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Lecture notes chapter 4, WS 2015-2016 (Weiss): Vector bundles, J-homomorphism & Adams conjecture

4.1. Infinite loop spaces and generalized cohomology theories

Why is Bott periodicity in K-theory so important? There are two answers. The first answer, which is the more obvious and less profound of the two:

- it enables us to calculate all the homotopy groups of BU and BO.

Indeed, it tells us that $\pi_k(BU) \cong \pi_{k-2}(BU)$ for $k > 2$, so that we get away with the calculation of $\pi_1(BU)$ and $\pi_2(BU)$. That is easy. And in the real case it tells us that $\pi_k(BO) \cong \pi_{k-8}(BO)$ for $k > 8$, so that we get away with the calculation of $\pi_k(BO)$ for $k = 1, 2, 3, 4, 5, 6, 7, 8$, which is manageable. This is quite remarkable. Homotopy theorists like to point out for comparison that there is no example known of a *compact* based CW-space X with is simply connected, not contractible and has all its homotopy groups $\pi_k(X)$ computed. (For example, the homotopy groups of S^2 are not really all known, although a curious general description in rather knot-theoretic terms emerged about 20 years ago.)

The second answer:

- Bott periodicity creates two important generalized cohomology theories (complex K-theory and real K-theory).

Without trying to give a precise definition of the term *generalized cohomology theory*, let me try to explain nevertheless. Suppose that Y is a fixed space with base point. I will argue that Y gives rise to something like a generalized cohomology theory on compact CW-spaces; let me denote that by h^* (although it depends strongly on the choice of Y). Namely, for a compact CW-space X and $n \geq 0$, we put

$$h^{-n}(X) := \pi_n(\mathrm{map}(X, Y))$$

where $\mathrm{map}(X, Y)$ is the mapping space with the compact-open topology. Note that $\mathrm{map}(X, Y)$ has a preferred base point, since we selected a base point in Y; this is good news as it helps us to make sense of $\pi_n(\mathrm{map}(X, Y))$. For a compact CW pair (X, A) we put

$$h^{-n}(X, A) := \pi_n(\mathrm{map}_*(X/A, Y))$$

where $\mathrm{map}_*(X/A, Y)$ is the space of based maps from X/A to Y. In order to make long exact sequences, we need the following lemma or exercise:

Exercise 4.1.1. For a compact CW-pair (X, A), the restriction map

$$\mathrm{map}(X, Y) \longrightarrow \mathrm{map}(A, Y)$$

is a Serre fibration. The fiber over the base point is $\mathrm{map}_*(X/A, Y)$.

From the fibration sequence in the exercise, we get a long exact sequence of homotopy groups and we can use our "cohomological" notation to write it in the form

$$\cdots \longrightarrow h^{-n}(X,A) \longrightarrow h^{-n}(X) \longrightarrow h^{-n}(A) \longrightarrow h^{1-n}(X,A) \longrightarrow \cdots$$

This is, incidentally, called the *Barratt-Puppe sequence* (except for the cohomological notation), discovered by Barratt and Puppe independently as far as I know. It looks a lot like one of these long exact sequences that one wants in a cohomology theory. There is some form of excision, too. Suppose for example that $X = A \cup B$ where A and B are CW-subspaces of the compact CW-space X. Then we clearly have

$$h^{-n}(X,A) \cong h^{-n}(B, A \cap B)$$

because $X/A \cong B/(A \cap B)$.

Exercise 4.1.2. Given a CW-space $X = A \cup B$ as above, and the above definition of h^*, try to construct a long exact Mayer-Vietoris sequence

$$\cdots \longrightarrow h^{-n}(X) \longrightarrow h^{-n}(A) \times h^{-n}(B) \longrightarrow h^{-n}(A \cap B) \longrightarrow h^{1-n}(X) \longrightarrow \cdots$$

(Hint: if you know the concept of a homotopy pullback square, it might help.)

The one weakness with our candidate h^* for a generalized cohomology theory is that the long exact sequences are not as long as they should be. It is a problem that $h^k(-)$ is only defined for $k \leq 0$. Moreover $h^0(-)$ gives us only sets rather than abelian groups, and $h^{-1}(-)$ gives us groups which need not be abelian.

The situation is a little better if Y is homotopy equivalent to the loop space $\Omega Y^{(1)}$ of another based space $Y^{(1)}$. Then we can redefine

$$h^{-n}(X) := \pi_{n+1}(\mathrm{map}(X, Y^{(1)}))$$
$$h^{-n}(X,A) := \pi_{n+1}(\mathrm{map}_*(X/A, Y^{(1)}))$$

which is consistent with the earlier definition since for example

$$\pi_{n+1}(\mathrm{map}(X, Y^{(1)})) \cong \pi_n(\Omega\mathrm{map}(X, Y^{(1)})) \cong \pi_n(\mathrm{map}(X, \Omega Y^{(1)})).$$

But with the new formula, $h^k(X)$ and $h^k(X,A)$ are defined for $k \leq 1$ (and they are groups for $k \leq 0$, and abelian groups for $k \leq -1$). This is an improvement. And if $Y^{(1)}$ is homotopy equivalent to the loop space $\Omega Y^{(2)}$ of another based space $Y^{(2)}$, then we can make a further improvement by redefining

$$h^{-n}(X) := \pi_{n+2}(\mathrm{map}(X, Y^{(2)}))$$
$$h^{-n}(X,A) := \pi_{n+2}(\mathrm{map}_*(X/A, Y^{(2)}))$$

so that h^k is defined for $k \leq 2$. And so on. This leads us mechanically to the following definitions.

Definition 4.1.3. An Ω*-spectrum* is a sequence of based spaces $Y^{(k)}$ where $k = 0, 1, 2, \ldots$ together with based maps $j_k \colon Y^{(k)} \to \Omega Y^{(k+1)}$ which are based weak homotopy equivalences.

The Ω-spectrum $(Y^{(k)}, j_k)_{k \geq 0}$ determines a generalized cohomology theory h^* on compact CW-spaces X and compact CW-pairs (X, A) by

$$h^r(X) = \pi_{k-r}(\mathrm{map}(X, Y^{(k)}))$$

$$h^r(X, A) = \pi_{k-r}(\mathrm{map}_*(X/A, Y^{(k)}))$$

for $r \in \mathbb{Z}$. (The right-hand side is an abelian group, independent of k up to unique isomorphism; choose any $k \geq r$ to make sense of it, but take $k \geq r+1$ to see a group structure and $k \geq r+2$ to see that it is an abelian group.)

Definition 4.1.4. A space Y is called an *infinite loop space* if it is the space $Y^{(0)}$ in some Ω-spectrum $(Y^{(k)}, j_k)_{k \geq 0}$.

Example 4.1.5. Setting $Y^{(k)} = \mathrm{BU} \times \mathbb{Z}$ for even $k \geq 0$ and $Y^{(k)} = \mathrm{U}$ for odd $k \geq 0$ (where $U = \bigcup_n \mathrm{U}(n)$ is the union of the unitary groups $\mathrm{U}(n)$), we have an Ω-spectrum. Indeed there is an obvious weak homotopy equivalence $\mathrm{U} \to \Omega(\mathrm{BU}) = \Omega(\mathrm{BU} \times \mathbb{Z})$ which we can call j_k for $k = 1, 3, 5, 7, \ldots$. And there is the Bott map $\mathrm{BU} \times \mathbb{Z} \to \Omega U = \Omega^2(\mathrm{BU} \times \mathbb{Z})$ which we can call j_k for $k = 0, 2, 4, \ldots$. (Therefore $\mathrm{BU} \times \mathbb{Z}$ is an infinite loop space.) The generalized cohomology theory corresponding to this Ω-spectrum has the name K^*; so that, for every CW-space X, we have abelian groups $K^r(X)$ (which depend only on r mod 2). This is a little confusing because what we previously called $K(X)$ or $K_{\mathbb{C}}(X)$ must now be called $K^0(X)$ (although I am not planning to do this consistently).

Similarly, setting $Y^{(8m-k)} = \Omega^k(\mathrm{BO} \times \mathbb{Z})$ for $m > 0$ and $k = 0, 1, 2, \ldots, 7$, and also for $m = k = 0$, gives us an Ω-spectrum. (It follows that $\mathrm{BO} \times \mathbb{Z}$ is an infinite loop space.) The generalized cohomology theory corresponding to this Ω-spectrum has the name KO^*; so that, for every CW-space X, we have abelian groups $KO^r(X)$ (which depend only on r mod 8).

