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Lecture notes chapter 6, WS 2015-2016 (Weiss): Vector bundles, J-homomorphism & Adams conjecture

6.1. Localization in homotopy theory

Let A be a subring of $\mathbb{Q}$. Since A is a subring, we have $\mathbb{Z} \subset A \subset \mathbb{Q}$.

Lemma 6.1.1. *A is determined by the set $S = \{p \in \mathbb{N} \mid p$ prime, $p^{-1} \in A\}$, or alternatively by $T = \{p \in \mathbb{N} \mid p$ prime, $p^{-1} \notin A\}$. Indeed A consists of all $a/b \in \mathbb{Q}$ where $a, b \in \mathbb{Z}$ and b is a product of primes in S.* □

In algebra, tensoring with A over $\mathbb{Z}$ (where applicable) has the effect of suppressing p-torsion for all $p \in S$; more precisely, if C is an abelian group then $C_A := C \otimes_{\mathbb{Z}} A$ is an A-module, and so multiplication by p is an isomorphism $C_A \to C_A$ if $p \in S$ (notation of the lemma). Therefore we also say that C_A is the localization of C at the set of primes T. *Example*: tensoring with $A = \mathbb{Z}[2^{-1}]$ over $\mathbb{Z}$ is *localizing away from* 2. Tensoring with $A = \mathbb{Z}[p^{-1} \,|\, p$ odd prime$]$ is *localizing at* 2. I believe at any rate that this is how homotopy theorists use the word *localization*.

Definition 6.1.2. A based connected CW-space X is A-*local* if, for every $k \geq 2$, the canonical homomorphism $\pi_k(X) \to \pi_k(X) \otimes_{\mathbb{Z}} A$ (given by $z \mapsto z \otimes 1$) is an isomorphism. (Then we can say that $\pi_k(X)$ is an A-module.) Equivalently, X is A-local if for every $k \geq 2$ and every prime p which is invertible in A, multiplication by p is an isomorphism $\pi_k(X) \to \pi_k(X)$.

Example 6.1.3. Take $A = \mathbb{Z}[2^{-1}]$. If X is A-local, then $\pi_k(X)$ for $k \geq 2$ has no 2-torsion elements. If $\pi_k(X)$ for some $k \geq 2$ contains an element z which has infinite order, then we get an injection $A \to \pi_k(X)$ taking $a \in A$ to $z \otimes a \in \pi_k(X) \otimes_{\mathbb{Z}} A \cong \pi_k(X)$. It follows that in such a case $\pi_k(X)$ is not a finitely generated abelian group (because $\mathbb{Z}$ is noetherian while A is not finitely generated as an abelian group). For that reason, A-local CW-spaces tend to be somewhat artificial constructions.

Example 6.1.4. Take $A = \mathbb{Q}$. If X is A-local, then $\pi_k(X)$ for $k \geq 2$ is a vector space over $\mathbb{Q}$.

Definition 6.1.5. Let X and Y be based connected CW-spaces, $f\colon X \to Y$ a based map. The map f is an A-*equivalence* if

- $f_*\colon \pi_1(X) \to \pi_1(Y)$ is an isomorphism;
- for all $k \geq 2$, the homomorphism $\pi_k(X) \otimes_{\mathbb{Z}} A \to \pi_k(Y) \otimes_{\mathbb{Z}} A$ induced by f is an isomorphism (of A-modules).

The map f is said to be an A-*localization* (in these notes) if it is an A-equivalence and Y is A-local.

