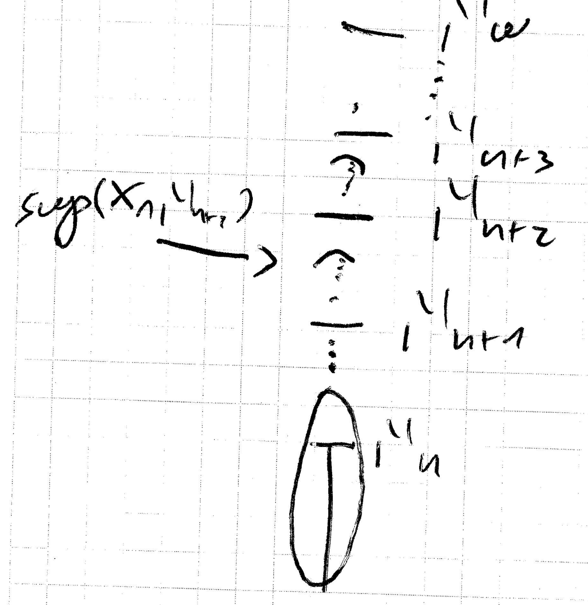
Introduction



~~What~~

Let $X \subseteq \aleph_\omega$

What is $\operatorname{cof}(\sup(X \cap \aleph_n))$

$\qquad\qquad\qquad\qquad\qquad \| \qquad \chi_X(n)$

$\operatorname{cof}(\chi_X(n))$ constant?

(easy, internally approachable)

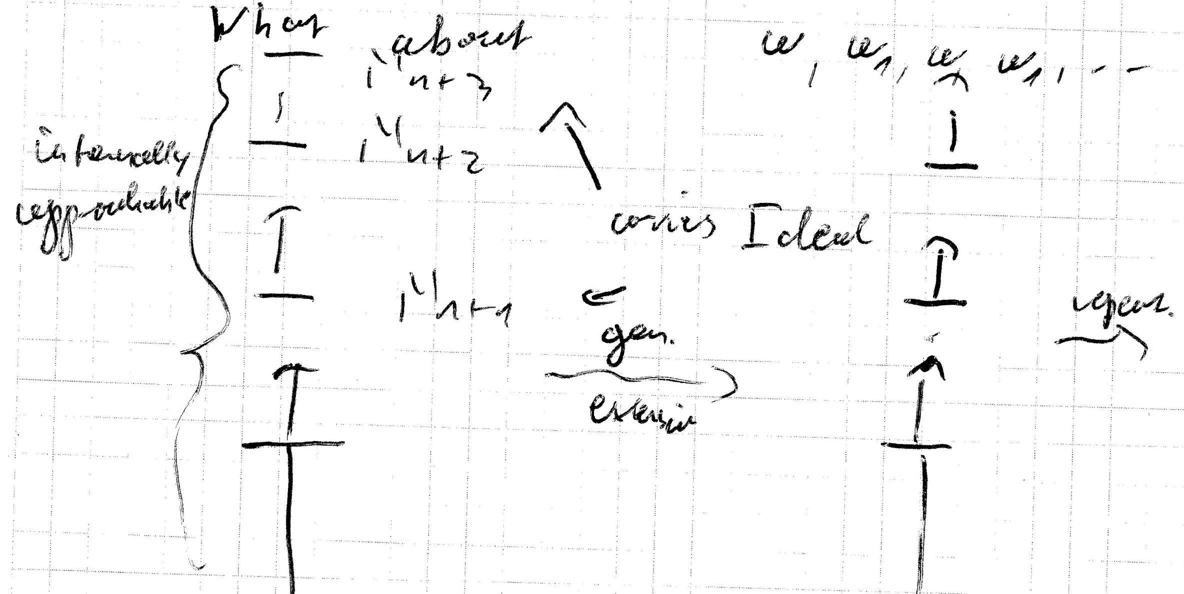
(see Mutual stationarity paper)

Also, finite changes are possible

$\sup(X \cap \kappa) = \sup(X^* \cap \kappa)$

if $X^* = Sk^{\alpha}(X \cup A)$, $A \subseteq \lambda$

$\lambda < \kappa$, κ regular.



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So infinitely many measurables

$\rightsquigarrow$ ~~ag~~ sequence of alternating cofinalities

$\longrightarrow$ a measurable.