4.2. The Adams operations

Remark 4.2.1. Please do not write *the Adam's operations*. His name was Frank Adams.

The Adams operations are natural transformations

$$\Psi^k \colon K(X) \to K(X)$$

where $K(X)$ can be read (consistently) as $K_{\mathbb{C}}(X)$ or $K_{\mathbb{R}}(X)$, unless otherwise specified. They are defined mainly for $k \geq 0$ although there is a cheap way to extend the definition to all $k \in \mathbb{Z}$. Adams introduced these operations in his paper *Vector fields on spheres* (1962). The guiding principle is as follows.

If $z \in \mathsf{K}(X)$ is an element which can be written as a sum $z = z_1 + z_2 + \cdots + z_r$ where each z_i is represented by a *line bundle* alias 1-dimensional vector bundle, then we should have

$$(*) \qquad\qquad \Psi^k(z) = z_1^k + z_2^k + \cdots + z_r^k$$

in the ring $\mathsf{K}(X)$. Let's note that the assumption z *can be written as a sum* $z = z_1 + z_2 + \cdots + z_r$ *where each* z_i *is represented by a line bundle* is awfully restrictive if we are talking about real vector bundles, but less so if we are talking about complex vector bundles. For example, $\mathsf{K}_{\mathbb{C}}(S^2)$ is generated as an abelian group by elements which are represented by line bundles: the trivial line bundle and the tautological line bundle on $\mathbb{C}P^1 \cong \mathbb{S}^2$.

Adams found a magic way to make sense of the prescription $(*)$ in the general case, i.e., without having to assume that z is represented by a Whitney sum of r line bundles. This is based on the observation that the expression $z_1^k + z_2^k + \cdots + z_r^k$ viewed as a function of "abstract" variables $z_1, \ldots, z_r$ is a *symmetric* polynomial. It is a well-known theorem in foundational algebra that symmetric polynomials can always be written as polynomials in the so-called *elementary symmetric polynomials.*

Theorem 4.2.2. *Let* A *be a commutative ring (with unit). Let* $t_1, \ldots, t_r$ *be variables. Let* Σ_r *be the symmetric group on letters or numbers* $\{1, 2, \ldots, r\}$*. Let* $A[t_1, \ldots, t_r]$ *be the polynomial ring. Then for the subring of polynomials which are symmetric, i.e., invariant under the action of* Σ_r *by permutation of the variables* $t_1, \ldots, t_r$*, we have*

$$\big(A[t_1, \ldots, t_r]\big)^{\Sigma_k} = A[s_1, s_2, \ldots, s_r]$$

where $s_1, \ldots, s_r$ *are the elementary symmetric polynomials. More precisely,*

$$s_j := \sum_f t_{f(1)} t_{f(2)} \cdots t_{f(j)}$$

where f *runs over all order-preserving injections from the set* $\{1, 2, \ldots, j\}$ *to the set* $\{1, 2, \ldots, r\}$*.* (For example $s_1 = t_1 + t_2 + \cdots + t_r$ and $s_r = t_1 t_2 \cdots t_{r-1} t_r$.)

It must be emphasized that $A[s_1, s_2, \ldots, s_r]$ in this formulation is meant to be *the* polynomial ring in the variables $s_1, \ldots, s_r$. Therefore the content of the theorem is that every symmetric polynomial in the variables $t_1, \ldots, t_r$ with coefficients in A can be written *uniquely* as a polynomial in the elementary symmetric polynomials $s_1, \ldots, s_r$ with coefficients in A.

Example 4.2.3. Take $r = 3$ and $k = 2$ and $A = \mathbb{Z}$. Then $t_1^2 + t_2^2 + t_3^2 = s_1^2 - 2s_2$.

Let's follow Adams and manoeuver between the relatively weak assumption that z is represented by a vector bundle $E \to X$ of fiber dimension r on X, and the stronger assumption that E is a Whitney sum of line bundles $E_1, \ldots, E_r$ on X. Write Λ^j for the j-th alternating power. This is an operation on vector spaces or, if we wish, an operation which can be performed fiberwise on vector bundles (over X). The operation Λ^j performed on r-dimensional vector bundles has properties which are curiously reminiscent of the elementary symmetric function s_j (in variables $t_1, \ldots, t_r$). Namely, if E happens to be a Whitney sum of line bundles $E_1 \oplus E_2 \oplus \cdots \oplus E_r$, then

$$\Lambda^j E \;\cong\; s_j(E_1, E_2, \ldots, E_r)$$

where we can make sense of the right-hand side by interpreting sums as Whitney sums $\oplus$ and products as tensor products $\otimes$. The isomorphism sign in the middle means *isomorphism of vector bundles.*

But the expression $\Lambda^j E$ is defined for any r-dimensional vector bundle E on X; we do not need a splitting into line bundles for that. Therefore we can proceed as follows. Write the symmetric polynomial $t_1^k + t_2^k + \cdots + t_r^k$ as a polynomial in the elementary symmetric polynomials:

$$t_1^k + t_2^k + \cdots + t_r^k = P_{k,r}(s_1, s_2, \ldots, s_r)\,.$$

Then define

$$\Psi^k(E) := P_{k,r}(\Lambda^1 E, \Lambda^2 E, \ldots, \Lambda^r E)\,.$$

Again, we make sense of the right-hand side by interpreting sums as Whitney sums $\oplus$ and products as tensor products $\otimes$.

Example 4.2.4. If E has dimension 3, then $\Psi^2(E) = \Lambda^1 E \otimes \Lambda^1 E - 2\Lambda^2 E = E \otimes E - 2\Lambda^2 E$ (using example 4.2.3).

As the example shows, we may view E as a vector bundle on X or, more generously, as an isomorphism class of vector bundles on X, but we must view $\Psi^k(E)$ as an element of $K(X)$ because formal differences of vector bundles are involved.

Proposition 4.2.5. $\Psi^k(E) \in K(X)$ *can be calculated as*

$$P_{k,r}(\Lambda^1 E, \Lambda^2 E, \ldots, \Lambda^r E)$$

for any r *which is at least as big as the (maximal) fiber dimension of* E*. Furthermore,* Ψ^k *takes Whitney sums of vector bundles on* X *to sums in* $K(X)$*, and tensor products of vector bundles on* X *to products in* $K(X)$*.*

Given the very algebraic definition of Ψ^k, this looks as it should have a proof which is mostly algebraic. I am trying to make such a proof, and as part of that I need a corollary to theorem 4.2.2.

Corollary 4.2.6. *Let* A *be a commutative ring. Let* $t_1, \dots, t_r, u_1, u_2, \dots, u_q$ *be variables. Let* $s'_1, \dots, s'_r$ *be the elementary symmetric polynomials in the variables* $t_1, \dots, t_r$ *and let* $s''_1, \dots, s''_q$ *be the elementary symmetric polynomials in* $u_1, \dots, u_q$. *Then*

$$\big(A[t_1, t_2, \dots, t_r, u_1, u_2, \dots, u_q]\big)^{\Sigma_r \times \Sigma_q} = A[s'_1, s'_2, \dots, s'_r, s''_1, \dots, s''_q].$$

Proof. For $\big(A[t_1, t_2, \dots, t_r, u_1, u_2, \dots, u_q]\big)^{\Sigma_r \times \Sigma_q}$ we can write

$$\begin{aligned} & \Big(\big(A[t_1, t_2, \dots, t_r]\big)^{\Sigma_r}[u_1, u_2, \dots, u_q]\Big)^{\Sigma_q} \\ = \ & \big(A[s'_1, \dots, s'_r][u_1, \dots, u_q]\big)^{\Sigma_q} \\ = \ & A[s'_1, \dots, s'_r][s''_1, \dots, s''_q] \\ = \ & A[s'_1, \dots, s'_r, s''_1, \dots, s''_q]. \quad \square \end{aligned}$$

Proof of proposition 4.2.5. For the first part we can assume that X is connected. If r is greater than the fiberwise dimension d of $E \to X$, then $\Lambda^r E, \dots, \Lambda^{d+1} E$ are zero. Therefore we only have to verify that $P_{k,r}$ turns into $P_{k,d}$ if we substitute $s_j \in \mathbb{Z}[t_1, \dots, t_d]$ for $s_j \in \mathbb{Z}[t_1, \dots, t_r]$; this means in practice that we let s_j be s_j for $j \le d$ and substitute 0 for s_j if $j > d$. Making these substitutions can be simulated by substituting 0 for $t_r, \dots t_{d+1}$ in polynomials in the variables $t_1, \dots, t_r$, which may be symmetric or not. If we do this to $t_1^k + \dots + t_r^k$, we obtain $t_1^k + \dots + t_d^k$.

