10.01.2017

Lecture notes chapter 5, WS 2015-2016 (Weiss): Vector bundles, J-homomorphism & Adams conjecture

5.1. The Poincaré-Hopf index theorem for a closed manifold

Let M be a closed smooth m-dimensional manifold; *closed* means compact, with empty boundary. Let ξ be a smooth vector field on M with isolated zeros.

Definition 5.1.1. For $x \in M$ where $\xi(x) = 0$ the *index* of ξ at x is the integer defined as follows. Choose a chart or local coordinate system $\varphi\colon U \to \mathbb{R}^m$ taking x to 0; here U is an open neighborhood of x in M. In the local coordinates, ξ is a vector field on the open set $\varphi(U) \subset \mathbb{R}^m$ which has a zero at the origin, and we can also view it as a map $\xi_\varphi\colon \varphi(U) \to \mathbb{R}^m$ taking 0 to 0. For sufficiently small $r > 0$, the restriction of ξ_φ to the sphere of radius r about the origin is a map $S^{m-1}(r) \to \mathbb{R}^m \smallsetminus \{0\}$ by our assumption (isolated zeros). The degree of that map is the index of ξ at x, denoted $\mathrm{index}_x(\xi)$. It does not depend on the choice of local coordinates.

Theorem 5.1.2 (Poincaré-Hopf index theorem)**.** $\sum_x \mathrm{index}_x(\xi) = \chi(M)$, *where* $\chi(M)$ *is the Euler characteristic. The sum is to be taken over all* $x \in M$ *where* $\xi(x) = 0$.

Wikipedia tells me that Poincaré proved this for 2-dimensional M and Heinz Hopf generalized it to higher dimensions. There is a variant for compact smooth manifolds M *with boundary*, too. In that case it is a condition that $\xi(y) \neq 0$ for every $y \in \partial M$, and moreover $\xi(y)$ should belong to the *outwards* half of the tangent space T_yM. (The tangent space T_yM is m-dimensional but comes with an $(m-1)$-dimensional linear subspace $T_y\partial M$; therefore we can speak of an outward halfspace and an inward halfspace.)

The theorem is considered easier in the cases where ξ is *transverse to the zero section*. That is to say, for $x \in M$ where $\xi(x) = 0$, the composition of linear maps

$$T_xM \xrightarrow{d\xi} T_{(x,0)}(TM) \longrightarrow \frac{T_{(x,0)}TM}{T_xM} \cong T_xM$$

is a linear isomorphism. The index of ξ at x is equal to $+1$ if that linear map has positive determinant, and to -1 if it has negative determinant.

(We have used the canonical inclusion of vector spaces $T_yM \to T_{(y,0)}(TM)$, for any $y \in M$. This is determined by the zero section $M \to TM$. The quotient alias cokernel of this linear map $T_yM \to T_{(y,0)}(TM)$ is always identified with T_yM.)

Exercise 5.1.3. How is that cokernel identified with T_yM ?

Although I assume the theorem is well known, let me outline the main steps of the proof as I know it. (This may be a little conservative.) In the first step, perturb ξ slightly to ensure that the perturbed thing is transverse to the zero section, taking care to ensure that the sum of the indices does not change. In the second step, show that the expression $\sum \mathrm{index}_x(\xi)$ is independent of ξ as long as ξ is transverse to the zero section. In the third step, choose a smooth triangulation of M and use it to construct a vector field ξ on M which is transverse to the zero section and which has $\xi(x) = 0$ if and only if x is the barycenter of a simplex in the triangulation. More precisely, we need $\mathrm{index}_x(\xi) = (-1)^k$ if x is the barycenter of a k-simplex. Then for this vector field ξ, obviously, $\sum_x \mathrm{index}_x(\xi)$ is equal to $\chi(M)$.

Let's follow Becker and Gottlieb in giving a rather ingenious homotopical interpretation of the Poincaré-Hopf index theorem. For this purpose suppose that M is embedded in $\mathbb{R}^k$ as a smooth submanifold; k might be very large. Let $p\colon E \to M$ be a tubular neighborhood for M in $\mathbb{R}^k$; that is to say, $p\colon E \to M$ has the structure of a smooth vector bundle but at the same time E comes with a smooth (codimension zero) embedding $E \to \mathbb{R}^k$ which extends the embedding $M \to \mathbb{R}^k$. Then we have the Pontryagin collapse map

$$c\colon \mathbb{R}^k \cup \infty \longrightarrow E \cup \infty =: \mathrm{thom}(E).$$

(This was introduced in chapter 2, with slightly different conventions. It agrees with the identity on $E \subset \mathbb{R}^k$ and takes all other elements of $\mathbb{R}^k \cup \infty$ to the point ∞ in $\mathrm{thom}(E)$. In chapter 2 it seemed convenient to use disk bundles, but here I find it more convenient not to use disk bundles. The Thom space $\mathrm{thom}(V)$ of a vector bundle V on a compact Hausdorff space is the one-point compactification of the total space.) We compose the Pontryagin collapse map with some other obvious maps as follows:

$$\begin{tikzcd}
\mathbb{R}^k \cup \infty \arrow[d, "c"] & & \mathbb{R}^k \cup \infty = \mathrm{thom}(\mathbb{R}^k \times *) \\
\mathrm{thom}(E) \arrow[r, "j"] & \mathrm{thom}(E \oplus TM) \cong & \mathrm{thom}(\mathbb{R}^k \times M) \arrow[u]
\end{tikzcd}$$

(This uses an identification of $E \oplus TM$ with a trivial vector bundle $\mathbb{R}^k \times M$. The map j is the obvious inclusion.)

Lemma 5.1.4 (Becker-Gottlieb). *The degree of the composite map from* $S^k \cong \mathbb{R}^k \cup \infty$ *to* $S^k \cong \mathbb{R}^k \cup \infty$ *is* $\chi(M)$.

