Lecture notes chapter 1, WS 2015-2016 (Weiss): Vector bundles, J-homomorphism & Adams conjecture

1.1. Vector bundles and K-theory

Let X be a compact space. For finite dimensional real vector bundles $E \to X$ and $E' \to X$, the Whitney sum $E \oplus E' \to X$ is the vector bundle on X defined by

$$E \oplus E' := \{(v, w) \in E \times E' \mid v, w \text{ have same value in } X\}$$

etc., so that the fiber $(E \oplus E')_x$ over $x \in X$ is identified as a vector space with $E_x \oplus E'_x$ for $x \in X$. The Whitney sum makes the set of isomorphism classes of (f.d.) real vector bundles on X into an abelian monoid (=semigroup) with neutral element.

Note: A f.d. real vector bundle on X has a fiber dimension which is locally constant as a function of $x \in X$.

Definition 1.1.1. For a compact Hausdorff space X, let $K_{\mathbb{R}}(X)$ be the Grothendieck group associated with the abelian monoid (semigroup) of isomorphism classes of (f.d.) real vector bundles on X. Similarly let $K_{\mathbb{C}}(X)$ be the Grothendieck group associated with the abelian monoid of isomorphism classes of (f.d.) complex vector bundles on X.

Remark 1.1.2. Suppose that X is compact Hausdorff as above. It is easy to show that for every (f.d. real) vector bundle $E \to X$, there exists another (f.d. real) vector bundle $E' \to X$ such that $E \oplus E'$ is isomorphic to a trivial vector bundle, i.e., a product $X \times \mathbb{R}^p$ for some p. This has the following consequence. We say that two (f.d. real) vector bundles E and E' on X are *stably* isomorphic if the vector bundle $E \times \mathbb{R}^m$ is isomorphic to $E' \times \mathbb{R}^n$, for some $m, n \geq 0$. Let

$$\tilde{K}_{\mathbb{R}}(X)$$

be the set of stable isomorphism classes of f.d. real vector bundles. This is then an abelian group (using the Whitney sum — no Grothendieck construction needed!) and if X is nonempty we have a short exact sequence

$$\mathbb{Z} \longrightarrow K_{\mathbb{R}}(X) \longrightarrow \tilde{K}_{\mathbb{R}}(X)$$

If X is path connected, this has a preferred splitting. — Similar remarks apply to $K_{\mathbb{C}}(X)$.

Remark 1.1.3. It is mildly wrong but very convenient to confuse a vector bundle $p \colon E \to X$ with its total space E. This convention is much used by Atiyah (not so much by Adams, who would normally write something like $\xi \colon E_\xi \to X$). I will try to follow the Atiyah convention when it is not too confusing. The previous remark illustrates that! For example $E \times \mathbb{R}^m$ is (if we are totally honest) the *total space* of a certain vector bundle on X which we make from $E \to X$; you are supposed to guess the details.

Although we are generally happy to assume that "all" spaces X are CW-spaces, it is not a good idea to assume that they are all compact. The above definition of $K_{\mathbb{R}}(X)$ and $K_{\mathbb{C}}(X)$ would make sense for a noncompact X, but it is not the agreed definition in such a case. We will give a better (more correct) definition for noncompact X later.

Exercise 1.1.4. Compute $K_{\mathbb{R}}(S^1)$ and $K_{\mathbb{C}}(S^2)$ using bare hands.

1.2. Spherical fibrations

Definition 1.2.1. A spherical fibration on a space X is a fibration $E \to X$ where each fiber E_x is homotopy equivalent to a sphere, S^{n-1}, for some n (which may depend on $x \in X$).

Definition 1.2.2. The *join* $U * V$ of two spaces U and V is the space obtained from the topological disjoint union

$$U \amalg (U \times [0,1] \times V) \amalg V$$

by making the identifications $(u, 0, v) \sim u$ and $(u, 1, v) \sim v$ for $u \in U$ and $v \in V$. In particular $S^{m-1} * S^{n-1}$ is homeomorphic to S^{m+n-1}.

Definition 1.2.3. The Whitney sum (more descriptively, the fiberwise join) of two spherical fibrations $E \to X$ and $E' \to X$ is the spherical fibration $E \oplus E' \to X$ obtained by taking the fiberwise join of E and E', so that the fiber of $E \oplus E'$ over $x \in X$ is identified with the join $E_x * E'_x$.

Exercise 1.2.4. $E \oplus E' \to X$ in the previous definition is indeed a spherical fibration.

In the case of spherical fibrations on X, the appropriate equivalence relation is *fiberwise homotopy equivalence.* More generally, if $p\colon E \to X$ and $q\colon E' \to X$ are fibrations, then we say that they are fiberwise homotopy equivalent if there exist maps

$$u\colon E \to E', \quad v\colon E' \to E$$

such that $qu = p$ and $pv = q$ and vu, uv are homotopic to the respective identity maps by homotopies which are *over* X. (For example in the first homotopy required, $(h_t\colon E \to E)_{t\in[0,1]}$ from vu to id_E, each map h_t must satisfy $ph_t = p$.) Here is a "funny" lemma related to this concept.

Lemma 1.2.5. *Let $p\colon E \to X$ and $q\colon E' \to X$ be fibrations, where E, E' and X are all homotopy equivalent to CW-spaces. Let $f\colon E \to E'$ be a map over X. Then the following are equivalent:*

(i) f *is a homotopy equivalence;*
(ii) f *is a fiberwise homotopy equivalence;*
(iii) *For every* $\mathsf{x} \in \mathsf{X}$*, the restriction* $\mathsf{E}_\mathsf{x} \to \mathsf{E}'_\mathsf{x}$ *of* f *is a homotopy equivalence.*

In Adams' paper *On the groups* $\mathsf{J}(\mathsf{X})$ *- I*, this is described as *Dold's theorem* (in the case where the fibers are homotopy equivalent to spheres). I found this disturbing since it looks like a profoundly "formal" statement. (I would not have been disturbed if Adams had attributed it to Bourbaki.) But let's see whether we can prove it using formal arguments only.

Proof. Clearly (ii) $\Rightarrow$ (iii). Also (iii) implies (i) using the long exact sequence of homotopy groups of a fibration and JHC Whitehead's theorem relating homotopy groups to homotopy equivalences. Therefore the only interesting part is the implication (i) $\Rightarrow$ (ii).

Let $\mathrm{map}(\mathsf{E}, \mathsf{E}')$ be the space of maps from E to E' and let $\mathrm{map}_\mathsf{X}(\mathsf{E}, \mathsf{E}')$ be the subspace of $\mathrm{map}(\mathsf{E}, \mathsf{E}')$ consisting of those maps $\mathsf{f}\colon \mathsf{E} \to \mathsf{E}'$ which satisfy $\mathsf{qf} = \mathsf{p}$. I will also need $\mathrm{map}(\mathsf{E}, \mathsf{E})$ and $\mathrm{map}_\mathsf{X}(\mathsf{E}, \mathsf{E})$, etc. Mutatis mutandis. There is a commutative diagram of mapping spaces

$$\begin{array}{ccccc}
\mathrm{map}_\mathsf{X}(\mathsf{E},\mathsf{E}) & \xrightarrow{\text{inc.}} & \mathrm{map}(\mathsf{E},\mathsf{E}) & \xrightarrow{\mathrm{p}\circ} & \mathrm{map}(\mathsf{E},\mathsf{X}) \\
\uparrow{\scriptstyle \circ\mathsf{f}} & & \uparrow{\scriptstyle \circ\mathsf{f}} & & \uparrow{\scriptstyle \circ\mathsf{f}} \\
\mathrm{map}_\mathsf{X}(\mathsf{E}',\mathsf{E}) & \xrightarrow{\text{inc.}} & \mathrm{map}(\mathsf{E}',\mathsf{E}) & \xrightarrow{\mathrm{p}\circ} & \mathrm{map}(\mathsf{E}',\mathsf{X})
\end{array}$$

where the rows are fibration sequences. (This is supposed to mean that the horizontal arrows on the right are fibrations, and the horizontal arrows on the left are the inclusions of the fibers over the base point, which is $\mathsf{p} \in \mathrm{map}(\mathsf{E}, \mathsf{X})$ for the top row and $\mathsf{q} \in \mathrm{map}(\mathsf{E}', \mathsf{X})$ for the bottom row.) Now assumption (i) implies that the vertical arrows in the middle and to the right of the diagram are homotopy equivalences, and it follows (using long exact sequences of homotopy groups and JHC Whitehead) that the vertical arrow on the left is a homotopy equivalence. In particular there exists $\mathsf{g} \in \mathrm{map}_\mathsf{X}(\mathsf{E}', \mathsf{E})$ such that $\mathsf{g} \circ \mathsf{f} = \mathsf{gf}$ is in the same path component of $\mathrm{map}_\mathsf{X}(\mathsf{E}, \mathsf{E})$ as id_E. In other words, $\mathsf{g}\colon \mathsf{E}' \to \mathsf{E}$ is a *fiberwise* left homotopy inverse for f. Repeating this argument with g instead of f (interchanging the roles of E and E', we can find a fiberwise left homotopy inverse for g (and there is also a silly formal argument showing that this is fiberwise homotopic to f). □

Exercise 1.2.6. Is there anything wrong or incomplete with this sketch proof, and if so, how would you repair it?

