1.12.2016/4.1.2017

Lecture notes chapter 3, WS 2015-2016 (Weiss): Vector bundles, J-homomorphism & Adams conjecture

This chapter will be, at best, a review/extract/summary of Atiyah's article *Algebraic topology and operators in Hilbert space* (1969). The article has a proof of Bott periodicity in complex K-theory which is based on elementary functional analysis. I am planning to add some remarks of a general nature and some remarks on Bott periodicity for real K-theory if I can sort it out.

3.1. Products in K-theory

For compact CW-spaces X and Y, the (external) tensor product of vector bundles leads to a natural homomorphism

$$\mathsf{K}(X) \otimes \mathsf{K}(Y) \longrightarrow \mathsf{K}(X \times Y)$$

called *external product*. (I am now using $\mathsf{K}(X)$ as an abbreviation for $\mathsf{K}_{\mathbb{C}}(X)$, and where I write *vector bundle* I probably mean *complex vector bundle*, unless otherwise specified.) In more detail, if $E \to X$ and $F \to Y$ are vector bundles, then we can make a vector bundle

$$E \boxtimes F \longrightarrow X \otimes Y$$

whose fiber over $(x, y) \in X \times Y$ is the vector space $E_x \otimes_{\mathbb{C}} F_y$. (The reason for not writing $E \otimes F$ is that we may want to reserve that notation for an *internal* tensor product.) In these terms, the above homomorphism of K-groups is defined and well defined by

$$[E, E'] \otimes [F, F'] \ \mapsto \ [\,(E \boxtimes F) \oplus (E' \boxtimes F'), (E \boxtimes F') \oplus (E' \boxtimes F)]$$

where E, E' are vector bundles on X and F, F' are vector bundles on Y. (Remember that we should think of $[E, E']$ as a formal *difference* $E - E'$. This justifies the surprising complexity of the formula, which is felt in practice as much as in theory.)

Next, suppose that X is a *based* compact CW-space; that is, X comes equipped with a base point which is also a 0-cell in the CW-structure. Previously we defined $\tilde{\mathsf{K}}(X)$ as the *cokernel* of the homomorphism from $\mathbb{Z} = \mathsf{K}(\star)$ to $\mathsf{K}(X)$ induced by the unique map $X \to \star$. Now we prefer to view it as the *kernel* of the homomorphism

$$\mathsf{K}(X) \to \mathsf{K}(\star) = \mathbb{Z}$$

induced by the inclusion of the base point, $* \hookrightarrow X$. This does not matter much — the two descriptions are related by a preferred isomorphism. But it is worth noting that for a based CW-space X we have a canonical splitting

$$\mathsf{K}(X) \cong \tilde{\mathsf{K}}(X) \oplus \mathsf{K}(\star) = \tilde{\mathsf{K}}(X) \oplus \mathbb{Z}$$

even if X is not (path) connected.

If Y is a also a based compact CW-space, then the external tensor product of vector bundles leads to a natural homomorphism

$$\tilde{K}(X) \otimes \tilde{K}(Y) \longrightarrow \tilde{K}(X \wedge Y).$$

(which could be called the *reduced* external product in K-theory) where $X \wedge Y$ is the quotient CW-space $(X \times Y)/(X \vee Y)$ (*smash product*). This homomorphism can be defined by "diagram chasing" in the commutative diagram of abelian groups

$$\begin{array}{ccc} & & K(X \vee Y) \\ & & \uparrow{\scriptstyle \text{restr.}} \\ K(X) \otimes K(Y) & \longrightarrow & K(X \times Y) \\ {\scriptstyle \text{incl.}}\uparrow & & \uparrow \\ \tilde{K}(X) \otimes \tilde{K}(Y) & \dashrightarrow & \tilde{K}(X \wedge Y) \end{array}$$

whose right-hand column is short exact.

Exercise 3.1.1. Explain why the right-hand column is short exact and do the diagram chase. *Some instructions*:

(i) Let P be a compact based CW-space, $Q \subset P$ a based CW-subspace which is a retract of P (i.e., there exists $r \colon P \to Q$ such that $r|_Q = \mathrm{id}_Q$). Let P/Q be the CW-quotient. Then

$$\tilde{K}(P) \cong \tilde{K}(Q) \oplus \tilde{K}(P/Q).$$

(ii) In particular $\tilde{K}(X \vee Y) \cong \tilde{K}(X) \oplus \tilde{K}(Y)$.

(iii) Show that $\tilde{K}(X \wedge Y) \to \tilde{K}(X \times Y) \to \tilde{K}(X \vee Y)$ is short exact.

3.2. The Bott element

The Bott element is an element $b \in \tilde{K}(S^2)$. It is represented by the formal difference

$$L - T$$

of 1-dimensional (complex) vector bundles on $S^2 = \mathbb{C}P^1$ where L is the *tautological line bundle* on $\mathbb{C}P^1$ and $T = S^2 \times \mathbb{C}$ is a trivial 1-dimensional vector bundle. (If we think of the points of $\mathbb{C}P^1$ as 1-dimensional $\mathbb{C}$-linear subspaces of $\mathbb{C}^2$, then the fiber of L over such a point V is the 1-dimensional complex vector space V itself. If we think of points of $\mathbb{C}P^1$ as unit vectors v in $\mathbb{C}^2$ modulo an equivalence relation, $v \sim cv$ for scalars $c \in \mathbb{C}$ of modulus 1, then the fiber of L over the point represented by v should be thought of as the 1-dimensional linear subspace of $\mathbb{C}^2$ spanned by v.)

Let's note that $\tilde{K}(S^2)$ is isomorphic to $\mathbb{Z}$ and b is a generator. The best way to see that may be to write

$$\tilde{K}(S^2) \cong \pi_2(\mathrm{BU}) \cong \pi_1\mathrm{U} \cong \mathbb{Z}$$

where $\mathrm{U} = \bigcup_{n\geq 0} \mathrm{U}(n)$ and $\mathrm{U}(n)$ is the group of unitary $\mathbb{C}$-linear automorphisms of $\mathbb{C}^n$.

Exercise 3.2.1. Why is $\pi_1\mathrm{U} \cong \mathbb{Z}$? More to the point, why is $\pi_1\mathrm{U}(n) \cong \mathbb{Z}$ for all $n \geq 1$?

Again let X be a based compact CW-space. External tensor product with $b \in \tilde{K}(S^2)$ gives us a homomorphism

$$\beta\colon \tilde{K}(X) \cong \tilde{K}(S^2) \otimes \tilde{K}(X) \longrightarrow \tilde{K}(S^2 \wedge X).$$

Theorem 3.2.2. (Bott periodicity in complex K-theory). *This homomorphism β is always an isomorphism.*

The proof, or rather the outline of a proof, will take up most of this chapter. — From a homotopical point of view, we should think of β not so much as a natural homomorphism relating reduced K-groups, but as a *map*. This can be written either in the form

$$S^2 \wedge \mathrm{BU} \longrightarrow \mathrm{BU}$$

or in the adjoint form

$$(*) \qquad\qquad \mathrm{BU} \longrightarrow \Omega^2\mathrm{BU}\,.$$

(*Notation*: $\Omega^n(-)$ is standard notation for a space of base-point preserving maps, $\mathrm{map}_*(S^n, -)$, so that the blank "$-$" is to be filled by a based space. Use the compact-open topology on $\mathrm{map}_*(S^n, -)$. We often identify S^n with the n-fold smash product $S^1 \wedge S^1 \wedge \cdots \wedge S^1$; then we may also think of Ω^n as the n-fold iteration of Ω^1.) The map $(*)$ should make the following diagram(s) commutative (for any based *connected* CW-space X):

$$(**) \qquad \begin{array}{ccc} \tilde{K}(X) & \xrightarrow{\quad\beta\quad} & \tilde{K}(S^2 \wedge X) \\ \cong\uparrow & & \uparrow\cong \\ [X, \mathrm{BU}]_* & \dashrightarrow & [S^2 \wedge X, \mathrm{BU}]_* \cong [X, \Omega^2\mathrm{BU}]_* \end{array}$$

where $[-,-]_*$ denotes homotopy classes of based maps. The dotted horizontal arrow is given by composition with $(*)$.

Corollary 3.2.3. *The map $(*)$ gives a homotopy equivalence from BU to the base point (path) component of $\Omega^2\mathrm{BU}$. The homotopy groups of BU are given by $\pi_k(\mathrm{BU}) \cong \mathbb{Z}$ for even $k > 0$, whereas $\pi_k(\mathrm{BU}) = 0$ for odd $k > 0$.*

Proof. ... modulo Bott perodicity. The Bott periodicity theorem together with diagram $(**)$ implies that $(*)$ induces isomorphisms on all homotopy groups except π_0. (Take $X = S^1, S^2, S^3, \ldots$ in the diagram, but not $X = S^0$ because we must have a connected X.) Therefore the map $(*)$ is a homotopy equivalence (after selecting the base point component in the target) by the JHC Whitehead theorem. The computation of the homotopy groups of BU follows by induction on k. We begin with the knowledge that $\pi_1(BU) \cong \pi_0 U \cong 0$ and $\pi_2(BU) \cong \pi_1 U \cong \mathbb{Z}$. □

Remark 3.2.4. The map $(*)$ is not immensely difficult to understand. If we think of BU as a direct limit of (complex) Grassmannians $\mathrm{Grm}(p,q)$, then we should construct compatible maps $S^2 \wedge \mathrm{Grm}(p,q) \to BU$. The tautological vector bundle of fiber dimension p on $\mathrm{Grm}(p,q)$ defines an element

$$z_{p,q} \in \tilde{K}(\mathrm{Grm}(p,q))$$

(after formal subtraction of a trivial vector bundle of fiber dimension p). This is permitted notation because $\mathrm{Grm}(p,q)$ is compact. Then we have

$$\beta(z_{p,q}) \in \tilde{K}(S^2 \wedge \mathrm{Grm}(p,q)) \cong [S^2 \wedge \mathrm{Grm}_{p,q}, BU]_* \cong [\mathrm{Grm}_{p,q}, \Omega^2 BU]_* .$$

This gives us a distinguished homotopy class of maps

$$\mathrm{Grm}_{p,q} \longrightarrow \Omega^2 BU$$

for each p, q. These homotopy classes satisfy the compatibility conditions that one might reasonably wish for. Unfortunately this fact does not give us a well defined map (or even a well defined homotopy class of maps) from the union or direct limit $BU = \bigcup_{p,q} \mathrm{Grm}(p,q)$ to $\Omega^2 BU$, but it does suggest that there should be such a map. We shall return to this slightly tricky issue later in this chapter with another "model" for BU.