Proof. The idea is to replace the inclusion $j\colon \mathrm{thom}(E) \longrightarrow \mathrm{thom}(E \oplus TM)$ in the above composition of maps by a perturbed variant, homotopic to j of course. To this end choose a smooth vector field ξ on M which is transverse to the zero section of TM. Let $x_1, \ldots, x_r$ be the finitely many points of M where ξ takes the value 0. Replace j by the map

$$j_\xi \colon \mathrm{thom}(E) \longrightarrow \mathrm{thom}(E \oplus TM)\ ;\ (x, v) \mapsto (x, v + \xi(x)).$$

Here $x \in M$ and $v \in E_x$; then (x, v) is supposed to have meaning as a point in $E \subset \mathrm{thom}(E)$ and $v + \xi(x) \in E_x \oplus T_xM$ and so $(x, v + \xi(x))$ should have meaning as a point in $E \oplus TM \subset \mathrm{thom}(E \oplus TM)$. The composition

$$\begin{tikzcd}
\mathbb{R}^k \cup \infty \arrow[d, "c"] & & \mathbb{R}^k \cup \infty = \mathrm{thom}(\mathbb{R}^k \times *) \\
\mathrm{thom}(E) \arrow[r, "j_\xi"] & \mathrm{thom}(E \oplus TM) \cong & \mathrm{thom}(\mathbb{R}^k \times M) \arrow[u]
\end{tikzcd}$$

is a map g which is smooth on $E \subset \mathbb{R}^k$, therefore on a neighborhood of the preimage of 0. Moreover

$$g^{-1}(0) = \{x_1, x_2, \ldots, x_r\} \subset M \subset E \subset \mathbb{R}^k$$

and the linear maps $dg(x_i) \colon \mathbb{R}^k \to \mathbb{R}^k$ are invertible by our assumption on ξ. Their orientation behavior is encoded in the indices of ξ at the points $x_1, \ldots, x_r$. Therefore the degree of g can be calculated as the sum of the local degrees of g at the points $x_1, \ldots, x_r$, which is

$$\sum_{i=1}^{r} \mathrm{index}_{x_i}(\xi),$$

and that is equal to $\chi(M)$. □

Exercise 5.1.5. Set up a variant of lemma 5.1.4 where M is a compact smooth manifold with boundary. (*Hint*: Begin with a smooth embedding $M \to \mathbb{R}^{k-1} \times [0, \infty)$ taking ∂M to $\mathbb{R}^{k-1} \times 0$.)

5.2. The Becker-Gottlieb transfer

Let X be a compact CW-space and let $p \colon Y \to X$ be a fiber bundle with smooth compact manifold fibers. (More precisely, it is assumed that we can find a covering of X by open subsets U_i and bundle charts

$$\varphi_i \colon Y|_{U_i} \xrightarrow{\cong} M_i \times U_i$$

(over U_i) such that $\varphi_j \varphi_i^{-1} \colon M_i \times (U_i \cap U_j) \to M_j \times (U_i \cap U_j)$ is adjoint to a continuous map from $U_i \cap U_j$ to the space of diffeomorphisms from M_i to M_j. That space of diffeomorphisms is an open subspace of $C^\infty(M_i, M_j)$ equipped with the compact-open C^∞ topology. Such a covering together with the charts φ_i would be called a *smooth bundle atlas*. If X is connected, then all M_i are diffeomorphic and we may assume that they are all the same: $M_i = M$ for all i.)

The compactness of X guarantees that for sufficiently large k we can find a smooth bundle embedding of $p\colon Y \to X$ in a trivial bundle $X \times \mathbb{R}^k \to X$:

$$e\colon Y \to \mathbb{R}^k \times X\,.$$

(In more detail, it is required that, for every bundle chart φ_i in the selected atlas, the composition

$$M_i \times U_i \xrightarrow{\varphi_i^{-1}} Y|_{U_i} \xrightarrow{e} \mathbb{R}^k \times U_i$$

is adjoint to a continuous map from U_i to the space of smooth embeddings from M_i to $\mathbb{R}^k$.) In this situation we can do the Becker-Gottlieb construction (of lemma 5.1.4) fiberwise, viewing X as a parameter space. There is hardly anything left to prove but there is a lot to state.

Notation: Suppose that A is any compact Hausdorff space with a map $q_1\colon A \to X$, and $q_2\colon V \to A$ is a vector bundle. Let

$$\mathrm{thom}_X(V)$$

be the disjoint union of V and X, not with the disjoint union topology but with the following: a subset W of $\mathrm{thom}_X(V) = V \amalg X$ is considered to be *open* if

- $W \cap V$ is open in V
- every $z \in W \cap X$ has an open neighborhood U in X such that $U \subset W$ and $(q_1 q_2)^{-1}(U) \smallsetminus W$ is compact (as a subset of V).

Briefly, $\mathrm{thom}_X(V)$ has a projection map to X (given by the identity on X and by $q_1 q_2$ on A), and the fiber of that map over $y \in X$ is exactly the ordinary Thom space $\mathrm{thom}(V|_{A_y})$ of the vector bundle $V|_{A_y} \to A_y$. If $q_1\colon A \to X$ is a fiber bundle, which is the typical situation here, then $\mathrm{thom}_X(V) \to X$ is also a fiber bundle.

(We return to the situation where $p\colon Y \to X$ is a bundle of smooth manifolds etc., as above.)

Theorem 5.2.1. *The Becker-Gottlieb construction produces a map*

$$p^!\colon \mathrm{thom}_X(\mathbb{R}^k \times X) \longrightarrow \mathrm{thom}_X(\mathbb{R}^k \times Y)$$

which respects the base sections. The composition

$$\mathrm{thom}_X(\mathbb{R}^k \times X) \xrightarrow{p^!} \mathrm{thom}_X(\mathbb{R}^k \times Y) \xrightarrow{p_*} \mathrm{thom}_X(\mathbb{R}^k \times X)$$

is a map over X which, on the fiber $\mathbb{R}^k \cup \infty \cong S^k$ over $x \in X$, has degree equal to $\chi(Y_x)$.

The two spaces in the theorem are spaces with a preferred map to X. Their fibers over $x \in X$ are, respectively, $\mathrm{thom}(\mathbb{R}^k \times \{x\}) \cong \mathbb{R}^k \cup \infty$ and $\mathrm{thom}_X(\mathbb{R}^k \times Y_x)$, the Thom space of the trivial vector bundle

$$\mathbb{R}^k \times Y_x \to Y_x$$

The map $p^!$ in the theorem is just the Becker-Gottlieb construction of the previous section done fiberwise.

Exercise 5.2.2. For a compact CW-space Z with base point, $\pi_k^{\mathrm{sta}}(Z)$ should be thought of as $\pi_{k+\ell}(S^\ell \wedge Z)$ where ℓ is a sufficiently large positive integer. These groups are called the *stable homotopy groups* of Z. Famous examples: $\pi_0^{\mathrm{sta}}(S^0) \cong \mathbb{Z}$ and $\pi_1^{\mathrm{sta}}(S^0) \cong \mathbb{Z}/2$. — The covering projection $S^n \to \mathbb{R}P^n$ induces a homomorphism of stable homotopy groups

$$\pi_k^{\mathrm{sta}}(S^n_+) \to \pi_k^{\mathrm{sta}}(\mathbb{R}P^n_+).$$

(The $+$ subscript means: take the disjoint union with a one-point space and declare the new point to be the base point.) Can theorem 5.2.1 be used to show that the cokernel of this homomorphism is a group of exponent 2 ? Is the kernel a group of exponent 2 ? (*Hint*: a covering projection is also a fiber bundle.)

