menachem magidor

(joint with j. kennedy + j. väänänen)

L – the const. universe

$L_0 = \emptyset$, $L_{\alpha+1} = \mathrm{Def}(L_\alpha; \in)$, α lim :

$L_\alpha = \bigcup_{\beta<\alpha} L_\beta$.

generalized logics $\mathcal{L}$

$L_0^{\mathcal{L}} = \emptyset$, $L_{\alpha+1}^{\mathcal{L}} = \mathrm{Def}^{\mathcal{L}}(L_\alpha^{\mathcal{L}}; \in)$, ...

$\mathcal{L}$ 2nd order logic :

th. (myhill–scott) $L^{\mathcal{L}} = \mathrm{HOD}$.

examples :

(1) cofinality ω logic, $\mathcal{L}(\mathrm{cof}\,\omega)$:

1st order logic with the quantifier

$Qxy\, \varphi(x,y)$: $\varphi(-,-)$ defines a lin. order which has ctble. cofinality .

skolem–löwenheim :

$\mathcal{A}$ a structure, there is then

$\mathcal{B} \prec_{\mathcal{L}(\mathrm{cof}\,\omega)} \mathcal{A}$, $|\mathcal{B}| = \omega_1$.

the inner model $L^{\mathrm{cof}\,\omega}$ =

$L[\{\alpha : \mathrm{cf}(\alpha) = \omega\}]$.

② <u>stationary logic</u>:

second order variables

& meaning: $P \in \mathcal{P}_{\omega_1}(M)$ ≡ P is a ctbl. subset of M

$(\mathrm{aa}P)\, \varphi(P)$:

$\{P \in \mathcal{P}_{\omega_1}(M) : M \models \varphi(P)\}$ contains a club in $\mathcal{P}_{\omega_1}(M)$.

③ κ is regular, $S \subset \kappa$, $\alpha \in S \Rightarrow \mathrm{cf}(\alpha) = \omega$:

we can express "S is stationary in λ" iff

$\{P : \sup(P) \in S\}$ is stationary in $\mathcal{P}_{\omega_1}(\lambda)$

③ <u>hartig-quantifier</u>

$Qxy\, \varphi(x)\, \psi(y)$ means

$|\{x : \varphi(x)\}| = |\{x : \psi(y)\}|$

$L^Q = L[\mathrm{Card}]$.

let $a \subset \omega$, $L[a]$

can force over $L[a]$ with a variant of namba forcing s.t.

$a = \{ n : \mathrm{cf}(\aleph^L_{n+2}) = \omega \}$ in the exten.

let $(a_\alpha : \alpha < \kappa)$ be a seq. of cohen reals (over L).

can force s.t. in the exten

$$a_\alpha = \{ n : \mathrm{cf}(\aleph^L_{\omega\alpha + n + 2}) = \omega \} .$$

in this exten, $L^{\mathrm{cof}(\omega)} = L[(a_\alpha : \alpha < \kappa)]$.

in $L^{\mathrm{cof}(\omega)} \models 2^{\aleph_0} = \kappa$.

$L[(a_\alpha : \alpha < \kappa)]$.

can define $S_\alpha \subset \aleph_{\alpha+1}$,

$\beta \in S_\alpha \Rightarrow \mathrm{cf}(\beta) = \omega$.

S_α stat., S_α does not reflect.

(e.g. use $\square$ in L)

can shoot a club thr. ~~the~~ complement of S_α without killing the stat. of the others, etc.

can then force to get

$a_\alpha = \{n : S_{\alpha\omega+n+1}$ is not stat. in $\aleph_{\alpha\omega+n+1}\}$

so $(a_\alpha : \alpha < \kappa) \in L^{aa}$

stat. logic

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if $L[a]$, L have the same cardinals,
can force to have
$n \in a$ iff $|\aleph^L_{2n+1}| = |\aleph^L_{2n}|$,

so $a \in L^Q$

Härtig logic

in the same way arrange $(a_\alpha : \alpha < \kappa) \in L^Q$

$L^Q \models 2^{\aleph_0} = \kappa$.

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theorem. δ woodin, $\mathbb{P} \in V_\delta$ forcing,

$g \subset \mathbb{P}$-gen.

$Th((L^{cf(\omega)})^V) = Th((L^{cf(\omega)})^{V[g]})$.

but may change the models:

let $(\kappa_n : n<\omega)$ is a seq. of measurable cardinals.

let $a \subset \omega$ be a Cohen real over V.

force with a iteration of prikry forcings s.t.

$a = \{n : cf(\kappa_n) = \omega\}$.

let $\lambda = \sup_{n<\omega} \kappa_n$.