For the second part we can also assume that X is connected. We take two vector bundles $E \to X$ and $F \to X$, of fiber dimension r and q, respectively. Then we have

$$\Psi^k(E \oplus F) = P_{k,r+q}(\Lambda^1(E \oplus F), \Lambda^2(E \oplus F), \dots, \Lambda^{r+q}(E \oplus F)).$$

Here we view $P_{k,r+q}$ as a polynomial in the elementary symmetric polynomials $s_1, s_2, \dots, s_{r+q}$ in the variables $t_1, \dots, t_r, u_1, \dots, u_q$. Now we make a leap of faith to the statement that each $\Lambda^j(E \oplus F)$ has an expression as a polynomial in exterior powers of E and F, matching the expression of s_j as a polynomial in $s'_1, \dots, s'_r$ and $s''_1, \dots, s''_q$ which we must have according to corollary 4.2.6. (See exercise 4.2.7 below.) If we plug these expressions for $\Lambda^j(E \oplus F)$ into the formula for $\Psi^k(E \oplus F)$ just above, then we get

$$\Psi^k(E \oplus F) = \text{polynomial in } \Lambda^1 E, \dots, \Lambda^r E, \Lambda^1 F, \dots, \Lambda^q F$$

where that polynomial (in the right-hand side) is the unique element of

$$\mathbb{Z}[s'_1, \dots, s'_r, s''_1, \dots, s''_q]$$

which turns into $t_1^k + \dots + t_r^k + u_1^k + \dots + u_q^k$ if we view $s'_1, \dots, s'_r, s''_1, \dots, s''_q$ as elements of $\mathbb{Z}[t_1, \dots, t_r, u_1, \dots, u_q]$. But then we recognize that polynomial as $P'_{k,r} + P''_{k,q}$ where the dashes in P' and P'' indicate that we use variables $s'_1, \dots, s'_r$ for one and $s''_1, \dots, s''_q$ for the other. — The proof for product compatibility is (apparently) *not* similar. It needs more machinery (representation theory). See section 4.6 (which was added much later). $\square$

Exercise 4.2.7. Replace the leap of faith in the above proof by a better argument.

Corollary 4.2.8. Ψ^k *is well defined as a group homomorphism from* $K(X)$ *to* $K(X)$*, and in fact it is a ring homomorphism. We have* $\Psi^1 = \mathrm{id}$*. For connected* X*, the homomorphism* $\Psi^0 \colon K(X) \to K(X)$ *is given by the virtual dimension function* $K(X) \to \mathbb{Z}$*, followed by the unique ring homomorphism* $\mathbb{Z} \to K(X)$*.*

This is a good opportunity to say or write something about λ-rings and special λ-rings. Initially I wanted to quote mainly from Wikipedia, but then I found *Group representations,* λ*-rings and the* J*-homomorphism* by Atiyah and Tall (1969). A λ*-ring* is a commutative ring R together with maps $\lambda^k \colon R \to R$ for $k = 0, 1, 2, 3, \dots$ which imitate the behavior (on direct sums) of the exterior powers Λ^i on (real/complex) vector spaces. The exterior powers of vector spaces $U, V, \dots$ satisfy $\Lambda^0(U) =$ ground field ($\mathbb{R}$ or $\mathbb{C}$) and $\Lambda^1(U) = U$, and

$$\Lambda^k(U \oplus V) \cong \bigoplus_{j=0}^{k} \Lambda^j(U) \otimes \Lambda^{k-j} V.$$

Therefore it is a condition in λ-rings that $\lambda^0(x) = 1$ and $\lambda^1(x) = x$ for all x, and

$$\lambda^k(x + y) = \sum_{j=0}^{k} \lambda^j(x) \cdot \lambda^{k-j}(y)$$

for all $x, y \in R$ and $k \geq 0$. These are all the conditions for a λ-ring, according to Atiyah and Tall.

The exterior powers of vector spaces satisfy

$$\Lambda^k(U \otimes V) \cong Q_k(\Lambda^0(U), \Lambda^1(U), \dots, \Lambda^k(U), \Lambda^0(V), \Lambda^1(V), \dots, \Lambda^k(V))$$

where Q_k is a certain polynomial in $2k+2$ variables, with integer coefficients. (This polynomial Q_k is extremely hard to determine for larger k, as we noted in connection with exercise 4.2.7. It may have some negative coefficients, so that the isomorphism can only be made sense of in a stable way: add something to both sides to cancel out the negative contributions in Q_k.) Also, it appears that the exterior powers satisfy

$$\Lambda^j(\Lambda^k(U)) \cong Q'_{j,k}(\Lambda^1(U), \dots, \Lambda^{jk}(U))$$

for a certain polynomial $Q'_{j,k}$ with integer coefficients. (At the moment I find this even more perplexing than the previous formula.) Therefore in a *special* λ*-ring* we impose the conditions

$$\lambda^k(xy) = Q_k(\lambda^0(x), \lambda^1(x), \dots, \lambda^k(x), \lambda^0(y), \lambda^1(y), \dots, \lambda^k(y)),$$

$$\lambda^j(\lambda^k(x)) = Q'_{j,k}(\lambda^1(x), \dots, \lambda^{jk}(x)).$$

Example 4.2.9. We promote the ring $\mathbb{Z}$ to a λ-ring in such a way that $\lambda^k(0) = 0$ for $k > 0$ and $\lambda^k(1) = 0$ for $k > 1$. This is surprisingly intricate. We immediately obtain $\lambda^k(n+1) = \lambda^k(n) + \lambda^{k-1}(n)$ for $k > 0$ and $n \in \mathbb{Z}$; therefore

$$\lambda^k(n) = \binom{n}{k}$$

for all $k \geq 0$ and $n > 0$. It follows also that $\lambda^k(-1) + \lambda^{k-1}(-1) = 0$ for $k > 0$, so that $\lambda^k(-1) = (-1)^k$ for all $k \geq 0$. Therefore

$$\lambda^k(n-1) = \lambda^k(n) - \lambda^{k-1}(n) + \lambda^{k-2}(n) - \lambda^{k-3}(n) + \cdots$$

for all $n \in \mathbb{Z}$ and all $k \geq 0$. This allows a recursive determination of $\lambda^k(n)$ for all n and $k \geq 0$. Briefly, we can find $\lambda^k(n)$ by making a Taylor series expansion

$$(1+t)^n = \sum_{k=0}^{\infty} \lambda^k(n) \cdot t^k$$

where t should be thought of as a small real or complex number.

Example 4.2.10. Let X be a compact CW-space. We promote $K(X)$ to a λ-ring. (This works for complex K-theory as well as for real K-theory, but let us take real K-theory here.)

If $z \in K(X)$ is represented by a vector bundle $E \to X$, then we want to ensure that $\lambda^k(z)$ is represented by the vector bundle $\Lambda^k E \to X$. In general, we can assume that z has the form $z_1 - n$ where z_1 is represented by a vector bundle $E \to X$ and n is a non-negative integer, alias trivial vector bundle on X of fiber dimension n. Then we let

$$\lambda^k(z) = \lambda^k(z_1 - n) := \sum_{j=0}^{\infty} \lambda^j(z_1) \cdot \lambda^{k-j}(-n)$$

where $\lambda^j(z_1)$ is the class of $\Lambda^j E \to X$ and $\lambda^{k-j}(-n) \in \mathbb{Z} \subset K(X)$ is to be determined as in example 4.2.9. (Note that the sum is finite, although it looks infinite, because $\lambda^j(z_1)$ is zero for sufficiently large j.) I hope it is not difficult to verify that this is well defined. (The question is how the right-hand side changes if we replace z_1 by $z_1 + 1$ and n by $n+1$. It should not change at all.)

Remark 4.2.11. Of course we would like to know that $K(X)$ is a *special* λ-ring. With the above definition of special λ-ring, this may be obvious, but the definition is rather sketchy. With a more explicit definition (as given in Atiyah-Tall), it is unfortunately not very obvious.

Proposition 4.2.12. $\Psi^{k\ell} = \Psi^k \circ \Psi^\ell$.