5.3. Some notions from stable homotopy theory

Suppose that P and Q are based CW-spaces, P compact for simplicity. A *stable map* from P to Q is a based map

$$S^k \wedge P \to S^k \wedge Q$$

for some non-negative integer k, which we should imagine as large. This notion deserves to be made a little more precise.

Definition 5.3.1. We write $[P, Q]^{\mathrm{sta}}$ for the set of homotopy classes of stable maps from P to Q. In more detail,

$$[P, Q]^{\mathrm{sta}} := \mathrm{colim}\ [S^k \wedge P, S^k \wedge Q]_*$$

where $[S^k \wedge P, S^k \wedge Q]_*$ is the set of based homotopy classes of based maps from $S^k \wedge P$ to $S^k \wedge Q$. The colim notation means that we identify any element of $[S^k \wedge P, S^k \wedge Q]_*$ with its image in $[S^{k+1} \wedge P, S^{k+1} \wedge Q]_*$ under the map obtained by $S^1 \wedge$, smashing with (the identity map of) S^1.

While this looks clean, a slightly better way to understand stable maps is as follows. We can write

$$[S^k \wedge P, S^k \wedge Q]_* :=\cong [P, \Omega^k(S^k \wedge Q)]_*.$$

Instead of $S^k \wedge Q$, it is also customary to write $\Sigma^k Q$. Letting k tend to infinity, we understand:

Definition 5.3.2. A stable map from P to Q is a based map from P to $\Omega^\infty\Sigma^\infty Q$. (But what is $\Omega^\infty\Sigma^\infty Q$? It is best interpreted as the mapping telescope of

$$Q \mapsto \Omega(S^1 \wedge Q) \to \Omega^2(S^2 \wedge Q) \to \Omega^3(S^3 \wedge Q) \to \cdots .$$

And what is a mapping telescope? Form the mapping cylinder of each map in the string separately; identify the rim (target end) of cylinder number n with the top (source end) of cylinder number $n+1$.) We note that $[P, \Omega^\infty\Sigma^\infty Q]_* \cong [P, Q]^{\mathrm{sta}}$; the set of homotopy classes of based maps from P to $\Omega^\infty\Sigma^\infty Q$ is identified with the set of homotopy classes of stable maps from P to Q.

Remark 5.3.3. $[P, Q]^{\mathrm{sta}}$ is always an abelian group. This should be reasonably clear from the definitions because

$$[S^k \wedge P, S^k \wedge Q]_* \cong \pi_k(\mathrm{map}_*(P, S^k \wedge Q))$$

which is a group if $k \geq 1$ and abelian if $k \geq 2$.

Proposition 5.3.4. *Composition of stable maps makes* $[P, P]^{\mathrm{sta}}$ *into a ring. Moreover* $[P, S^0]^{\mathrm{sta}}$ *has the structure of a commutative ring and* $[P, P]^{\mathrm{sta}}$ *is an algebra over* $[P, S^0]^{\mathrm{sta}}$ *, in the weak sense that there is a preferred ring homomorphism from* $[P, S^0]^{\mathrm{sta}}$ *to the center of* $[P, P]^{\mathrm{sta}}$*).*

But ... *while* $[P, P]^{\mathrm{sta}}$ *has a 1 (represented by the identity), it can happen and it will often happen that* $[P, S^0]^{\mathrm{sta}}$ *has no unit.*

Sketch of a proof. The ring structure on $[P, P]^{\mathrm{sta}}$ uses composition of (homotopy classes of) stable maps $P \to P$ as the multiplication, as promised. The ring axioms are not hard to verify (verification omitted). The ring structure on $[P, S^0]^{\mathrm{sta}}$ is perhaps a little more surprising. An element of $[P, S^0]^{\mathrm{sta}}$ can be represented by a based map

$$S^k \wedge P \longrightarrow S^k \wedge S^0 = S^k .$$

By repeating the second coordinate, we can also write this as a map

$$f\colon S^k \times P \longrightarrow S^k \times P$$

which respects the projection to P and respects base points in each fiber S^k (and moreover takes $S^k \times *$ to $* \times *$). If we represent elements of $[P, S^0]^{\mathrm{sta}}$ by maps f in this way, then it emerges that we have a way to compose. This gives the multiplication of the ring structure. It also explains why we have a preferred ring homomorphism from $[P, S^0]^{\mathrm{sta}}$ to $[P, P]^{\mathrm{sta}}$. Namely, a map $f\colon S^k \times P \longrightarrow S^k \times P$ as above determines a based map $S^k \wedge P \to S^k \wedge P$ (by passing to quotients) which can be interpreted as an element of $[P, P]^{\mathrm{sta}}$.

Alternatively, the ring structure on $[P, S^0]^{\mathrm{sta}}$ can be understood as follows. Since

$$[P, S^0]^{\mathrm{sta}} \cong [P, \Omega^\infty \Sigma^\infty S^0]_*$$

it suffices to show that the based space $R := \Omega^\infty \Sigma^\infty S^0$ admits a structure of *ring object* in the homotopy category of based spaces. That is to say, we need certain maps

$$\alpha\colon R \times R \to R\ , \qquad \mu\colon R \times R \to R$$

which can play the role of addition and multiplication (and we need two elements in $\pi_0 R$ which can be called 0 and 1). Without going into details, let's approximate R by $R_k := \Omega^k \Sigma^k S^0 = \mathrm{map}_*(S^k, S^k)$. Then we can define α as the map from

$$R_k \times R_k \cong \mathrm{map}_*(S^k \vee S^k, S^k)$$

to $\mathrm{map}_*(S^k, S^k)$ given by pre-composition with a selected map $S^k \to S^k \vee S^k$ which has degree 1 on each wedge summand of $S^k \vee S^k$. We can define μ as the map from $R_k \times R_k$ to R_k given by composition. □

Exercise 5.3.5. Fill in your favorite missing details of this proof. In particular, why is the ring structure on $[P, S^0]^{\mathrm{sta}}$ commutative? Why is it that the ring homomorphism $[P, S^0]^{\mathrm{sta}} \longrightarrow [P, P]^{\mathrm{sta}}$ has image contained in the center of $[P, P]^{\mathrm{sta}}$?

Lemma 5.3.6. *Suppose that $P = P^\circ_+$ for some compact CW-space P° (see Ex. 5.2.2 for notation). Then $[P, S^0]^{\mathrm{sta}}$ is a ring with 1, and the canonical ring homomorphism*

$$[P, S^0]^{\mathrm{sta}} \longrightarrow [P, P]^{\mathrm{sta}}$$

is split injective, i.e., has a preferred left inverse $[P, P]^{\mathrm{sta}} \to [P, S^0]^{\mathrm{sta}}$, a homomorphism of abelian groups.

