/`
arXiv:1506.07094

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Example: 2D-Shifts

The Method of Freezing for 2D-Shifts

Solve

$$\partial_t v_\mu(t) + \mathcal{L}_\mu(v_\mu(t)) - \mathfrak{g}_\mu(t) \cdot \nabla v_\mu(t) = 0$$
$$\left[(\partial_{x_i} v_\mu,\, \partial_{x_j} v_\mu)\right]_{i,j} \cdot \left[\mathfrak{g}_\mu\right]_j = \left[(\mathcal{L}_\mu(v_\mu),\, \partial_{x_i} v_\mu)\right]_i$$

and

$$\partial_t g_\mu(t) = \mathfrak{g}_\mu(t)$$

with initial conditions $v_\mu(0) = u(0)$, $g_\mu(0) = (0,0)^T$.

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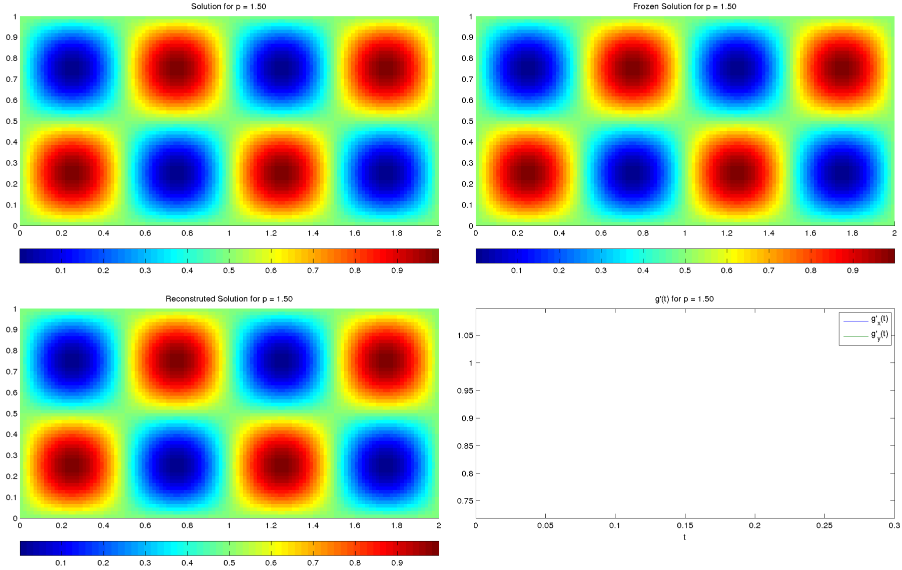
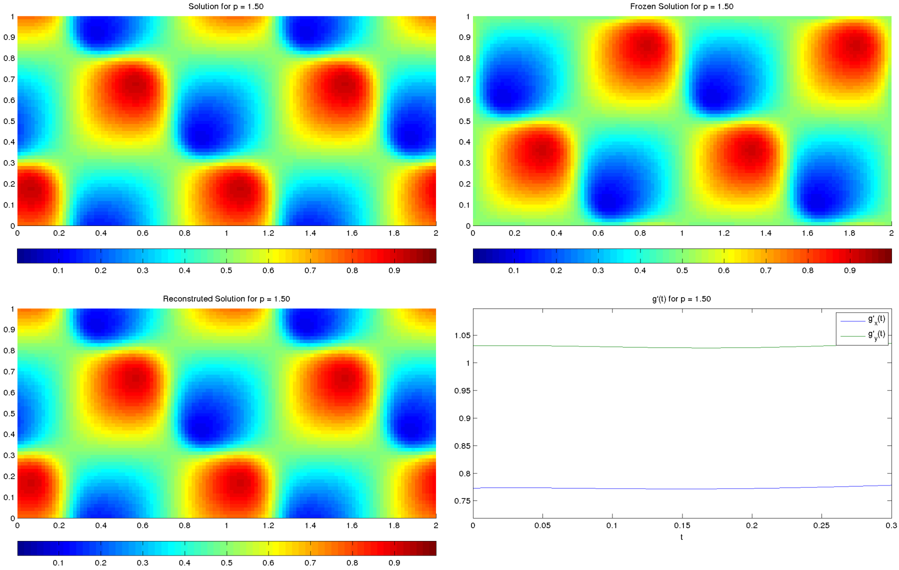
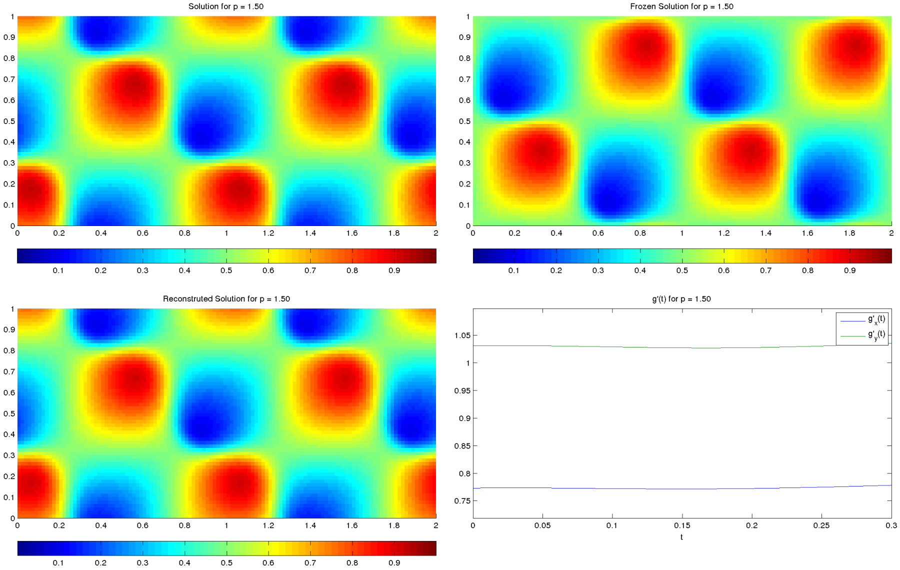
Frozen vs. Non-frozen Solution (μ=1.5, b=(0.75,1))



