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Lecture notes chapter 2, WS 2015-2016 (Weiss): Vector bundles, J-homomorphism & Adams conjecture

This chapter can be regarded as a digression. The purpose of the digression is to explain why spherical fibrations and the Adams conjecture matter in the classification theory of manifolds.

2.1. Poincaré duality spaces

Definition 2.1.1. A compact 1-connected CW-space X is a *Poincaré duality space* of formal dimension n if there exists an element $\varphi \in H_n(X;\mathbb{Z})$ such that the homomorphisms

$$H^k(X;\mathbb{Z}) \longrightarrow H_{n-k}(X;\mathbb{Z})\ ;\ a \mapsto a \frown \varphi$$

(cap product with φ) are isomorphisms for all k. The element φ is called a *fundamental class* for X.

Example 2.1.2. Every 1-connected compact orientable n-manifold (without boundary) is a Poincaré duality space. This follows from the Poincaré duality theorem.

Remark 2.1.3. Let X be a PD space as in definition 2.1.1, and suppose in addition that it is *connected*. Then $H^0(X;\mathbb{Z}) \cong \mathbb{Z}$ and we deduce $H_n(X;\mathbb{Z}) \cong \mathbb{Z}$ by Poincaré duality. It is clear that φ must be a generator of the group $H_n(X;\mathbb{Z}) \cong \mathbb{Z}$. Therefore there are exactly two choices for a fundamental class φ. (A choice of fundamental class can also be called an *orientation*.)

A more general definition of *Poincaré duality space* is available. We will not need this, but it is worth knowing anyway. The standard version is as follows: A compact CW-space X (which need not be 1-connected) is an *orientable Poincaré duality space* of formal dimension n if there exists an element $\varphi \in H_n(X;\mathbb{Z})$ such that the homomorphisms

$$H^k(X;J) \longrightarrow H_{n-k}(X;J)\ ;\ a \mapsto a \frown \varphi$$

are isomorphisms for all k and every *local coefficient system* J on X. — To explain what a *local coefficient system* is, let me assume that X is a connected and based CW-space (no Poincaré duality whatsoever required here), so that we have a universal covering

$$\tilde{X} \to X$$

and $\pi_1 := \pi_1(X)$ acts on the left of $\tilde{X}$ by deck transformations. Then the local coefficient system J is nothing but a π_1-module, in other words an abelian group with a left action of π_1 which respects the addition (so $g(x+y) =$

$gx + gy$ for $x, y \in J$ and $g \in \pi_1$). We can define $H^k(X; J)$ as H^k of the cochain complex

$$\hom_{\pi_1}(C(\tilde{X}), J)$$

and $H_\ell(X; J)$ as the ℓ-th homology of the chain complex $C(\tilde{X}) \otimes_{\pi_1} J$. Here $C(-)$ denotes the singular or cellular chain complex (it does not matter which). There is a slight subtlety in the definition of

$$C(\tilde{X}) \otimes_{\pi_1} J.$$

Since $C(\tilde{X})$ is a chain complex of left π_1-modules and J is also a left π_1-module, the construction $\otimes_{\pi_1}$ is to be interpreted in such a way that we enforce the relations $a \otimes b \sim ga \otimes gb$ in the ordinary tensor product $\otimes$ for all $g \in \pi_1$. Equivalently, we can make left π_1-modules into *right* π_1-modules by defining $ag := g^{-1}a$, in which case $a \otimes b \sim ga \otimes gb$ turns into $a \otimes b \sim ag^{-1} \otimes gb$ which may look more familiar (to algebraists). With these definitions, there is a "refined" cap product which takes the form of a bi-additive (essentially bilinear) map $H^k(X; J) \times H_\ell(X; \mathbb{Z}) \to H_{k-\ell}(X; J)$.

This form of Poincaré duality, with local coefficient systems J, still holds for compact orientable manifolds without boundary. The standard proof is actually not very different from the standard proof of Poincaré duality for ordinary coefficients $\mathbb{Z}$. (Note that $\mathbb{Z}$ can be viewed as a π_1-module with the trivial action of $\pi_1 = \pi_1(X)$.) Also, it should be mentioned that if X is 1-connected, then all local coefficient systems on X are just "coefficients" and it is easy to show that Poincaré duality for coefficients $\mathbb{Z}$ implies Poincaré duality for all coefficients J in such a case.

Exercise 2.1.4. Let X be a connected based CW-space and write $\pi_1 := \pi_1(X)$. Write $\tilde{X}$ for the universal cover.

(i) Take $J = \mathrm{map}(\pi_1, \mathbb{Z})$, the abelian group of all functions from π_1 to $\mathbb{Z}$. There is a nearly-obvious left action of π_1 on J by translation: for $g \in \pi_1$ and $f \in J$ let $g \cdot f$ be defined by $(g \cdot f)(h) = f(hg^{-1})$. So J is a π_1-module. Show that

$$H^k(X; J) \cong H^k(\tilde{X}; \mathbb{Z})$$

for all k.