Furthermore, $[P, S^0]^{\mathrm{sta}}$ comes with a preferred surjective ring homomorphism to $\mathrm{map}(P^\circ, \mathbb{Z})$. An element of $[P, S^0]^{\mathrm{sta}}$ is invertible if and only if its image in $\mathrm{map}(P^\circ, \mathbb{Z})$ is invertible.

Proof. The element 1 of $[P, S^0]^{\mathrm{sta}}$ is represented by the (honest) based map $P \to S^0$ which takes all of P° to the non-base point of S^0. It is easy to verify that it is a neutral element for the multiplication in $[P, S^0]^{\mathrm{sta}}$. Post-composition with that element $1 \in [P, S^0]^{\mathrm{sta}}$ gives a group homomorphism

$$[P, P]^{\mathrm{sta}} \to [P, S^0]^{\mathrm{sta}}$$

which is left inverse to the ring homomorphism $[P, S^0]^{\mathrm{sta}} \to [P, P]^{\mathrm{sta}}$ defined previously.

The ring homomorphism from $[P, S^0]^{\mathrm{sta}}$ to $\mathrm{map}(P^\circ, \mathbb{Z})$ is as follows:

$$\begin{array}{c} [P, S^0]^{\mathrm{sta}} \cong \pi_0(\mathrm{map}_*(P, \Omega^\infty\Sigma^\infty S^0)) \\ \Big\downarrow \\ \mathrm{map}(P^\circ, \pi_0(\Omega^\infty\Sigma^\infty S^0)). \end{array}$$

Now we note that $\pi_0(\Omega^\infty\Sigma^\infty S^0) = [S^0, S^0]^{\mathrm{sta}} \cong \mathbb{Z}$.

Finally, suppose that some element of $[P, S^0]^{\mathrm{sta}}$ has image in $\mathrm{map}(P^\circ, \mathbb{Z})$ which is invertible (i.e., all the values of this map $N \to \mathbb{Z}$ are ± 1). Represent this element by

$$f \colon S^k \times P \longrightarrow S^k \times P$$

(for some $k > 0$) respecting the projection to P and the base point in each fiber, etc. This is worth as much as a map

$$S^k \times P^\circ \longrightarrow S^k \times P^\circ$$

which respects the projection to P° and respects the base point in each fiber S^k. Our assumption means that it has degree ± 1 in each fiber S^k. Therefore it is invertible by general facts about fibrations. $\square$

5.4. Becker-Gottlieb transfer as a tool in homotopy theory

Using the stable language/notation, we can reformulate theorem 5.2.1 as follows. (I downgrade this formulation to a proposition because it is weaker than the original statement. But it may be easier to remember and it is strong enough for our purposes.)

Proposition 5.4.1. *The fiber bundle* $p \colon Y \to X$ *(with closed smooth manifold fibers) determines a stable map*

$$p^! \colon X_+ \longrightarrow Y_+ .$$

The composition

$$X_+ \xrightarrow{p^!} Y_+ \xrightarrow{p} X_+$$

is an element of $[X_+, X_+]^{\mathrm{sta}}$ *which belongs to the subring* $[X_+, S^0]^{\mathrm{sta}}$. *The image of that element under the standard ring homomorphism*

$$[X_+, S^0]^{\mathrm{sta}} \to \mathrm{map}(X, \mathbb{Z})$$

is the map defined by $x \mapsto \chi(Y_x)$ *for* $x \in X$.

Remark 5.4.2. To obtain $p^!$ in proposition 5.4.1 from $p^!$ in theorem 5.2.1, pass from the fiberwise Thom spaces $\mathrm{thom}_X(...)$ to the ordinary Thom spaces $\mathrm{thom}(...)$. Note that $\mathrm{thom}(\mathbb{R}^k \times X) = S^k \wedge X_+$ and $\mathrm{thom}(\mathbb{R}^k \times Y) = S^k \wedge Y_+$ (if we write $\mathbb{R}^k \cup \infty = S^k$). In the passage from fiberwise Thom spaces to ordinary Thom spaces, some information is lost.

Example 5.4.3. Let X be a compact CW-space and let $E \to X$ be a real vector bundle of even fiber dimension $2n$. Suppose that E is equipped with a Riemannian metric (i.e., each fiber E_x comes equipped with an inner product). Let Y be the following space: an element of Y is a point $x \in X$ together with a selection of n pairwise orthogonal 2-dimensional linear subspaces of E_x. (I am trying to emphasize that the selected linear subspaces are not enumerated, i.e., they are not labeled with numbers from 1 to n.) The forgetful map

$$p\colon Y \longrightarrow X$$

is clearly a fiber bundle. Let us examine the fibers. For a selected $x \in X$, we have an obvious left action of the orthogonal group $O(E_x)$ on the fiber Y_x : a splitting of E_x into 2-dimensional linear subspaces, pairwise orthogonal, can be transformed by a linear automoprhism of E_x . The action is clearly transitive. Therefore Y_x can be identified with the coset space

$$O(E_x)/H(z_0)$$

where $H(z_0)$ is the stabilizer (isotropy group for the action) of our favorite element $z_0 \in Y_x$. Furthermore, if we now choose an orthonormal basis of E_x with basis vectors $e_1, e_2, \dots, e_{2n}$, then our favorite element z_0 is the selection of 2-dimensional linear subspaces

$$\mathbb{R}e_1 + \mathbb{R}e_2 \ , \ \mathbb{R}e_3 + \mathbb{R}e_4 \ , \ \dots,$$

and we can make identifications

$$O(E_x) \cong O(2n), \ H(z_0) \cong \Sigma_n \ltimes (O(2))^n =: \Sigma_n \wr O(2).$$

Here Σ_r is the symmetric group on n letters, acting on $O(2)^n$ by permuting the n factors. Therefore, in wreath product notation:

$$Y_x \cong \frac{O(2n)}{\Sigma_n \wr O(2)} \, .$$

This is a smooth manifold and, fortunately for us, it has Euler characteristic ± 1; see exercise 5.6.1. We want to deduce the following:

($) *the homomorphism* $p^*\colon K_F(X) \longrightarrow K_F(Y)$ *is injective.*

This is a key step towards the proof of the Adams conjecture. There are two proofs (known to me). The first follows Becker and Gottlieb. In addition to the Becker-Gottlieb transfer $p^!$, it uses another argument which is more formal, more difficult, too, but less surprising. The other is more direct and uses a shortcut due to E.H. Brown (of the Brown representation theorem). I am very grateful to Johannes Ebert for telling me about this alternative. I must admit, I have been slow to take it on board, though not catastrophically slow.