Exercise 1.2.7. Suppose that $E \to B$ is a fibration. Let $f_0, f_1 \colon X \to B$ be two maps which are homotopic. Show that f_0^*E and f_1^*E are fiberwise homotopy equivalent.

Remark 1.2.8. Although I suggested it *is* sometimes alright to confuse a vector bundle with its total space, I do not recommend using the word *fibration* to mean *fibration sequence* or *homotopy fiber sequence*. Somebody whose authority I do not want to question has told me that this is unacceptable. A fibration sequence (or homotopy fiber sequence) is a diagram involving two composable arrows, and perhaps additional structures or conditions. A *fibration* is a single map with a good property.

1.3. The groups $J(X)$

For a compact Hausdorff space X, the fiberwise homotopy equivalence classes of spherical fibrations on X form an abelian monoid (with neutral element) under Whitney sum.

Definition 1.3.1. Let $K_F(X)$ be the Grothendieck group of this abelian monoid. (Here the subscript F is meant to remind us of *fibrations* — it is not another funny field.)

Remark 1.3.2. If X is a compact CW-space, then for every spherical fibration $E \to X$ there exists another spherical fibration $E' \to X$ such that the Whitney sum $E \oplus E'$ (fiberwise join of E and E') is fiberwise homotopy trivial. (The proof will be given later.) It follows that we can define an abelian group

$$\tilde{K}_F(X)$$

whose elements are the *stable* fiberwise homotopy equivalence classes of spherical fibrations on X, and where the addition is Whitney sum (fiberwise join). This is analogous to remark 1.1.2. Then, if X is nonempty, there is a short exact sequence

$$\mathbb{Z} \to K_F(X) \to \tilde{K}_F(X).$$

A real finite dimensional vector bundle $E \to X$ on (a paracompact space) X determines a spherical fibration in one of several equivalent ways. The fastest way is to delete the zero section from E; the result is

$$E \smallsetminus \text{(zero section)} \longrightarrow X$$

which is a fiber bundle with fibers homeomorphic to $\mathbb{R}^n \smallsetminus \{0\}$ for some n. Since $\mathbb{R}^n \smallsetminus \{0\}$ is homotopy equivalent to S^{n-1}, this is good enough as a spherical fibration. (It seems I have to assume that X is paracompact because it is a nontrivial theorem that a fiber bundle over a paracompact base space is a fibration.) An alternative is to equip E with a fiberwise scalar product (again I am using paracompactness of X for this) and to replace E by its *unit sphere bundle* $S(E) \to X$; so that for $x \in X$, the fiber of $S(E) \to X$ over x is the unit sphere of the vector space E_x. Although it does not make a big difference (the two variants are fiberwise homotopy equivalent), the second method has the following advantage:

$$S(E \oplus E') \cong S(E) \oplus S(E').$$

That is, the unit sphere bundle of a Whitney sum of vector bundles E, E' is identified with the Whitney sum (=fiberwise join) of the unit sphere bundles of E and E'. Therefore this procedure $E \mapsto S(E)$ determines a homomorphism

$$K_{\mathbb{R}}(X) \longrightarrow K_F(X).$$

Definition 1.3.3. (Atiyah, Adams, early 1960s.) Let $J(X)$ be the image of this homomorphism of abelian groups.

Remark 1.3.4. If X is a compact nonempty CW-space, we have a short exact sequence

$$\mathbb{Z} \longrightarrow J(X) \longrightarrow \tilde{J}(X)$$

in the style of remarks 1.1.2 and 1.3.2. Here $\tilde{J}(X)$ is the image of

$$\tilde{K}_{\mathbb{R}}(X) \to \tilde{K}_F(X).$$

1.4. Universal vector bundles

Let $\mathrm{Grm}_{\mathbb{R}}(p, q)$, or $\mathrm{Grm}(p, q)$ for short, be the space of p-dimensional linear subspaces of $\mathbb{R}^{p+q}$. It is named after Grassmann. For the topology, I like to think of $\mathrm{Grm}(p, q)$ as a subspace of the vector space of symmetric real $(p + q) \times (p + q)$-matrices: those which are idempotent and have rank p. (That is to say, a p-dimensional linear subspace of $\mathbb{R}^{p+q}$ determines a linear projection, which we view as a linear map from $\mathbb{R}^{p+q}$ to $\mathbb{R}^{p+q}$; it is idempotent, has rank p and is self-adjoint with respect to the standard scalar product.)

Proposition 1.4.1. *For CW-spaces of dimension* $< q$, *there is a natural bijection from the set of isomorphism classes of* p*-dimensional vector bundles on* X *to* $[X, \mathrm{Grm}(p, q)]$.

Proof. The trivial vector bundle $\mathrm{Grm}(p, q) \times \mathbb{R}^{p+q}$ on $\mathrm{Grm}(p, q)$ has a well-known vector subbundle E consisting of the elements

$$(V, w) \in \mathrm{Grm}(p, q) \times \mathbb{R}^{p+q}$$

which satisfy $w \in V$. This is the *tautological* vector bundle on $\mathrm{Grm}(p, q)$. It gives us a natural map

$$[X, \mathrm{Grm}(p, q)] \longrightarrow \{\text{iso. classes of } p\text{-dim. vect. bundles on } X\}$$

which takes the homotopy class of $f\colon X \to \mathrm{Grm}(p,q)$ to the isomorphism class of the vector bundle f^*E on X. Now we need to show that this map is a bijection for $\dim(X) < q$.

So let $W \to X$ be any p-dimensional vector bundle on X. For each $x \in X$, the space L^W_x of linear injections from the vector space fiber W_x to $\mathbb{R}^{p+q}$ is easily seen to be $(q-1)$-connected. Therefore the fiber bundle $L^W \to X$ with fiber L^W_x over $x \in X$ admits a section and that is unique up to vertical homotopy. Having such a section amounts to identifying W with a vector subbundle of the trivial vector bundle $X \times \mathbb{R}^{p+q}$. In this way we get a map from the set of isomorphism classes of p-dimensional vector bundles on X to the set $[X, \mathrm{Grm}(p,q)]$. It is clearly inverse to the other one. □

Let $\mathrm{Grm}(p,\infty)$ be the direct limit (for $q \to \infty$) or union of the spaces $\mathrm{Grm}(p,q)$ with the direct limit topology, where we interpret $\mathbb{R}^{p+q}$ as $\mathbb{R}^p \times \mathbb{R}^q$ and use the standard inclusions $\mathbb{R}^q \to \mathbb{R}^{q+1}$. The tautological vector bundles on $\mathrm{Grm}(p,q)$ for varying q are compatible so that we can view the union of their total spaces as the total space of a p-dimensional vector bundle on $\mathrm{Grm}(p,\infty)$. This is still called the *tautological* vector bundle on $\mathrm{Grm}(p,q)$.

Exercise 1.4.2. For a compact CW-space X, a continuous map from X to $\mathrm{Grm}(p,\infty)$ will always have image contained in $\mathrm{Grm}(p,q)$ for some q.

Corollary 1.4.3. *For a compact CW-space X, the set of isomorphism classes of p-dimensional real vector bundles on X is in natural bijection with the set of homotopy classes $[X, \mathrm{Grm}(p,\infty)]$.*

Exercise 1.4.4. Show this is also valid for an arbitrary CW-space X.

We express the corollary informally by saying that the tautological vector bundle on $\mathrm{Grm}(p,\infty)$ is a *universal* p-dimensional vector bundle; and also by saying that $\mathrm{Grm}(p,\infty)$ is a classifying space for p-dimensional vector bundles.

Let $\mathrm{Grm}(\infty,\infty)$ be the direct limit (for $p \to \infty$) or union of the spaces $\mathrm{Grm}(p,\infty)$, using the standard inclusions $\mathbb{R}^p \to \mathbb{R}^{p+1}$.

Corollary 1.4.5. *For a compact connected CW-space X, there is a natural bijection between the set of stable isomorphism classes of vector bundles on X, also known as $\tilde{K}_{\mathbb{R}}(X)$, and the set*

$$[X, \mathrm{Grm}(\infty,\infty)].$$

We express this corollary informally by saying that $\mathrm{Grm}(\infty,\infty)$ is a classifying space for stable real vector bundles. (But there is no universal "stable" vector bundle in this case.)