(Fact: can get some sequence from a measurable but (Koepke) with limitations

Question:

Let the sequence

$\langle S^2_0, S^3_1, S^4_1, S^5_0, S^6_{11}, S^7_1, \ldots \rangle$

be mutually stationary.

Is there an inner model with infinitely many measurables?)

Here we are interested in sequences with longer blocks

i.e. $\omega_1, \omega_1, \omega_2, \omega_2, \omega_1, \omega_1, \omega_2, \omega_2, \ldots$

$\neg 0^{P}$

Let $\Omega \gg \aleph_\omega$, $V_\Omega \models ZFC - Rep$!

Notation:

$\exists^* X\ \varphi(X, \vec{a})$ with $\vec{a} \in V_\Omega$ $:\Leftrightarrow$ $\forall F : [V_\Omega]^{<\omega} \to V_\Omega$ $\ V_\Omega \models \exists X \subseteq V_\Omega : F''[X]^{<\omega} \subseteq X \wedge \varphi(X, \vec{a})$

$\forall^* X\ \varphi(X, \vec{a})$ with $\vec{a} \in V_\Omega$ $:\Leftrightarrow$ $\exists F : [V_\Omega]^{<\omega} \to V_\Omega$ $\ \forall X \subseteq V_\Omega : F''[X]^{<\omega} \subseteq X \to \varphi(X, \vec{a})$

Definition: $(\ell > 1)$

$S \subseteq \mathcal{P}(V_\Omega)$ has blocks of size ℓ iff:

- S is stationary $-$ $\forall X \in S : \overline{\overline{X}} < \aleph_\omega$
- $\forall n < \omega : \forall X \in S\ \aleph_n \notin X \Rightarrow \mathrm{cof}(\sup(X \cap \aleph_\omega)) = \mu_n$
- $\exists \langle n_k : k < \omega \rangle :$

  $n_{k+1} \geq n_k + \ell$, $\mu_{n_k} = \mu_{n_k + i}$ $(i < \ell)$
- $\langle \mu_{n_k} : k < \omega \rangle$ is not eventually const.
- $\forall n$ ~~[illegible]~~ $\mu_n \geq \aleph_1$ $(*)$

~~What is known:~~

~~$\exists \kappa : o(\kappa) \geq \kappa^{+(\ell+1)} + 1$ (+ GCH)~~

$\xrightarrow{\text{ext. Prikry}}$ (A. Sharon)

~~$2^{\aleph_\omega} > \aleph_{\omega+1}$ + PCF configuration~~

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What is known:

$\bullet\ \exists \kappa: o(\kappa) \geq \kappa^{+\omega}$ + GCH $\left(\geq \kappa^{+(\ell+1)}+1\right)$

est. Pri. $\rightsquigarrow$ $2^{\aleph_\omega} > \aleph_{\omega+1}$ + PCF conf.

A. Sharon (O. B-Neria)

PCF magic $\rightsquigarrow$ Shelah: $\exists\ \cancel{\ }\ S$: S has blocks of size ℓ (*)

core models $\rightsquigarrow$ A.-Cox-Welch: $\exists \langle \kappa_n : n<\omega\rangle : \kappa_n \nearrow \aleph_\omega$, $\overset{\forall n:}{o^K(\kappa_n)} \geq \kappa^{+(\ell+1)}$

$\left(+\ (o(\kappa) \geq \kappa^{+(\ell+1)})^{+}\right)$

New stuff: (A.)

1. $\exists S$: S has blocks of size ℓ (*)
$\Rightarrow \exists \langle \kappa_n : n<\omega\rangle$: κ_n ~~almost~~ "close to" Prikry sequence

2. $\exists S$: S has blocks of size ℓ
$\Rightarrow \exists \langle \kappa_n : n<\omega\rangle$: $\underbrace{\kappa_n \nearrow \aleph_\omega}_{\text{"new"}}$ + $\overset{\forall n:}{o^K(\kappa_n)} \geq \kappa^{+(\ell+1)}$

Section ①:

Def.: M some trans. Model of a fragment of ZFC. E an M-extender.
E is locally $o^{\ell}(\kappa) \geq \kappa^{+(\ell+1)}+1$ iff
$\mathrm{ult}(M,E) \models_{\mathrm{crit}(E)} o(\kappa) \geq \kappa^{+(\ell+1)}$

(Note: possibly $(\kappa^{+(\ell+1)})^{\mathrm{ult}(M,E)} < (\kappa^{+(\ell+1)})^M$)

E is fully locally $o(\kappa) \geq \kappa^{+(\ell+1)}+1$
iff E is locally $o(\kappa) \geq \kappa^{+(\ell+1)}+1$
and sup of gen. of E is cofinal in $(\kappa^{+(\ell+1)})^{\mathrm{ult}(M;E)}$.