But first let us see what we can do with $p^!$. In order to show that p^* in ($) is injective, we write it in the form

$$[X_+, BG \times \mathbb{Z}]_* \xrightarrow{\circ p} [Y_+, BG \times \mathbb{Z}]_*$$

and place it in a bigger commutative diagram

$$\begin{tikzcd}
{[X_+, BG \times \mathbb{Z}]_*} \arrow[r, "\circ p"] \arrow[d] & {[Y_+, BG \times \mathbb{Z}]_*} \arrow[d] & \\
{[X_+, BG \times \mathbb{Z}]^{\mathrm{sta}}} \arrow[r, "\circ p"] & {[Y_+, BG \times \mathbb{Z}]^{\mathrm{sta}}} \arrow[r, "\circ p^!"] & {[X_+, BG \times \mathbb{Z}]^{\mathrm{sta}}}
\end{tikzcd}$$

where the vertical arrows are obvious stabilization maps. By Becker-Gottlieb, the composition of the two lower horizontal arrows is bijective. Therefore, since we want to show that the upper horizontal arrow is injective, it suffices to show that the left-hand vertical arrow is injective. This is taken care of in the following statements, proposition 5.4.4 and theorem 5.4.5.

Proposition 5.4.4. *Let Z be an infinite loop space. Then for any based compact CW-space P, the stabilization map $[P, Z]_* \to [P, Z]^{\mathrm{sta}}$ is injective.*

Theorem 5.4.5. *$BG \times \mathbb{Z}$ is an infinite loop space.*

Proof of prop. 5.4.4. The stabilization map that we are investigating can be written in the form

$$[P, Z]_* \longrightarrow [P, \Omega^\infty \Sigma^\infty Z]_* .$$

As such it is induced by the inclusion of Z in the mapping telescope $\Omega^\infty\Sigma^\infty Z$. It is therefore enough to show that for every $k \geq 1$ the partial stabilization map $[P, Z]_* \to [P, \Omega^k\Sigma^k Z]_*$ is injective. For that it will suffice to produce a based map from $\Omega^k\Sigma^k Z$ to Z making the following homotopy commutative:

$$\begin{tikzcd}
Z \arrow[r, "\text{incl.}"] \arrow[dr, "\text{id}"'] & \Omega^k\Sigma^k Z \arrow[d, dotted] \\
& Z
\end{tikzcd}$$

Our assumption on Z implies that instead of Z we can write $\Omega^k Z_k$ for some (other) based CW-space Z_k. Now we are looking for the dotted arrow in

$$\begin{tikzcd}
\Omega^k Z_k \arrow[r, "\text{incl.}"] \arrow[dr, "\text{id}"'] & \Omega^k\Sigma^k\Omega^k Z_k \arrow[d, dotted] \\
& \Omega^k Z_k
\end{tikzcd}$$

There is an easy solution as follows:

$$\begin{array}{c} \Omega^k(\Sigma^k\Omega^k)Z_k \\ \Big\downarrow{\scriptstyle \Omega^k\nu} \\ \Omega^k Z_k \end{array}$$

where $\nu\colon (\Sigma^k\Omega^k)Z_k \to Z_k$ is obtained by adjunction from the identity map $\Omega^k Z_k \to \Omega^k Z_k$. □

Exercise 5.4.6. Verify that $\Omega^k\nu$ makes that triangle strictly commutative.

I am not planning to give a proof of theorem 5.4.5. Instead I hope to explain (after the Christmas break) the more direct argument for statement (\$) due to Brown. As indicated above, I could have been more efficient by explaining this right away, but I have an interesting excuse! Adams wrote in his book *Infinite loop spaces* (1978): *Finally I come to the proof of Becker and Gottlieb ... Of course this involves quoting substantial results from infinite-loop-space theory ... ; but there is no help for it — it is essential to the argument.*

5.5. Finite covering spaces and associated transfer maps

Let $q\colon Y^\natural \to Y$ be a covering space with finite fibers, where Y is a compact CW-space. (The Y which I have in mind is the Y from example 5.4.3, but for the moment there is no need to be so specific.

Definition 5.5.1. The covering space $q : Y^\natural \to Y$ determines an elementary transfer map (wrong-way map)

$$q^\oplus\colon K_{\mathbb{R}}(Y^\natural) \to K_{\mathbb{R}}(Y)$$

as follows. A vector bundle $V \to Y^\natural$ determines a vector bundle $V^\sharp \to Y$ by the rule

$$V^\sharp_y := \bigoplus_{z\in q^{-1}(y)} V_z$$

for $y \in Y$. In words, the fiber of $V^\sharp$ over $y \in Y$ is the direct sum of the fibers of V over the n points in $q^{-1}(y) \subset Y^\natural$. If an element of $K_{\mathbb{R}}(Y^\natural)$ can be written as $[V] - [W]$ where V and W are vector bundles over $Y^\natural$, then we let

$$q^\oplus([V] - [W]) = [V^\sharp] - [W^\sharp].$$

It is straightforward verify that this gives a well defined homomorphism of abelian groups from $K_{\mathbb{R}}(Y^\natural)$ ro $K_{\mathbb{R}}(Y)$.

Exercise 5.5.2. Make this explicit (compute $q^{\oplus}$) in the case where $Y = S^1$ and q is the 2-sheeted covering space determined by the subgroup of index 2 in $\pi_1(S^1) \cong \mathbb{Z}$.

Proposition 5.5.3. *The map* $q^{\oplus}\colon K_{\mathbb{R}}(Y^\natural) \to K_{\mathbb{R}}(Y)$ *agrees with the homomorphism of* K*-groups induced by the stable map* $q^!\colon Y_+ \to Y^\natural_+$ *of Becker-Gottlieb.*

Proof. Write $K = K_{\mathbb{R}}$. For sufficiently large r, the Becker-Gottlieb map

$$q^!\colon S^{8r} \wedge Y_+ \longrightarrow S^{8r} \wedge Y^\natural_+$$

is defined as an honest based map. We ought to show that the following commutes:

$$\begin{tikzcd} K(Y^\natural) \arrow[r, "\cong"] \arrow[d, "q^{\oplus}"] & \tilde{K}(Y^\natural_+) \arrow[r] & \tilde{K}(S^{8r} \wedge Y^\natural_+) \arrow[d, "(q^!)^*"] \\ K(Y) \arrow[r, "\cong"] & \tilde{K}(Y_+) \arrow[r] & \tilde{K}(S^{8r} \wedge Y_+) \end{tikzcd}$$

where the right-hand horizontal arrows are given by multiplication with a generator b of $\tilde{K}(S^{8r})$. But multiplication with b is somewhat inexplicit, inscrutable. Therefore I prefer the following setup. We view the Becker-Gottlieb map as an honest based map

$$q^!\colon S^{8r-1} \wedge Y_+ \longrightarrow S^{8r-1} \wedge Y^\natural_+$$

and we write

$$[S^{8r-1} \wedge Y^\natural_+, \mathrm{SO}]_* \,, \quad [S^{8r-1} \wedge Y_+, \mathrm{SO}]_*$$

instead of $\tilde{K}(S^{8r} \wedge Y^\natural_+)$, $\tilde{K}(S^{8r} \wedge Y_+)$.