(ii) Take $J = \bigoplus_{g \in \pi_1} \mathbb{Z}$, with the left action of π_1 by translation. (Details as in (i); this J here is a π_1-submodule of the J in (i).) Show that

$$H_k(X; J) \cong H_k(\tilde{X}; \mathbb{Z}).$$

(iii) Taking J as in (ii), show that $H^0(X; J) = 0$ if π_1 is an infinite group.

2.2. Normal bundles and Spivak normal fibrations

Let M be a smooth compact manifold of dimension n, without boundary, embedded smoothly in $\mathbb{R}^k$ for some k, possibly quite large. Then M has a normal disk bundle $E \to M$ of fiber dimension $k-n$.

In more detail, without too much differential topology jargon: for each $x \in M$ we have the tangent space T_xM which can be viewed as a linear subspace (!) of $\mathbb{R}^k$. The orthogonal complement $T_x^\perp M$ of T_xM in $\mathbb{R}^k$ is the fiber of the *normal bundle* of M at x, another vector bundle on M. The map

$$TM \longrightarrow \mathbb{R}^k$$

given by $T_xM \ni v \mapsto x+v$ is far from being an embedding (make a drawing, taking for example $M = S^1$ and $\mathbb{R}^k = \mathbb{R}^2$). The map

$$T^\perp M \longrightarrow \mathbb{R}^k$$

given by $T_x^\perp M \ni v \mapsto x+v$ is usually still far from being an embedding, but if we restrict it by allowing only vectors v of norm $\leq \varepsilon$ (for small enough ε), then it is a smooth embedding. So we think of $E \to M$ as the disk bundle of fiber radius ε associated with the normal bundle $T^\perp M \to M$, and then we have a canonical smooth embedding $E \hookrightarrow \mathbb{R}^k$ by the formula just given. Let $\partial E \to M$ be the boundary sphere bundle (with fibers $\cong S^{k-n-1}$). Clearly $(E, \partial E)$ is a smooth manifold with boundary, of dimension k and contained in $\mathbb{R}^k$ as a compact codimension 0 submanifold (with boundary).

The *Pontryagin collapse map*

$$c\colon S^k \cong \mathbb{R}^k \cup \infty \longrightarrow E/\partial E$$

is defined by $c(z) = z$ if $z \in E \smallsetminus \partial E \subset \mathbb{R}^k$ and $c(z) = \partial E/\partial E$ otherwise (also when $z = \infty$). Note that it is continuous! It is easy to see that c takes the fundamental class in $H_k(S^k;\mathbb{Z})$ to a fundamental class for the manifold-with-boundary $(E, \partial E)$.

Exercise 2.2.1. Prove this "easy" statement about fundamental classes.

We can formulate this observation as follows. Recall that we have $M \subset \mathbb{R}^k$ with normal vector bundle $T^\perp M \to M$ and associated disk bundle $E \to M$. Then E is a compact manifold with boundary ∂E, no surprise here; but remarkably, the fundamental class $\in H_k(E, \partial E; \mathbb{Z})$ is *in the image of the Hurewicz homomorphism* from $\pi_k(E/\partial E)$ to $H_k(E, \partial E; \mathbb{Z})$. Indeed it is the image of the element $[c] \in \pi_k(E/\partial E)$.

It turns out that something similar is true for Poincaré duality spaces. In this situation we should not be looking for a vector bundle playing the role of normal bundle, but for a spherical fibration.

So let X be a 1-connected Poincaré duality space of formal dimension n. For simplicity we assume that X is a compact simplicial complex (not really an additional condition, since every compact CW-space is homotopy equivalent to a simplicial complex). Then we can always find an embedding

$$X \longrightarrow \mathbb{R}^k$$

(for some $k \gg 0$) which is linear on each simplex of X. Let's use this to think of X as a simplicial subcomplex of $\mathbb{R}^k$ (in some triangulation of $\mathbb{R}^k$). Then $X \subset \mathbb{R}^k$ admits a *regular neighborhood* E which can also be described as a compact simplicial subcomplex in $\mathbb{R}^k$. I am not planning to give many details; I think it is customary and safe to define E as the union of all simplices in the two-fold barycentric subdivision of (the given triangulation of) $\mathbb{R}^k$ which have nonemtpy intersection with X.

(i) E is a compact k-dimensional manifold with boundary ∂E.
(ii) There is a preferred projection $r\colon E \to X$ (continuous, at least) which is a homotopy equivalence. The restriction of r to X is the identity id_X. We write $r_\partial\colon \partial E \to X$ for the restriction of r to ∂E.