3.3. Fredholm operators on Hilbert space

Hilbert space will mean a complex vector space $\mathbb{H}$ with hermitian inner product $\langle -, - \rangle$ such that

- $\mathbb{H}$ is a Banach space (complete normed vector space) with the norm
$$\|v\| = \sqrt{\langle v, v \rangle};$$
- $\mathbb{H}$ is infinite dimensional and *separable*. This means that there is a subset S of $\mathbb{H}$ which is countably infinite, satisfies $\|s\| = 1$ for all $s \in S$ and $\langle s, t \rangle = 0$ whenever $s, t \in S$ are distinct, and the $\mathbb{C}$-linear subspace of $\mathbb{H}$ spanned by S is dense in $\mathbb{H}$. (Such a subset S is probably called a *Hilbert basis* for $\mathbb{H}$, but it is obviously not a vector space basis in the usual sense.)

These conditions imply that every $v \in \mathbb{H}$ can be uniquely written in the form

$$v = \sum_{s \in S} a_s s$$

with coefficients $a_s \in \mathbb{C}$ which satisfy $\sum_s |a_s|^2 < \infty$. To put it somewhat differently, the conditions imply that $\mathbb{H}$ is isomorphic (as a complex vector space with inner product) to the vector space ℓ^2 of all sequences $(a_k)_{k=0,1,2,\ldots}$ with $a_k \in \mathbb{C}$, subject to the condition that $\sum_k |a_k|^2 < \infty$ and with hermitian inner product given by

$$\langle (a_k), (b_k) \rangle := \sum_k a_k \bar{b}_k \ \in \mathbb{C}\,.$$

Example 3.3.1. Let $V = C^0([0,1])$ be the vector space of all complex valued continuous functions from $[0,1]$ to $\mathbb{C}$. This has a hermitian inner product given by

$$\langle f, g \rangle := \int_0^1 f(x) \cdot (g(x))^- \, dx\,.$$

That does not make V into a Hilbert space, but it does make V into a normed vector space. The completion of V with respect to that norm is a Hilbert space $L^2([0,1])$. It has a Hilbert basis given by the functions g_k where $k \in \mathbb{Z}$ and $g_k(x) = \exp(2k\pi i x)$. (This seems to be the most basic theorem of Fourier analysis.)

Exercise 3.3.2. (*The old exercise that Euler failed to do.*) Let $f(x) = x$ for $x \in [0,1]$. Express f in the Hilbert basis of example 3.3.1 and use this expression to calculate $\|f\|^2$ in a roundabout way. Deduce a famous formula (due to Euler, who used a more complicated argument).

A *linear operator* on $\mathbb{H}$ will usually mean a continuous linear map A from $\mathbb{H}$ to $\mathbb{H}$. It is well known that continuity of A is equivalent to *boundedness* (on the unit disk of $\mathbb{H}$). In other words, A is continuous iff there exists $r \geq 0$ such that $A(v) \leq r$ for all $v \in \mathbb{H}$ of norm ≤ 1. The minimal such r is the *norm* of A. This notion of norm makes the (complex) vector space of all linear operators on $\mathbb{H}$ into a Banach space (complete normed vector space). Standard notation: $\mathcal{B}(\mathbb{H})$ (maybe other fonts).

The *adjoint* of an operator $A \in \mathcal{B}(\mathbb{H})$ is $A^* \in \mathcal{B}(\mathbb{H})$ determined uniquely by the equation $\langle Av, w \rangle = \langle v, A^* w \rangle$, valid for all $v, w \in \mathbb{H}$. Note that $A^{**} = A$ and $(AB)^* = B^* A^*$ for $A, B \in \mathcal{B}(\mathbb{H})$. An operator $A \in \mathcal{B}(\mathbb{H})$ is *self-adjoint* if $A^* = A$.

Proposition 3.3.3. $(\ker A)^\perp =$ *closure of* $\mathrm{im}(A^*)$ *in* $\mathbb{H}$. □

Definition 3.3.4. An operator $A \in \mathcal{B}(\mathbb{H})$ is *of finite rank* if its image is a finite-dimensional linear subspace of $\mathbb{H}$. An operator $A \in \mathcal{B}(\mathbb{H})$ is *compact* if it belongs the closure of the linear subspace of $\mathcal{B}(\mathbb{H})$ given by the operators of finite rank. (In other words, A is a compact operator iff, for every $\varepsilon > 0$, there exists a finite rank operator $B \in \mathcal{B}(\mathbb{H})$ such that $\|A - B\| < \varepsilon$. An equivalent condition: the closure of $\{A(v) \mid \|v\| \leq 1\}$ in $\mathbb{H}$ is compact. That's a little theorem.)

Exercise 3.3.5. (Fredholm.) Let $\kappa\colon [0,1] \times [0,1] \to \mathbb{C}$ be a continuous function. Define an operator A on the Hilbert space $L^2([0,1])$ by

$$A(f)(x) = \int_0^1 \kappa(x,y) \cdot f(y)\ dy$$

for $f \in L^2([0,1])$ and $x \in [0,1]$. Show that A is a compact operator and that its adjoint A^* is given by

$$A^*(f)(x) = \int_0^1 \kappa(y,x)^- \cdot f(y)\ dy.$$

(Such an operator A is called an *integral operator* and the function κ is often called a *kernel*, although that's a little unhelpful for us.)

Definition 3.3.6. An operator $A \in \mathcal{B}(\mathbb{H})$ is a *Fredholm operator* if $\mathrm{im}(A)$ is a closed (linear) subspace of $\mathbb{H}$ and $\ker(A), \ker(A^*)$ are both finite dimensional. The integer

$$\dim(\ker(A)) - \dim(\ker(A^*))$$

is the *index* of the Fredholm operator A.

Exercise 3.3.7. The *closed graph* theorem for Banach spaces states that a linear map $A : V_1 \to V_2$ between (complex) Banach spaces V_1, V_2 is continuous if its graph is closed as a subset of $V_1 \times V_2$. (No proof given here. Note that the converse is obvious: if A continuous, then the graph is closed.) Use this to show that a (continuous) operator A on $\mathbb{H}$ which has a finite dimensional kernel and a finite dimensional cokernel has a closed image. — It follows that $A \in \mathcal{B}(\mathbb{H})$ is a Fredholm operator if and only if it has finite dimensional kernel and finite dimensional cokernel. The index of A is the dimension of the kernel minus the dimension of the cokernel.

Example 3.3.8. In $\mathbb{H} = \ell^2$, the "shift" operator P_n given by

$$P_n(a_0, a_1, a_2, \dots) = (a_n, a_{n+1}, \dots)$$

is clearly surjective and has adjoint P_n^* given by

$$P_n^*(a_0, a_1, a_2, a_3, \dots) = (0, 0, ..., 0, 0, a_0, a_1, a_2, a_3, \dots).$$

Then $\ker(P_n^*) = 0$ while $\ker(P_n)$ is n-dimensional. Therefore P_n is a Fredholm operator and the index of P_n is n.

Exercise 3.3.9. Let A be the operator on the Hilbert space $\mathbb{H}$ given by $A(f)(x) := xf(x)$ for $f \in L^2([0,1])$ and $x \in [0,1]$. This is self-adjoint. Is it a Fredholm operator? If so, what is its index?

Let A be a Fredholm operator on $\mathbb{H}$. Let $V = \ker(A)$ and $W = \ker(A^*)$. Then A is determined by its restriction to $V^\perp$, which we can view as a continuous linear bijection

$$A_{22} : V^\perp \to W^\perp .$$

By the closed graph theorem (see exercise 3.3.7), that continuous linear bijection A_{22} is continuously invertible. Now let B be any operator on $\mathbb{H}$ whose norm is less than $1/\|(A_{22})^{-1}\|$. If we use the splittings to write A and B in block/matrix form as linear maps from $V \oplus V^\perp$ to $W \oplus W^\perp$, then we have

$$A = \begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \qquad A + B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & A_{22} + B_{22} \end{bmatrix}$$

where $A_{22} + B_{22}$ is still invertible. This makes it clear that the restriction of $A + B$ to $V^\perp$ is still injective, and the composition of $A + B \colon \mathbb{H} \to \mathbb{H}$ with the projection $\mathbb{H} \to \mathbb{H}/W$ is still surjective. It follows that $A + B$ is still a Fredholm operator.

To solve $(A + B)(v) = 0$ we write $v = v_1 + v_2$ where $v_1 \in V$ and $v_2 \in V^\perp$, so that we get a system of two equations

$$B_{11}(v_1) + B_{12}(v_2) = 0 \in W, \qquad B_{21}(v_1) + (A_{22} + B_{22})(v_2) = 0 \in W^\perp .$$

This teaches us that $v_2 = -(A_{22} + B_{22})^{-1} B_{21}(v_1)$, and then

$$\big(B_{11} - B_{12}(A_{22} + B_{22})^{-1} B_{21}\big)(v_1) = 0 .$$

Therefore $\ker(A + B)$ is isomorphic (not claimed to be equal) to

$$\ker\Big(B_{11} - B_{12}(A_{22} + B_{22})^{-1} B_{21} \colon V \to W\Big).$$

A similar calculation shows that $\ker(A^* + B^*)$ is isomorphic to

$$\ker\Big(\big(B_{11} - B_{12}(A_{22} + B_{22})^{-1} B_{21}\big)^* \colon W \to V\Big).$$

Now it follows easily that the index of $A + B$ is $\dim(V) - \dim(W)$, and therefore equal to the index of A. Therefore we have shown:

Proposition 3.3.10. *The space of Fredholm operators is an open subspace of $\mathcal{B}(\mathbb{H})$. The index is a continuous function from that subspace to $\mathbb{Z}$.* □

Theorem 3.3.11. (Fredholm.) *Let A be a compact operator on $\mathbb{H}$. Then $\mathrm{id}_{\mathbb{H}} - A$ is a Fredholm operator and its index is 0.*

Proof. Write $W = \ker(\mathrm{id} - A)$. Assume for a contradiction that W is infinite dimensional. It is a closed subspace of $\mathbb{H}$. Let $D(\mathbb{H})$ be the unit disk of $\mathbb{H}$. Then $A(D(\mathbb{H}))$ contains $D(W) = D(\mathbb{H}) \cap W$, a closed subset of $\mathbb{H}$ which is not compact (theorem of Riesz). This contradicts one of the definitions of a compact operator.