Proof. This should also go into section 4.6 (but it has not been done in full yet). □

4.3. Some easy computations with Adams operations

We start by computing the Adams operations in $K(S^2)$, using complex K-theory. Let L be the tautological line bundle over $S^2 = \mathbb{C}P^1$. Then

$$K(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$$

generated (as an abelian group) by 1 in the first summand and by $[L]$ in the other summand. Therefore

$$\tilde{K}(S^2) \cong \mathbb{Z}$$

with generator $[L] - 1$. We can determine the ring structure, too, by noting that

$$([L] - 1)^2 = 0 \,.$$

Indeed, $([L]-1)^2$ is the image of the external square $([L]-1) \boxtimes ([L]-1) \in \tilde{K}(S^2 \wedge S^2)$ under the homomorphism of reduced K-groups

$$\tilde{K}(S^2 \wedge S^2) \longrightarrow \tilde{K}(S^2)$$

induced by the diagonal map $S^2 \to S^2 \wedge S^2$, where $S^2 \wedge S^2 \simeq S^4$. But any based map from S^2 to S^4 is based nullhomotopic. — Therefore we have

$$[L]^k = (1 + ([L]-1))^k = 1 + k([L]-1)$$

by the binomial theorem; terms involving higher powers of $[L]-1$ can be dropped because. Therefore, by the guiding principle,

$$\Psi^k([L]) = [L]^k = 1 + k([L]-1)$$

which implies:

Proposition 4.3.1. $\Psi^k([L]-1) = k([L]-1)$ *in* $\tilde{K}_{\mathbb{C}}(S^2) \cong \mathbb{Z}$*, where* $[L]-1$ *is the standard generator of that group.*

Corollary 4.3.2. *For the standard generator* $z \in \tilde{K}(S^{2n}) \cong \mathbb{Z}$ *we have*

$$\Psi^k(z) = k^n z \,.$$

Proof. For $n = 1$, this is what we just calculated. For larger n, the standard generator is $z = ([L] - 1)^{\boxtimes n}$, the n-fold external power of

$$[L] - 1 \in \tilde{K}(S^2).$$

Now Ψ^k preserves external products; this follows in a formal way from the fact that Ψ^k preserves (internal) products. □

Next, let's find out how Ψ^k acts in $\tilde{K}_{\mathbb{R}}(S^m)$. Here we can get a lot for free by using the fact that the following always commutes,

$$\begin{tikzcd} K_{\mathbb{R}}(X) \arrow[r] \arrow[d] & K_{\mathbb{R}}(X) \arrow[d] \\ K_{\mathbb{C}}(X) \arrow[r] & K_{\mathbb{C}}(X) \end{tikzcd}$$

where the vertical arrows are induced by complexification $\otimes_{\mathbb{R}}\mathbb{C}$ of real vector bundles. Namely, complexification gives an injective homomorphism

$$\tilde{K}_{\mathbb{R}}(S^4) \to \tilde{K}_{\mathbb{C}}(S^4) \cong \mathbb{Z}$$

with image $2\mathbb{Z}$, and an isomorphism

$$\tilde{K}_{\mathbb{R}}(S^8) \to \tilde{K}_{\mathbb{C}}(S^8) \cong \mathbb{Z}.$$

This last one is particularly important because it is related to the fact that Bott periodicity in real and complex K-theory are compatible (although the period is 2 in complex K-theory, 8 in real K-theory. Therefore (and using the periodicity) we can immediately deduce:

Corollary 4.3.3. *For even $n > 0$ and for the standard generator z in $\tilde{K}_{\mathbb{R}}(S^{2n}) \cong \mathbb{Z}$ we have $\Psi^k(z) = k^n z$.*

The remaining nontrivial groups $\tilde{K}_{\mathbb{R}}(S^n)$ where $1 \le n \le 8$ are those where $n = 1$ and $n = 2$:

$$\tilde{K}_{\mathbb{R}}(S^1) \cong \mathbb{Z}/2, \qquad \tilde{K}_{\mathbb{R}}(S^2) \cong \mathbb{Z}/2.$$

(These are easily calculated because they correspond to $\pi_0(O)$ and $\pi_1(O)$, respectively. See exercise 4.3.5.) A generator for $\tilde{K}(S^1)$ is $([L] - 1)$, where L is the canonical *real* line bundle on $\mathbb{R}P^1 \cong S^1$; this L is also known as the nontrivial line bundle on S^1. A generator for $\tilde{K}(S^2)$ is $([L] - 1)^{\boxtimes 2}$, in the same notation. By the guiding principle, we get:

$$\Psi^k([L] - 1) = [L]^k - 1 = \begin{cases} 0 & \text{if } k \text{ is even} \\ [L] - 1 & \text{if } k \text{ is odd.} \end{cases}$$

This leads us to the following.

Proposition 4.3.4. *Suppose that $n \equiv 1, 2$ modulo 8 and $n > 0$. Then Ψ^k is the identity on $\tilde{K}(S^n)$ if k is odd, and the zero homomorphism if k is even.*

Exercise 4.3.5. Easy arguments with fibration sequences show that the inclusion $\mathrm{O}(3) \to \mathrm{O}$ induces an isomorphism on π_1 (with the standard choice of base points). The *plate trick* is allegedly a way to prove that $\pi_1(\mathrm{O}(3))$ is isomorphic to $\mathbb{Z}/2$. From Wikipedia (Dec 2016): *One way of doing the trick is to rest a small plate flat on the palm, then perform two rotations of one's hand while keeping the plate upright, ending in the original position. The hand makes one rotation passing over its shoulder, twisting the arm, and then another rotation passing under, untwisting it.* Finding somebody who will perform the plate trick should not be a problem, but how exactly does this show that $\pi_1(\mathrm{O}(3)) \cong \mathbb{Z}/2$? (Hint: easy arguments with fibration sequences also show that the inclusion $\mathrm{O}(2) \to \mathrm{O}(3)$ induces a surjective homomorphism in π_1 .)

4.4. The Adams conjecture (as a statement)

The Adams conjecture as stated in the article *On the groups* $J(X)$ *— I* by Adams is as follows. Let X be a compact CW-space. In section 1.3 we defined $J(X)$ as the image of the forgetful map $K_{\mathbb{R}}(X) \to K_F(X)$ where $K_F(X)$ is the K-group based on spherical fibrations, join instead of Whitney sum, etc.

Statement 4.4.1. *Let* X *be a compact CW-space. Then for any integer* $k > 0$ *and* $y \in K_{\mathbb{R}}(X)$, *there exists a positive integer* e *such that*

$$k^e(\Psi^k(y) - y) = 0 \in J(X) \subset K_F(X).$$

Equivalently, the composition

$$K_{\mathbb{R}}(X) \xrightarrow{\Psi^k - \mathrm{id}} K_{\mathbb{R}}(X) \longrightarrow K_F(X) \otimes_{\mathbb{Z}} \mathbb{Z}[k^{-1}]$$

is zero, where $\mathbb{Z}[k^{-1}]$ *is the subring of* $\mathbb{Q}$ *consisting of all rational numbers whose denominator divides* k^e *for some integer* $e \geq 0$.

Remark 4.4.2. If I understand correctly, Adams also shows in his series of four articles on the groups $J(X)$ that, if this is true, then it is best possible. More precisely, the lower bound on the kernel of $K_{\mathbb{R}}(X) \to J(X) \subset K_F(X)$ given by the statement is a *precise* description of the kernel. We should beware that this is a complicated statement since all $k \geq 1$ need to be taken into account. (And it is not clear how much of this we are going to reach in this course.)

Let us try to unravel this statement (including the sharpening mentioned in the remark) when X is a sphere. First suppose $X = S^{2n}$ where n is even. Write K for $K_{\mathbb{R}}$. Then $\tilde{K}(S^{2n})$ is infinite cyclic; write y for the standard generator. It follows that $\tilde{J}(S^{2n})$ is also cyclic, with the same generator. We calculated

$$\Psi^k(y) = k^n y \in \tilde{K}(S^{2n})$$

so that the Adams equation $k^e(\Psi^k(y)-y)=0$ simplifies to $k^e(k^n-1)y=0$. (This must hold for some $e \ge 0$, possibly large, depending on k and n.) It follows immediately that $\tilde{J}(S^{2n})$ is *finite* cyclic, and its order $\omega(n)$ divides $k^e(k^n-1)$. This can also be expressed as follows:

$$\forall k \ge 1: \ \frac{k^n-1}{\omega(n)} \in \mathbb{Z}[k^{-1}].$$

Now, in the spirit of remark 4.4.2, the number $\omega(n)$ must be *the largest of all positive integers* x *which satisfy*

$$\forall k \ge 1: \ \frac{k^n-1}{x} \in \mathbb{Z}[k^{-1}].$$

(We can take that as a number-theoretic re-definition of $\omega(n)$ for even n, independent of the Adams conjecture. The word *largest* can be interpreted in the usual sense or in the *greatest common multiple* sense; it does not matter.) At first sight it is not obvious that such a "largest" exists, but it is rather easy to see after all.