Similar considerations apply to complex vector bundles. Later we will often write $BO(p)$ instead of $\mathrm{Grm}_{\mathbb{R}}(p,\infty)$ and BO instead of $\mathrm{Grm}_{\mathbb{R}}(\infty,\infty)$ and $BU(p)$ instead of $\mathrm{Grm}_{\mathbb{C}}(p,\infty)$ and BU instead of $\mathrm{Grm}_{\mathbb{C}}(\infty,\infty)$. This has something to do with different ways of constructing these classifying spaces.

1.5. Technical points about fibrations

Example 1.5.1. The property of being a fibration is *not* invariant under fiberwise homotopy equivalence. Here is an example. Let $\alpha\colon [0,1] \to [0,1]$ be the identity map (a fibration). Let $E := \{\,(x,y) \in \mathbb{R}^2 \mid x,y \in [0,1],\ xy = 0\,\}$. The projection $\beta\colon E \to [0,1]$, where $\beta(x,y) = x$, is not a fibration. But it is easy to see that α and β are fiberwise homotopy equivalent.

As an answer to many prayers prompted by this disturbing (counter)example, Dold introduced the concept of a *weak fibration* (and the *weak homotopy lifting property*, WHLP, which he calls WCHP as in weak covering homotopy property).

Definition 1.5.2. A map $\gamma\colon E \to B$ is a *weak fibration*, or has the WHLP, if the following holds. For every map $f\colon X \to E$ and homotopy

$$(h_t\colon X \to B)_{t\in[0,1]}$$

which satisfies $h_0 = \gamma f$ and which is stationary near $t = 0$, there exists a homotopy $(H_t\colon X \to E)_{t\in[0,1]}$ such that $H_0 = f$ and $\gamma H_t = h_t$ for all $t \in [0,1]$.

Stationary near $t = 0$ means that $h_t = h_0$ for all t in some interval $[0,\varepsilon]$, where $\varepsilon > 0$. So in the WHLP, we allow only homotopy lifting problems where the homotopy to be lifted is stationary at first, leaving a time interval $[0,\varepsilon]$ in which f can be adjusted by a *vertical* homotopy before the more serious homotopy lifting begins.

Let's note that the pullback of a weak fibration (along a map to the base space) is again a weak fibration.

Exercise 1.5.3. Show that if $u\colon E \to B$ and $v\colon E' \to B$ are fiberwise homotopy equivalent (over B) and one of them has the WHLP, then the other has the WHLP.

In particular, β in example 1.5.1 has the WHLP.

Exercise 1.5.4. Let $E \to B$ and $E' \to B$ be any maps and let $f\colon E \to E'$ be a map over B which is a fiberwise homotopy equivalence. The mapping cylinder $\mathrm{cyl}(f)$ of f comes with a canonical projection map to $B \times [0,1]$, the mapping cylinder of $\mathrm{id}\colon B \to B$. Show that $\mathrm{cyl}(f)$ as a space over $B \times [0,1]$ is fiberwise homotopy equivalent to $E \times [0,1]$.

Corollary: if $E \to B$ is a weak fibration in the above circumstances, then so is $\mathrm{cyl}(f) \to B \times [0,1]$.

The WHLP is so close to the HLP that many of the standard consequences of the HLP can also be deduced from the WHLP. For example, if $u\colon E \to B$ is a weak fibration and $* \in E$ and $Y = u^{-1}(u(*))$ is the fiber over $u(*) \in B$, then $\pi_k(E, Y; *) \cong \pi_k(B; u(*))$ for all k.

Theorem 1.5.5. *The WHLP is a local property for paracompact base spaces.* (In more detail, if $f\colon E \to B$ is a map where B is paracompact, and B has an open covering $(U_i)_{i\in\Lambda}$ such that the restrictions $f^{-1}(U_i) \to U_i$ have the WHLP, then f itself has the WHLP.)

Dold proves this (and the analogous statement for the HLP) in *Partitions of unity in the theory of fibrations* (Annals of Math. 78 (1963)). It looks like a long (although self-contained) proof, and I propose that we accept the statement.

Example 1.5.6. Let

$$\begin{array}{ccccc} X & \xleftarrow{u} & Y & \xrightarrow{v} & Z \\ \downarrow & & \downarrow & & \downarrow \\ A & \xleftarrow[p]{} & B & \xrightarrow[q]{} & C \end{array}$$

be a commutative diagram of spaces and maps where A, B, C are CW-spaces and the maps p, q are cellular. Suppose that the vertical maps are weak fibrations and that the maps

$$Y \to p^*X, \qquad Y \to q^*Z$$

determined by the diagram are fiberwise homotopy equivalences. Let $I = [-1, 1]$ for this example. I write $A \cup_p (B \times I)_q \cup C$ for the double mapping cylinder (points in $B \times \{-1\} \cong B$ are identified with their values in A under p and points in $B \times \{1\}$ are identified with their values in C under q). We cannot be certain that the induced map

$$\begin{array}{c} X \cup_u (Y \times I)_v \cup Z \\ \Big\downarrow \alpha \\ A \cup_p (B \times I)_q \cup C \end{array}$$

is again a weak fibration, but it is very close to that. Let $\varphi\colon I \to I$ be defined by $t \mapsto -1$ for $t < -1/2$, $t \mapsto 1$ for $t > 1/2$ and $t \mapsto 2t$ for $-1/2 \le t \le 1/2$. Let $f\colon A \cup_p (B \times I)_q \cup C \to A \cup_p (B \times I)_q \cup C$ be the identity on A and C and take elements of the form (b, t) where $b \in B$ to

$(b, \varphi(t))$. Make a pullback square

$$\begin{array}{ccc} f^*(X \cup_u (Y \times I)_v \cup Z) & \longrightarrow & X \cup_u (Y \times I)_v \cup Z \\ \downarrow{\scriptstyle f^*\alpha} & & \downarrow{\scriptstyle \alpha} \\ A \cup_p (B \times I)_q \cup C & \xrightarrow{f} & A \cup_p (B \times I)_q \cup C \end{array}$$

Now the composite map from upper left to lower right is a weak fibration. (This can be deduced from Dold's locality theorem 1.5.5. Use the open covering of $A \cup_p (B \times I)_q \cup C$ by two open subsets $A \cup_p (B \times [-1, 1/2))$ and $(B \times (-1/2, 1]) \cup_q C$.))

Let's use all this to prove a useful homotopy theoretic statement which is sometimes called the theorem of the cube (or similar - can't remember the precise name). It is really just a formal way to express what we have just seen in example 1.5.6. Again let

$$\begin{array}{ccccc} X & \longleftarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ A & \xleftarrow[p]{} & B & \xrightarrow[q]{} & C \end{array}$$

be a commutative diagram of spaces and maps where all spaces shown are homotopy equivalent to CW-spaces. (It follows that the homotopy fibers of the vertical maps are also homotopy equivalent to CW-spaces. This will play a role. Consult Milnor's wonderful article *On spaces having the homotopy type of a CW-complex*.) We assume in addition that the horizontal arrows in the right-hand square are *cofibrations*. This implies that the union $X \sqcup_Y Z$ has somewhat predictabe homotopy and homology properties — for example there is long exact Mayer-Vietoris sequence relating the homology groups of X, Y, Z and $X \sqcup_Y Z$. A similar remark applies to $A \sqcup_B C$.

Theorem 1.5.7. *Suppose in addition that the left-hand and right-hand square in the diagram above are homotopy pullback squares* (see definition below). *Then the left-hand and right-hand square in*

$$\begin{array}{ccccc} X & \xrightarrow{\text{incl.}} & X \sqcup_Y Z & \longleftarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{\text{incl.}} & A \sqcup_B C & \longleftarrow & C \end{array}$$

are also homotopy pullback squares.

Definition 1.5.8. Given a commutative square of spaces and maps

$$\begin{array}{ccc} P & \xrightarrow{u} & Q \\ \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle f} \\ A & \xrightarrow{e} & B \end{array}$$

we obtain a map from P to the space

$$A \times^{h}{}_{B} Q := \{(y, \omega, z) \in A \times \mathrm{map}([0,1], B) \times Q \mid \omega(0) = e(y),\ \omega(1) = f(z)\}$$

by $x \mapsto (u(x), \omega_x, v(x))$ where ω_x is the constant path in B with value $ev(x) = fu(x)$. If that map is a homotopy equivalence, then we say that the square is a homotopy pullback square. In the case where all spaces involved are CW-spaces, that condition is equivalent to each of following:

- For each $z \in Q$, the homotopy fiber of $P \to Q$ over z maps by a homotopy equivalence to the homotopy fiber of $A \to B$ over the image of z.
- For each $y \in A$, the homotopy fiber of $P \to A$ over y maps by a homotopy equivalence to the homotopy fiber of $Q \to B$ over the image of y.