Example 6.1.6. Take $A = \mathbb{Z}[3^{-1}]$ and $X = S^5$. Choose a based map $g\colon S^5 \to S^5$ of degree 3. Let Y be the reduced mapping telescope (iterated mapping cylinder) associated with the sequence of based maps

$$S^5 \xrightarrow{g} S^5 \xrightarrow{g} S^5 \xrightarrow{g} S^5 \xrightarrow{g} S^5 \xrightarrow{g} \cdots$$

Then we find

$$\pi_k(Y) = \mathrm{colim}\big[\ \pi_k(S^5) \xrightarrow{\cdot 3} \pi_k(S^5) \xrightarrow{\cdot 3} \pi_k(S^5) \xrightarrow{\cdot 3} \pi_k(S^5) \xrightarrow{\cdot 3} \cdots\ \big]$$

which is the same as $\pi_k(S^5) \otimes_{\mathbb{Z}} A$. In other words, the inclusion $X \to Y$ is an A-localization.

The space Y can also be described as follows. It is (homotopy equivalent to) the pushout of a diagram of based maps

$$\bigvee_{j=1}^{\infty} S^5 \longleftarrow \bigvee_{j=1}^{\infty} S^5 \longrightarrow \bigvee_{j=1}^{\infty} D^6$$

where the arrow on the right is the standard inclusion and the arrow on the left has degree matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

In this description, Y is a CW-space with one 0-cell (base point), a countable infinity of 5-cells and a countable infinity of 6-cells.

Similarly, take $A = \mathbb{Q}$ and $X = S^5$. Let $g_j\colon S^5 \to S^5$ be a based map of degree j, where $j = 2, 3, 4, 5, \ldots$. Let Y be the reduced mapping telescope associated with the sequence of based maps

$$S^5 \xrightarrow{g_2} S^5 \xrightarrow{g_3} S^5 \xrightarrow{g_4} S^5 \xrightarrow{g_5} S^5 \xrightarrow{g_6} \cdots$$

Then we find

$$\pi_k(Y) = \mathrm{colim}\big[\ \pi_k(S^5) \xrightarrow{\cdot 2} \pi_k(S^5) \xrightarrow{\cdot 3} \pi_k(S^5) \xrightarrow{\cdot 4} \pi_k(S^5) \xrightarrow{\cdot 5} \cdots\ \big]$$

which is the same as $\pi_k(S^5) \otimes_{\mathbb{Z}} A$. In other words, the inclusion $X \to Y$ is an A-localization.

Exercise 6.1.7. Compute or describe $[Y, S^6]_*$ (set of based homotopy classes of based maps) for the two instances of Y in example 6.1.6.

The second Y in example 6.1.6 is an Eilenberg-MacLane space. That is, it has just one nontrivial homotopy group: $\pi_5(Y) \cong \mathbb{Q}$ and $\pi_k(Y) = 0$ for all $k \neq 5$. Why ?

Theorem 6.1.8. *For any subring $A \subset \mathbb{Q}$ and based connected CW-space X, there exists an* (inclusion) *map $f \colon X \to Y$ of based connected CW-spaces which is an A-localization. This has the following universal property:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, dotted, "g^\sharp"] \\ & Z \end{tikzcd}$$

for every based map $g \colon X \to Z$, where Z is an A-local connected based CW-space, there exists a map $g^\sharp \colon Y \to Z$ such that $g^\sharp f = g$. The map $g^\sharp$ is unique up to a homotopy $(h_t \colon Y \to Z)_{t \in [0,1]}$ which is stationary on X, so that $h_t|_X$ is independent of t.

Proof of existence of the A-localization. We construct Y in the form of an ascending union of CW-spaces

$$\bigcup_{j=0}^{\infty} X_j$$

where $X = X_0 \subset X_1 \subset X_2 \subset X_3 \subset \cdots$ and X_{j+1} is obtained from X_j by certain elementary gluing-on operations. These gluing-on operations are as follows.