Note that we have a Pontryagin collapse map

$$c\colon S^k \cong \mathbb{R}^k \cup \infty \longrightarrow E/\partial E$$

defined much as before; and again, this takes fundamental class to fundamental class. Now we would like to say that $(E, \partial E)$ behaves like the total space (or total pair) of a disk bundle.

Theorem 2.2.2. *Each homotopy fiber of* $r_\partial\colon \partial E \to X$ *has the homology of a sphere of dimension* $k-n-1$.

This is due to M Spivak (his Princeton PhD thesis, supervised by J Milnor) and it is therefore customary to say *Spivak normal fibration of* X for the fibration associated with $\partial E \to X$. As a rule we are not averse to stabilization (taking fiberwise join with S^0, several times if required) and in that sense we can say that the Spivak normal fibration is a spherical fibration. See the following remark.

Remark 2.2.3. Replacing the inclusion $X \hookrightarrow \mathbb{R}^k$ by the composition

$$X \hookrightarrow \mathbb{R}^k \cong \mathbb{R}^k \times \{0\} \hookrightarrow \mathbb{R}^{k+1},$$

a new regular neighborhood is $E \times D^1$, and for the new retraction we may take the composition

$$E \times D^1 \xrightarrow{\text{proj.}} E \xrightarrow{r} X.$$

Restricting that to $\partial(E \times D^1)$ we have a new map $\partial(E \times D^1) \longrightarrow X$. As an application of the "cube theorem" we get

$$\mathrm{hofiber}_x[\partial(E \times D^1) \to X] \simeq \big(\mathrm{hofiber}_x[r_\partial\colon \partial E \to X]\big) * S^0$$

where $\text{hofiber}_x[...]$ is short for *homotopy fiber of "..." over* $x \in X$. If $\text{hofiber}_x[r_\partial\colon \partial E \to X]$ has the homology of S^{k-n-1} as claimed in theorem 2.2.2, then the join of it with S^0 is homotopy equivalent to S^{k-n}. (See the exercise which follows.)

Exercise 2.2.4. Let F be a space ($\simeq$ CW-space) which has the homology of a sphere S^ℓ. Show that $F * S^0 \simeq S^{\ell+1}$. (Use fundamental theorems of homotopy theory: W Hurewicz and G Whitehead).

The proof of theorem 2.2.2 reduces easily to the following statement.

Proposition 2.2.5. *Let* $p\colon A \to B$ *be a map of spaces ($\simeq$ CW-spaces) and let* $B^\natural$ *be the mapping cylinder of* p*, so that there is a pair* $(B^\natural, A)$*. Let* R *be any commutative ring (with 1). Suppose that* $H^*(B^\natural, A; R)$ *is free on one generator* $u \in H^j(B^\natural, A; R)$ *as a module over the ring* $H^*(B; R)$*. If* B *is 1-connected, then the homotopy fibers of* p *have the cohomology (with coefficients* R*) of* S^{j-1}*.*

Reduction of theorem 2.2.2 to proposition 2.2.5. Apply the proposition with $p = r_\partial$, so that $A = \partial E$ and $B = X$. Then we can identify $(B^\natural, A)$ with $(E, \partial E)$, by a homotopy equivalence of pairs. Poincaré duality for the oriented manifold pair $(E, \partial E)$ gives an isomorphism

$$H^*(E, \partial E) \cong H_{k-*}(E)$$

of graded $H^*(E)$-modules; cohomology taken with coefficients in any commutative ring R. But $H_{k-*}(E)$ is free on one generator (in degree $k - * = n$) as an $H^*(E)$-module, since $E \simeq X$ and X is a Poincaré duality space. Therefore $H^*(E, \partial E)$ is also free on one generator u as an $H^*(E)$-module. This u lives in degree $k - n$; so we take $j = k - n$.

Now the proposition implies that the homotopy fibers of $p = r_\partial$ have the cohomology of a sphere S^{j-1}, for *any* choice of coefficient ring R. It follows that they have the $\mathbb{Z}$-homology of a sphere. (See exercise just below.) □

Exercise 2.2.6. Show that if a space Y satisfies $H^*(Y; R) \cong H^*(S^{j-1}; R)$ for any commutative ring R, then it satisfies $H_*(Y; \mathbb{Z}) \cong H_*(S^{j-1}; \mathbb{Z})$. (*Hint*: reduce as fast as possible to a statement about chain complexes of free abelian groups. *Hint*: Exercise 5 in §VI.6 of Dold's book *Lectures on algebraic topology* is close to this one and comes with helpful instructions.)