This is probably easier to remember, although less symmetrical. So informally, a square is homotopy pullback square if the left-hand vertical homotopy fibers map by homotopy equivalences to the corresponding right-hand vertical homotopy fibers. (Equivalently: the upper horizontal homotopy fibers map by homotopy equivalences to the corresponding lower horizontal homotopy fibers.)

Exercise 1.5.9. Here are a few examples which illustrate the power of the theorem well, although at the same time they underline how obvious it is. (Forgive if you have seen these examples; I tend to over-sell them.)

(i) By taking $A = C = \mathbb{R}P^\infty$ and $B =$ point, and by making inspired choices of X, Y, Z etc., show that the free product $\mathbb{Z}/2 * \mathbb{Z}/2$ (a non-commutative group) fits into a short exact sequence of groups and homomorphisms $\mathbb{Z} \to \mathbb{Z}/2 * \mathbb{Z}/2 \to \mathbb{Z}/2$.

(ii) By taking $A = C = \mathbb{C}P^\infty$ and $B =$ point, and by making inspired choices of X, Y, Z etc., show that $\pi_k(\mathbb{C}P^\infty \vee \mathbb{C}P^\infty) \cong \pi_k(S^2)$ for $k \geq 3$, whereas $\pi_2(\mathbb{C}P^\infty \vee \mathbb{C}P^\infty) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(iii) Returning to (i): can you make a similar statement for the free product $\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$?

Helpful remark (which could even spoil the exercise). The construction which to a diagram $U \to V \to W$ of spaces associates the space $U \cup_V W$ is *homotopy invariant* if the second arrow, $V \to W$, is a closed cofibration. The preferred interpretation of *homotopy invariant* is as follows. If we have a commutative diagram

$$\begin{array}{ccccc} U & \longleftarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ U' & \longleftarrow & V' & \longrightarrow & W' \end{array}$$

(where the horizontal arrows on the right are closed cofibrations and) where the vertical arrows are homotopy equivalences, then the induced map from $U \cup_V W$ to $U' \cup_{V'} W'$ is again a homotopy equivalence.

Sketch proof of the theorem. (Do not read this sketch proof before you have had a go at exercise 1.5.9.) We may assume that the vertical arrows in the original diagram (with X, Y, Z and A, B, C) are fibrations. Then our assumption on homotopy pullbacks means that the maps from Y to p^*X and from Y to q^*Z are fiberwise homotopy equivalences (over B). We may assume that A, B, C are CW-spaces and that p, q are cellular. We may drop the cofibration assumptions but then, instead of using $X \sqcup_Y Z$ and $A \sqcup_B C$, we should use the double mapping cylinders as in example 1.5.6. So, as an acceptable substitute for the diagram given in the theorem, we get

$$\begin{array}{ccccc} X & \longrightarrow & f^*(X \cup_u (Y \times I)_v \cup Z) & \longleftarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & A \cup_p (B \times I)_q \cup C & \longleftarrow & C \end{array}$$

where the middle vertical arrow is the weak fibration which we found in example 1.5.6. Here all the vertical arrows are weak fibrations and the little squares are (strict) pullback squares, therefore also homotopy pullback squares (!) by the weak fibration properties. □

Definition 1.5.10. Instead of *double mapping cylinder*, we also say *homotopy pushout*.

1.6. Simplicial spaces and the bar construction

The category Δ has objects $[n] = \{0, 1, \ldots, n\}$ where n runs through the non-negative integers; a morphism from $[m]$ to $[n]$ is an order-preserving map from $[m] = \{0, 1, \ldots, m\}$ to $[n] = \{0, 1, \ldots, n\}$.

There is an important (covariant) functor from Δ to the category of spaces given by $[n] \mapsto \Delta^n$ where Δ^n is the space of functions s from $[n]$ to $[0, 1]$ such that $\Sigma_{i \in [n]} s(i) = 1$. The value of $s \in \Delta^n$ on $i \in [n]$ is called the i-th barycentric coordinate of s. (*Little exercise*: explain in detail how this is a covariant functor.)

A *simplicial space* is a *contravariant* functor X from Δ to the category of spaces. The *geometric realization* of such an X is the space $|X|$ obtained from the disjoint union

$$\coprod_{n\geq 0} X[n]\times\Delta^n$$

by making the identifications

$$X[m]\times\Delta^m \ni (f^*z,s)\ \sim\ (z,f_*s)\in X[n]\times\Delta^n$$

for morphisms $f\colon [m]\to[n]$ in Δ and $z\in X[n]$, $s\in\Delta^m$.

I assume that this is (somewhat) known. The geometric realization is occasionally not well behaved and for such cases one has substitutes which are typically bigger, but in some respects better behaved. Here is one substitute which I shall denote by $\|X\|$. The *fat realization* $\|X\|$ of a simplicial space X is obtained from the disjoint union

$$\coprod_{n\geq 0} X[n]\times\Delta^n$$

by making the identifications

$$(f^*z,s)\sim(z,f_*s)$$

only for *injective* morphisms $f\colon [m]\to[n]$ in Δ and $z\in X[n]$, $s\in\Delta^m$. (So the construction $\|X\|$ does not use any information about $f^*\colon X[n]\to X[m]$ for non-injective morphisms $f\colon [m]\to[n]$ in Δ.) There is a quotient map (identification map)

$$\|X\|\to|X|.$$

Exercise 1.6.1. Show that this quotient map has contractible point inverses.

(This suggests that it has a good chance to be a homotopy equivalence. But it is not always a homotopy equivalence. G Segal, in *Categories and cohomology theories*, Topology 13 (1974), gives the following sufficient condition: if the maps $f^*\colon X[n]\to X[m]$ associated with *surjective* morphisms $f\colon [m]\to[n]$ in Δ are always (closed) cofibrations, then ... it is a homotopy equivalence.)

The geometric realization $|X|$ has a *formal* k-*skeleton*: the image of

$$\coprod_{n\leq k} X[n]\times\Delta^n$$

in $|X|$ under the defining identification map. Similarly, $\|X\|$ has a formal k-skeleton. (Most people use an expression like *horizontal* k-*skeleton* or *vertical* k-*skeleton*, but it is always impossible to remember which of the two, so I have decided to use the word *formal* instead.) Beware: the quotient map $\|X\|\to|X|$ restricts to a map from the formal k-skeleton of $\|X\|$ to the formal

k-skeleton of $|X|$, but that map is typically not a homotopy equivalence, even if X is very well behaved. Think of that as a weakness of the fat realization. There is a fix which I may mention later.

Exercise 1.6.2. Let $g\colon X \to Y$ be a map (= natural transformation) between simplicial spaces X and Y. Suppose that each $g_{[n]}\colon X[n] \to Y[n]$ is a homotopy equivalence (of ordinary spaces). Show that the map from $\|X\|$ to $\|Y\|$ determined by g is also a homotopy equivalence. (This gives meaning to the vague statement *the fat realization is always well behaved.*)

Hint: not really difficult, but possibly tiresome. Induction over formal skeletons might be a good idea. I found the following principle useful: a map $f\colon A \to B$ of spaces is a homotopy equivalence iff the standard inclusion of A into the mapping cylinder $\mathrm{cyl}(f)$ admits a strong deformation retraction.

Example 1.6.3. Let Q be a nonempty space and let X be the simplicial space defined by $X[n] := \mathrm{map}([n], Q)$. (You can also write $X[n] = Q^{n+1}$, but the description $X[n] := \mathrm{map}([n], Q)$ describes better how we think of $[n] \mapsto X[n]$ as a contravariant functor from Δ to spaces.) Then $\|X\|$ is contractible.

Sketch proof: choose a point z in Q. For each $k \geq 0$, the inclusion of the formal k-skeleton of $\|X\|$ into the formal $(k+1)$-skeleton is nullhomotopic. More precisely, if a point p in the formal k-skeleton of $\|X\|$ has coordinates $(y_0, y_1, ..., y_\ell\ ;\ t_0, t_1, \dots, t_\ell)$ where $\ell \leq k$ and $y_0, \dots, y_\ell \in Q$ and $(t_0, \dots, t_\ell) \in \Delta^\ell \smallsetminus \partial\Delta^\ell$, then we can make path of the form

$$(y_0, y_1, ..., y_\ell, z\ ;\ tt_0, tt_1, \dots, tt_\ell, 1-t)$$

in the $(k+1)$-skeleton, where t runs from 1 to 0. This path takes us from p to a point whose coordinate description, due to the relations in the description of $\|X\|$, has the simple form $(z\ ;\ 1)$ because it is in the formal 0-skeleton.