- For every $[z] \in \pi_k(X_j)$ where $k \geq 2$ and every $q \in \mathbb{Z}$ which has $q^{-1} \in A$, we form a reduced mapping cylinder

  $$\text{cyl}[\, S^k \xrightarrow{\deg q} S^k \,]$$

  and glue it to X_j using the map z for gluing (on the source end of the mapping cylinder).
- For every $[z] \in \pi_k(X_j)$ where $k \geq 2$ and every $q \in \mathbb{Z}$ which has $q^{-1} \in A$ and $q[z] = 0 \in \pi_k(X_j)$, we choose a factorization

  $$\begin{tikzcd} S^k \arrow[rr, "z"] \arrow[dr, "\text{incl.}"'] & & X_j \\ & C_{k,q} := \text{cone}[S^k \xrightarrow{\deg q} S^k] \arrow[ur, dotted, "\bar{z}"'] & \end{tikzcd}$$

(which we can since $q[z] = 0$). Then we glue a copy of $\mathrm{cone}(C_{k,q})$ to X_j using the map $\bar{z}$ for gluing. (Beware: $\mathrm{cone}(C_{k,q})$ is the cone of a cone, yes, but the word cone in $C_{k,q} = \mathrm{cone}[S^k \to S^k]$ means a *mapping cone*, while the word cone in $\mathrm{cone}(C_{k,q})$ means an ordinary cone, a.k.a. mapping cone of a map to a point.)

The first of these two gluing operations has the following effect: *every* $[z] \in \pi_k(X_j)$ *becomes divisible by* q *in* $\pi_k(X_{j+1}$*, assuming* q^{-1} *belongs to* A. The second operation has the following effect: *every* $[z] \in \pi_k(X_j)$ *which satisfies* $q[z] = 0$ *in* $\pi_k(X_j)$ *maps to* 0 *in* $\pi_k(X_{j+1})$*, assuming* q^{-1} *belongs to* A. As a consequence, for the ascending union

$$Y = \bigcup_j X_j$$

we have: every $[z] \in \pi_k(Y)$ is divisible by q in $\pi_k(X_{j+1}$, assuming q^{-1} belongs to A; and if $[z] \in \pi_k(X)$ satisfies $q[z] = 0$, then already $[z] = 0 \in \pi_k(Y)$. (This is claimed for $k \geq 2$.) Therefore Y is A-local. Also, it is fairly clear from the construction (using Seifert-vanKampen) that the inclusion $X \to Y$ induces an isomorphism on fundamental groups, since we attached only cells of dimension ≥ 2, and where we attached 2-cells we did so only by forming a wedge with S^2.

But it remains to be shown that the inclusion $X \to Y$ induces an isomorphism in $\pi_k \otimes_{\mathbb{Z}} A$ for all $k \geq 2$. Of course it is enough to show that the inclusion $X_j \to X_{j+1}$ induces an isomorphism in $\pi_k \otimes_{\mathbb{Z}} A$ for all $k \geq 2$.

Here (I am sorry to say) I will use Serre theory. A good reference is the old book by Spanier. One of the major results of Serre theory says that if a based map $V \to W$ of based simply connected CW-spaces induces an isomorphism in $H_*(-) \otimes_{\mathbb{Z}} A$, then it induces an isomorphism in $\pi_* \otimes_{\mathbb{Z}} A$. We cannot apply this directly to the inclusion $X_j \to X_{j+1}$ since we did not try to ensure that X_j and X_{j+1} are simply connected. But we can pass to the universal covers:

$$\tilde{X}_j \hookrightarrow \tilde{X}_{j+1} \,.$$

So it remains to show that this inclusion, $\tilde{X}_j \hookrightarrow \tilde{X}_{j+1}$, induces an isomorphism in $H_*(-) \otimes_{\mathbb{Z}} A$. But this is easy. We can think of

$$\tilde{X}_{j+1}$$

as being obtained from $\tilde{X}_j$ by certain gluing/attaching operations, essentially the same operations that produced X_{j+1} from X_j , just more of them. Therefore we have attached copies of

$$\mathrm{cyl}[\, S^k \xrightarrow{\deg q} S^k \,]$$

(along the cylinder top) and copies of $\mathrm{cone}(C_{k,q})$ (along the boundary of the cone). This does not change $H_*(-)\otimes_{\mathbb{Z}} A$ since, in both cases, the inclusion of the cylinder top (in the cylinder) and the inclusion of the cone boundary (in $\mathrm{cone}(C_{k,q})$) induce isomorphisms in $H_*(-)\otimes_{\mathbb{Z}} A$. This is easy to verify. □