Thm:

~~Assum.~~ Let $S \subseteq \mathcal{P}(V_\Omega)$ have blocks of size ℓ. Then $\exists \langle \kappa_n : n<\omega \rangle$ s.t. $\forall n$:
$\exists m<\omega$: $(\kappa_n^+)^K < \aleph_m^V$, $o^K(\kappa_n) \geq \aleph^V_{m+(\ell-1)}$,
the set $\{\nu < \aleph^V_{m+(\ell-1)} \mid E^K_\nu$ is fully locally $o(\kappa) \geq \kappa^{+(\ell+1)}+1\}$
is stationary.

Preliminaries:

Let $X \subseteq V_\Omega$, $\sigma_X : H_X \cong X$, $\beta_n^X := \sigma_X^{-1}(\aleph_n^V)$,
$\beta_\omega^X := \sup \beta_n^X$, $K_X = \sigma_X^{-1}(K \| \aleph_\omega^V)$
$\langle M_i^X, E_i^X, \kappa_i^X, \nu_i^X, \pi_{ij}^X : i \leq j \leq \theta_X \rangle$

Thm (Zeman)
$S \subseteq \mathcal{P}(V_\Omega)$ stat. s.t. $\forall^* X \in S$: $X \cap On$ ω-closed (set of ordinals).
$\forall^* X \in S$: $\mathbb{R}$ side of co-it. is trivial and K-side drops at stage 0 or 1.

~~Lx~~ Core Lemma:

Let $\mathcal{M}$ be a fine-structural model, $\kappa \in \mathcal{M}$ s.t. $\mathcal{M} \models$ "κ reg." $\rho_n^{\mathcal{M}} \geq \kappa > \alpha \geq \rho_{n+1}^{\mathcal{M}}$ and $\mathcal{M}$ is ~~sou~~ n-sound above α.
Then $\operatorname{cof}(\kappa) = \operatorname{cof}(\rho_n^{\mathcal{M}})$.

Proof:

(n=0) $\beta \longmapsto \sup(\operatorname{Hull}_1^{\mathcal{M}\|\beta}(\{p_{n+1}^{\mathcal{M}}\} \cup \alpha) \cap \kappa)$ ⊣

Prop. (A.-Cox-Welch)

- $\forall X \in S$: θ_X is a limit
- $\exists n^* < \theta_X$: $\nu_i^X \geq \beta_{n^*}^X \Rightarrow$ no drop at i
- ~~&~~ $\exists\, m_{(X)} < \omega$ & $\exists\, \mu^*_{(X)}$ s.t.
  $\forall i < \theta_X$ $\nu_i^X \geq \beta_{n^*}^X \Rightarrow \operatorname{cof}(\rho_m^{\mathcal{M}_i^X}) = \mu^*$
- $n_k \geq n^* \wedge \mu_{n_k} \neq \mu^*$
  $\Rightarrow$ $(\kappa^+_{i_k^X})^{K_X} < \beta^X_{n_k}$
  $\wedge$ $\nu^X_{i_k^X} \geq \beta^X_{n_k+(l-1)}$

where i_k^X least s.t.
$\nu^X_{i_k^X} \geq \beta_{n_k}$

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Let $F_k^X := E^X_{i^X_k}$, $\lambda_k^X := \kappa^X_{i^X_k}$, $\eta_k^X := \nu^X_{i^X_k}$

2.) $(N_k^X)^- := M^X_{i^X_k} \| \eta_k^X$ (k s.t. $\beta^{n_k} \geq n^*$ and $\mu_{n_k} \neq \mu^*$)

$N_k^X :=$ expansion of $(N_k^X)^-$ by F_k^X

1.) Assume w.l.o.g. $\nu^X_{i^X_k} = \beta^X_{n_k+(e-1)}$ $\forall k$.
Fix k s.t. $n_k \geq n^*$ and $\mu_{n_k} \neq \mu^*$.

Prop.:
$E^X_{i^X_k}$ is fully locally $o(X) \geq \kappa^{+(e+1)}+1$

Proof:

Assume not. Say $\eta < \nu^X_{i^X_k}$ is sup. of generators.
Then $\nu^X_{i^X_k}$ is reg. in $M := M^X_{i^X_k+1}$.
(because of agreement in iteration trees and $\nu^X_{i^X_k} = \beta^X_{n_k+(e-1)}$.)