Definition 1.6.4. Let $\mathcal{C}$ be a small category. The *nerve* of $\mathcal{C}$ is the simplicial set $N\mathcal{C}$ given (as a contravariant functor from Δ to sets) by

$$[m] \mapsto N_m\mathcal{C} := \mathrm{fun}([m]^{\mathrm{op}}, \mathcal{C})$$

where $\mathrm{fun}([m]^{\mathrm{op}}, \mathcal{C})$ is the set of contravariant functors from $[m]$ to $\mathcal{C}$. Here I need to explain that we can view $[m]$ as a category: the objects are $0, 1, 2, \dots, m$, and the set $\mathrm{mor}(i, j)$ has exactly one element if $i \leq j$, otherwise none. Therefore a contravariant functor from $[m]$ to $\mathcal{C}$ is exactly the same thing as a diagram in $\mathcal{C}$ of the shape

$$c_0 \xleftarrow{f_1} c_1 \xleftarrow{f_2} c_2 \longleftarrow \cdots \xleftarrow{f_{m-1}} c_{m-1} \xleftarrow{f_m} c_m$$

The map $N_q\mathcal{C} \to N_p\mathcal{C}$ induced by a morphism $f\colon [p] \to [q]$ in Δ is obtained by viewing f as a functor, and pre-composing with that functor.

The geometric realization of $N\mathcal{C}$ (fat or not) is denoted $B\mathcal{C}$ and is sometimes called the *bar construction* applied to $\mathcal{C}$.

This construction is also available if $\mathcal{C}$ is a small category *enriched in spaces*, which means that each morphism set $\mathrm{mor}(c, d)$ in $\mathcal{C}$ comes equipped with a topology and composition of morphisms is continuous. In this case, clearly, $N_m\mathcal{C}$ (defined as above) also has the structure of a topological space and $N\mathcal{C}$ as a whole becomes a simplicial space. Again the geometric realization is denoted $B\mathcal{C}$. In this case it can make a difference whether we use the fat realization or the standard one.

The reasons for defining $B\mathcal{C}$ as we have done will only emerge gradually. A confusing aspect of the bar construction is that it has several uses. In the early days, before 1970, everybody seemed to think that $B\mathcal{C}$ is useful as a target: maps from other spaces to $B\mathcal{C}$ have interesting interpretations. After 1970, this was forgotten to some extent and instead most people seemed to think that $B\mathcal{C}$ is useful as a source: maps from $B\mathcal{C}$ to other spaces have interesting interpretations. But here we take the old-fashioned view; so we look for interesting interpretations of maps from (other) spaces to $B\mathcal{C}$.

Example 1.6.5. Let Q be a topological monoid with neutral element $*$. This means that Q is a topological space with a base point $*$, and that it comes with a map $\mu\colon Q \times Q \to Q$ (a "multiplication") which has $*$ as a two-sided neutral element and is associative. Then we can view Q as a category enriched in spaces. This category has only one object, call it e; the space of morphisms $\mathrm{mor}(e, e)$ is Q and the identity morphism id_e is the base point $* \in Q$. Composition of morphisms is given by the multiplication μ, so that $f \circ g := \mu(f, g)$. For the nerve of this enriched category we write simply NQ and we write BQ for $|NQ|$ or for $\|NQ\|$.

For example, $N_kQ = Q \times Q \times \cdots \times Q \times Q$ (k factors) and the injective morphism $[5] \to [6]$ in Δ which omits the element 3 induces a map from N_6Q to N_5Q which is given by

$$(q_1, q_2, \ldots, q_5, q_6) \mapsto (q_1, q_2, q_3q_4, q_5, q_6)$$

where q_3q_4 must be decoded using the multiplication in Q. The injective morphism $[5] \to [6]$ in Δ which omits the element 6 induces a map from N_6Q to N_5Q given by

$$(q_1, q_2, \ldots, q_5, q_6) \mapsto (q_1, q_2, q_3, q_4, q_5).$$

The injective morphism $[5] \to [6]$ in Δ which omits the element 0 induces a map from N_6Q to N_5Q given by $(q_1, q_2, \ldots, q_5, q_6) \mapsto (q_2, q_3, q_4, q_5, q_6)$.

The surjective morphism $g\colon [6] \to [5]$ in Δ which takes the value 3 twice induces a map $N_5Q \to N_6Q$ given by $(q_1, q_2, \dots, q_5) \mapsto (q_1, q_2, q_3, *, q_4, q_5)$.

Example 1.6.6. This is a variation on the previous example where we start with a monoid Q and a left action of Q on a space T. Let $N(Q;T)$ be the simplicial space which takes the object $[k]$ of Δ to

$$N_k(Q;T) = Q^k \times T = Q \times Q \times \cdots \times Q \times Q \times T.$$

Instead of defining the induced maps properly, I will give a few special cases. The injective morphism $[5] \to [6]$ in Δ which omits the element 3 induces a map from $N_6(Q;T)$ to $N_5(Q;T)$ which is given by

$$(q_1, q_2, \dots, q_5, q_6, t) \mapsto (q_1, q_2, q_3q_4, q_5, q_6, t).$$

The injective morphism $[5] \to [6]$ in Δ which omits the element 6 induces a map from $N_6(Q;T)$ to $N_5(Q;T)$ given by

$$(q_1, q_2, \dots, q_5, q_6, t) \mapsto (q_1, q_2, q_3, q_4, q_5, q_6t)$$

where q_6t must be decoded using the left action of Q on T. The injective morphism $[5] \to [6]$ in Δ which omits the element 0 induces a map from N_6Q to N_5Q given by $(q_1, q_2, \dots, q_5, q_6, t) \mapsto (q_2, q_3, q_4, q_5, q_6, t)$. The surjective morphism $g\colon [6] \to [5]$ in Δ which takes the value 3 twice induces a map $N_5Q \to N_6Q$ given by $(q_1, q_2, \dots, q_5, t) \mapsto (q_1, q_2, q_3, *, q_4, q_5, t)$.

(This construction can *almost* be viewed as a special case of definition 1.6.4. The monoid Q and the action of it on T allow us to construct a category $\mathcal{C}$ where an object is an element of T and a morphism from $t \in T$ to $t' \in T$ is an element $q \in Q$ such that $t' = qt$. Then $N(Q;T)$ is $N\mathcal{C}$, if we disregard the topologies. To get the topologies right, we would have to view $\mathcal{C}$ as a category with a *space* of objects T. This goes slightly beyond the concept of a category enriched in spaces. Let's not go into that.)

Theorem 1.6.7. *Suppose that the topological monoid* Q *is grouplike* (see definition below) *and homotopy equivalent to a CW-space.*

- (i) *Then there is a homotopy fiber sequence* $T \to B(Q;T) \to BQ$.
- (ii) *Take* $T := Q$ *with the action of* Q *by left translation. Then* $B(Q;Q)$ *is contractible.*
- (iii) *In the situation of* (ii)*, the map* $\pi_1(BQ) \to \pi_0(Q)$ *from the long exact sequence of homotopy groups associated with the homotopy fiber sequence of* (i) *is an isomorphism of groups.*

Definition 1.6.8. A topological monoid Q (with neutral element $*$) is *grouplike* if one of the following equivalent conditions is satisfied:

- (i) for every $x \in Q$, left multiplication by x is a homotopy equivalence from Q to Q;

(ii) for every $y \in A$, right multiplication by y is a homotopy equivalence from Q to Q;
(iii) the monoid structure on $\pi_0 Q$ determined by the monoid structure on Q makes $\pi_0 Q$ into a group.

Exercise 1.6.9. Prove that the three conditions in 1.6.8 are equivalent.

Sketch proof of thm 1.6.7 part (i). We use the *fat* version of geometric realization throughout this proof. — There is an obvious projection map $N(Q;T) \to N(Q;*) = NQ$ which induces a map of geometric realizations $B(Q;T) \to BQ$. The fiber of that over the base point (the unique element of $N_0 Q \times \Delta^0 \subset BQ$) is precisely $N_0(Q;T) = T$. So we have to show that, for the projection map

$$B(Q;T) \to BQ\ ,$$

the inclusion of the fiber over the base point (which is T) in the homotopy fiber over the base point is a homotopy equivalence.