Proof of the universal property. We constructed Y as $\bigcup X_j$. Suppose per induction that $g^\sharp$ is already defined on X_j; then we need to extend to X_{j+1}. (The induction begins with $X_0 = X$, where $g^\sharp = g$ by definition.) Since X_{j+1} was constructed from X_j by attaching certain cylinders and cones, we need to ask whether $g^\sharp$, already defined on X_j, can be extended when we attach such a cylinder or cone. The map is prescribed on the top of the cylinder, or on the boundary of the cone (because these are glued to X_j) and we need to extend to the entire cylinder/cone. This is straightforward in each case since Z is A-local.

For the uniqueness, we can reason in a similar step-by-step way. Suppose that we have found two solutions. We need a homotopy $h\colon Y\times[0,1]\to Z$ connecting these two, and stationary on $X\subset Y$. Per induction we can suppose that h is already defined on $X_j\times[0,1]$. We wish to extend to $X_{j+1}\times[0,1]$. Since X_{j+1} was constructed from X_j by attaching certain cylinders and cones, we need to ask whether h, already defined on the union of $X_j\times[0,1]$ and $Y\times\{0,1\}$, can be extended when we attach such a cylinder or cone to X_j. In the cylinder case, h is prescribed on the union of (cylinder top)$\times[0,1]$ and (all of cylinder)$\times\{0,1\}$; we need to extend to (all of cylinder)$\times[0,1]$. The extension is easy to construct since the target space Z is A-local. The cone case is similar. In this case h is prescribed on the union of (cone boundary)$\times[0,1]$ and (all of cone)$\times\{0,1\}$. We need to extend to (all of cone)$\times[0,1]$. □

Exercise 6.1.9. Let X,Y,Z be based connected CW-spaces. Suppose that $X\subset Y$ as a based CW-subspace and suppose that the inclusion is an A-equivalence. Suppose that Z is A-local. Show that any based map $g\colon X\to Z$ can be extended to a map $Y\to Z$. The extension is unique up to a homotopy (stationary on X).

6.2. Localization and mapping spaces

Suppose that P and X are based connected CW-spaces, P compact. Let X_A be the connected CW-space obtained from X by A-localization as in theorem 6.1.8. We may write $X\subset X_A$.

Let $\mathrm{map}_*(P,X)$ be the space of based maps from P to X. (If we want to insist that $\mathrm{map}_*(P,X)$ is strictly a CW-space, then we can define it as the geometric realization of the simplicial set of based maps from P to X; a k-simplex is a based map from $P\wedge\Delta^k_+$ to X.)

Let us ask whether $\mathrm{map}_*(P, X_A)$ has a good chance of being the A-localization of $\mathrm{map}_*(P, X)$. More precisely the inclusion $X \to X_A$ determines an inclusion

$$\mathrm{map}_*(P, X) \longrightarrow \mathrm{map}_*(P, X_A).$$

We ask whether this satisfies the conditions for an A-localization (if we discard the non-base-point components).

Example 6.2.1. If $P = X = S^5$ and $A = \mathbb{Q}$ then

$$\pi_0 \mathrm{map}_*(P, X) = \pi_0 \mathrm{map}_*(S^5, S^5) \cong \mathbb{Z}$$

whereas $\pi_0 \mathrm{map}_*(S^5, S^5_{\mathbb{Q}}) = \pi_5(S^5_{\mathbb{Q}}) \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$. So on π_0 , the inclusion

$$\mathrm{map}_*(P, X) \longrightarrow \mathrm{map}_*(P, X_A)$$

induces an injection (but not a bijection) which looks like $\mathbb{Z} \hookrightarrow \mathbb{Q}$. With the same choices for P and X we have $\pi_1 \mathrm{map}_*(P, X) = \pi_1 \mathrm{map}_*(S^5, S^5) \cong \pi_6(S^5) \cong \mathbb{Z}/2$ whereas

$$\pi_1 \mathrm{map}_*(S^5, S^5_{\mathbb{Q}}) \cong \pi_6(S^5_{\mathbb{Q}}) \cong \pi_6(S^5) \otimes \mathbb{Q} \cong \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

Therefore the inclusion $\mathrm{map}_*(P, X) \longrightarrow \mathrm{map}_*(P, X_A)$ does not induce a bijection on π_1.