In more detail, we select a "random" vector bundle $V \to Y^\natural$ and obtain $V^\sharp \to Y$. We choose an embedding (unnamed) of $V^\sharp \to Y$ into a trivial vector bundle $\mathbb{R}^N \times Y \to Y$. This will automatically give us an embedding of $V \to Y^\natural$ into a trivial vector bundle $\mathbb{R}^N \times Y^\natural \to Y^\natural$. Namely, we have the composition

$$V \hookrightarrow q^*(V^\sharp) \hookrightarrow q^*(\mathbb{R}^N \times Y) = \mathbb{R}^N \times Y^\natural\,.$$

We view Ωb as a based map $S^{8r-1} \to \mathrm{SO}(M)$ for a large integer M. Now we have

$$\Omega b \boxtimes V\colon S^{8r-1} \wedge Y^\natural_+ \longrightarrow \mathrm{SO}(MN),$$

$$\Omega b \boxtimes V^\sharp\colon S^{8r-1} \wedge Y_+ \longrightarrow \mathrm{SO}(MN)$$

in the notation of section 3.6 (taking S^{8r-1} for P and Y_+ or $Y^\natural_+$ for Q). We need to show

$$\Omega b \boxtimes V^\sharp \simeq (\Omega b \boxtimes V) \circ q^!\,.$$

Here we go. The definition of $q^!$ uses an embedding

$$Y^\natural \hookrightarrow Y \times \mathbb{R}^{8r-1}$$

such that the composition $Y^\natural \hookrightarrow Y \times \mathbb{R}^{8r-1} \to Y$ is q. We fix this. Now we argue pointwise, so we select $y \in Y$. Then $q^{-1}(y)$ is a finite set embedded in $\mathbb{R}^{8r-1}$. We also need a tubular neighborhood U_y for $q^{-1}(y)$ in $\mathbb{R}^{8r-1}$. Imagine this as the union of small disjoint metric open balls about the points of $q^{-1}(y)$. Now

$$(\Omega b \boxtimes V) \circ q^! \big|_{S^{8r-1} \wedge \{y\}_+}$$

has the following form:

$$\begin{tikzcd}
& S^{8r-1} \wedge \{y\}_+ \arrow[r, "\cong"] & S^{8r-1} \arrow[d, "\text{Pontr. collapse}"'] & \\
U_y \cup \infty \arrow[rr, phantom, ""] & & \displaystyle\bigvee_{s \in q^{-1}(y)} S^{8r-1} \arrow[r, "(\Omega b \boxtimes V_s)_{s \in q^{-1}(y)}"] & SO(MN)
\end{tikzcd}$$

$$U_y \cup \infty \xrightarrow{\cong} \bigvee_{s \in q^{-1}(y)} S^{8r-1}$$

We can also write this in the form

$$\begin{tikzcd}
S^{8r-1} \arrow[d, "\text{(Pontr. collapse for each conn. component of } U_y)"'] & \\
\displaystyle\prod_{s \in q^{-1}(y)} S^{8r-1} \arrow[r, "\prod_s \Omega b \boxtimes V_s"] & SO(MN)
\end{tikzcd}$$

using the multiplication in SO(MN). (Although SO(MN) is not commutative, there is no need for an ordering of factors here because the specific elements of SO(MN) that we need to multiply happen to commute pairwise.) Now it only remains to note that the vertical arrow in the last diagram is homotopic (by a specific homotopy $(h_{t,y})_{t \in [0,1]}$ whose construction is left to the reader) to the diagonal map

$$S^{8r-1} \longrightarrow \prod_{s \in q^{-1}(y)} S^{8r-1}.$$

These homotopies $(h_{t,y})_{t \in [0,1]}$, taken together for all $y \in Y$, induce a homotopy from $(\Omega b \boxtimes V) \circ q^!$ to $\Omega b \boxtimes V^\sharp$. $\square$

Corollary 5.5.4. *The following diagram is commutative:*

$$\begin{tikzcd}
\mathsf{K}_{\mathbb{R}}(\mathsf{Y}^{\natural}) \arrow[r, "\Psi^k"] \arrow[d, "q^{\oplus}"] & \mathsf{K}_{\mathbb{R}}(\mathsf{Y}^{\natural}) \otimes \mathbb{Z}[k^{-1}] \arrow[d, "q^{\oplus}"] \\
\mathsf{K}_{\mathbb{R}}(\mathsf{Y}) \arrow[r, "\Psi^k"] & \mathsf{K}_{\mathbb{R}}(\mathsf{Y}) \otimes \mathbb{Z}[k^{-1}]
\end{tikzcd}$$

Proof. We compare that diagram with

$$\begin{tikzcd}
\tilde{\mathsf{K}}_{\mathbb{R}}(\mathsf{S}^{8r} \wedge \mathsf{Y}^{\natural}_{+}) \arrow[r, "k^{-4r}\Psi^k"] \arrow[d, "(q^!)^*"] & \tilde{\mathsf{K}}_{\mathbb{R}}(\mathsf{S}^{8r} \wedge \mathsf{Y}^{\natural}_{+}) \otimes \mathbb{Z}[k^{-1}] \arrow[d, "(q^!)^*"] \\
\tilde{\mathsf{K}}_{\mathbb{R}}(\mathsf{S}^{8r} \wedge \mathsf{Y}_{+}) \arrow[r, "k^{-4r}\Psi^k"] & \tilde{\mathsf{K}}_{\mathbb{R}}(\mathsf{S}^{8r} \wedge \mathsf{Y}_{+}) \otimes \mathbb{Z}[k^{-1}]
\end{tikzcd}$$

for large $r \geq 0$. Since r is large, we can think of $q^!$ as an honest based map

$$\mathsf{S}^{8r} \wedge \mathsf{Y}_{+} \longrightarrow \mathsf{S}^{8r} \wedge \mathsf{Y}^{\natural}_{+}\,.$$

Therefore this new diagram is commutative. By Bott periodicity, real case, and by proposition 5.5.3, external product with a generator of $\tilde{\mathsf{K}}(\mathsf{S}^{8r}) \cong \mathbb{Z}$ gives a natural isomorphism from the old diagram to the new diagram. (We are exploiting the fact that Ψ^k respects products. But since Ψ^k applied to a generator of $\tilde{\mathsf{K}}(\mathsf{S}^{8r})$ multiplies that generator with k^{4r}, we have to use $k^{-4r}\Psi^k$ instead of Ψ^k in the new diagram.) □

Remark 5.5.5. The map $q^{\oplus}\colon \mathsf{K}_{\mathbb{R}}(\mathsf{Y}^{\natural}) \to \mathsf{K}_{\mathbb{R}}(\mathsf{Y})$ has an analogue for spherical fibrations:

$$q^{\oplus}\colon \mathsf{K}_{\mathsf{F}}(\mathsf{Y}^{\natural}) \to \mathsf{K}_{\mathsf{F}}(\mathsf{Y})$$

(replace direct sums by joins). We will also need this. We will not need the analogue of proposition 5.5.3 for this situation, although I believe it holds.