Proof of proposition 2.2.5. Without loss of generality, B is a CW-space and $p\colon A \to B$ is a fibration. (If not, we can use the Serre construction to turn it into one.) Without loss of generality, and comes with a chosen base point. The cylinder projection $(B^\natural, A) \to B$, which we should strictly speaking write in the form $(B^\natural, A) \to (B, B)$, is a fibration pair. Let $(K, \partial K)$ be the fiber pair over the base point of B.

Note that K is contractible, being the (homotopy) fiber of $B^\natural \to B$. We want to show that ∂K has the cohomology (with coefficients R) of S^{j-1}. Equivalently, we want to show that $H^*(K, \partial K) = R$ if $* = j$ and $H^*(K, \partial K) = 0$ if $* \neq j$.

Let us now use the cohomology Serre spectral sequence (with coefficients R throughout) for the fibration pair $(B^\natural, A) \to B$. It has the form

$$E_2^{s,t} = H^s(B; H^t(K, \partial K)) \;\Rightarrow\; H^{s+k}(B^\natural, A).$$

The spectral sequence comes with cup products. Using these gives me a guilty conscience, because the cohomology Serre spectral sequence with cup products is hard to set up (and I did not do it convincingly in my topology course of years ago). But here we only need cup products in the following sense: we want to regard the spectral sequence as a spectral sequence of graded modules over the graded ring $H^*(B)$. This is much easier to set up. (Consider it done, therefore.)

The differentials in E_2^{**} go from position (s, t) to $(s+2, t-1)$; in E_3^{**}, from (s, t) to $(s+3, t-2)$; in E_4^{**}, from (s, t) to $(s+4, t-3)$; and so on.

If $H^{t_0}(K, \partial K)$ is nontrivial for some $t_0 < j$, then we can choose this minimal and the spectral sequence shows us that the corresponding term

$$E_2^{0,t_0} = H^0(B; H^{t_0}(K, \partial K)) \cong H^{t_0}(K, \partial K)$$

survives to the infinity page, i.e., maps injectively to $H^{t_0}(B^\natural, A)$. But since $t_0 < j$, that cohomology group is zero by assumption; contradiction.

It follows that the term $E_2^{0,j} = H^j(K, \partial K)$ survives unharmed to the infinity page, and by our assumption must map isomorphically to $H^j(B^\natural, A) \cong \mathbb{Z}$. The $H^*(B)$ module structure, along with our assumption, now implies that all terms

$$E_2^{s,j} = H^s(B; H^j(K, \partial K))$$

survive to the infinity page and map isomorphically to the corresponding groups $H^{s+j}(B^\natural, A)$.

If $H^{t_1}(K, \partial K)$ is nonzero for some $t_1 > j$, then we can take this minimal again, and we find that the corresponding term

$$E_2^{0,t_1} = H^0(B; H^{t_1}(K, \partial K)) \cong H^{t_1}(K, \partial K)$$

survives unharmed to the infinity page. This contradicts the fact that we have already exhausted $H^*(B^\natural, A)$ with the terms coming from row j of the E_2^{**}-page. □

Now I want to indicate briefly how theorem 2.2.2 can still be proved if we drop the assumption that X be 1-connected. Let's assume nevertheless that X is connected (= path connected, since X is a CW-space) and equipped with a base point. Let $\tilde{X} \to X$ be the universal covering.

We use proposition 2.2.5 again, but this time we choose $B := \tilde{X}$ (which is 1-connected) and for A we take the pullback of

$$\begin{array}{ccc} & & \tilde{X} \\ & & \downarrow \\ \partial E & \longrightarrow & X \end{array}$$

so that we have a projection $A \to \partial E$ which is again a covering space (fiber bundle with discrete fibers). In order to use the proposition, we need to know that $H^*(B^\natural, A; R)$ is free on one generator as a graded module over the graded ring $H^*(B; R)$. To that end we write

$$H^*(B^\natural, A; R) = H^*(E, \partial E; J)$$

where $J = \mathrm{map}(\pi_1(X), R)$, viewed as a left module over $\pi_1(X)$ in the usual manner. (Use exercise 2.1.4.) Then we have

$$\begin{aligned} H^*(E, \partial E; J) &\cong_{(a)} H_{k-*}(E; J) \\ &\cong H_{k-*}(X; J) \\ &\cong_{(b)} H^{n-k+*}(X; J) \\ &\cong_{(c)} H^{n-k+*}(\tilde{X}; R) \\ &= H^{n-k+*}(B; R). \end{aligned}$$

(The isomorphism with label (a) is Poincaré duality for $(E, \partial E)$; the one with label (b) is Poincaré duality for X, which is quite a different thing; and the one labelled (c) comes from exercise 2.1.4.) So we see that $H^*(E, \partial E; J)$ is isomorphic to $H^{n-k+*}(B; R)$, and it is straightforward to verify that this isomorphism (with a shift by $n - k$) is one of graded modules over $H^{n-k+*}(B; R)$. □