Example 4.4.3. Take $n=10$. For a prime p and a nonzero integer x let $\nu_p(x)$ be the exponent of the highest power of p dividing x.

If x divides $2^e(2^{10}-1) = 2^e \cdot 3 \cdot 11 \cdot 31$ for some e, then $\nu_3(x) \le 1$, $\nu_{11}(x) \le 1$, $\nu_{31}(x) \le 1$, $\nu_p(x)=0$ for all $p \ne 2,3,11,31$. If x also divides $3^e(3^{10}-1) = 3^e \cdot 2^3 \cdot 11^2 \cdot 61$ for some e, then in addition $\nu_2(x) \le 3$. Therefore such an x must divide $2^3 \cdot 3 \cdot 11 = 264$. Therefore $\omega(10)$ must divide 264.

Exercise 4.4.4. (i) Show that $\omega(10)=264$ by investigating the multiplicative group of the ring $\mathbb{Z}/264$. (ii) Determine $\omega(16)$ by the same method. (iii) Show that $\omega(n)$, in the number-theoretic definition, is the largest (and also the gcm) of the positive integers x such that the multiplicative group of the ring $\mathbb{Z}/x$ has n for an exponent. (In words, $k^n \equiv 1$ mod x for every integer k which is relatively prime to x.)

Next let $X=S^m$ where $m \equiv 1$ or $m \equiv 2$ mod 8. Then $\tilde{K}(S^m) \cong \mathbb{Z}/2$. If we take k even in the statement of the Adams conjecture, then we have to invert k and we will have inverted 2 and there is no interesting information left. Suppose then that k is odd. Then $\Psi^k = \mathrm{id}$ on $\tilde{K}(S^m)$ and so $k^e(\Psi^k - \mathrm{id})$ is already zero as a homomorphism from $\tilde{K}(S^m)$ to itself. Therefore we have to rely on remark 4.4.2 rather than statement 4.4.1. Then we learn that $\tilde{K}(S^m)$ maps injectively to $\tilde{K}_F(S^m)$. So the order of $\tilde{J}(S^m)$ ought to be 2 in this case.

***Summary*:** The Adams conjecture in the strong form (4.4.1 and 4.4.2) implies that the groups $\tilde{J}(S^m)$ are finite cyclic of the following orders:

- for $m = 2n$ where n is even, order $\omega(n)$, the gcm (and therefore largest) of all positive integers x which satisfy

$$\forall \text{ positive integers } k : \qquad \frac{k^n - 1}{x} \in \mathbb{Z}[k^{-1}] \ ;$$

- order 2 for $m \equiv 1$ or $m \equiv 2$ mod 8;
- order 1 in the remaining cases.

(Adams writes $m(n)$ instead of $\omega(n)$. I find this too inconspicuous. The number which I have denoted $\omega(n)$ is better known as the *denominator* of $\beta_n/2n$ where β_n is defined *for all* $n \geq 0$ by the power series expansion

$$\frac{t}{\exp(t) - 1} = \sum_{n=0}^{\infty} \beta_n \frac{t^n}{t!} \, .$$

It turns out that $\beta_n = 0$ for odd $n > 1$. Furthermore $\pm\beta_n$ is also known as the *Bernoulli* number $B_{n/2}$ for even $n \geq 0$. The sign depends a little on conventions.)

4.5. Confirming the Adams conjecture in low-dimensional cases

In the article *On the groups* $J(X)$ *— I*, Adams confirms the Adams conjecture in the cases where $y \in K_{\mathbb{R}}(X)$ is represented by a vector bundle of small fiber dimension.

Proposition 4.5.1. *If* $y \in K_{\mathbb{R}}(X)$ *is represented by a vector bundle of fiber dimension* 1 *or* 2*, then*

$$k^e(\Psi^k(y) - y) = 0 \in J(X) \subset K_F(X)$$

for some positive integer e.

The proof falls into two parts, corresponding to 1-dimensional vector bundles and 2-dimensional vector bundles. In both cases we argue by reduction to universal examples.

Any 1-dimensional vector bundle $E \to X$ is the pullback of the canonical line bundle on a (real) Grassmannian $\mathrm{Grm}(1, n-1) = \mathbb{R}P^{n-1}$ along some map $X \to \mathbb{R}P^{n-1}$. Therefore we may as well assume that $X = \mathbb{R}P^{n-1}$ for some $n \gg 0$, and $E \to \mathbb{R}P^{n-1}$ *is* the canonical line bundle, and $y \in K(\mathbb{R}P^{n-1})$ is the class of E.

We found that $\Psi^k(y) = y$ if k is odd and $\Psi^k(y) = 1$ if k is even. Therefore the case of odd k is trivial here. For even k we obtain $\Psi^k(y) - y = 1 - y$. So it suffices to show $2^e(1 - y) = 0 \in K_{\mathbb{R}}(\mathbb{R}P^{n-1})$ for some $e \gg 0$. This is a consequence of the following lemma, which says that we can take $e = n$.

Lemma 4.5.2. *The Whitney sum of* 2^n *copies of the canonical line bundle* $E \to \mathbb{R}P^{n-1}$ *is a trivial vector bundle.*

Proof. Let Cl_n be the Clifford algebra of the vector space $V = \mathbb{R}^n$ with the standard scalar product. The algebra Cl_n is obtained from the free algebra on V, which is the tensor algebra

$$T(V) := \mathbb{R} \ \oplus \ V \ \oplus \ (V \otimes V) \ \oplus \ (V \otimes V \otimes V) \ \oplus \ \dots$$

by imposing the relations

$$w \cdot w = -\|w\|^2 \cdot 1$$

for all w in the summand $V \subset T(V)$. More explicitly, if we use the standard orthonormal basis $e_1, e_2, \dots, e_n$ for V, then Cl_n is generated as an algebra by (the images of the) vectors $e_1, e_2, \dots, e_n$ and the only relations we need are

$$e_j^2 = -1$$

for $j = 1, 2, 3, \dots, n$ as well as $e_j e_k + e_k e j = 0$ for $j \neq k$. (The latter relation holds because we imposed $(e_j + e_k)^2 = -2 = e_j^2 + e_k^2$.) It follows that Cl_n as a real vector space has a basis with basis vectors corresponding to the subsets of $\{1, 2, \dots, n\}$; namely, every monomial

$$e_{j_1} e_{j_2} \dots e_{j_k} \in \mathrm{Cl}_n$$

where $j_1 < j_2 < \dots < j_k$ qualifies as a basis element. (For example the empty subset of $\{1, 2, \dots, n\}$ corresponds to the "empty" monomial, which stands for $1 \in \mathrm{Cl}_n$.) A multiplication table for the basis elements is easy to make using the above relations, $e_j^2 = -1$ and $e_j e_k + e_k e j = 0$ for $j \neq k$. It follows that Cl_n is noncommutative for $n > 0$. It follows also that Cl_n has dimension 2^n as a real vector space; in particular it is a finite dimensional real vector space, which was not completely obvious from the outset since the tensor algebra $T(V)$ is infinite dimensional (assuming $n > 0$).

Now we make use of Cl_n in the following way. We identify it with the vector space $\mathbb{R}^{2^n}$ using the standard monomial basis, and note that for every unit vector

$$v \in S^{n-1} \subset \mathbb{R}^n = V \subset \mathrm{Cl}_n = \mathbb{R}^{2^n}$$

we have an $\mathbb{R}$-linear map

$$f_v \colon \mathbb{R}^{2^n} \longrightarrow \mathbb{R}^{2^n}$$

given by left multiplication with v in the Clifford algebra. Since $v^2 = -1$ in the Clifford algebra, we have $f_v f_v = -\mathrm{id}$, so that f_v is a linear isomorphism, and also

$$f_v = -f_{-v} .$$

This last equation in particular means that the linear isomorphisms f_v can be used to give a trivialization of the vector bundle

$$\mathbb{R}^{2^n} \times_{\mathbb{Z}/2} S^{n-1} \longrightarrow * \times_{\mathbb{Z}/2} S^{n-1} = \mathbb{R}P^{n-1}$$

where the generator of $\mathbb{Z}/2$ acts on $\mathbb{R}^{2^n}$ by $-\mathrm{id}$, and on S^{n-1} by the antipodal map. Namely,

$$\text{orbit of } (w,v) \mapsto (f_v(w), \text{orbit of } v) \in \mathbb{R}^{2^n} \times \mathbb{R}P^{n-1}.$$

Since that vector bundle (which we have just trivialized) on $\mathbb{R}P^{n-1}$ is evidently the Whitney sum of 2^n copies of the canonical line bundle, this completes the proof. $\square$

Remark 4.5.3. Lemma 4.5.2 is not optimal. A better estimate can be obtained by working with a an irreducible (nonzero) *module* over the Clifford algebra, instead of using the free module on one generator as we have done.