But ~~$M^X_{i^X_k}$~~ M is ^{m}-sound$_{e-1}$ above η.
Therefore $cof(\beta^X_{n_k+(e-1)}) = \mu^*$

$$\begin{array}{c} \beta^X_{n_k+(e-1)} \\ \| \\ \mu_{n_k+(e-1)} \\ \| \\ \mu_{n_k} \end{array}$$

blocks

$\lightning$

$$N_k^x := \left(Ult\left(\mathbb{D}^{(N_k^x)^-}_{(\lambda_k^x)^+} ; F_k^x \right) ; \in , E^{Ult\ldots} , \ell_{F_k^x} \wedge \mathbb{D}^{(W_e^+)^-}_{(\lambda_k^x)^+} \right)$$

$\uparrow$
fine-structured model.

Prop: N_k^x is 1-sound.

Proof:

1st case: $\eta_k^x < On \wedge M^x_{i_k^x}$ ... $\square$

2nd case: $\eta_k^x = On \wedge M^x_{i_k^x} \wedge m>0$ .. $\square$

3rd case: $\eta_k^x = On \wedge M^x_{i_k^x} \wedge m=0$

Note that η_k^x is regular in ~~W_k^x (it is regular in M^x~~

N_k^x because it is in K^x

and this represents the minimal disagreement.

But N_k^x is sound above $\lambda_k^x < \nu_k^x$

and therefore $cof(\eta_x^k) = \mu^*$ ↯

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Prop: $\rho_1^{N_x^k} \geq \nu_k^X$

Proof: Assume not.

For $\tilde{\iota}_k^X \leq \alpha < \nu_k^X$ let

$$j_\alpha : \mathrm{Ult}(N_k^X \| \tilde{\iota}_k^X; F_k^X \upharpoonright \alpha) \longrightarrow (N_k^X)^-$$

$$\parallel \qquad [\, \iota_{F_k^X \upharpoonright \alpha}(f)(a) \longmapsto \iota_{F_k^X}(f)(a) \,]$$

$$N_\alpha^*$$

Let α be such that
$\{p_1^{N_x^k}\} \cup \rho_1^{N_x^k} \subseteq \mathrm{ran}(j_\alpha)$.

Note: j_α is Σ_0 (Łoś)

it is also Δ_0 as
$j_\alpha : (N_\alpha^*; \iota_{F_k^X \upharpoonright \alpha}) \longrightarrow (N_k^X; \iota_{F_k^X})$

[
$\varphi(\vec{a})$ ~~can be~~ is equivalent to
$\varphi^*(\vec{a}, \iota_{F_k^X} \upharpoonright \beta)$ and j_α respects $\iota \upharpoonright \beta$]

But j_α is cofinal

$$\begin{tikzcd}
\cdot \arrow[rr, "j_\alpha"] & & \cdot \\
& N_k^X \upharpoonright \alpha \arrow[ul, "\iota_{F_k^X \upharpoonright \alpha}", "\mathrm{cof}"'] \arrow[ur, "\mathrm{cof}", "\iota_{F_k^X}"'] &
\end{tikzcd}$$

$\Rightarrow$ j_α is Σ_1 on $(N_\alpha^*; \iota_{F_k^X \upharpoonright \alpha})$

Let $x \in N_x^k$, $x = \tau^{N_x^k}(\vec{\beta}, p_1^{N_k^X})$, $\vec{\beta} < \alpha$

$\bar{x} := \tau^{(N_\alpha^*)^+}(\vec{\beta}, j_\alpha^{-1}(p_1^{N_k^X}))$; $j_\alpha(\bar{x}) = x$

$\Rightarrow$ j_α surj. $\lightning$

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Finally so can form

$\mathrm{ult}(N^x_k, \sigma_x \upharpoonright \eta^x_k)$

$\sigma^*: N^{*x}_k \longrightarrow N^*$ is Σ_2 (N^* is a premouse)

w.l.o.g. N^* it.

Top extender of N^* is fully locally $o(\kappa) \geq \kappa^{+(e+1)} + 1$

and N^* extends K past

$\sigma_x{}'' \eta^x_k = \sigma^*(\eta^x_k)$ $(= X \cap i''_{n_k+(e-1)})$

and is sound above it, so $N^* \in K$

Q.E.D.

Part 2