5.6. Proof of the Adams conjecture

We return to the situation and notation of example 5.4.3. The commutative diagram

$$\begin{tikzcd}
\mathsf{K}_{\mathbb{R}}(\mathsf{Y}) \arrow[r, "\Psi^k-\mathrm{id}"] & \mathsf{K}_{\mathbb{R}}(\mathsf{Y}) \arrow[r] & \mathsf{K}_{\mathsf{F}}(\mathsf{Y}) \otimes \mathbb{Z}[k^{-1}] \\
\mathsf{K}_{\mathbb{R}}(\mathsf{X}) \arrow[r, "\Psi^k-\mathrm{id}"] \arrow[u, "p^*"] & \mathsf{K}_{\mathbb{R}}(\mathsf{X}) \arrow[r] \arrow[u, "p^*"] & \mathsf{K}_{\mathsf{F}}(\mathsf{X}) \otimes \mathbb{Z}[k^{-1}] \arrow[u, "p^*"]
\end{tikzcd}$$

and statement ($) from example 5.4.3 imply that the element $[\mathsf{E}] \in \mathsf{K}_{\mathbb{R}}(\mathsf{X})$ goes to zero in $\mathsf{K}_{\mathsf{F}}(\mathsf{X}) \otimes \mathbb{Z}[k^{-1}]$ provided $[p^*\mathsf{E}] \in \mathsf{K}_{\mathbb{R}}(\mathsf{Y})$ goes to zero in $\mathsf{K}_{\mathsf{F}}(\mathsf{Y}) \otimes \mathbb{Z}[k^{-1}]$.

By construction, the vector bundle $p^*E \to Y$ of fiber dimension $2n$ comes with additional data as follows. For each $y \in Y$ the fiber $(p^*E)_y \cong E_{p(y)}$ comes with a preferred splitting into n linear subspaces of dimension 2. (Selecting $y \in Y$ amounts to selecting $x = p(y) \in X$ and a splitting of the vector space E_x into pairwise perpendicular 2-dimensional linear subspaces.) Now let

$$q\colon Y^\natural \longrightarrow Y$$

be the covering space obtained by saying that $q^{-1}(y)$ for $y \in Y$ is the set of those n selected 2-dimensional linear subspaces of $(p^*E)_y = E_{p(y)}$. (The set $q^{-1}(y)$ has n elements. Therefore $q\colon Y^\natural \to Y$ is an n-sheeted covering, fiber bundle with fibers homeomorphic to $\{1, 2, \dots, n\}$, but we should not assume that it is a trivial fiber bundle.) We set up another diagram

$$\begin{tikzcd}
K_{\mathbb{R}}(Y^\natural) \arrow[r, "\Psi^k - \mathrm{id}"] \arrow[d, "q^\oplus"] & K_{\mathbb{R}}(Y^\natural) \arrow[r] \arrow[d, "q^\oplus"] & K_F(Y^\natural) \otimes \mathbb{Z}[k^{-1}] \arrow[d, "q^\oplus"] \\
K_{\mathbb{R}}(Y) \arrow[r, "\Psi^k - \mathrm{id}"] & K_{\mathbb{R}}(Y) \otimes \mathbb{Z}[k^{-1}] \arrow[r] & K_F(Y) \otimes \mathbb{Z}[k^{-1}]
\end{tikzcd}$$

It is commutative by corollary 5.5.4. By construction, the element $[p^*E]$ in $K_{\mathbb{R}}(Y)$ (bottom left-hand term of the diagram) is the image of an element

$$[V] \in K_{\mathbb{R}}(Y^\natural)$$

(top left-hand term of the diagram), where $V \to Y^\natural$ is a two-dimensional vector bundle. (For $x \in X$ and $y \in Y$ with $p(y) = x$ and $w \in Y^\natural$ with $q(w) = y$, the fiber V_w is one of the 2-dimensional summands of E_x in the splitting of E_x determined by y.)

By Adams' own pre-proof of the Adams conjecture, we know that $[V]$ goes to 0 under the composite map in the top row of the diagram. Therefore $[p^*E]$ goes to 0 under the composite map in the bottom row of the diagram. This completes the proof. □

Exercise 5.6.1. We used a special case of the following general fact. Let G be a compact connected Lie group, T a maximal torus in G, and $N(T)$ the normalizer of T in G. *Then the coset manifold* $G/N(T)$ *has Euler characteristic 1.* (We used this for $G = \mathrm{SO}(2n)$ only.) Here is an outline of a proof which follows Adams (proof of thm 4.21 in his book *Lectures on Lie groups*). Fill in details as far as possible.

(i) It suffices to show that $N(T)/T$ is finite and that the Euler characteristic of G/T is $|N(T)/T|$. Why?

(ii) In the tangent space of T at the identity element e, choose a nonzero tangent vector v such that the corresponding one-parameter subgroup L of T is *dense* in T. This is always possible. Why?

(iii) For every element $xT \in G/T$ (meaning $x \in G$) let $\beta_{xT}\colon T \to G/T$ be the map
$$s \mapsto sxT$$
and let $\xi(xT)$ be the tangent vector at $xT \in G/T$ obtained by applying the derivative $d\beta_{xT}$ to ν.

(iv) Now we have a smooth vector field ξ on G/T. Show that $\xi(xT) = 0$ if and only if $x \in N(T)$.

(v) It only remains to show that the index of ξ at each point $xT \in G/T$ where $\xi(xT) = 0$ is $+1$. (Maybe more instructions on that later.)

5.6. The shortcut due to Brown

E.H.Brown found a more direct way to prove the decisive statement (\$) in section 5.4. Let's recall the assumptions: $p\colon Y \to X$ is a fiber bundle with smooth compact manifold fibers, and these fibers have Euler characteristic ± 1. We are supposed to show that
$$p^*\colon \tilde{K}_F(X) \to \tilde{K}_F(Y)$$
is injective. It will not hurt to assume that X is a connected CW-space. More surprisingly, it seems to me that for Brown's argument we also have to assume that the *fibers* of $p\colon Y \to X$ are connected. (In the application to the Adams conjecture these fibers were diffeomorphic to $O(2n)/\Sigma_n \wr O(2)$ which is indeed connected.)