It is enough to show that, for every $k \geq 0$, for the projection map

$$\text{formal } k\text{-skeleton of } B(Q;T) \longrightarrow \text{formal } k\text{-skeleton of } BQ$$

the inclusion of the fiber over the base point (which is T) in the homotopy fiber over the base point is a homotopy equivalence. We proceed by induction on k. The induction beginning is the case $k = 0$ and this is obvious. The induction step is an application of theorem 1.5.7. We apply it with

$$\begin{aligned} X &= \text{formal } k\text{-skeleton of } B(Q;T) \\ Y &= N_{k+1}(Q;T) \times \partial\Delta^k \\ Z &= N_{k+1}(Q;T) \times \Delta^k \\ A &= \text{formal } k\text{-skeleton of } BQ \\ B &= N_{k+1}Q \times \partial\Delta^k \\ C &= N_{k+1}Q \times \Delta^k. \end{aligned}$$

The inductive assumption tells us what the homotopy fiber of the projection from X to A (over the base point) is — briefly, it is the fiber, T — and so allows us to verify that the conditions in theorem 1.5.7 are satisfied. The conclusion of the theorem is that the homotopy fiber of $X \sqcup_Y Z \to A \sqcup_B C$ over the base point is still the same thing, the fiber T. But $A \sqcup_B C$ is precisely the formal $(k+1)$-skeleton of BQ and $X \sqcup_Y Z$ is precisely the formal $k+1$-skeleton of $B(Q;T)$. □

Exercise 1.6.10. Where did this sketch proof use the assumption that Q is grouplike? (Hint: the first induction step, from $k = 0$ to $k = 1$, is more serious than the later steps because $\partial\Delta^1$ is not path connected. Verify how this step fails when Q is not grouplike; for example Q could be the discrete monoid $\mathbb{N} = \{0, 1, 2, \dots, \}$ with the usual addition.)

Sketch proof of thm 1.6.7 part (ii). We compare the simplicial space $N(Q;Q)$ with the simplicial space X of example 1.6.3 which has $X[n] = \mathrm{map}([n], Q)$ (using the same Q). There is a map $N(Q;Q) \to X$ (natural transformation between contravariant functors ...) given at the object $[m]$ of Δ by

$$(q_0, q_1, \dots, q_m) \mapsto (q_0 q_1 \cdots q_m, q_1 q_2 \cdots q_m, \cdots, q_{m-1} q_m, q_m).$$

The grouplike condition (plus *homotopy equivalent to CW-space*) should (!) imply that this is a homotopy equivalence for every $m \geq 0$. Then exercise 1.6.2 applies to show $B(Q;Q) \simeq \|X\|$ and example 1.6.3 tells us that $\|X\|$ is contractible. □

Corollary 1.6.11. *We have* $\pi_k(BQ) \cong \pi_{k-1}(Q)$ (group isomorphism) *for* $k > 0$ (and $\pi_0(BQ)$ is a singleton).

Remark. Both BQ and Q have preferred base points (which should be indicated when we write about homotopy groups ... but have not been indicated for the sake of brevity).

Proof. The homotopy fiber sequence $Q \to B(Q;Q) \to BQ$ determines a long exact sequence of homotopy groups. Since $B(Q;Q)$ is contractible, this collapses to a collection of bijections from $\pi_k(BQ)$ to $\pi_{k-1}(Q)$ for $k > 0$. These bijections are automatically group isomorphisms for $k > 1$, because that is part of the long-exact-sequence-of-homotopy-fiber-sequence package. If $k = 1$, we still have a group structure on $\pi_0(Q)$ (since Q is a grouplike topological monoid). It can be verified manually that the boundary map $\pi_1(BQ) \to \pi_0(Q)$ is not just a bijection, but an isomorphism in this case, too. □

Example 1.6.12. If Q is a discrete group (i.e., a group with the discrete topology), then clearly $\pi_k(Q)$ is trivial for $k > 0$. Therefore we get

$$\pi_1(BQ) \cong Q$$

and $\pi_k(BQ)$ is trivial for all $k \neq 1$.

Sketch proof of thm 1.6.7 part (iii). Each element $g \in Q$ determines an element of $\pi_1(BQ)$ as follows. The formal 1-skeleton of BQ (fat realization for simplicity) is the quotient space of $Q \times \Delta^1$ obtained by collapsing $Q \times \partial\Delta^1$ to a single point. Therefore every $g \in Q$ determines a loop $\Delta^1 \to BQ$ by $x \mapsto (g, x) \in Q \times \Delta^1/\sim \hookrightarrow BQ$. This gives a map

$$\lambda \colon \pi_0(Q) \longrightarrow \pi_1(BQ).$$

It is easy to check that λ is a homomorphism. It is easy to check the the composition of λ with the map $\pi_1(BQ) \to \pi_0(Q)$ from the long exact sequence is the identity map. Exactness of the long exact sequence, together with contractibility of the total space $B(Q;Q)$, also implies that $\pi_1(BQ) \to \pi_0(Q)$ is injective. (This requires some good understanding of what exactness at the end of the sequence means.) □

1.7. Monoids of homotopy automorphisms

Let T be a compact CW-space. The set of maps $\mathrm{map}(T,T)$ is a topological space with the compact-open topology. Let $Q_T \subset \mathrm{map}(T,T)$ be the subspace consisting of all $f\colon T \to T$ which are homotopy equivalences (which admit an inverse up to homotopy). Composition of maps makes Q_T into a grouplike topological monoid with neutral element $\mathrm{id}_T \in Q_T$. This monoid comes with an obvious left action on T given by evaluation.

Theorem 1.7.1. *The space* BQ_T *is a classifying space for fibrations with fibers homotopy equivalent to* T. *More precisely, for compact CW-spaces X there is a natural bijection*

$$[X, BQ_T] \longrightarrow \frac{\{\text{fibrations } E \to X \text{ where all fibers are} \simeq T\}}{\text{fiber homotopy equivalence}}$$

where $[X, BQ_T]$ *is the set of homotopy classes of maps from* X *to* BQ_T.

Proof. The map in the theorem is easy to describe. From the previous section we have a projection map

$$p\colon B(Q_T;T) \to BQ_T\,.$$

We can use the Serre construction to factorize this as follows,

$$B(Q_T;T) \hookrightarrow B(Q_T;T)^\sharp \xrightarrow{p^\sharp} BQ_T$$

so that the second arrow is a fibration and the first arrow is a homotopy equivalence. By theorem 1.6.7, the fibers of the right-hand map are all homotopy equivalent to T. (It is enough to check this for the fiber over the base point since BQ_T is path connected.) So

$$p^\sharp\colon B(Q_T;T)^\sharp \longrightarrow BQ_T$$

is a fibration with all fibers $\simeq T$. Now suppose that X is a compact CW-space and $g\colon X \to BQ_T$ is any map. Then the pullback $g^*p^\sharp$ is a fibration on X with fibers $\simeq T$. Moreover, if $g_0, g_1\colon W \to BQ_T$ are homotopic, then $g_0^*p^\sharp$ and $g_1^*p^\sharp$ are fiberwise homotopy equivalent as fibrations on X (an exercise of long long ago). Therefore

$$(*) \qquad [g] \mapsto \text{fiber homotopy equivalence class of } g^*p^\sharp$$

is a well defined map. This is the map which the theorem refers to.

In order to be able to show that $(*)$ is a bijection for every X, we make a few observations of a technical nature. (They have much in common with an argument used in the proof of proposition 1.4.1.)

(i) The space of maps $g\colon T \to B(Q_T;T)$ which map T by a homotopy equivalence to a single fiber of $B(Q_T;T) \to BQ_T$ is contractible.

(ii) The space of maps $g\colon T \to B(Q_T;T)^\sharp$ which map T by a homotopy equivalence to a single fiber of $B(Q_T;T)^\sharp \to BQ_T$ is contractible.

(iii) Every commutative diagram

$$\begin{array}{ccc} S^{k-1} \times T & \longrightarrow & B(Q_T;T)^\sharp \\ \downarrow{\scriptstyle \text{proj}} & & \downarrow \\ S^{k-1} & \longrightarrow & BQ_T \end{array}$$

which restricts to homotopy equivalences of the (vertical) fibers admits an extension as indicated, still commutative:

$$\begin{array}{ccc} D^{k} \times T & \dashrightarrow & B(Q_T;T)^\sharp \\ \downarrow{\scriptstyle \text{proj}} & & \downarrow \\ D^{k} & \dashrightarrow & BQ_T \end{array}$$

(iv) For any weak fibration $E \to D^k$ with fibers $\simeq T$, every commutative diagram

$$\begin{array}{ccc} E|_{S^{k-1}} & \longrightarrow & B(Q_T;T)^\sharp \\ \downarrow{\scriptstyle \text{proj}} & & \downarrow \\ S^{k-1} & \longrightarrow & BQ_T \end{array}$$

which restricts to homotopy equivalences of the (vertical) fibers admits an extension as indicated, still commutative:

$$\begin{array}{ccc} E & \dashrightarrow & B(Q_T;T)^\sharp \\ \downarrow{\scriptstyle \text{proj}} & & \downarrow \\ D^{k} & \dashrightarrow & BQ_T \end{array}$$

Proof of (i): That space of maps is just $B(Q_T;Q_T)$.
Proof of (ii): we have a commutative square

$$\begin{array}{ccc} B(Q_T;Q_T) = \text{that space in (i)} & \longrightarrow & \text{that space in (ii)} \\ \downarrow & & \downarrow \\ BQ_T & = & BQ_T \end{array}$$