Now we turn to the case of 4.5.1 where y is represented by a 2-dimensional vector bundle. The proof uses the following proposition.

Proposition 4.5.4. *Let E and E' be spherical fibrations on X, where X is a compact CW-space as before; assume that the fibers are homotopy equivalent to S^{n-1} in both cases. Suppose that there is a map $E \to E'$ over X which has degree $\pm k$ on the fibers. Then $[E] = [E'] \in K_F(X) \otimes_{\mathbb{Z}} \mathbb{Z}[k^{-1}]$.*

(Let's postpone the proof. I am not planning to skip it, but it seems to me that it should be part of a chapter on *localization*, the art of making selected prime numbers invertible in homotopy theory.)

In addition to that, we need the following fact:

Lemma 4.5.5. *Let L be a complex line bundle on X. Then the forgetful homomorphism of abelian groups*

$$\varphi \colon K_{\mathbb{C}}(X) \to K_{\mathbb{R}}(X)$$

satisfies $\varphi\Psi^k_{\mathbb{C}}[L] = \Psi^k_{\mathbb{R}}\varphi[L]$.

Since $\Psi^k_{\mathbb{C}}[L] = [L]^k$, we get $\Psi^k_{\mathbb{R}}\varphi[L] = \varphi([L]^k)$, but φ is not a ring homomorphism; so do not confuse with $(\varphi[L])^k$.

This lemma could be true in greater generality, and perhaps there is an algebraic proof, but I don't know how that would go. It suffices to look at universal examples, i.e., it suffices to confirm the cases where X is $\mathbb{C}P^{n-1}$ and L is the canonical complex line bundle. (We should allow any n.)

Proof. Adams gives a proof of lemma 4.5.5 using elementary representation theory. I don't see any other way, but I have made a heroic effort to avoid *characters* of representations. — Let $W = \mathbb{C}$, viewed as a real vector space with an action of S^1, the unit circle in $\mathbb{C}$, by scalar multiplication. We need to look at the tensor powers of W, taken over $\mathbb{R}$, with the induced action of S^1. More precisely, $z \in S^1 \subset \mathbb{C}$ acts on an element of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ (in the k-fold tensor power of W) by taking it to

$$zv_1 \otimes zv_2 \otimes \cdots \otimes zv_k\,.$$

Pour fixer les idées, let's try the 6-th power: $W \otimes W \otimes W \otimes W \otimes W \otimes W$. For every subset A of $\{1,2,3,4,5,6\}$ we make an $\mathbb{R}$-linear map

$$g_A \colon W \otimes W \otimes W \otimes W \otimes W \otimes W \longrightarrow \mathbb{C}$$

by composing

$$\begin{tikzcd} W \otimes W \otimes W \otimes W \otimes W \otimes W \arrow[d, "u_1\otimes u_2\otimes u_3\otimes u_4\otimes u_5\otimes u_6"'] & \mathbb{C} \arrow[d, equal] \\ W \otimes W \otimes W \otimes W \otimes W \otimes W \arrow[r] & W \otimes_{\mathbb{C}} W \otimes_{\mathbb{C}} W \otimes_{\mathbb{C}} W \otimes_{\mathbb{C}} W \otimes_{\mathbb{C}} W \end{tikzcd}$$

where $u_j \colon W \to W$ is complex conjugation if $j \in A$, and the identity otherwise. Then we note the following:

(i) The map g_A respects the actions of S^1 if we agree that $z \in S^1$ acts on the target $\mathbb{C}$ of g_A by scalar multiplication with $z^{6-2|A|}$.
(ii) If $B = \{1,2,3,4,5,6\} \smallsetminus A$, then g_B equals g_A followed by conjugation.
(iii) The maps g_A for A not containing 1 define an $\mathbb{R}$-linear isomorphism

$$W \otimes W \otimes W \otimes W \otimes W \otimes W \longrightarrow \prod_{A \subset \{2,3,4,5,6\}} \mathbb{C}\,.$$

(In (iii) we selected the subsets A not containing 1 because of (ii); we need to select one subset from each pair of complementary subsets.) Next, by writing the canonical complex line bundle L on $\mathbb{C}P^{n-1}$ in the form

$$W \otimes_{S^1} S^{2n-1} \longrightarrow * \times_{S^1} S^{2n-1} = \mathbb{C}P^{n-1}$$

and applying the splitting of $W \otimes W \otimes W \otimes W \otimes W \otimes W$ just obtained, we get the following equation in $K_{\mathbb{R}}(\mathbb{C}P^{n-1})$:

$$\begin{aligned} (\varphi[L])^6 &= \varphi\big([L]^6 + 5[L]^4 + 10[L]^2 + 10[L]^0 + 5[L]^{-2} + [L]^{-4}\big) \\ &= \varphi\big([L]^6 + 6[L]^4 + 15[L]^2 + 10[L]^0\big) \\ &= \varphi\big(\Psi^6_{\mathbb{C}}[L] + 6\Psi^4_{\mathbb{C}}[L] + 15\Psi^2_{\mathbb{C}}[L] + 10\Psi^0_{\mathbb{C}}[L]\big). \end{aligned}$$

This story has an analogue in the world of symmetric polynomials:

$$(t_1 + t_2)^6 = t_1^6 + 6t_1^5t_2 + 15t_1^4t_2^2 + 20t_1^3t_2^3 + 15t_1^2t_2^4 + 6t_1t_2^5 + t_2^6$$

$$= (t_1^6 + t_2^6) + 6(t_1t_2)(t_1^4 + t_2^4) + 15(t_1^2t_2^2)(t_1^2 + t_2^2) + 20t_1^3t_2^3\,.$$

Let's read this as an equation for operations on 2-dimensional real vector bundles – more accurately, for operations taking us from isomorphism classes of such vector bundles E on X to $K_{\mathbb{R}}(X)$:

$$[E]^6 = \Psi^6_{\mathbb{R}}E + 6(\Lambda^2 E)(\Psi^4_{\mathbb{R}}E) + 15(\Lambda^2 E)^2(\Psi^2_{\mathbb{R}}E) + 20(\Lambda^2 E)^3\,.$$

If E happens to be the underlying real vector bundle of a complex line bundle, then $\Lambda^2 E$ is a trivial line bundle and our formula simplifies to

$$[E]^6 = \Psi^6_{\mathbb{R}}E + 6\Psi^4_{\mathbb{R}}E + 15\Psi^2_{\mathbb{R}}E + 20 = \Psi^6_{\mathbb{R}}E + 6\Psi^4_{\mathbb{R}}E + 15\Psi^2_{\mathbb{R}}E + 10\Psi^0_{\mathbb{R}}E\,.$$

If E is the underlying real vector bundle of the complex line bundle L on $X = \mathbb{C}P^{n-1}$, we can combine this with the earlier computation and we get

$$\big(\varphi(\Psi^6_{\mathbb{C}} + 6\Psi^4_{\mathbb{C}} + 15\Psi^2_{\mathbb{C}} + 10\Psi^0_{\mathbb{C}})\big)[L] = \big((\Psi^6_{\mathbb{R}} + 6\Psi^4_{\mathbb{R}} + 15\Psi^2_{\mathbb{R}} + 10\Psi^0_{\mathbb{R}})\varphi\big)[L].$$

Using this equation, we can deduce $\varphi\Psi^k_{\mathbb{C}}[L] = \Psi^k_{\mathbb{R}}\varphi[L]$ for $k = 6$ if we know $\varphi\Psi^k_{\mathbb{C}}[L] = \Psi^k_{\mathbb{R}}\varphi[L]$ for $k < 6$. More generally, we can use this method to prove $\varphi\Psi^k_{\mathbb{C}}[L] = \Psi^k_{\mathbb{R}}\varphi[L]$ by induction on k. The induction beginning has to deal with the cases $k = 0$ and $k = 1$; these are easy. This completes the proof of lemma 4.5.5. $\square$

Now for the proof of proposition 4.5.1 in the case where y is represented by a 2-dimensional vector bundle. If that vector bundle is orientable, then we can write $y = \varphi(z)$ where z is represented by a complex line bundle L on X. Then $\Psi^k_{\mathbb{C}}([L]) = [L^{\otimes k}]$ and so $\Psi^k_{\mathbb{R}}(y)$ is also represented by $L^{\otimes k}$ (tensor power taken over $\mathbb{C}$) by lemma 4.5.5. There is a map

$$L \smallsetminus \text{zero section} \quad\longrightarrow\quad L^{\otimes k} \smallsetminus \text{zero section}$$

over X defined by $v \mapsto v \otimes v \otimes \cdots \otimes v$. This is of degree k in the fibers. Therefore proposition 4.5.4 can be applied. The result is that the images of y and $\Psi^k(y)$ in $K_F(X) \otimes_{\mathbb{Z}} \mathbb{Z}[k^{-1}]$ are the same, which confirms proposition 4.5.1 in this case.