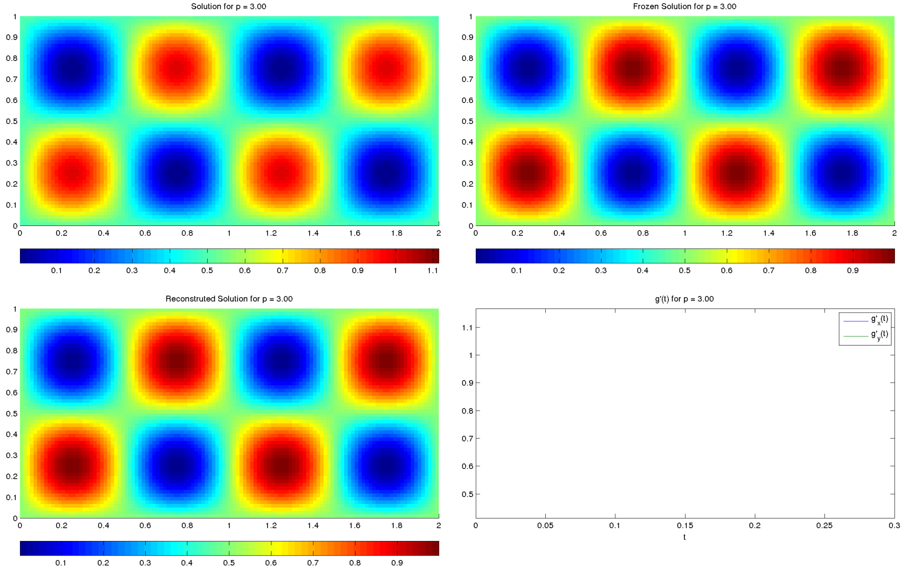
Frozen vs. Non-frozen Solution (μ=1.5, b=(0.75,1))



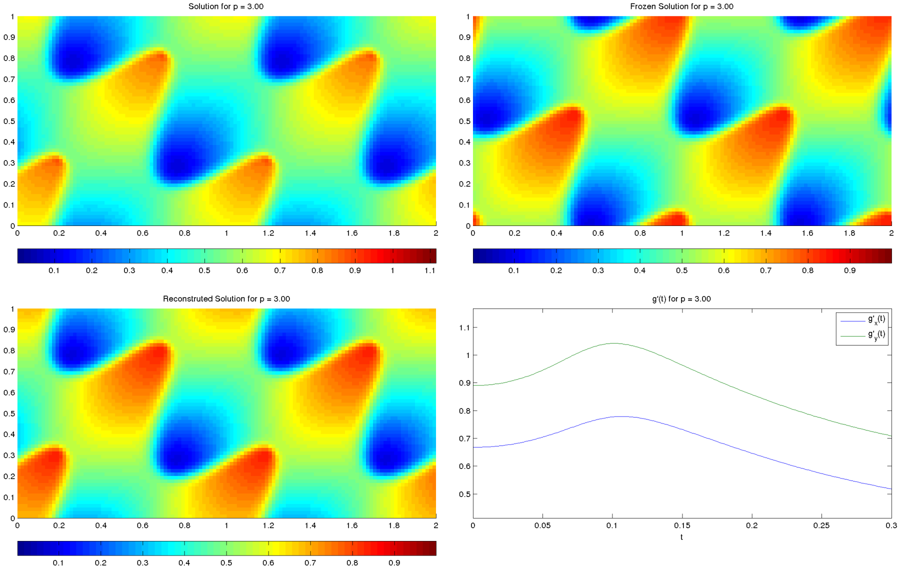
Frozen vs. Non-frozen Solution (μ=1.5, b=(0.75,1))