$\{\delta<\lambda : cf(\delta) = \omega\} \in L^{cf(\omega)}$.

if $\{\delta<\lambda : cf^{V[G]}(\delta) = \omega\} \in (L^{cf(\omega)})^V \subset V$.

then $a \in V$ $\lightning$.

proof of the theorem:

force with $\mathbb{Q}_{<\delta}$ = ctm. tower.

$G \subset \mathbb{Q}_{<\delta}$ gen.

in $V[G] \models \delta = \omega_1$.

have $j : V \longrightarrow M$, $j(\omega_1) = \delta$,

${}^{\omega}M \subset M$ in $V[g]$.

$(L^{cf(\omega)})^V$

$j : (L^{cf(\omega)})^V \longrightarrow (L^{cf(\omega)})^M = (L^{cf(\omega)})^{V[G]}$.

for which ordinals α is $cf(\alpha)^{V[G]} = \omega$?

$\iff cf^V(\alpha) < \delta$.

so $(L^{cf(\omega)})^{V[G]} = (L^{cf(<\delta)})^V$.

so they have the same theories.

if h is $\mathbb{P}$-gen., $\mathbb{P} \in V_\delta$,
δ is still wood. so same argument shows

$$Th\,(L^{cf(\omega)})^{V[h]} = Th\,((L^{cf(<\delta)})^V,$$

so $Th(L^{cf(\omega)})^{V[h]} = Th\,(L^{cf(\omega)})^V$. $\dashv$

So with a proper class of woodins,
one cannot change the theory of $L^{cf(\omega)}$.

<u>th.</u> $|\mathcal{P}(\omega) \cap L^{cf(\omega)}| = \aleph_0$, assume δ wood.

proof: assume $(a_\alpha : \alpha < \omega_1)$ is a list
of pairwise diff. reals in $L^{cf(\omega)}$.

let $j : V \longrightarrow M \subset V[g]$, g $\mathbb{Q}_{<\delta}$-generic.

$$L^{cf(\omega)} \longrightarrow \left(L^{cf(\omega)}\right)^M$$
$$\|$$
$$\left(L^{cf(<\delta)}\right)^V$$

so $j(a_\alpha : \alpha < \omega_1)$ is a list of V-reals of length δ. $\not\downarrow$

~~Conj.~~ Thm. for a cone of x, $L^{cf(\omega)}(x) \models CH$.

~~Thm~~ $L^{cf(\omega)} \subset L^{cf(\omega)}(x)$.

Lem. Th ~~...~~ $L^{cf(\omega)}$ is a cdbl. Σ^1_3 set.

Prf.: may pick $\beta < \omega_2$ s.t. $y \in L^{cf(\omega)}_\beta$, say β least.
δ woodin, κ measurable.

$y \in M \prec V_\theta$, M countable.

let $\pi : M \longrightarrow \overline{M}$ transitive collapse.

$\bar\delta = \pi(\delta)$, $\bar\beta = \pi(\beta)$.

$j : \overline{M} = M_0 \longrightarrow M_1$ generic ultrapower.

$\left(L^{cf(<\kappa)}\right)^{M_1}$ iterate.

gen
$$\begin{tikzcd} M_0 \arrow[r] \arrow[rrr, bend right, "j"'] & M_1 \arrow[r] & \cdots \arrow[r] & M_{\omega_1} \end{tikzcd}.$$

$j(\omega_1^{M_0}) = \omega_1^V$, $\quad \beta^* = j(\bar\beta)$.

$M_{\omega_1} \models \overline{\overline{\beta^*}} = \omega_1$.

Im $\eta < \beta^*$.

$\quad cf(\eta) = \omega$ iff $M_{\omega_1} \models cf(\eta) = \omega$.

$$\Rightarrow \quad \underset{\substack{\cup\!\!\!\!| \\ y}}{L^{cf(\omega)}_{\beta^*}} = \left(L^{cf(\omega)}_{\beta^*}\right)^{M_{\omega_1}}.$$

Supp. on the other hand

$\exists$ ctbl M with a Woodin which is generically ω_1-iterable, $y \in \left(L^{cf(\omega)}_{\omega_2}\right)^M$

then $y \in L^{cf(\omega)}$.

So $\mathbb{R} \cap L^{cf(\omega)}$ is a ctbl. Σ^1_3 set. $\dashv$

$\mathbb{R} \cap L^{q(\omega)}(x)$ is $\Sigma^1_3(x)$, unif., by same argt.

"$L^{q(\omega)}(x) \models CH$" is a projective statement.

turing inv.

by turing determinacy,

either there is a cone of reals x s.t.

$L^{q(\omega)}(x) \models CH$ or

s.t. $L^{q(\omega)}(x) \not\models CH$.

let x be any real.

force $\omega_1 \to 2^{\aleph_0}$, code it by a real, z.