If the 2-dimensional vector bundle which we have chosen to represent y is not orientable, then $y = y_1 y_2$ where y_1 is represented by a real line bundle $H \to X$ and y_2 is represented by the underlying real vector bundle of a complex line bundle $L \to X$. Then

$$\Psi^k(y) = \Psi^k(y_1)\Psi^k(y_2).$$

If k is odd, then $\psi^k(y_1) = y_1$ and so $\Psi^k(y)$ is represented by $H \otimes_{\mathbb{R}} L^{\otimes k}$, where $L^{\otimes k}$ denotes the complex tensor power. As before we can construct a map

$$H \otimes_{\mathbb{R}} L \smallsetminus \text{zero section} \quad\longrightarrow\quad H \otimes_{\mathbb{R}} L^{\otimes k} \smallsetminus \text{zero section}$$

over X, fiberwise of degree $\pm k$. It is defined by $w \otimes v \mapsto w \otimes v \otimes v \otimes \cdots \otimes v$. So proposition 4.5.4 can be applied again. — If k is even, then $\Psi^k(y_1) = 1$ but we are allowed to multiply by high powers of 2:

$$2^e(\Psi^k(y) - y) = 2^e(\Psi^k(y_2) - y_1 y_2) = 2^e(\Psi^k(y_2) - y_2)$$

since we have $2^e y_1 = 2^e$ for $e \gg 0$. Multiplying the right-hand expression by a suitable power of k, we get 0 in $J(X) \subset K_F(X)$ by what we have already shown (case of an orientable 2-dimensional vector bundle). This completes the proof. $\square$

4.6. Adams operations and representation theory

In the definition of the Adams operations, we used certain functors from finite dimensional vector spaces (over $\mathbb{R}$ or $\mathbb{C}$) to finite dimensional vector spaces, such as the i-th exterior power, $V \mapsto \Lambda^i V$. For a better understanding of the properties of the Adams operations, such as multiplicativity, we need to undertake a more systematic investigation of such functors from finite dimensional vector spaces to finite dimensional vector spaces. It should be general enough to include the examples $V \mapsto \Lambda^i V$, but also, for any two examples which are allowed, their direct sum (value-wise), their tensor product (value-wise), and maybe their composition. Let us concentrate on the real case to start with.

Definition 4.6.1. A k-*symmetric functor* from finite dimensional real vector spaces to finite dimensional real vector spaces (my terminology) is a rule which to every f.d. real vector space V associates a f.d. real vector space $F(V)$, and to every pair of such vector spaces V, W, a linear map

$$\Big(\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(V, W)\Big)^{\Sigma_k} \otimes F(V) \longrightarrow F(W)$$

such that an obvious associativity condition and an obvious unit condition are satisfied. See also remark 4.6.3 below. (The symbol Σ_k in the superscript position means that we are taking the fixed points for the action of Σ_k which permutes the tensor factors.)

Example 4.6.2. Take $k = 2$ and let $f, g \colon V \to W$ be linear maps. Then

$$f \otimes g + g \otimes f \in \Big(\bigotimes_{j=1}^{2} \hom_{\mathbb{R}}(V, W)\Big)^{\Sigma_2}$$

so that, if F is any 2-symmetric functor as in definition 4.6.1, then it has to give us a linear map named

$$(f \otimes g + g \otimes f)_* \colon F(V) \longrightarrow F(W).$$

Remark 4.6.3. A k-symmetric functor F can also be viewed as a functor in the ordinary sense. Indeed, a linear map $f\colon V \to W$ gives rise to an element

$$f \otimes f \otimes \cdots \otimes f \in \Big(\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(V, W) \Big)^{\Sigma_k}$$

which induces $(f \otimes f \otimes \cdots \otimes f)_*\colon F(V) \to F(W)$. (It is tempting, but it would not be helpful, to rename this f_* .)

But in some sense we are taking the opposite view here. The finite dimensional vector spaces over $\mathbb{R}$ can be made into a category so that the set of morphisms from V to W is

$$\Big(\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(V, W) \Big)^{\Sigma_k}$$

(with an obvious convention for composition of such morphisms). A k-symmetric functor is an endofunctor of this enlarged category, satisfying an additional multilinearity condition.

Example 4.6.4. The functor $V \mapsto F(V) := \bigotimes_{j=1}^{k} V$ has the structure of a k-symmetric functor. Indeed every element of

$$\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(V, W)$$

induces a linear map $F(V) \to F(W)$ in an obvious way; a fortiori this can be said for elements of

$$\Big(\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(V, W) \Big)^{\Sigma_k}.$$

Example 4.6.5. The functor $V \mapsto F(V) := \bigotimes_{j=1}^{k} V$ has the structure of a k-symmetric functor. Indeed every element of

$$\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(V, W)$$

induces a linear map $F(V) \to F(W)$ in an obvious way; a fortiori this can be said for elements of

$$\Big(\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(V, W) \Big)^{\Sigma_k}.$$

Example 4.6.6. The functor

$$V \mapsto F(V) := \mathrm{sym}^k V := \Big(\bigotimes_{j=1}^{k} V\Big)^{\Sigma_k}$$

(k-th symmetric power) has the structure of a k-symmetric functor. It can be viewed as a k-symmetric *subfunctor* of the functor in example 4.6.5.

Example 4.6.7. The functor $V \mapsto F(V) := \Lambda^k V$, k-th exterior power, has the structure of a k-symmetric functor. It can be viewed as a k-symmetric quotient functor of the functor in example 4.6.5. (It can also be viewed as a subfunctor of the functor in example 4.6.5, but here I prefer to take the quotient functor view.) This may seem counterintuitive; we may feel tempted to say that it is k-*antisymmetric*. But let's do an experiment. Take $k = 3$ and take vector spaces V, W and take $v_1, v_2, v_3 \in V$ and take linear maps $f_1, f_2, f_3 \colon V \to W$. We need to make sense of

$$\Big(\sum_{\sigma \in \Sigma_3} f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes f_{\sigma(3)}\Big)(v_1 \wedge v_2 \wedge v_3)$$

as an element of $\Lambda^k W$. We obviously try to make sense of it by writing

$$(*) \qquad \sum_{\sigma \in \Sigma_3} f_{\sigma(1)}(v_1) \wedge f_{\sigma(2)}(v_2) \wedge f_{\sigma(3)}(v_3)$$

instead. But is this well defined? In $\Lambda^3 V$ we have certain relations, for example, $v_1 \wedge v_2 \wedge v_3 = -v_2 \wedge v_1 \wedge v_3$. What if we plug in $-v_2 \wedge v_1 \wedge v_3$ instead of $v_1 \wedge v_2 \wedge v_3$? Then we get

$$(**) \qquad -\sum_{\sigma \in \Sigma_3} f_{\sigma(1)}(v_2) \wedge f_{\sigma(2)}(v_1) \wedge f_{\sigma(3)}(v_3).$$

Fortunately this is the same as $(*)$, due to the fact that certain relations hold in $\Lambda^3 W$.