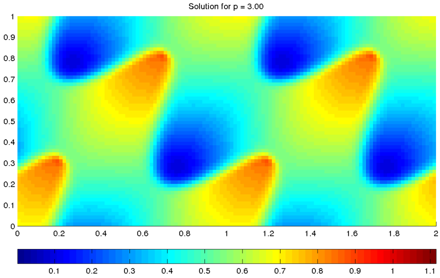
Frozen vs. Non-frozen Solution (μ=3, b=(0.75,1))

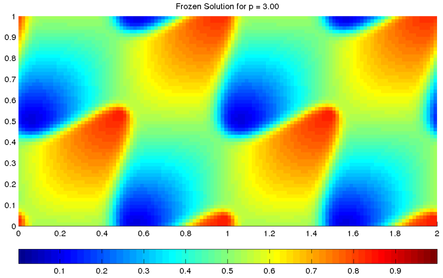


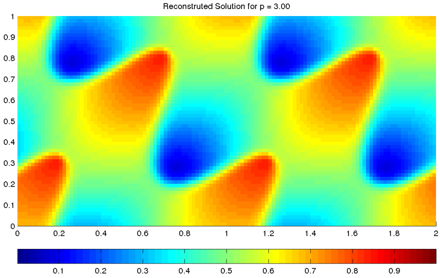
Frozen vs. Non-frozen Solution (μ=3, b=(0.75,1))

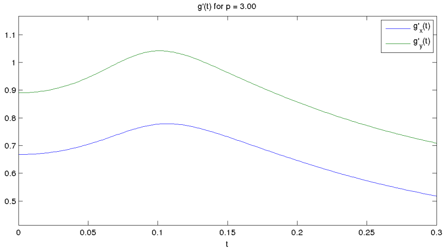


Frozen vs. Non-frozen Solution (μ=3, b=(0.75,1))

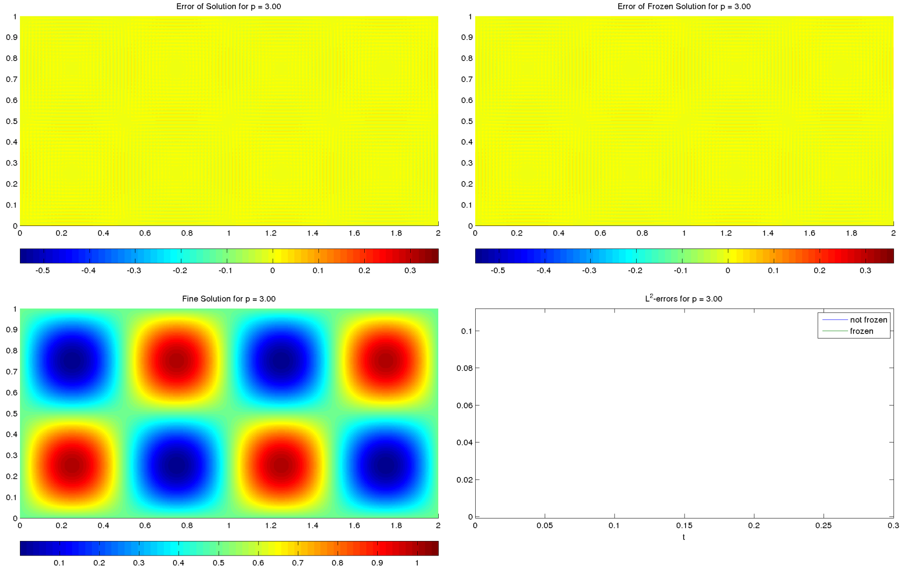








Errors of Frozen and Non-frozen Solution (μ=3)