Therefore $W = \ker(\mathrm{id} - A)$ is finite dimensional. Similarly, the kernel of $(\mathrm{id} - A)^* = \mathrm{id} - A^*$ is finite dimensional (since A^* is again a compact operator). It remains to show that $\mathrm{im}(\mathrm{id} - A)$ is a closed linear subspace of $\mathbb{H}$. Choose a sequence

$$v_0, v_1, v_2, v_3, \ldots$$

in $W^\perp$ such that $((\mathrm{id} - A)(v_i))_{i \geq 0}$ converges to some $u \in \mathbb{H}$. We need to show $u \in \mathrm{im}(\mathrm{id} - A)$. If the sequence of numbers $\|v_i\|$ is not bounded, then we can assume wlog that

$$\lim_{k \to \infty} \|v_k\| = \infty$$

(by selecting a subsequence) and we find that

$$\lim_{k \to \infty} (\mathrm{id} - A)(v_k / \|v_k\|) = 0\,.$$

Wlog, the limit

$$\lim_{k \to \infty} A(v_k / \|v_k\|)$$

exists (otherwise select a subsequence, using compactness of operator A) and it follows that

$$\lim_{k \to \infty} v_k / \|v_k\|$$

also exists, and belongs to $W = \ker(\mathrm{id} - A)$. This contradicts the assumption that $v_k / \|v_k\|$ are unit vectors in $W^\perp$. Therefore our assumption that the sequence of numbers $\|v_i\|$ is not bounded was erroneous. It is a bounded sequence. Therefore we may assume wlog that

$$\lim_{k \to \infty} A(v_i)$$

exists, using the compactness of operator A to select a subsequence if necessary. But then

$$v_\infty = \lim_{k \to \infty} v_i$$

also exists and we have $u = (\mathrm{id} - A)(v_\infty)$.

Finally we have to show that the index of $\mathrm{id} - A$ is zero. But this is clear since we have a continuous function taking $t \in [0, 1]$ to the index of $\mathrm{id} - tA$, an integer. □

Fredholm's theorem 3.3.11 has a more abstract formulation which also looks more general. For that we observe that $\mathcal{B}(\mathbb{H})$ is a Banach *algebra*: operators from $\mathbb{H}$ to $\mathbb{H}$ can be composed (the product in $\mathcal{B}(\mathbb{H})$ is composition). In the Banach algebra $\mathcal{B}(\mathbb{H})$, the compact operators form a two-sided ideal $\mathcal{K}$. This is fairly clear from any of the two definitions of *compact operator*. Therefore we can form the quotient algebra $\mathcal{B}(\mathbb{H})/\mathcal{K}$, for the time being without any discussion of a topology. This is known as the *Calkin* algebra.

Corollary 3.3.12. *An operator* $A \in \mathcal{B}(\mathbb{H})$ *is a Fredholm operator if and only if its class in* $\mathcal{B}(\mathbb{H})/\mathcal{K}$ *is invertible.*

Exercise 3.3.13. Deduce corollary 3.3.12 from theorem 3.3.11. (Note also that theorem 3.3.11 is a trivial consequence of corollary 3.3.12.)

Exercise 3.3.14. Let $A, B \in \mathcal{B}(\mathbb{H})$ be Fredholm operators. Show that $A \circ B$ is a Fredholm operator whose index is index of A plus index of B. (*Hint*: arrange the kernels and cokernels of A, B and $A \circ B$ in an exact sequence.)

3.4. The theorems of Kuiper and Atiyah-Jänich

Let $\mathrm{GL}(\mathbb{H}) \subset \mathcal{B}(\mathbb{H})$ be the subspace consisting of the *invertible* operators. We can think of $\mathrm{GL}(\mathbb{H})$ as a topological group.

Theorem 3.4.1. (N Kuiper.) $\mathrm{GL}(\mathbb{H})$ *is contractible.*

This is considered easy, but I will not attempt a proof. Let us instead use the theorem to learn something about Fredholm operators. A Fredholm operator $A\colon \mathbb{H} \to \mathbb{H}$ determines finite-dimensional linear subspaces $V = \ker(A)$ and $W = \ker(A^*)$ and splittings $\mathbb{H} = V \oplus V^\perp$, $\mathbb{H} = W \oplus W^\perp$ as in the proof of proposition 3.3.10. Therefore, viewing A as a linear map from $V \oplus V^\perp$ to $W \oplus W^\perp$, we have

$$A = \begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix}$$

where $A_{22}\colon V^\perp \to W^\perp$ is (continuous and) invertible. Since $V^\perp$ and $W^\perp$ are Hilbert spaces in their own right (they are as such isomorphic to our preferred $\mathbb{H}$, whatever that may be), the *space* of such invertible continuous linear maps $V^\perp \to W^\perp$ is contractible according to Kuiper's theorem. Therefore, if we feel like "making" a Fredholm operator A, our task is mainly to choose finite dimensional linear subspaces $\ker(A) = V$ and $\ker(A^*) = W$ of $\mathbb{H}$; after that, it remains only to choose $A_{22}\colon V^\perp \to W^\perp$, a *contractible choice*. The more serious choices V and W remind us of the definition of K-theory in terms of a Grothendieck group, of Grassmannians and all that. In this way Kuiper's theorem leads rather inexorably to a comparison between $\mathcal{F} \subset \mathcal{B}(\mathbb{H})$, the space of Fredholm operators, and BU.

Theorem 3.4.2. *(Atiyah-Jänich.) The space* $\mathcal{F}$ *of Fredholm operators is homotopy equivalent to* $\mathrm{BU} \times \mathbb{Z}$.

(I shall only prove *weakly homotopy equivalent*. To get the stronger statement from the weaker one, we would need to know in addition that $\mathcal{F}$ is homotopy equivalent to a CW-space.)

Proof. I have decided to give a baaad proof: details obscure, intention clear. It took all my criminal energy. — Let $\mathbb{H}$ be our standard Hilbert space with Hilbert basis $\{e_0, e_1, e_2, \dots\}$ and let $\mathbb{H}'$ be another Hilbert space with Hilbert basis $\{e_{-1}, e_{-2}, e_{-3}, \dots\}$. Let $\mathcal{F}$ be the space of Fredholm operators on $\mathbb{H}$ and let $\mathcal{F}^\sharp$ be the space of all *surjective* continuous linear maps

$$A\colon \mathbb{H}' \oplus \mathbb{H} \to \mathbb{H}$$

with the following additional property:

$A|_{\mathbb{H}}$ is Fredholm.

Our model for $BU \times \mathbb{Z}$ is as follows: it is the space of all closed linear subspaces L of $\mathbb{H}' \oplus \mathbb{H}$ such that the composition

$$\mathbb{H} \xrightarrow{\text{incl.}} \mathbb{H}' \oplus \mathbb{H} = L \oplus L^\perp \xrightarrow{\text{proj.}} L^\perp$$

is Fredholm (has finite dimensional kernel and cokernel). This opportunistic choice of model makes it immediately clear that we have a forgetful map

$$\varphi\colon \mathcal{F}^\sharp \longrightarrow BU \times \mathbb{Z}$$

given by $A \mapsto \ker(A)$. (Write $\ker(A) =: L$ and $A|_{\mathbb{H}} = A_1 Q$ where A_1 is the restriction of A to $L^\perp$ and Q is the restriction to $\mathbb{H}$ of the orthogonal projection to $L^\perp$. Since $A_1\colon L^\perp \to \mathbb{H}$ is invertible and $A|_{\mathbb{H}}$ is Fredholm, it follows that Q is Fredholm.) It is not hard to show that φ is a fibration and, by Kuiper's theorem and the closed graph theorem, the fibers are contractible. (The fiber $\varphi^{-1}(L)$ is identified with the space of continuous linear bijections from the orthogonal complement of L in $\mathbb{H}' \oplus \mathbb{H}$ to $\mathbb{H}$.) Therefore φ is a homotopy equivalence.

There is another forgetful map $\rho\colon \mathcal{F}^\sharp \to \mathcal{F}$ given by restriction: $A \mapsto A|_{\mathbb{H}}$. We need to show that this is a weak homotopy equivalence. Let us look at the fibers. For $A \in \mathcal{F}$, to construct an element in $\rho^{-1}(A)$ we need to construct a continuous linear $B\colon \mathbb{H}' \to \mathbb{H}$ such that $\mathrm{im}(B) + \mathrm{im}(A) = \mathbb{H}$. Write $B = B_1 + B_2$ where $B_1\colon \mathbb{H}' \to \mathrm{im}(A)$ and $B_2\colon \mathbb{H}' \to \mathrm{im}(A)^\perp \subset \mathbb{H}$. There is no condition on B_1. It follows that the choice of B_1 is a contractible choice. But B_2 needs to be onto. The adjoint of B_2 is an *injective* linear map from the finite dimensional $\mathrm{im}(A)^\perp$ to the infinite dimensional $\mathbb{H}'$. Now we see that B_2 is also a contractible choice. So the fibers of ρ are contractible.

Now there is a magic little criterion which says: *a Serre microfibration with contractible fibers is a Serre fibration — still with contractible fibers, therefore a weak equivalence.* Let me decode the jargon. When we say that a map $p\colon E \to X$ is a *fibration*, we mean to say that it has the homotopy lifting property:

$$\begin{array}{ccc} Y \times \{0\} & \longrightarrow & E \\ {\scriptstyle \text{incl.}}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ Y \times [0,1] & \longrightarrow & X \end{array}$$

When we say *Serre* fibration, it means that we are satisfied if the homotopy lifting property can be established in the cases where Y is a compact CW-space. When we say *Serre microfibration*, it means that we are already satisfied if Y is a compact CW-space and the dotted arrow is only defined on $Y \times [0, \varepsilon]$ for some $\varepsilon > 0$, which may depend on Y and the other data in the homotopy lifting problem.