The fiber of the left-hand map over the base point of BQ_T is Q_T and we showed that the inclusion of that fiber into the corresponding homotopy fiber is a homotopy equivalence. The right-hand map is a fibration whose fiber over the base point of BQ_T is identified with the space of homotopy equivalences from T to the fiber of $B(Q_T;T)^\sharp \to BQ_T$. Since the fiber of $B(Q_T;T)^\sharp \to BQ_T$ contains T, and the inclusion of T in that fiber is a homotopy equivalence, we can say that the fiber (also homotopy fiber) of the right-hand map is again identified (up to homotopy equivalence) with Q_T. It follows that the left-hand vertical homotopy fiber over the base point maps by a weak homotopy equivalence to the right-hand homotopy fiber over the base point, i.e., that square is a homotopy pullback square. Therefore the top horizontal arrow is a homotopy equivalence. Therefore the top right-hand term is contractible.
Proof of (iii): this is just another way to say that every map from S^{k-1} to "that space in (ii)" extends to D^k; and it is true because that space in (ii) is contractible.
Proof of (iv): this should follow from (iii) because $E \to D^k$ is fiberwise homotopy equivalent to a trivial bundle $D^k \times T \to D^k$. (Details left to the gentle reader.)

Now we return to the map $(*)$ in order to show that it is surjective. Let $E \to X$ be a fibration with fibers $\simeq T$, where X is a compact CW-space. Suppose that X has r cells, $r \geq 0$. We want to prove by induction on r that there exists a map $g\colon X \to B(Q_T;T)$ and a fiberwise homotopy equivalence of $g^*p^\sharp$ with E (over X). Equivalently, we want to show that there exists a commutative diagram

$$\heartsuit \qquad \begin{array}{ccc} E & \dashrightarrow & B(Q_T;T)^\sharp \\ \downarrow & & \downarrow \\ X & \dashrightarrow & BQ_T \end{array}$$

where the top horizontal arrow restricts to homotopy equivalences between corresponding (vertical) fibers. Therefore suppose that $X = X' \cup \varphi(D^k)$ where X' is a CW-subspace of X with only $r-1$ cells, and φ is a characteristic map (from D^k to X) for the missing cell. Write E' for the restriction of E to X' and E'' for φ^*E. By inductive assumption, there exists a commutative square

♠
$$\begin{array}{ccc} E' & \longrightarrow & B(Q_T;T)^\sharp \\ \downarrow & & \downarrow \\ X' & \longrightarrow & BQ_T \end{array}$$

where the top horizontal map restricts to homotopy equivalences between corresponding vertical fibers. We can enlarge that to

♣
$$\begin{array}{ccccc} (\partial\varphi)^*E' & \longrightarrow & E' & \longrightarrow & B(Q_T;T)^\sharp \\ \downarrow & & \downarrow & & \downarrow \\ S^{k-1} & \xrightarrow{\partial\varphi} & X' & \longrightarrow & BQ_T \end{array}$$

where $\partial\varphi$ is the restriction of φ to S^{k-1}. If we delete the middle column, then we are in the situation of (iv) just above, although E in (iv) is now called E'' and we should note that

$$E''|_{S^{k-1}} = (\partial\varphi)^*E'.$$

Therefore the (composed) horizontal arrows in ♣ admit extensions to D^k and E'' respectively, as in (iv). Using these extensions together with the horizontal arrows in ♠, we obtain the horizontal arrows in ♡. This proves the surjectivity of $(*)$.

The proof of injectivity is similar, except for a small additional precaution that we need to take. Suppose that g_0 and g_1 are two maps from X to BQ_T such that the fibrations $g_0^*p^\sharp$ and $g_1^*p^\sharp$ are fiberwise homotopy equivalent. Does it follow that there is a fibration on the CW-space $X \times [0,1]$ whose restriction to $X \times \{0\} \cong X$ is fiberwise homeomorphic to $g_0p^\sharp$ and whose restriction to $X \times \{1\} \cong X$ is fiberwise homeomorphic to $g_1p^\sharp$? This may not be easy to arrange, but if we are content with a *weak* fibration on $X \times [0,1]$ having these specified restrictions over $X \times \{0\}$ and $X \times \{1\}$, then we know that the answer is yes. So let $E \to X \times [0,1]$ be such a weak fibration. As in the proof of surjectivity of $(*)$, we can now construct maps from $X \times [0,1]$ to BQ_T and from E to $B(Q_T;T)^\sharp$, compatibly, and already prescribed on/over $X \times \{0\}$ and $X \times \{1\}$. We proceed in steps, one step for each cell of $X \times [0,1]$ which is not in $X \times \{0,1\}$. At the end, we have in particular a map from $X \times [0,1]$ to BQ_T which restricts to g_0 on $X \times \{0\}$ and to g_1 on $X \times \{1\}$. This shows that g_0 and g_1 are homotopic, and so achieves the proof of injectivity. □

Remark 1.7.2. There is a variant of theorem 1.7.1 for based maps, but it is slightly more complicated. Let X be a compact CW-space with base point

$*$ which we assume to be a 0-cell. Then there is a natural bijection

$$[X, BQ_T]_* \longrightarrow \frac{\{\text{fibrns. } E \to X, \text{ fibers } \simeq T, \text{ fiber over } * \text{ trivialized}\}}{\text{fiber homotopy equivalence respecting triv. over } *}.$$

Here $[X, BQ_T]_*$ is the set of based homotopy classes of based maps from X to BQ_T. In more detail: in the right-hand side we allow fibrations $p\colon E \to X$ where all fibers are homotopy equivalent to T, but where *in addition* a map $u\colon T \to E$ has been selected which lands in the fiber $p^{-1}(*)$ and amounts to a homotopy equivalence from T to $p^{-1}(u)$. By a *fiber homotopy equivalence respecting trivializations over* $*$ between two such, say $E \to X$ and $E' \to X$ with $u\colon T \to E$ and $u'\colon T \to E'$, we mean a map $f\colon E \to E'$ over X which is a fiberwise homotopy equivalence, together with a vertical homotopy from fu to u'.

1.8. BG(n) and BG

Specializing the definitions and results of the previous section, let's take $T = S^{n-1}$ for some fixed $n > 0$. It is customary to write $G(n)$ instead of Q_T in that case. So $G(n)$ is the grouplike monoid of homotopy automorphisms of S^{n-1}. Then theorem 1.7.1 specializes to the following statement.

- $BG(n)$ is a classifying space for fibrations with fibers homotopy equivalent to S^{n-1}; so for any compact CW-space X, we have a natural bijection relating $[X, BG(n)]$ to the set of fiber homotopy equivalence classes of fibrations on X with fiber S^{n-1}.

We also obtain from theorem 1.6.7 and/or corollary 1.6.11:

$$\pi_k(BG(n)) \cong \pi_{k-1}(G(n)) \text{ for all } k > 0.$$

Here it is important to remember that $\pi_k(BG(n))$ is $[S^k, BG(n)]_*$, not to be confused with $[S^k, BG(n)]$. In terms of spherical fibrations, we have to use the formulation of remark 1.7.2: so $\pi_{k-1}(G(n))$ is in bijection with fiber homotopy equivalence classes of fibrations $E \to S^k$ with fibers $\simeq S^{n-1}$ and with a specified trivialization of the fiber over the base point $* \in S^k$.

Exercise 1.8.1. Let n be an even integer, $n > 0$.

(i) The *Euler class* of *oriented* n-dimensional vector bundles on S^n gives a homomorphism from $\pi_n(\mathrm{BSO}(n)$ to $\mathbb{Z}$. Is it surjective?

(ii) The standard action of $\pi_1(\mathrm{BO}(n))$ on $\pi_n(\mathrm{BO}(n))$ is nontrivial, so that the forgetful map from $\pi_n(\mathrm{BO}(n))$ to $[S^n, \mathrm{BO}(n)]$ is not injective (although clearly surjective).

(iii) The standard action of $\pi_1(\mathrm{BG}(n))$ on $\pi_n(\mathrm{BG}(n))$ is also nontrivial.

Taking join with the identity map of S^0 determines a map from $\mathrm{G}(n)$ (the space of homotopy automorphisms of S^{n-1}) to $\mathrm{G}(n+1)$ (the space of homotopy automorphisms of $S^n \cong S^{n-1} * S^0$). We tend to think of that as an inclusion and write

$$\mathrm{G} = \bigcup_{n>0} \mathrm{G}(n)$$

for the union in that sense, with the direct limit topology (meaning that a subset of G is considered *closed* if and only if its intersection with $\mathrm{G}(n)$ is closed for all n). This implies for example that a continuous map from a compact CW-space to G will always have image contained in $\mathrm{G}(n)$ for some n, possibly large.