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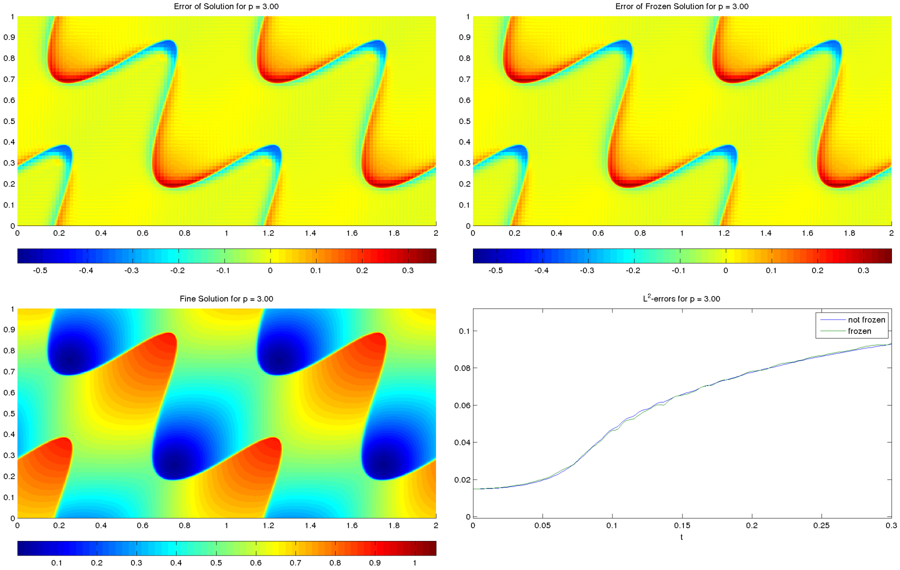

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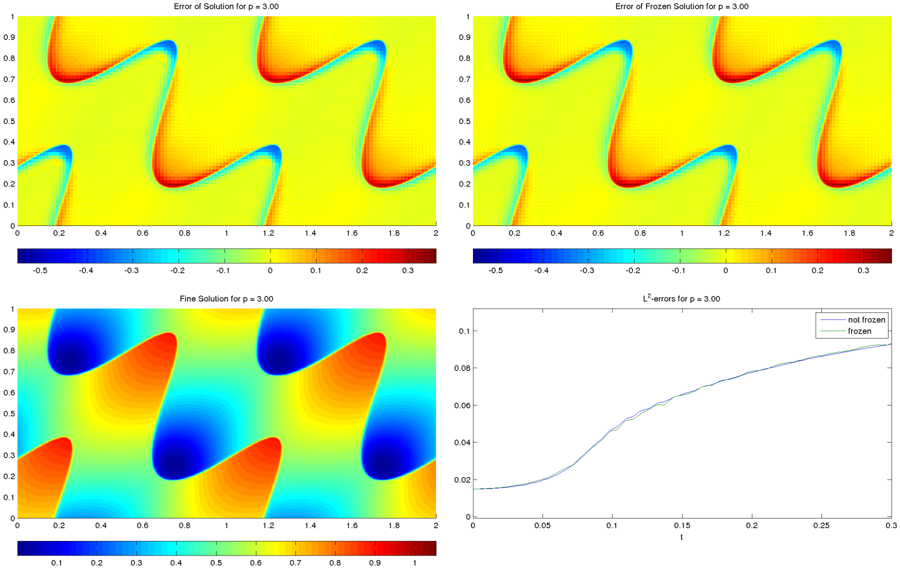
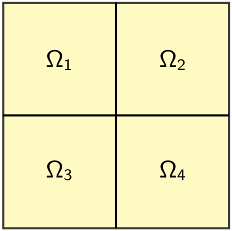


Model Problem



$$\Omega = \bigcup_{i=1}^{4} \Omega_i, \quad \mathcal{P} = [\alpha, 1]^4, \ \alpha > 0$$

$$a_\mu(x) = \sum_{i=1}^{4} \mu_i \cdot \chi_{\Omega_i}(x), \qquad x \in \Omega, \mu \in \mathcal{P}$$

$$f \in L^2(\Omega)$$

Thermal block problem

For $\mu \in \mathcal{P}$, find $u_\mu \in H_0^1(\Omega)$ s.t.

$$-\nabla \cdot (a_\mu \nabla u_\mu) = f$$

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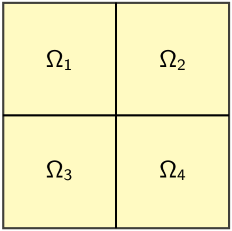
Model Problem



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$$a_\mu(x) = \sum_{i=1}^{4} \mu_i \cdot \chi_{\Omega_i}(x), \qquad x \in \Omega, \mu \in \mathcal{P}$$

$$f \in L^2(\Omega)$$

Thermal block problem

For $\mu \in \mathcal{P}$, find $u_\mu \in H_0^1(\Omega)$ s.t.

$$\sum_{k=1}^{4} \mu_k \int_{\Omega_k} \nabla u_\mu \cdot \nabla v = \int_\Omega f \cdot v \qquad \forall v \in H_0^1(\Omega)$$

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