So it only remains to show that ρ is a Serre microfibration. But this is easy. Let us try a homotopy lifting problem

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\;g\;} & \mathcal{F}^\sharp \\ {\scriptstyle \text{incl.}}\downarrow & \nearrow & \downarrow \\ Y \times [0,1] & \xrightarrow{\;f\;} & \mathcal{F} \end{array}$$

where Y is a compact CW-space. For $y \in Y$ write $g(y,0) = e(y) + f(y,0)$ where $e(y)\colon \mathbb{H}' \to \mathbb{H}$ is a continuous linear map (so that $f(y,0)\colon \mathbb{H} \to \mathbb{H}$ and $e(y)$ together determine a continuous linear map $\mathbb{H}' \oplus \mathbb{H} \to \mathbb{H}$ which satisfies the conditions for membership in $\mathcal{F}^\sharp$). We try to define the dotted arrow by

$$(y,t) \mapsto e(y) + f(y,t)$$

for $y \in Y$ and $t \in [0,1]$. Clearly $e(y) + f(y,t)$ is still a continuous linear map from $\mathbb{H}' \oplus \mathbb{H}$ to $\mathbb{H}$. Of the conditions for membership in $\mathcal{F}^\sharp$, only the surjectivity is in doubt; but this will also hold for small enough t since it holds for $t = 0$ by assumption. □

Exercise 3.4.3. (i) The above proof used a certain model for $BU \times \mathbb{Z}$. The points are closed linear subspaces L of $\mathbb{H}' \oplus \mathbb{H}$ such that $\dim(L \cap \mathbb{H}) < \infty$ and $\dim(L^\perp \cap \mathbb{H}') < \infty$. Show that this model has a structure of (infinite dimensional) Banach manifold, with charts in the Banach space $\mathcal{B}(\mathbb{H})$.

(ii) A related question: given a map from X to the Banach manifold in (i), where X is a compact CW-space, is there a practical way to associate with that one or two honest vector bundles on X, hence an element of $K(X)$?

Exercise 3.4.4. In the proof above it was claimed that for a surjective continuous linear map $A\colon \mathbb{H}' \oplus \mathbb{H} \to \mathbb{H}$, the following are equivalent: (i), the restriction of A to $\mathbb{H}$ is Fredholm; (ii), the vector spaces $\ker(A) \cap \mathbb{H}$ and $\ker(A)^\perp \cap \mathbb{H}'$ are finite dimensional.

We can use the Atiyah-Jänich theorem to make some attractive models for addition and subtraction (etc.) in $BU \times \mathbb{Z}$. (I don't want to exaggerate the importance of such models, but I promised something along these lines.) To start with let $\mathbb{H}$ be a Hilbert space with an orthogonal splitting $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$, where both $\mathbb{H}_1$ and $\mathbb{H}_2$ are infinite dimensional (therefore again Hilbert spaces in the narrow sense). Let $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ be the corresponding spaces of Fredholm operators. We have an "inclusion" $\mathcal{F}_1 \to \mathcal{F}$ which takes a Fredholm operator $A\colon \mathbb{H}_1 \to \mathbb{H}_1$ to

$$\begin{bmatrix} A & 0 \\ 0 & \mathrm{id} \end{bmatrix} : \mathbb{H}_1 \oplus \mathbb{H}_2 \to \mathbb{H}_1 \oplus \mathbb{H}_2 \,.$$

Corollary 3.4.5. *This "inclusion" $\mathcal{F}_1 \to \mathcal{F}$ is a homotopy equivalence.* □

Corollary 3.4.6. (Informal statement) *Composition of Fredholm operators on $\mathbb{H}$ is a good model for (direct sum) addition in $BU \times \mathbb{Z}$.*

Proof. We can assume $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$ as in corollary 3.4.5. By that corollary, it does not matter much whether we view composition of Fredholm operators as a map $(A, B) \mapsto A \circ B$ from $\mathcal{F} \times \mathcal{F}$ to $\mathcal{F}$, or as a map $(A, B) \mapsto A \circ B$ from $\mathcal{F}_1 \times \mathcal{F}_2$ to $\mathcal{F}$. (In the latter case, we regard $\mathcal{F}_1$ and $\mathcal{F}_2$ as subspaces of $\mathcal{F}$ in the usual way.) But if we view it as a map $\mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{F}$, then it is given in block matrix notation by

$$\mathcal{F}_1 \times \mathcal{F}_2 \ni (A, B) \mapsto \begin{bmatrix} A & 0 \\ 0 & \mathrm{id} \end{bmatrix} \begin{bmatrix} \mathrm{id} & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathcal{F} \,.$$

In this form we can also view it as a map $\mathcal{F}_1^\sharp \times \mathcal{F}_2^\sharp \to \mathcal{F}^\sharp$ (notation as in the proof of theorem 3.4.2). This is more obviously an implementation of Whitney sum. □

Definition 3.4.7. A self-adjoint (continuous linear) operator $A\colon \mathbb{H} \to \mathbb{H}$ is *positive* if $\langle v, A(v) \rangle \geq 0$ for all $v \in \mathbb{H}$.

Lemma 3.4.8. *The space $\mathcal{P}$ of (self-adjoint) positive operators on $\mathbb{H}$, a subspace of $\mathcal{B}(\mathbb{H})$, is contractible. The intersection $\mathcal{P} \cap \mathcal{F}$ is also contractible.*

Proof. The space $\mathcal{P}$ is star-shaped with $\mathrm{id}_{\mathbb{H}}$ as the center of the star. The same argument works for $\mathcal{P} \cap \mathcal{F}$. □

Corollary 3.4.9. (Informal statement) *The map $\mathcal{F} \to \mathcal{F}$ given by $A \mapsto A^*$ is a good model for* additive inverse (wrt direct sum) *on* $BU \times \mathbb{Z}$.

Proof. We regard $\mathcal{P}\cap\mathcal{F}$ as an inflated base point for $\mathcal{F}$. (It is a contractible subspace.) Suppose that we agreed to view composition of Fredholm operators as a good model for the Whitney sum in $BU\times\mathbb{Z}$. Then we observe that the map $A\mapsto A\circ A^*$ takes all of $\mathcal{F}$ to the inflated base point $\mathcal{P}\cap\mathcal{F}$. □

Corollary 3.4.10. (Informal statement) *Let V be a p-dimensional linear subspace of $\mathbb{C}^n$, where $n>0$. For $A\in\mathcal{F}$ let $A_V\colon \mathbb{H}\otimes_{\mathbb{C}}\mathbb{C}^n\to\mathbb{H}\otimes_{\mathbb{C}}\mathbb{C}^n$ be defined so that A_V agrees with $A\otimes\mathrm{id}_V$ on $\mathbb{H}\otimes V$ and with the identity on $\mathbb{H}\otimes V^\perp$. Then A_V is a Fredholm operator on $\mathbb{H}\otimes_{\mathbb{C}}\mathbb{C}^n$. The map*

$$(V,A)\mapsto A_V$$

is a good model for the map $\mathrm{Grm}(p,n-p)\times(BU\times\mathbb{Z})\longrightarrow BU\times\mathbb{Z}$ *which corresponds to the* tensor product *of (virtual) vector bundles.* □

Now we can give a description of the Bott map in Fredholm operator terms. Write $S^2=\mathbb{C}P^1$ and imagine points of $\mathbb{C}P^1$ as 1-dimensional linear subspaces $V\subset\mathbb{C}^2$. One of these is the base point: $V_0=\mathbb{C}^1\subset\mathbb{C}^2$. Then the Bott map can be described by

$$A\mapsto(V\mapsto A_V\circ(A_{V_0})^*)\,.$$

Let us see what this could mean. We are dealing with two spaces of Fredholm operators: $\mathcal{F}$, space of Fredholm operators on $\mathbb{H}$, and $\mathcal{F}^e$, space of Fredholm operators on $\mathbb{H}\otimes_{\mathbb{C}}\mathbb{C}^2$; the e superscript is for *enlarged.* The formula gives a map

$$\mathcal{F}\longrightarrow \mathrm{map}(\mathbb{C}P^1,\mathcal{F}^e)\,.$$

But the maps from $\mathbb{C}P^1$ to $\mathcal{F}^e$ that we obtain here take the base point to the inflated base point of $\mathcal{F}_e$ (consisting of the positive self-adjoint operators in $\mathcal{F}^e$). Therefore it should be permitted to describe our map in the form

$$\beta:\mathcal{F}\longrightarrow \mathrm{map}_*(\mathbb{C}P^1,\mathcal{F}^e)=\Omega^2\mathcal{F}^e\,.$$

3.5. Töplitz operators

Let $\mathbb{H}$ be the Hilbert space $L^2(S^1)$. It is the completion of $C^0(S^1)$, space of complex-valued continuous functions on S^1, with respect to the norm

$$\|f\|:=\frac{1}{\sqrt{2\pi}}\left(\int_{S^1}|f(z)|^2\,dz\right)^{1/2}.$$

(The factor $1/\sqrt{2\pi}$ is for easier bookkeeping.) It has a Hilbert basis

$$\{f_k\mid k\in\mathbb{Z}\}$$

where $f_k(z):=z^k$ for $z\in S^1$. There is an orthogonal sum splitting

$$\mathbb{H}=\mathbb{H}(-)\oplus\mathbb{H}(+)$$

where $\mathbb{H}(-)$ is the closure of the span of $\{f_k \mid k < 0\}$ and consequently $\mathbb{H}(+)$ is the closure of the span of $\{f_k \mid k > 0\}$. An element of $\mathbb{H}(+)$, say

$$f := \sum_{k \geq 0} a_k f_k$$

has a canonical extension to a function (in the L^2-sense) defined on the unit disk in $\mathbb{C}$ by the formula

$$z \mapsto \sum_{k \geq 0} a_k z^k .$$

This is in fact a convergent power series in the open unit disk, so defines a *holomorphic* function in the open unit disk. So we should see $\mathbb{H}(+)$ as the linear subspace of $\mathbb{H}$ consisting of those L^2-functions on S^1 which extend "nicely" to the open unit disk. (This is only a remark for motivation.)