Since the inclusion $\mathrm{G}(n) \to \mathrm{G}(n+1)$ is a map of monoids (a homomorphism), it induces an inclusion

$$\mathrm{BG}(n) \hookrightarrow \mathrm{BG}(n+1).$$

Then we can write

$$\mathrm{BG} = \bigcup_{n\geq 1} \mathrm{BG}(n)$$

where the right-hand side has the direct limit topology (strictly speaking, something to verify here).

- BG is a classifying space for stable spherical fibrations in the following sense. For compact *connected* CW-spaces X there is a natural bijection between $[X, \mathrm{BG}]$ and the set of stable fiber homotopy equivalence classes of spherical fibrations on X. In the notation of definition 1.3.1, the set $\tilde{K}_F(X)$ is in natural bijection with $[X, \mathrm{BG}]$.

  (We have to insist on a connected CW-space X here because we started with an interpretation of $\mathrm{BG}(n)$ as classifying space for fibrations with fibers homotopy equivalent to S^{n-1}, where n is fixed; i.e., strictly speaking not a classification of spherical fibrations on X if we allow fibers $\simeq S^{n-1}$ with unspecified n.)
- $\pi_k(\mathrm{BG}) = \pi_{k-1}\mathrm{G}$.

Exercise 1.8.2. $\pi_1(\mathrm{BG}) \cong \mathbb{Z}/2$ and the standard action of $\pi_1(\mathrm{BG})$ on $\pi_k(\mathrm{BG})$ (for $k \geq 1$) is trivial. Therefore $\pi_k(\mathrm{BG})$ can be identified with $[S^k, \mathrm{BG}]$, and also with $\tilde{K}_F(S^k)$, and of course also with $\pi_{k-1}\mathrm{G}$.

The bijection $\tilde{K}_F(S^k) \to \pi_{k-1}\mathrm{G}$ is actually an isomorphism of abelian groups. (Remember that we used *fiberwise join* to make $\tilde{K}_F(S^k)$ into a group.)

The isomorphism $\tilde{K}_F(S^k) \cong \pi_{k-1}\mathrm{G}$ (of abelian groups) prompts one more remark. Let us write

$$S^{n-1} = S^0 * S^0 * \cdots S^0$$

($n-1$ copies of S^0) and let us fix one point in the first copy of S^0 as the base point, both for that copy of S^0 and also for S^{n-1}. Let $\mathrm{G}'(n) \subset \mathrm{G}(n)$ be the sub-monoid consisting of the maps $S^{n-1} \to S^{n-1}$ which are invertible up to homotopy *and* fix the base point. Now we have the following commutative diagram

$$\begin{array}{ccccccccc}
G'(1) & \longrightarrow & G'(2) & \longrightarrow & G'(3) & \longrightarrow & G'(4) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
G(1) & \longrightarrow & G(2) & \longrightarrow & G(3) & \longrightarrow & G(4) & \longrightarrow & \cdots
\end{array}$$

where all vertical arrows are inclusions, and all horizontal maps are monoid homomorphisms given by join with the identity of (the "last" join factor) S^0. It is easy to show that the map from $\mathrm{G}(n)$ to $\mathrm{G}(n+1)$ in that diagram factors up to (based) homotopy through $\mathrm{G}'(n+1)$:

$$\begin{array}{ccccccccc}
G'(1) & \longrightarrow & G'(2) & \longrightarrow & G'(3) & \longrightarrow & G'(4) & \longrightarrow & \cdots \\
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & & \\
G(1) & \longrightarrow & G(2) & \longrightarrow & G(3) & \longrightarrow & G(4) & \longrightarrow & \cdots
\end{array}$$

It follows that the map from

$$\pi_{k-1}(\mathrm{G}') = \pi_{k-1}(\bigcup_n \mathrm{G}'(n)) = \mathrm{colim}_n \pi_{k-1}(\mathrm{G}'(n))$$

to $\pi_{k-1}\mathrm{G}$ (determined by the vertical arrows) is an isomorphism. But now $\mathrm{G}'(n)$ is the space of based maps $S^{n-1} \to S^{n-1}$ which have degree ± 1; we can also write

$$(\Omega^{n-1}S^{n-1})_{\pm 1}$$

for that. Therefore G' is $(\Omega^\infty S^\infty)_{\pm 1}$ (accepted, traditional notation). This has two path components which are both homotopy equivalent to the base point component of $\Omega^\infty S^\infty$ (easy exercise). In any case

$$\pi_{k-1}\mathrm{G}' = \pi_{k-1}\Omega^\infty S^\infty = \pi^s_{k-1}$$

for $k > 1$. So we arrive at

$$\tilde{K}_F(S^k) \cong \pi^s_{k-1}$$

for $k > 1$. Therefore the homomorphism $\tilde{K}_{\mathbb{R}}(X) \to \tilde{K}_F(X)$ of section 1.3 simplifies in the case $X = S^k$ (where $k > 1$) to a homomorphism

$$\tilde{K}_{\mathbb{R}}(S^k) = \pi_k BO = \pi_{k-1}(O) \quad \longrightarrow \quad \pi^s_{k-1}\,.$$

And so $\tilde{J}(S^k)$ is the image of that homomorphism from $\pi_{k-1}(O)$ to π^s_{k-1}. Since the abelian groups π^s_{k-1} are famously difficult, not well understood but very important in algebraic topology, this gives us an indication of the importance of $\tilde{J}(X)$ in the special case $X = S^k$.

Finally it must be mentioned that, for a compact connected CW-space X, there is a commutative diagram

$$\begin{array}{ccc} \tilde{K}_{\mathbb{R}}(X) & \longrightarrow & \tilde{K}_F(X) \\ \downarrow{\cong} & & \downarrow{\cong} \\ [X, BO] & \longrightarrow & [X, BG] \end{array}$$

where the upper horizontal arrow is as in section 1.3 and the lower horizontal arrow is determined by a map $BO \to BG$. That map $BO \to BG$ is fairly obvious if we interpret BO as $\bigcup_{n \geq 0} BO(n)$ and each $BO(n)$ as the bar construction for the topological group $O(n)$. The only problem is that we have already seen another definition of $BO(n)$, so that we need to make a connection between the two definitions. The following remark gives *some* instructions for that.

Remark 1.8.3. (i) If we re-define $BO(n)$ as the bar construction of $O(n)$, then we have a vector bundle $E^! \to BO(n)$ where $E^! = B(O(n); \mathbb{R}^n)$ (using the standard action of $O(n)$ on $\mathbb{R}^n$).

(ii) For any n-dimensional real vector space V, the space of pairs (x, f) where $x \in BO(n)$ and $f\colon V \to E^!_x$ is a linear isomorphism is contractible. Here $E^!_x$ is the fiber of $E^! \to BO(n)$ over x. (Without loss of generality, V is $\mathbb{R}^n$. Then that space of pairs (x, f) agrees with $B(O(n); GL(n))$ where $GL(n)$ is the (real) general linear group. Since the inclusion $O(n) \to GL(n)$ is a homotopy equivalence, the inclusion $B(O(n); O(n)) \to B(O(n); GL(n))$ is also a homotopy equivalence. But $B(O(n); O(n))$ is contractible.)

(iii) Suppose that X is a paracompact space and carries an n-dimensional vector bundle, $E \to X$. Let $E^\sharp$ be the space of triples (x, f, y) where $x \in X$, $y \in BO(n)$ (bar construction) and f is a linear isomorphism from the fiber E_x to the fiber $E^!_y$. Then the forgetful map $E^\sharp \to X$ is a fiber bundle with contractible fibers. Therefore it admits a section. Such a section means: a map $\nu\colon X \to BO(n)$ and an isomorphism between the vector bundles $f^*E^!$

and E on X. In particular, X could be the *old* incarnation of $BO(n)$, the Grassmannian, and $E \to X$ could be the tautological vector bundle. Then this procedure gives us a map from $BO(n)$ the Grassmannian to $BO(n)$ the bar construction. Since both spaces are classifying spaces for vector bundles, it is easy to see that that map induces an isomorphism on all homotopy groups. Therefore it is at least a *weak equivalence* (by definition of that expression). But in fact both spaces are homotopy equivalent to CW-spaces, so that we can use Whitehead's theorem to deduce that the comparison map is a genuine homotopy equivalence.

(iv) One should reason more carefully in (iii) to ensure that the comparison maps from $BO(n)$ the Grasmannian to $BO(n)$ the bar construction are compatible as n varies.

To sum up: questions about the homomorphism $\tilde{K}_{\mathbb{R}}(X) \to \tilde{K}_F(X)$ of section 1.3 and its image $\tilde{J}(X)$, for "general" X, are really questions about the inclusion map $BO \to BG$, seen from a homotopy theory point of view.