Lemma 6.2.2. *Every connected component of* $\mathrm{map}_*(P, X_A)$ *is* A*-local, for any choice of base point* f *in that component.*

Proof. Choose a based map $f \colon P \to X_A$, declare it to be the base point of $\mathrm{map}_*(P, X_A)$, and choose $k \geq 2$. Now $\pi_k(\mathrm{map}_*(P, X_A))$ is the abelian group of homotopy classes of maps from $S^k \times P$ to X_A which take $S^k \times *$ to the base point and which map $* \times P$ by f. In this description we can replace S^k by S^k_A without making a difference, since the target X_A is A-local (this is similar to exercise 6.1.9). Then we get a structure of module over the ring

$$A \cong [S^k_A, S^k_A]_* \cong [S^k, S^k_A]_* = \pi_k(S^k_A)$$

(on the abelian group $\pi_k(\mathrm{map}_*(P, X_A))$) by pre-composition: $g \in [S^k_A, S^k_A]_*$ acts on an element represented by $S^k_A \times P \to X_A$ by pre-composition with

$$g \times \mathrm{id}_P \colon S^k_A \times P \longrightarrow S^k_A \times P.$$

Here $[S^k_A, S^k_A]_*$ is a ring in the following way: the multiplication is given by composition, while addition is the usual addition in $\pi_k(S^k_A)$. $\square$

Proposition 6.2.3. *The inclusion* $\mathrm{map}_*(P, X) \to \mathrm{map}_*(P, X_A)$ *induces an isomorphism*

$$\pi_k(\mathrm{map}_*(P, X)) \otimes_{\mathbb{Z}} A \longrightarrow \pi_k(\mathrm{map}_*(P, X_A)) \otimes_{\mathbb{Z}} A \cong \pi_k(\mathrm{map}_*(P, X_A))$$

for every $k \geq 2$ *and every choice of base point* $f \in \mathrm{map}_*(P, X) \subset \mathrm{map}_*(P, X_A)$.

Proof. We assumed that P is a *compact* based CW-space. Therefore we can proceed by induction on the number of cells in P. The induction beginning is the case where P has no cells other than the base point. That case is clear. For the induction step, let P be P, of dimension d, and let Q be obtained from P by removing a d-dimensional cell (not the base point).

Let $u\colon S^k \times P \to X_A$ represent an element of $\pi_k(\mathrm{map}_*(P, X_A))$. It is understood that u takes $S^k \times *$ to the base point of X_A and maps $* \times P$ by f. Let z be an integer which is invertible in A. We want to show that $z^n[u]$ comes from $\pi_k(\mathrm{map}_*(P, X))$, for some $n \geq 0$. So we need a factorization up to homotopy

$$(*)\qquad \begin{tikzcd} S^k \times P \arrow[r, "(\deg z^n)\times \mathrm{id}_P"] \arrow[d, dotted] & S^k \times P \arrow[d, "u"] \\ X \arrow[r] & X_A \end{tikzcd}$$

(The homotopy is required to be stationary on $* \times P$.) By inductive assumption we can assume that such a factorization has already been found on $S^k \times Q$, with a specific factor z^m instead of the so far undetermined z^n. The problem of extending this factorization from $S^k \times Q$ to $S^k \times P$ is a problem concerning a single cell of top dimension $k + d$ in $S^k \times P$; the obstruction is therefore is an element $[w]$ of $\pi_{k+d}(X_A, X)$ (with a possibly new choice of base point in X, which does not matter since X is connected). By the properties of X_A we know that $z^\ell[w] = 0$ for some $\ell \gg 0$. It follows easily that the factorization $(*)$ exists (for P, not just Q) if we use a factor z^n where $n = m + \ell$.