Proposition 4.6.8. (i) *If* F_1 *and* F_2 *are* k*-symmetric functors, then the functor* $F_1 \oplus F_2$ *, defined more precisely by* $V \mapsto F_1(V) \oplus F_2(V)$*, is again a* k*-symmetric functor.*

(ii) *If* F_1 *is a* k*-symmetric functor and* F_2 *is an* ℓ*-symmetric functor, then the functor* $F_1 \otimes F_2$ *, defined more precisely by* $V \mapsto F_1(V) \otimes_{\mathbb{R}} F_2(V)$*, is a* $(k+\ell)$*-symmetric functor.*

(iii) *If* F_1 *is a* k*-symmetric functor and* F_2 *is an* ℓ*-symmetric functor, then* $F_2 \circ F_1$ *, defined more precisely by* $V \mapsto F_2(F_1(V))$*, is a* $k\ell$*-symmetric functor.*

Proof. Claim (i) should be clear. Claim (ii) comes from writing U for $\hom(V, W)$ and observing

$$\Big(\bigotimes_{j=1}^{k+\ell} U\Big)^{\Sigma_{k+\ell}} \subset \Big(\bigotimes_{j=1}^{k} U\Big)^{\Sigma_k} \otimes \Big(\bigotimes_{j=1}^{\ell} U\Big)^{\Sigma_\ell} \cong \Big(\bigotimes_{j=1}^{k+\ell} U\Big)^{\Sigma_k \times \Sigma_\ell}$$

which uses an inclusion or embedding $\Sigma_k \times \Sigma_\ell \hookrightarrow \Sigma_{k+\ell}$. Claim (iii) follows similarly from an inclusion $\Sigma_\ell \times \Sigma_k \hookrightarrow \Sigma_{k\ell}$. This is obtained by viewing $\Sigma_{k\ell}$ as the group of permutations of $\{1, 2, \ldots, k\} \times \{1, 2, \ldots, \ell\}$, and then viewing $\Sigma_\ell \times \Sigma_k$ as the subgroup consisting of permutations having the form $(t_1, t_2) \mapsto (\sigma_1(t_1), \sigma_2(t_2))$ for $\sigma_1 \in \Sigma_k$ and $\sigma_2 \in \Sigma_\ell$. $\square$

We turn to the classification of k-symmetric functors. The ring

$$A_k = \Big(\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^k)\Big)^{\Sigma_k}$$

(an algebra over $\mathbb{R}$, to be precise) is important for the classification. For a k-symmetric functor F, the vector space $F(\mathbb{R}^k)$ has the structure of a left module over A_k by means of the evaluation map, which we can write in the abbreviated form $A_k \otimes_{\mathbb{R}} F(\mathbb{R}^k) \to F(\mathbb{R}^k)$.

Lemma 4.6.9. *For any finite dimensional real vector space* V*, the adjoint of the composition map*

$$\Big(\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(V, \mathbb{R}^k)\Big)^{\Sigma_k} \otimes_{A_k} \Big(\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(\mathbb{R}^k, V)\Big)^{\Sigma_k} \longrightarrow A_k$$

is an isomorphism

$$\Big(\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(\mathbb{R}^k, V)\Big)^{\Sigma_k} \longrightarrow \hom_{A_k}\Big(\Big(\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(V, \mathbb{R}^k)\Big)^{\Sigma_k}, A_k\Big).$$

Proof. Write $G = \Sigma_k$ and write $\mathbb{R}G$ for the group algebra; elements are formal linear combinations with real coefficients of elements from G. Write

$$X := \bigotimes_{j=1}^{k} \mathbb{R}^k\,, \qquad Y := \bigotimes_{j=1}^{k} V$$

and view both of these as representations of G by permuting the tensor factors. Then we have $A_k = \hom_{\mathbb{R}G}(X, X)$ by definition. In this notation, the claim of the lemma is that the adjoint of the composition map

$$\hom_{\mathbb{R}G}(Y, X) \otimes_{\hom_{\mathbb{R}G}(X,X)} \hom_{\mathbb{R}G}(X, Y) \longrightarrow \hom_{\mathbb{R}G}(X, X)$$

is an isomorphism

$$\hom_{\mathbb{R}G}\big(X,Y\big) \longrightarrow \hom_{\hom_{\mathbb{R}G}(X,X)}\Big(\hom_{\mathbb{R}G}\big(Y,X\big),\hom_{\mathbb{R}G}\big(X,X\big)\Big).$$

This turns out to be a general fact for representations of G. In other words, we can simplify things here by assuming that X and Y are arbitrary (finite dimensional real) representations of $G = \Sigma_k$. Then we can quickly reduce to the situation where X and Y are both irreducible and isomorphic to each other. (In the case where they are both irreducible but not isomorphic to each other, we get $\hom_{\mathbb{R}G}(Y,X) = 0$ and $\hom_{\mathbb{R}G}(X,Y) = 0$.) If they are isomorphic to each other, then we can also assume that $X = Y$. We get $\hom_{\mathbb{R}G}(X,X) = \mathbb{R}$ as a special case of Schur's lemma. (For more general finite G and an irreducible real representation of G, Schur's lemma would say that $\hom_{\mathbb{R}G}(X,X)$ is a finite dimensional division algebra over $\mathbb{R}$, so it could also be $\mathbb{C}$ or $\mathbb{H}$; but in the case $G = \Sigma_k$, it has to be $\mathbb{R}$.) This makes the verification easy. □

Theorem 4.6.10. *A k-symmetric functor F is fully determined (up to unique natural isomorphism) by the vector space $F(\mathbb{R}^k)$, with the structure of left module over A_k. Indeed we have*

$$F(V) \cong \Big(\bigotimes_{j=1}^{k}\hom_{\mathbb{R}}(\mathbb{R}^k,V)\Big)^{\Sigma_k} \otimes_{A_k} F(\mathbb{R}^k)\,.$$

Proof. Let V be any finite dimensional real vector space. It is easy to verify that the composition map

$$\Big(\bigotimes_{j=1}^{k}\hom_{\mathbb{R}}(\mathbb{R}^k,V)\Big)\otimes\Big(\bigotimes_{j=1}^{k}\hom_{\mathbb{R}}(V,\mathbb{R}^k)\Big)\longrightarrow\Big(\bigotimes_{j=1}^{k}\hom_{\mathbb{R}}(V,V)\Big)$$

is surjective. Then it follows (reader, explain) that the composition map

$$\Big(\bigotimes_{j=1}^{k}\hom_{\mathbb{R}}(\mathbb{R}^k,V)\Big)^{\Sigma_k}\otimes\Big(\bigotimes_{j=1}^{k}\hom_{\mathbb{R}}(V,\mathbb{R}^k)\Big)^{\Sigma_k}\longrightarrow\Big(\bigotimes_{j=1}^{k}\hom_{\mathbb{R}}(V,V)\Big)^{\Sigma_k}$$

is also surjective. This implies (reader, explain) that the evaluation map

$$\Big(\bigotimes_{j=1}^{k}\hom_{\mathbb{R}}(\mathbb{R}^k,V)\Big)^{\Sigma_k}\otimes F(\mathbb{R}^k)\longrightarrow F(V)$$

is surjective. This last composition map can be more efficiently set up in the form

$$(\$)\qquad \Big(\bigotimes_{j=1}^{k}\hom_{\mathbb{R}}(\mathbb{R}^k,V)\Big)^{\Sigma_k}\otimes_{A_k} F(\mathbb{R}^k)\longrightarrow F(V)$$

where we view

$$\Big(\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(\mathbb{R}^k, V)\Big)^{\Sigma_k}$$

as a right A_k-module and $F(\mathbb{R}^k)$ as a left A_k-module and we take the tensor product over A_k . We can of course still say that (\$) is surjective. Now there is another useful map

$$(\$\$) \qquad F(V) \longrightarrow \hom_{A_k}\Big(\Big(\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(V, \mathbb{R}^k)\Big)^{\Sigma_k}, F(\mathbb{R}^k)\Big),$$

adjoint to an evaluation map; here $\hom_{A_k}(\ldots)$ refers to homomorphisms of *left* A_k-modules. The composition $(\$\$) \circ (\$)$ is an isomorphism. (My idea is that this follows from lemma 4.6.9 by tensoring the isomorphism of the lemma with $F(\mathbb{R}^k)$, over A_k.) Therefore (\$) must be an isomorphism, too, and we end up with the formula

$$F(V) \;\cong\; \Big(\bigotimes_{j=1}^{k} \hom_{\mathbb{R}}(\mathbb{R}^k, V)\Big)^{\Sigma_k} \otimes_{A_k} F(\mathbb{R}^k). \quad \square$$

According to theorem 4.6.10, we classify k-symmetric functors by classifying modules over A_k . The classification of modules over A_k reduces easily to elementary representation theory of Σ_k over the field $\mathbb{R}$.

(*to be continued*)