Töplitz operators arise as follows. Let $u \in C^0(S^1)$ be a *continuous* complex-valued function on S^1. Define a continuous linear map

$$T_u \colon \mathbb{H}(+) \longrightarrow \mathbb{H}(+)$$

by composing

$$\mathbb{H}(+) \xrightarrow{\text{incl.}} \mathbb{H} \xrightarrow{\text{mult. with } u} \mathbb{H} \xrightarrow{\text{proj.}} \mathbb{H}(+) .$$

Here *multiplication with* u means pointwise multiplication of elements in $\mathbb{H}(+)$, which are functions on S^1, with the fixed u, also a function on S^1.

Proposition 3.5.1. *If u is invertible, i.e. $u(z) \neq 0$ for all $z \in S^1$, then T_u is a Fredholm operator on $\mathbb{H}(+)$.*

Proof. Let $v(z) = 1/u(z)$ for $z \in S^1$. It suffices to show that $T_u T_v - \mathrm{id}$ is a compact operator on $\mathbb{H}(+)$. This is mainly a calculation with Fourier coeffients. Suppose therefore that

$$u = \sum_{k \in \mathbb{Z}} b_k f_k , \qquad v = \sum_{k \in \mathbb{Z}} c_k f_k$$

and let us apply $T_u T_v$ to f_j where $j \geq 0$. Multiplication of functions corresponds to *convolution of Fourier series*; this is actually obvious in our context since $f_k \cdot f_\ell = f_{k+\ell}$ almost by definition. Therefore

$$v \cdot f_j = \sum_{\ell \in \mathbb{Z}} c_{\ell - j} f_\ell , \qquad T_v(f_j) = \sum_{\ell \geq 0} c_{\ell - j} f_\ell ,$$

and similarly

$$T_u T_v(f_j) = \sum_{k, \ell \geq 0} b_{k-\ell} c_{\ell - j} f_k .$$

For fixed k we have $\sum_{\ell\in\mathbb{Z}} b_{k-\ell}c_{\ell-j} = 0$ if $j \neq k$ and $= 1$ if $j = k$ (since $u \cdot v \equiv 1$); therefore

$$(T_uT_v - \mathrm{id})(f_j) = -\sum_{k\geq 0}\sum_{\ell<0} b_{k-\ell}c_{\ell-j}\ f_k\,.$$

This is enough to tell us that $(T_uT_v - \mathrm{id})$ is a finite rank operator if u has a finite Fourier series. At this point it is wise to leave the Fourier theory alone. The function u can be uniformly approximated by finite $\mathbb{C}$-linear combinations of the functions f_k ; this follows from Stone-Weierstrass, not Fourier theory! Therefore, if we choose a finite $\mathbb{C}$-linear combination u_1 of the functions f_k which is sufficiently close to u in the uniform metric on $C^0(S^1)$, then $v_1 = 1/u_1$ is also in $C^0(S^1)$ and

$$T_{u_1}T_{v_1} - \mathrm{id}$$

is a finite rank operator which (by easy estimates not using Fourier theory) is as close as we wish to $T_uT_v - \mathrm{id}$, in the usual norm on $\mathcal{B}(\mathbb{H}(+))$. □

Exercise 3.5.2. Determine the index of T_u when $u = f_k$, that is, $u(z) = z^k$.

There is a more general version of Töplitz operator where we begin with a continuous map u from S^1 to the matrix ring $\hom_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n)$. In such a case T_u is a continuous operator on $\mathbb{H}(+) \otimes \mathbb{C}^n$, defined by composing

$$\mathbb{H}(+) \otimes \mathbb{C}^n \xrightarrow{\text{incl.}} \mathbb{H} \otimes \mathbb{C}^n \xrightarrow{\text{mult. with } u} \mathbb{H} \otimes \mathbb{C}^n \xrightarrow{\text{proj.}} \mathbb{H}(+) \otimes \mathbb{C}^n\,.$$

Here *multiplication with* u means pointwise multiplication of elements in $\mathbb{H}(+) \otimes \mathbb{C}^n$, which are functions on S^1 with values in $\mathbb{C}^n$, with the fixed u, which is a function on S^1 whose values are complex $(n \times n)$-matrices. (Place the matrix to the *left* of the vector.)

Theorem 3.5.3. *If u is invertible, i.e.* $\det(u(z)) \neq 0$ *for all $z \in S^1$, then T_u is a Fredholm operator on* $\mathbb{H}(+) \otimes \mathbb{C}^n$.

The proof is similar to the proof of proposition 3.5.1. □

3.6. A homotopy inverse for the Bott map

By theorem 3.5.3, a continuous map $u\colon S^1 \to \mathrm{U}(n) \subset \mathrm{GL}(n) \subset \hom_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n)$ determines a Fredholm operator T_u on $\mathbb{H}(+) \otimes \mathbb{C}^n$. It does not cost us anything to assume that u is a based map, i.e., that it takes the base point to the identity $n \times n$-matrix. Then we have

$$\alpha\colon \Omega\mathrm{U}(n) \longrightarrow \mathcal{F}'$$

where $\mathcal{F}'$ is (dreadfully improvised notation) the space of Fredholm operators on $\mathbb{H}(+) \otimes \mathbb{C}^n$. (We know from corollary 3.4.5 that we don't need to distinguish carefully between spaces of Fredholm operators on distinct Hilbert spaces, as long as these Hilbert spaces are separable and infinite dimensional.) We can make n as large as we like. We can say that $\Omega U(n)$ is the same as $\Omega^2 BU(n)$.

Remark 3.6.1. It is useful to know that α preserves addition in some form. If we write $n = n_1 + n_2$, then we have $U(n_1) \times U(n_2) \subset U(n)$. And if we restrict $\alpha\colon \Omega U(n) \to \mathcal{F}'$ to $\Omega(U(n_1) \times U(n_2))$, then we see that *blockwise* addition of (loops of) matrices corresponds to blockwise addition of Fredholm operators.

This map α is our candidate for a homotopy inverse for the Bott map β. We can view α and β as natural transformations

$$\alpha_*\colon \tilde{K}(S^2 \wedge X) \to \tilde{K}(X), \qquad \beta_*\colon \tilde{K}(X) \to \tilde{K}(S^2 \wedge X)$$

where X is a compact pointed CW-space. (Both are natural homomorphisms.) Let us try to show that $\alpha_*\beta_* = \mathrm{id}$ and $\beta_*\alpha_* = \mathrm{id}$ for every such X. (After that we can discuss whether that is enough.)

The following elementary lemma about products in K-theory and *clutching maps* will be needed. (The statement is elementary, but I am still working on the proof.) Let P and Q be based compact CW-spaces, $E \to S^1 \wedge P$ and $F \to Q$ complex vector bundles. We can write $S^1 \wedge P$ as a union of two copies of $D^1 \wedge P$, with intersection $S^0 \wedge P \cong P$. Since the two copies of $D^1 \wedge P$ are contractible, E can be trivialized over each and the difference of the trivializations is a based map (the clutching map)

$$\psi\colon P \longrightarrow U(n) \simeq GL(n)$$

where n is the fiber dimension of E. If we know the map ψ, we can recover $E \to S^1 \wedge P$. Now E and F determine elements in $\tilde{K}(S^1 \wedge P)$ and in $\tilde{K}(Q)$; we need to subtract certain trivial bundles formally, but they will not matter in this discussion. The product of these elements is an element of

$$\tilde{K}(S^1 \wedge P \wedge Q).$$

This can be represented by a vector bundle over $S^1 \wedge P \wedge Q$ which we should be able to describe by a clutching map $\Psi\colon P \wedge Q \longrightarrow U(m)$ for some m.

Is there a formula for Ψ in terms of ψ ? I believe yes. Choose a vector bundle embedding of $F \to Q$ in a trivial vector bundle $\mathbb{C}^k \times Q \to Q$. Take $m = kn$. Define

$$\psi \boxtimes F\colon P \times Q \longrightarrow U(kn) = U(\mathbb{C}^n \otimes \mathbb{C}^k)$$

by $(x, y) \mapsto \psi_x \otimes \mathrm{id}$ on $\mathbb{C}^n \otimes F_y$ and by $(x, y) \mapsto \mathrm{id} \otimes \mathrm{id}$ on $\mathbb{C}^n \otimes F_y^{\perp}$, where $x \in P$ and $y \in Q$. Similarly define $\psi \boxtimes F_0\colon P \times Q \longrightarrow U(kn)$ so that $(\psi \boxtimes F_0)(x, y) = (\psi \boxtimes F)(x, *)$, where $* \in Q$ is the base point.

Lemma 3.6.2. *Let* $\Psi := (\psi \boxtimes F)\cdot(\psi \boxtimes F_0)^{-1}$*, pointwise product. Then the element of* $\tilde{K}(S^1 \wedge P \wedge Q)$ *determined by* $\Psi\colon P \wedge Q \to U(kn)$ *is the reduced* $\boxtimes$ *product of the elements of* $\tilde{K}(S^1 \wedge P)$ *and* $\tilde{K}(Q)$ *described by* $E \to S^1 \wedge P$*, with clutching map* ψ*, and* $F \to Q$*, respectively.* □

Remark 3.6.3. Let $Q_+ = Q \amalg *$ where the extra point serves as the new base point (and the old base point is demoted to ordinary status). The based map $Q_+ \to Q$ which is the identity on $Q \subset Q_+$ induces a split injection

$$\gamma\colon \tilde{K}(S^1 \wedge P \wedge Q) \longrightarrow \tilde{K}(S^1 \wedge P \wedge Q_+).$$

(Use part (i) of exercise 3.1.1.) The maps $\psi \boxtimes F$ and $\psi \boxtimes F_0$ are both well defined as maps from $P \wedge Q_+ = (P\times Q)/(*\times Q)$ to $U(kn)$. Therefore, under the homomorphism γ, the element determined by Ψ maps to the difference of the elements determined by $\psi \boxtimes F$ and $\psi \boxtimes F_0$ respectively.