$L^{q(\omega)}(x \oplus z) \models CH$.

L^{aa} – model

L^{aa}_β . $\mathrm{Def}^{aa}_\beta(L^{aa}_\beta, \in)$

$\varphi($

is the a club ~~of~~ C in $P_{\omega_1}(L^{aa}_\beta)$ s.t. for $P \in C$,

$\Phi(P, a_1, \ldots, a_k)$

$\{P : L^{aa}_\beta \models \varphi(P)\}$ contains a club,

$\{P : L^{aa}_\beta \not\models \varphi(P)\}$ — " — , or

$\{P : L^{aa}_\beta \models \varphi(P)\}$ is stationary and costat.

theorem: assuming woodin cardinals, the third option never shows up.

theorem is ~~the~~ true even if we consider the set of P's s.t.

$$\{P : (L^{aa}_\beta, \in, a_1, \ldots, a_k) \models \varphi(I, a_1, \ldots, a_k)\}$$

$$\cap$$

$$P_{\omega_1}(L^{aa}_\beta)$$

$L^{aa} \models CH$.

$(L^{aa}_\beta ; \in) = N$.

defin $ult^*_{aa}(L^{aa}_\beta ; \in)$.

$F : \mathcal{P}_{\omega_1}(N) \longrightarrow N$ is semi-definable iff
there is a formula $\varphi(P, y, a_1, \ldots, a_n)$

$$(N, P) \models \phi(P, y, a_1, \ldots, a_n) \iff F(P) = y.$$

by club determinacy, for any semi-definable F, G.

$\{P : F(P) = G(P)\}$ contains a club or
is disjoint from a club.

So we can define $ult^*_{aa}(N)$ as
the coll. of all $[F]$ = equivalence class of F.

we get $j : N \longrightarrow \underbrace{ult^*_{aa}(N)}_{\shortparallel \atop N^*}$ inside L^{aa}.

$N^* = L_{\beta^*}[T]$, T is a theory which
gives you answers to
certain questions ...

$$j : (L^{aa}_\beta ; \in) \longrightarrow (N^* ; \in) \quad \text{in} \quad L^{aa} .$$

$$j'' \beta \in L^{aa} .$$

~~Loo~~ consider $(N^* ; \in, j''\beta) \models$

T^* theory of $(N^* ; \in, j''\beta)$:

$\Phi([F], j''\beta) \in T^*$ iff

$\{ p : (L^{aa}_\beta ; \in, p) \models \Phi(F(p), p) \}$.

T^* gives a way of defining

$$j_1 : (N^* ; \in, j''\beta) \longrightarrow N^{**} .$$

can iterate.

Say $y \in L^{aa}$, $y \in L^{aa}_\beta$, β minimal.

cth., $\rightarrow M \prec (L^{aa}_\beta, E)$ $\quad$ E extender derived for ω_1-iteration

inside L^{aa} $\quad$?||

$\bar{M} = M_0$.

$M_0 \to M_1 \to \cdots$

$M_{\alpha+1} = (\mathrm{ult}^*(M_\alpha); \in, j_{\alpha,\alpha+1}``, M_\alpha \cap OR)$

$\underline{M_{\omega_1}}$.

<u>clm</u>. $M_{\omega_1} \models$ ~~…~~

quantifier for the thy ass. to M_{ω_1} $\to$ $aa\, P\ \phi(P, \ldots)$ iff

$\{P : M_{\omega_1} \models \varphi(P, -)\}$ contains the clb.

So M_{ω_1} constructs the true L^{aa} (up to its height)

$y \in (L^{aa})^{M_{\omega_1}}$.

$\Rightarrow L^{aa} \models$ "each real has at most ctbly many pred."

<u>get</u> a Δ^1_3 w.o.

L^Q $\qquad L[\mathrm{Card}]$.

supp. there is an iterable mouse M with a measurable limit of measurables.

iterate M through the universe, using the top measure, call M_∞ the iterate.

f.a. cardinal $\kappa \geq \aleph_1$,

$$\mathrm{Card}\left(\{\alpha < \kappa : \alpha \text{ measurable in } M_\infty\}\right) = \kappa .$$

now iterate s.t. the measurable cardinals of the new iterate M^*_∞ are the

$$\aleph^V_{\omega\cdot(\alpha+1)} .$$

$$((\aleph_{\omega\cdot\alpha+n} : n<\omega) : \alpha \in OR)$$

is a seq. of Magidor it. of adding Prikry seq. to M^*_∞.

<u>claim</u>. $L[M^*_\infty | OR, ((\aleph_{\omega\cdot\alpha+n}) : n<\omega) : \alpha \in OR]$

$= L[\mathrm{Card}]$.

NTS: $M^*_\infty | OR$ can be def. over $L[\mathrm{Card}]$.