Lemma 5.6.1. *Let $g\colon E \to X$ be a spherical fibration with fibers $\simeq S^{k-1}$. Let $c_X E$ be the mapping cylinder of g, so that there are a pair $(c_X E, E)$ and a fibration pair $(c_X E, E) \to X$ with fiber pairs $\simeq (D^k, S^{k-1})$, and a Thom space $c_X E/E$. The following are equivalent:*

(i) *g is stably trivial;*

(ii) *there is a stable map $u\colon c_X E/E \to D^k/S^{k-1}$ such that the restriction of u to $c_z E_z/E_z$ (for each $z \in X$) is a stable map of degree ± 1.*

Proof. If g is stably trivial, then the trivialization induces a (stable) map from the Thom space of g to the Thom space of the spherical fibration $S^{k-1} \to *$. This clearly has the property mentioned in (ii).

Conversely, if we have the stable map u, then we can suppose that it is an honest map
$$S^j \wedge (c_X E/E) \longrightarrow S^j \wedge (D^k/S^{k-1}).$$
Restricted to $S^j \wedge (c_z E_z/E_z)$ this gives a map u_z of degree ± 1 from the sphere $S^j \wedge (c_z E_z/E_z)$ to the sphere $S^j \wedge (D^k/S^{k-1}) \cong S^{j+k}$. We may identify $S^j \wedge (c_z E_z/E_z)$ with the join $S^j * E_z$ (also $\simeq S^{j+k}$), so that u_z is a homotopy equivalence from $S^j * E_z$ to S^{j+k}. This depends continuously on $z \in X$ and so gives a stable trivialization of g. $\square$

Continuing in the notation of the lemma, let $g\colon E \to X$ be a spherical fibration. Then we have the pullback spherical fibration

$$p^*g\colon p^*E \to Y$$

and $c_Y p^*E$, the mapping cylinder of p^*g. Then there is a commutative diagram

$$\begin{tikzcd} (c_Y p^*E, p^*E) \arrow[r] \arrow[d, "q"'] & Y \arrow[d, "p"] \\ (c_X E, E) \arrow[r] & X \end{tikzcd}$$

where the horizontal arrows are fibration pairs with fibers (D^k, S^{k-1}) for some k. (The label q is a new name for an otherwise obvious map.) The vertical arrows are fiber bundle projections with smooth manifold fibers. The fibers have Euler characteristic ± 1.

Brown's idea is to apply the Becker-Gottlieb transfer not to p (right-hand column), but to q, the left-hand column of the square. What we should get as a result is a stable map of pairs

$$q^!\colon (c_X E_+, E_+) \longrightarrow (c_Y p^*E_+, p^*E_+).$$

But it is more efficient to pass to the quotients right away; so we write

$$q^!\colon c_X E/E \longrightarrow c_Y p^*E/p^*E\,.$$

As in the lemma above, these quotients can be described as the *Thom spaces* (disk fibration modulo boundary sphere fibration) of $g\colon E \to X$ and $p^*g\colon p^*E \to Y$, respectively.

Now suppose that $p^*g\colon p^*E \to Y$ is stably trivial as a spherical fibration. A stable trivialization leads to a stable map

$$\nu\colon c_Y p^*E/p^*E \longrightarrow D^k/S^{k-1}$$

(as in the lemma, but with Y and p^*g instead of X and g). The composition

$$c_X E/E \xrightarrow{q^!} c_Y p^*E/p^*E \xrightarrow{\nu} D^k/S^{k-1}$$

is a map reminiscent of u in lemma 5.6.1. If we can show that it satisfies the degree ± 1 condition in (ii) of lemma 5.6.1, then it follows that $g\colon E \to X$ is stably trivial as a spherical fibration.

When we try to verify this, we are automatically replacing X by a selected point $z \in X$. So we might start again (for this purpose) and assume that X is a single point. Then we may assume that g is the projection $S^{k-1} \to *$ and so p^*g is the projection $S^{k-1} \times Y \to Y$. Here Y is a compact smooth manifold and, according to what I said earlier, I allow myself to assume that it is connected. The map ν takes the form

$$\nu\colon (D^k/S^{k-1}) \wedge Y_+ \longrightarrow D^k/S^{k-1}.$$

(We can assume that it is an honest map, not just a stable map; otherwise increase k.) We should not assume that it is the standard projection, induced by the based map $Y_+ \to S^0$ which takes all of Y to the non-base point. Instead it corresponds to *some* fiber homotopy trivialization of the projection

$$S^{k-1} \times Y \to Y$$

(which is *already* a trivialized fibration). By contrast the map q takes the form

$$q\colon (D^k/S^{k-1}) \wedge Y_+ \longrightarrow D^k/S^{k-1}$$

and it *is* the standard projection. From lemma 5.1.4 or proposition 5.4.1 we know that the composition

$$D^k/S^{k-1} \xrightarrow{q^!} (D^k/S^{k-1}) \wedge Y_+ \xrightarrow{q} D^k/S^{k-1}$$

is a map of degree ± 1. What we need to know however is that the composition

$$D^k/S^{k-1} \xrightarrow{q^!} (D^k/S^{k-1}) \wedge Y_+ \xrightarrow{\nu} D^k/S^{k-1}$$

is a map of degree ± 1. For that we can write $\nu \simeq q \circ e$ where

$$e\colon (D^k/S^{k-1}) \wedge Y_+ \longrightarrow (D^k/S^{k-1}) \wedge Y_+$$

is defined by $e(x,y) = (\nu(x,y), y)$. Now it is clear that e is a stable homotopy equivalence. Since I am allowed to assume that Y is connected, I may conclude that the lowest nontrivial reduced homology group of $(D^k/S^{k-1}) \wedge Y_+$ is

$$\tilde{H}_k((D^k/S^{k-1}) \wedge Y_+) \cong \tilde{H}_0(Y_+) = H_0(Y) \cong \mathbb{Z}$$

and e will induce an automorphism of that group (which can only be multiplication by 1 or -1). Therefore I can calculate the degree of $\nu \circ q^!$ by reading off what it does on reduced homology $\tilde{H}_k$, and I find it does the same up to sign as $q \circ q^!$. This completes the verification. □

* * *

Plans for the weeks after the Christmas/New Year break:

- *No infinite loop space theory after all! Instead, another proof of* (\$) *in section 5.4. which uses the Becker-Gottlieb transfer in a slightly different way and thereby avoids infinite loop space theory. See updates in section 5.4. and the new section 5.6.*

- *Localization theory after all! I just realized that the proof of Proposition 4.5.4 ist still missing.*
- *Perhaps some words on some forms of representation theory; ideally this should allow me to fill in the missing details in the proof of Proposition 4.2.5.*
- *How did Adams show that the upper bound on* $\mathsf{J}(\mathsf{X})$ *given by the Adams conjecture is also a lower bound? (See remark 4.2.4.) Some indications — probably not the full story.*