Application. Let X and Y be based CW-spaces, $p \in \tilde{K}(S^2 \wedge X)$ and $q \in \tilde{K}(Y)$ so that $p \boxtimes q \in \tilde{K}(S^2 \wedge X \wedge Y)$. We show

$$(\diamondsuit) \qquad \alpha_*(p \boxtimes q) = \alpha_*(p) \boxtimes q \qquad \in \tilde{K}(X \wedge Y).$$

Suppose that p is described by a clutching map

$$\psi\colon S^1 \wedge X \longrightarrow U(n)$$

and $q \in \tilde{K}(Y)$ can be written $q_1 - q_2$ in $K(Y)$, where q_1 is represented by a vector bundle $F \to Y$ and q_2 is represented by a suitable trivial vector bundle. Let $Y_+ = Y \amalg *$. Then

$$p \boxtimes q \in \tilde{K}(S^2 \wedge X \wedge Y),$$

or preferably its image in $\tilde{K}(S^2 \wedge X \wedge Y_+)$, is the difference of two elements determined by clutching maps

$$\psi \boxtimes F\colon S^1 \wedge X \wedge Y_+ \longrightarrow U(kn)\ , \quad \psi \boxtimes F_0\colon S^1 \wedge X \wedge Y_+ \longrightarrow U(kn)$$

respectively. Applying α to these elements (and using the homomorphism property of α, remark 3.6.1), we get a contribution $\alpha_*(p) \boxtimes q_1$ for the first and a contribution $-\alpha_*(p) \boxtimes q_2$ for the second. (Use corollary 3.4.10 here.) The total is $\alpha_*(p) \boxtimes (q_1 - q_2) = \alpha_*(p) \boxtimes q$. □

Proof of $\alpha_*\beta_* = \mathrm{id}$. Take $X = S^0$ in equation $(\diamondsuit)$, so that $S^2 \wedge X = S^2$; let $p = b$ be the Bott element. Then

$$\alpha_*\beta_*(q) = \alpha_*(b \boxtimes q) = \alpha_*(b) \boxtimes q = 1 \boxtimes q = q\ .$$

Here we have used $\alpha_*(b) = 1$, which is contained in exercise 3.5.2. □

Proof of $\beta_*\alpha_* = \mathrm{id}$. We use a mirror image form of $(\diamondsuit)$, namely:

$$\alpha_*(q \boxtimes p) = q \boxtimes \alpha_*(p)$$

for $q \in \tilde{K}(Y)$ and $p \in \tilde{K}(X \wedge S^2)$. Take X and p arbitrary, $Y = S^2$ and $q = b$, the Bott element. Then $\beta_*\alpha_*(p) = b \boxtimes \alpha_*(p) = \alpha_*(b \boxtimes p) = \alpha_*\beta_*(p) = p$. Here we have used $\alpha_*\beta_* = \mathrm{id}$. □

(*Remark added later: Depending on conventions, we might find* $\alpha_*(b) = -1$ *instead of* $\alpha_*(b) = 1$, *but that does not matter much. Then we get* $\alpha_*\beta_* = -\mathrm{id}$ *and* $\beta_*\alpha_* = -\mathrm{id}$.)

3.7. Bott periodicity and complex conjugation

In his paper K-*theory and reality* (1966), Atiyah develops a general method by which, it seems, almost any good proof of Bott periodicity for complex K-theory can be refined to prove Bott periodicity for real K-theory, too. The plot is quite surprising. My description of it will remain very superficial. Regrettably.

Atiyah makes the following remark for motivation. A complex-valued $g \in L^2(S^1)$ (as in section 3.5) has a Fourier transform which is a two-sided sequence $(a_k)_{k\in\mathbb{Z}}$ of complex numbers. For us this is simply the expression of g in the Hilbert basis $\{f_k \mid k \in \mathbb{Z}\}$ where $f_k(z) = z^k$. If g happens to be real-valued, then the Fourier transform $(a_k)_{k\in\mathbb{Z}}$ of g is typically not a string of real numbers! Instead it has the more intriguing symmetry

$$a_{-k} = \bar{a}_k$$

where the bar is for complex conjugation.

It turns out that the Bott map β has a similar (unexpected) equivariance property. Suppose that $\mathbb{H}_\mathbb{R}$ is a *real* Hilbert space and let $\mathbb{H}$ be the tensor product of $\mathbb{H}_\mathbb{R}$ with $\mathbb{C}$ (over $\mathbb{R}$), which is then a complex Hilbert space in a rather obvious way. Let $\mathcal{F}$ be the space of Fredholm operators on $\mathbb{H}$ and let $\mathcal{F}_\mathbb{R}$ be the space of (real) Fredholm operators on $\mathbb{H}_\mathbb{R}$. There is an embedding $\mathcal{F}_\mathbb{R} \to \mathcal{F}$ (by extension of scalars) which we can regard as an inclusion.

Now $\mathbb{H}$ has a linear automorphism κ given by $\kappa(v \otimes z) = v \otimes \bar{z}$ for $v \in \mathbb{H}_\mathbb{R}$ and $z \in \mathbb{C}$; this is only $\mathbb{R}$-linear, though. It satisfies $\kappa\kappa = \mathrm{id}$. Then we have

$$\mathcal{F}_\mathbb{R} = \{A \in \mathcal{F} \mid \kappa A \kappa = A\}.$$

To make it more wordy: $\mathcal{F}$ comes with an involution given by

$$A \;\mapsto\; \kappa A \kappa = \kappa \circ A \circ \kappa$$

and the set of fixed points of that involution is exactly $\mathcal{F}_\mathbb{R}$. (*Involution* tends to mean *automorphism of order two*; and here automorphism means self-homeomorphism.) Atiyah's message is that we should direct our attention not so much to $\mathcal{F}_\mathbb{R}$, but to $\mathcal{F}$ which is now to be regarded as a space with involution (alias action of $\mathbb{Z}/2$).

Then we should look at the Bott map (as described at the end of section 3.4) from this point of view:

$$\beta\colon \mathcal{F} \longrightarrow \Omega^2\mathcal{F}\ .$$

Does it respect the involutions? At first sight, the answer appears to be no, but if we define the involution on the target space $\Omega^2\mathcal{F}$ correctly, then, miraculously, the map does respect the involutions.

To this end, suppose that X is any based space with an action of $\mathbb{Z}/2$ respecting the base point. Let $\Omega^{1,1}X$ be the space of (all continuous) based maps from $\mathbb{C}\cup\infty$ to X (where $\mathbb{C}\cup\infty$ has base point ∞). This is really the same as $\Omega^2 X$, inasmuch as $\mathbb{C}\cup\infty$ is not very different from $\mathbb{R}^2\cup\infty$ or S^2, for that matter, but we write $\Omega^{1,1}X$ in order to specify an action of $\mathbb{Z}/2$ on $\Omega^2 X$ as follows. The nontrivial element $T\in\mathbb{Z}/2$ acts by

$$f \mapsto \big(z \mapsto T\cdot(f(\bar z))\big)$$

for $f\colon \mathbb{C}\cup\infty \to X$ and $z\in\mathbb{C}\cup\infty$; if $z=\infty$, we set $\bar z = \infty$. With this notation, it turns out that the Bott map

$$\beta\colon \mathcal{F} \longrightarrow \Omega^{1,1}\mathcal{F}$$

as defined in section 3.4 respects the standard involutions. (Maybe we should write $\mathcal{F} \longrightarrow \Omega^{1,1}\mathcal{F}^e$ to be consistent.)

Theorem 3.7.1. *The Bott map $\mathcal{F} \longrightarrow \Omega^{1,1}\mathcal{F}$ is an equivariant (weak) homotopy equivalence.* Unraveled:

- it respects the actions of $\mathbb{Z}/2$;
- the underlying ordinary map is a (weak) homotopy equivalence;
- the induced map of the fixed point subspaces is also a (weak) homotopy equivalence.

Note that the induced map of the fixed point subspaces has the form

$$\mathcal{F}_{\mathbb{R}} \longrightarrow \mathrm{map}_*^{\mathbb{Z}/2}(\mathbb{C}\cup\infty, \mathcal{F})$$

where $\mathrm{map}_*^{\mathbb{Z}/2}(-)$ means *space of based maps respecting the actions of $\mathbb{Z}/2$ on source and target.* (The action on the source $\mathbb{C}\cup\infty$ is given by conjugation, $z\mapsto\bar z$.) That may seem disappointing since it does not resemble what we are after, a homotopy equivalence from $\mathcal{F}_{\mathbb{R}}$ to $\Omega^8\mathcal{F}_{\mathbb{R}}$ (which is what Bott found using different methods).

One might think that the proof of theorem 3.7.1 consists in noting that the map α that we constructed in section 3.6. also respects the standard involutions (if we apply it in a situation where $\mathbb{H}$ has the form $\mathbb{H}_{\mathbb{R}}$ tensored with $\mathbb{C}$, etc.). This may be so for the map α, but the geometry becomes hard to manage. Let us therefore switch from geometry to algebra, replacing *spaces* by their homotopy groups. Here we are dealing with spaces with an action with $\mathbb{Z}/2$ and we need to reconsider what we mean by *homotopy groups.*

Let $\mathbb{R}^{p,q} = \mathbb{R}^p \times \mathbb{R}^q$ with the linear action by $\mathbb{Z}/2$ where the nontrivial element acts by $-\mathrm{id}$ on the factor $\mathbb{R}^p$ and by id on $\mathbb{R}^q$. Let

$$S^{(p,q)} = \mathbb{R}^{p,q} \cup \infty$$

be the one-point compactification of $\mathbb{R}^{p,q}$, action of $\mathbb{Z}/2$ extended in the obvious way. The base point of this is always ∞. (Atiyah writes $S^{p,q}$ for the unit sphere of $\mathbb{R}^{p,q}$, so that I need to insist on the brackets in $S^{(p,q)}$ for distinction.) Let X be a based space with an action of $\mathbb{Z}/2$ which respects the base point. Then we define

$$\pi_{p,q}(X)$$

as the set of equivariant homotopy classes of equivariant based maps from $S^{(p,q)}$ to X. (*Equivariant* means *respecting actions* — of $\mathbb{Z}/2$, in our case.) For the usual reasons, $\pi_{p,q}(X)$ is a group if $q > 0$ and an abelian group if $q > 1$. Note that

$$\pi_{0,q}(X) = \pi_q(X^{\mathbb{Z}/2})$$

where $X^{\mathbb{Z}/2}$ is the set of fixed points of the action. Similarly, it is not difficult to see that

$$\pi_{1,q}(X) \cong \pi_{q+1}(X, X^{\mathbb{Z}/2}).$$

These properties can be used to show the following. Suppose that $f\colon X \to Y$ is an equivariant map between based spaces with an action of $\mathbb{Z}/2$. Suppose that f induces bijections $\pi_{p,q}(X) \to \pi_{p,q}(Y)$ whenever $p \geq 0$ and $q > 0$. *Then, under mild conditions,* f *is an ordinary weak homotopy equivalence and induces an ordinary weak equivalence of the fixed-point subspaces,*

$$X^{\mathbb{Z}/2} \to Y^{\mathbb{Z}/2}.$$

A sufficient mild condition is that X and Y and the subspaces $X^{\mathbb{Z}/2} \subset X$, $Y^{\mathbb{Z}/2} \subset Y$ are path connected. — This mild condition is unfortunately not satisfied in the case of interest to us, where $X = \mathcal{F}$ and $Y = \Omega^{1,1}\mathcal{F}$. In that case there is another feature which makes up for the nuisance: an addition map, related to Whitney sum of vector bundles (but I don't want to explain in detail how that makes up for the nuisance).

Previously we noted that based maps from a based compact CW-space X to $\mathcal{F}$ could be interpreted in vector bundle terms:

$$[X, \mathcal{F}]_* \cong \tilde{K}(X).$$

In the equivariant setting, this correspondence goes as follows. Let X be a compact Hausdorff space with an action of $\mathbb{Z}/2$. We are interested in *complex* *vector bundles* $E \to X$ with the following extra datum: an action of $\mathbb{Z}/2$ on E which

- makes the projection map $E \to X$ into a $\mathbb{Z}/2$-map;
- is such that the generator $T \in \mathbb{Z}/2$ acts in a conjugate-linear way, i.e., for all $x \in X$ the action of T gives an $\mathbb{R}$-linear isomorphism from E_x to $E_{T(x)}$ and we have $zTv = T\bar{z}v$ for all $v \in E_x$ and all $z \in \mathbb{C}$.

In particular, the restriction of $E \to X$ to $X^{\mathbb{Z}/2}$ is a complex vector bundle with an action of $\mathbb{Z}/2$ which is $\mathbb{R}$-linear, $\mathbb{C}$-conjugate linear and respects the fibers. It is easy to see that this is recoverable from $E^{\mathbb{Z}/2} \to X^{\mathbb{Z}/2}$, a real vector bundle.

Out of the isomorphism classes of complex vector bundles on X with these additional data we make a Grothendieck group which Atiyah denotes $KR(X)$. (He also writes *Real* K*-theory* with a capital R for that ... a habit which evolved after the article *K-theory and reality.*) If X comes with a chosen base point which is a fixed point for the action of $\mathbb{Z}/2$, then we have a reduced version

$$\tilde{K}R(X)$$

defined as the kernel of the restriction map $KR(X) \to KR(*) \cong \mathbb{Z}$.

We can think of $\tilde{K}R(X)$ *as the set of equivariant homotopy classes of equivariant based maps from* X *to* $\mathcal{F}$. (Of course we are assuming here that $\mathcal{F}$ is the space of Fredholm operators on $\mathbb{H}$, where $\mathbb{H}$ is the complexification of a real Hilbert space $\mathbb{H}_{\mathbb{R}}$.) In particular, taking $X = S^{(p,q)}$ we can write

$$\tilde{K}R(S^{(p,q)}) = \pi_{p+q}(\mathcal{F}).$$

Using this interpretation, we can say the following.

(i) $\pi_{p,q}(\mathcal{F})$ has an abelian group structure for all $p, q \geq 0$. (We can use the Whitney sum of vector bundles for that ... although for $q > 0$ this coincides with the standard group structure which we always have on $\pi_{p,q}(Y)$, for any based space Y with action of $\mathbb{Z}/2$.)

(ii) The groups $\pi_{p,q}(\mathcal{F})$ taken together form a bi-graded ring: there are bi-additive multiplication maps

$$\pi_{p,q}(\mathcal{F}) \times \pi_{r,s}(\mathcal{F}) \longrightarrow \pi_{p+r,q+s}(\mathcal{F})$$

(given by the tensor product of vector bundles). There is a unit

$$1 \in \pi_{0,0}(\mathcal{F}) \cong \mathbb{Z}.$$

(iii) $\pi_{0,q}(\mathcal{F}) = \pi_q(\mathcal{F}_{\mathbb{R}})$.

(iv) The forgetful maps $\pi_{p,q}(\mathcal{F}) \to \pi_{p+q}(\mathcal{F})$, taken for all $p, q \geq 0$ together, form a ring homomorphism (from a bigraded ring to a graded ring).

In (iv), forming the "ordinary" homotopy group $\pi_{p+q}(\mathcal{F})$ we should pay no attention to the action of $\mathbb{Z}/2$ on $\mathcal{F}$. Let's also note that $\pi_{p,q}(\Omega^{1,1}Y) \cong \pi_{p+1,q+1}(Y)$ for any based space Y with action of $\mathbb{Z}/2$ fixing the base point. Therefore the Bott map $\beta\colon \mathcal{F} \to \Omega^{1,1}\mathcal{F}$ induces homomorphisms

$$\beta_*\colon \pi_{p,q}(\mathcal{F}) \to \pi_{p+1,q+1}(\mathcal{F})$$

and we want to show that these are bijective (in order to prove theorem 3.7.1). Therefore we should ensure that we have

$$\alpha_*\colon \pi_{p+1,q+1}(\mathcal{F}) \to \pi_{p,q}(\mathcal{F})$$

or more generally, in K-theoretic notation,

$$\alpha_*\colon \tilde{K}R(S^{(1,1)} \wedge X) \to \tilde{K}R(X).$$

Let me give some evidence that we have such an α_*. As in section 3.5, a (continuous) map

$$u\colon S^1 \to \mathrm{GL}_n(\mathbb{C})$$

determines a Töplitz operator $T_u\colon \mathbb{H}(+) \otimes \mathbb{C}^n \longrightarrow \mathbb{H}(+) \otimes \mathbb{C}^n$. Suppose that $u, v\colon S^1 \to \mathrm{GL}_n(\mathbb{C})$ are two continuous maps related by

$$\overline{u(z)} = v(\bar{z})$$

where $S^1 = S^{(1,0)}$ is viewed as the unit circle in $\mathbb{C}$, with the action of $\mathbb{Z}/2$ by conjugation. Then $T_u = \kappa T_v \kappa$. I believe this follows from the definitions if we view $\mathbb{H}(+)$ as $\mathbb{H}_\mathbb{R}(+) \otimes_\mathbb{R} \mathbb{C}$ where $\mathbb{H}_\mathbb{R}(+)$ has the same Hilbert basis $\{f_0, f_1, f_2, \dots\}$ that we use normally for $\mathbb{H}(+)$ as a complex Hilbert space. *Example, for myself*: $n = 1$ and u has Fourier series $\sum_{k\in\mathbb{Z}} a_k f_k$ where $f_k(z) = z^k$, here for all $k \in \mathbb{Z}$; then v must have Fourier series $\sum_{k\in\mathbb{Z}} \bar{a}_k f_k$. For $j \geq 0$ and $c \in \mathbb{C}$ we get

$$T_u(cf_j) = \sum_{k\geq -j} a_k c f_{j+k}\ , \qquad T_v(\kappa(cf_j)) = T_v(\bar{c}f_j) = \sum_{k\geq -j} \bar{a}_k \bar{c} f_{j+k} = \kappa(T_u(f_j)).$$

It follows that if $u\colon S^{(1,0)} \wedge X \to \mathrm{GL}(n, \mathbb{C})$ is a (based) map which is equivariant, i.e., satisfies

$$\overline{u(z,x)} = u(\bar{z}, \bar{x})$$

(where I am using a conjugation bar to describe the involution on X), then the map $x \mapsto T_{u(-,x)}$ from X to $\mathcal{F}$ (the space of Fredholm operators on $\mathbb{H}(+) \otimes_\mathbb{C} \mathbb{C}^n$) is equivariant: $T_{u(-,x)} = \kappa T_{u(-,\bar{x})} \kappa$. Since these equivariant maps u are exactly the clutching functions that we would use to describe elements of the group

$$\tilde{K}R(S^{(1,1)} \wedge X),$$

that explains how we get from $\tilde{K}R(S^{(1,1)} \wedge X)$ to $\tilde{K}R(X)$, set or abelian group of homotopy classes of based $\mathbb{Z}/2$-maps from X to $\mathcal{F}$.

3.8. Computation of $\pi_* BO$ in a range

We can use theorem 3.7.1 to get information about the groups $\pi_k BO$ for $k \le 8$, and the homomorphisms $\pi_k BO \to \pi_k BU$ induced by the inclusion $BO \to BU$.

By specializing the theorem to the fixed point spaces, we obtain a homotopy equivalence

$$\mathcal{F}_{\mathbb{R}} \to \mathrm{map}_*^{\mathbb{Z}/2}(S^{(1,1)}, \mathcal{F})$$

where $\mathcal{F}$ is the space of Fredholm operators on $\mathbb{H} = \mathbb{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ equipped with the involution $A \mapsto \kappa A \kappa$. (And $\mathcal{F}_{\mathbb{R}}$ is the fixed point space of that involution on $\mathcal{F}$ which we can also identify with the space of real Fredholm operators on $\mathbb{H}_{\mathbb{R}}$.) Writing $S^{(1,1)}$ as the union of two disks which are interchanged by the involution, with intersection $S^{(0,1)}$, we get

$$\mathrm{map}_*^{\mathbb{Z}/2}(S^{(1,1)}, \mathcal{F}) = \mathrm{map}_*((D^2, S^1), (\mathcal{F}, \mathcal{F}_{\mathbb{R}})),$$

space of based maps of pairs from the pair (D^2, S^1) to the pair $(\mathcal{F}, \mathcal{F}_{\mathbb{R}})$. Using $(D^2, S^1) \cong (D^1 \wedge S^1, S^0 \wedge S^1)$, this space of maps of pairs turns into

$$\mathrm{map}_*((D^1, S^0), (\Omega\mathcal{F}, \Omega\mathcal{F}_{\mathbb{R}}));$$

it is the homotopy fiber (over the base point) of the inclusion

$$\Omega\mathcal{F}_{\mathbb{R}} \to \Omega\mathcal{F}.$$

So the specialization of theorem 3.7.1 to fixed points gives a homotopy equivalence

$$\mathcal{F}_{\mathbb{R}} \longrightarrow \mathrm{hofiber}[\Omega\mathcal{F}_{\mathbb{R}} \to \Omega\mathcal{F}]$$

which we can also write in the form of a homotopy equivalence

$$BO \times \mathbb{Z} \to \mathrm{hofiber}[O \hookrightarrow U].$$

Making this substitution in the long exact sequence relating the homotopy groups of O, U and $\mathrm{hofiber}[O \to U]$, we obtain a long exact sequence

$$\cdots \to \pi_k O \to \pi_{k+1} O \to \pi_{k+1} U \to \pi_{k-1} O \to \pi_k O \to \pi_k U \to \cdots$$

(ending in $\cdots \to \pi_2 U \to \pi_0 O \to \pi_1 O \to \pi_1 U$). To this we add some observations or known facts:

(i) $\pi_0 O \cong \mathbb{Z}/2$, $\pi_1 O \cong \mathbb{Z}/2$, $\pi_2 O = 0$.

(ii) $\pi_k U \cong \mathbb{Z}$ for odd $k \ge 0$ and $\pi_k U = 0$ for even $k \ge 0$.

(iii) By construction, the homomorphisms $\pi_{k-1} O \to \pi_k O$ in the LES, equivalently $\pi_k BO \to \pi_{k+1} BO$, are given by multiplication with the nonzero element h of

$$\tilde{K}_{\mathbb{R}}(S^1) = \pi_1 BO = \pi_0 O \cong \mathbb{Z}/2.$$

(iv) The generator of $\mathbb{Z}/2$ acts on $\tilde{K}_{\mathbb{C}}(S^{2k}) = \pi_{2k-1} U = \pi_{2k}\mathcal{F} \cong \mathbb{Z}$ by $x \mapsto (-1)^k x$. Therefore $\pi_{2k-1} O \to \pi_{2k-1} U$ has to be zero for odd k.

By (i), (ii) and (iv) the above long exact sequence breaks up into shorter exact sequences

(v) $0 \to \pi_3 O \to \pi_3 U \to \pi_1 O \to 0$
(vi) $0 \to \pi_5 U \to \pi_3 O \to \pi_4 O \to 0$
(vii) $0 \to \pi_4 O \to \pi_5 O \to 0$
(viii) $0 \to \pi_6 O \to \pi_7 O \to \pi_7 U \to \pi_5 O \to \pi_6 O \to 0$

We need an additional argument to show that $\pi_4 O = 0$. By (v) we have an injection $\pi_3 O \to \pi_3 U = \mathbb{Z}$ with cokernel of order 2, induced by the inclusion. It follows that the forgetful homomorphism

$$\pi_4 BU = \tilde{K}_{\mathbb{C}}(S^4) \longrightarrow \tilde{K}_{\mathbb{R}}(S^4) = \pi_4 BO$$

is an isomorphism. The commutative diagram

$$\begin{array}{ccc} \tilde{K}_{\mathbb{C}}(S^4) & \xrightarrow[\cong]{\text{forget}} & \tilde{K}_{\mathbb{R}}(S^4) \\ \downarrow \cdot h & & \downarrow \cdot h \\ \tilde{K}_{\mathbb{C}}(S^5) & \xrightarrow{\text{forget}} & \tilde{K}_{\mathbb{R}}(S^5) \end{array}$$

tells us that the right-hand vertical map is zero, since $\tilde{K}_{\mathbb{C}}(S^5) = \pi_4 U = 0$. But that map is surjective by (vi). Therefore $\pi_5 BO = 0$, meaning $\pi_4 O = 0$. — Now $\pi_4 O = 0$ implies $\pi_5 O = 0$ by (vii) and then $\pi_6 O = 0$ by (viii).

Corollary 3.8.1. (a) *The homomorphism $\pi_7 O \to \pi_7 U \cong \mathbb{Z}$ induced by the inclusion is an isomorphism.*

(b) *The homomorphism $\pi_3 O \to \pi_3 U \cong \mathbb{Z}$ induced by the inclusion is injective and its cokernel has order 2.*

(c) *The groups $\pi_4 O, \pi_5 O, \pi_6 O$ are zero.* □

3.9. Bott periodicity in the real case

Lemma 3.9.1. *There is a based $\mathbb{Z}/2$-homeomorphism*

$$g\colon S^{(4,4)}/S^{(0,4)} \to S^{(8,0)}/S^{(4,0)}.$$

Proof. We can view g as a $\mathbb{Z}/2$-homeomorphism

$$\mathbb{R}^{4,4} \setminus \mathbb{R}^{0,4} \longrightarrow \mathbb{R}^{8,0} \setminus \mathbb{R}^{4,0}$$

to be constructed — after that we can apply one-point compactification. For clarification: we shall interpret $\mathbb{R}^{0,4} \subset \mathbb{R}^{4,4}$ as the linear subspace spanned by the last 4 basis vectors, which may seem obvious, but we shall also read $\mathbb{R}^{4,0} \subset \mathbb{R}^{8,0}$ as the linear subspace spanned by the *last* 4 basis vectors. This is not in conflict with the $\mathbb{R}^{p,q}$ notation.

For this purpose we write the two 8-dimensional euclidean spaces (both) as $H \times H$, where H is the algebra of quaternions. Now let $g(v, w) := (v, vw)$, using quaternion multiplication. Then clearly $g(-v,w) = -g(v,w)$, which means that g is equivariant as a map from $\mathbb{R}^{4,4}$ to $\mathbb{R}^{8,0}$. As such it is not a homeomorphism, but if we delete the linear subspace $0 \times \mathbb{H}$ in source and target, we get a homeomorphism. ☐

Corollary 3.9.2. *The forgetful homomorphism* $\tilde{K}R(S^{(8,0)}) \to \tilde{K}_{\mathbb{C}}(S^8)$ *is surjective.*

Proof. There is a commutative diagram

$$\begin{array}{ccccc} \tilde{K}R(S^{(8,0)}/S^{(4,0)}) & \longrightarrow & \tilde{K}R(S^{(8,0)}) & \xrightarrow{\text{forget}} & \tilde{K}_{\mathbb{C}}(S^8) \\ \uparrow & & & & \Vert \\ \tilde{K}R(S^{(4,4)}/S^{(0,4)}) & \longrightarrow & \tilde{K}R(S^{(4,4)}) & \xrightarrow[\cong]{\text{forget}} & \tilde{K}_{\mathbb{C}}(S^8) \end{array}$$

where the left-hand vertical arrow is induced by the homeomorphism of lemma 3.9.1. The horizontal arrow decorated with a $\cong$ sign is an isomorphism as a consequence of theorem 3.7.1. Now it only remains to show that the other horizontal arrow in the lower row is surjective.

The group in the middle of the lower row is isomorphic to $\mathbb{Z}$ and has generator $b_\$^4$ where we think of $b_\$$ as an element of

$$\tilde{K}R(S^{(1,1)})$$

which lifts $b \in \tilde{K}_{\mathbb{C}}(S^2)$, in accordance with theorem 3.7.1. (I am trying to make a distinction between $b_\$$ and b.) Therefore it suffices to show that the restriction homomorphism

$$\tilde{K}R(S^{(4,4)}) \longrightarrow \tilde{K}R(S^{(0,4)}) = \tilde{K}_{\mathbb{R}}(S^4)$$

takes $b_\$^4$ to 0. We know that it takes $b_\$^4$ to h^4, where h is the unique nonzero element of

$$\tilde{K}_{\mathbb{R}}(S^1).$$

But h^4 is zero because already h^3 is zero; indeed h^3 lives in a group which is zero. ☐

By corollary 3.8.1, the group $\tilde{K}_{\mathbb{R}}(S^8)$ is isomorphic to $\mathbb{Z}$. More precisely, the complexification homomorphism

$$\tilde{K}_{\mathbb{R}}(S^8) \to \tilde{K}_{\mathbb{C}}(S^8)$$

is an isomorphism. Let $c \in \tilde{K}_{\mathbb{R}}(S^8)$ be the element taken to $b^4 \in \tilde{K}_{\mathbb{C}}(S^8)$.

Theorem 3.9.3. *For every compact based CW-space* X*, external multiplication with* c *is an isomorphism*

$$\tilde{K}(X) \to \tilde{K}(S^8 \wedge X).$$

Proof. Think of X as a space with trivial action of $\mathbb{Z}/2$. The homomorphism that we are investigating can be written as

$$\tilde{K}R(X) \longrightarrow \tilde{K}R(S^{(0,8)} \wedge X)$$

and is given by multiplication with $c \in \tilde{K}R(S^{(0,8)})$. Select

$$d \in \tilde{K}R(S^{(8,0)})$$

which maps to $b^4 \in \tilde{K}_{\mathbb{C}}(S^8)$ under the forgetful homomorphism; this can be done by corollary 3.9.2. Now

$$dc = b_{\$}^8 \in \tilde{K}R(S^{(8,8)}) \cong \mathbb{Z}$$

since dc maps forgetfully to $b^8 \in \tilde{K}_{\mathbb{C}}(S^8)$ by construction. Therefore multiplication with c has an inverse which is multiplication by d. In more detail, we have multiplication by c,

$$\text{(i)} \qquad \tilde{K}R(X) \longrightarrow \tilde{K}R(S^{(0,8)} \wedge X)$$

and multiplication by d,

$$\text{(ii)} \qquad \tilde{K}R(S^{(0,8)} \wedge X) \longrightarrow \tilde{K}R(S^{(8,8)} \wedge X)$$

and again multiplication by c,

$$\text{(iii)} \qquad \tilde{K}R(S^{(8,8)} \wedge X) \longrightarrow \tilde{K}R(S^{(8,16)} \wedge X).$$

The composition (ii)$\circ$(i) is a bijection since it is multiplication with $b_{\$}^4$ and the composition (iii)$\circ$(ii) is a bijection for the same reason. Therefore (i) is a bijection. $\square$