This shows that $\pi_k(\mathrm{map}_*(P, X)) \otimes_{\mathbb{Z}} A \longrightarrow \pi_k(\mathrm{map}_*(P, X_A)) \otimes_{\mathbb{Z}} A$ is onto. The argument for injectivity is similar and left to the reader, except for the following instructions. We begin with a map of pairs

$$v\colon (D^{k+1} \times P, S^k \times P) \longrightarrow (X_A, X).$$

This is meant to represent an element of $\pi_k(\mathrm{map}_*(P, X))$ which goes to zero in $\pi_k(\mathrm{map}_*(P, X_A))$; we have already selected a nullhomotopy which confirms that it goes to zero. It is understood that v takes $D^{k+1} \times *$ to the base point of X_A and maps $* \times P$ by f. We need a factorization up to homotopy

$$(**)\qquad \begin{tikzcd} (D^{k+1} \times P, S^k \times P) \arrow[r, "(\deg z^n)\times \mathrm{id}_P"] \arrow[d, dotted] & (D^{k+1} \times P, S^k \times P) \arrow[d, "v"] \\ (X, X) \arrow[r] & (X_A, X) \end{tikzcd}$$

This would show that $\partial v\colon S^k \times P \longrightarrow X$ becomes zero after multiplying with z^n. The factorization $(**)$ can be constructed by induction on the number of cells of P, as before. — Note that the argument for injectivity is actually an argument for both surjectivity and injectivity, but I felt it would be kinder to give the surjectivity argument first. □

6.3. Localization and spherical fibrations

For $n \geq 3$ let $G(n) \subset \mathrm{map}(S^{n-1}, S^{n-1})$ be the group-like topological monoid of homotopy automorphisms of S^{n-1} as in section 1.8. Let

$$G'(n) \subset \mathrm{map}_*(S^{n-1}, S^{n-1})$$

be the group-like topological monoid consisting of the degree ± 1 components in $\mathrm{map}_*(S^{n-1}, S^{n-1})$. We showed in section 1.8 that the inclusion

$$\bigcup_n G'(n) \simeq \bigcup_n G(n)$$

is a weak homotopy equivalence, i.e., induces an isomorphism on homotopy groups. (I propose that we accept that $G'(n)$ and $G(n)$ have the homotopy type of CW-spaces, and that the inclusions $G(n) \to G(n+1)$, $G'(n) \to G'(n+1)$ are cofibrations; then we may conclude that the inclusion $\bigcup G'(n) \to \bigcup G(n)$ is an honest homotopy equivalence.)

We fix a subring $A \subset \mathbb{Q}$ as before. Let

$$G'(n)_{A,\pm 1} \subset \mathrm{map}_*(S_A^{n-1}, S_A^{n-1})$$

be the union of the degree ± 1 connected components. (Notice that the space $\mathrm{map}_*(S_A^{n-1}, S_A^{n-1})$ has as many connected components as A has elements, since $\pi_0 \mathrm{map}_*(S_A^{n-1}, S_A^{n-1}) = \pi_{n-1}(S_A^{n-1}) \cong A$. For the present discussion it is clear that we should discard the non-invertible ones, but actually I want to discard some more, in case A has invertible elements other than ± 1.)

Lemma 6.3.1. *The inclusion $BG'(n) \to BG'(n)_{A,\pm 1}$ is an A-localization.*

Proof. By construction, this inclusion map is a based map of connected based spaces which induces an isomorphism on π_1. It remains to verify that the homomorphisms

$$\pi_k(BG'(n)) \otimes_{\mathbb{Z}} A \longrightarrow \pi_k(BG'(n)_{A,\pm 1}) \otimes_{\mathbb{Z}} A \longleftarrow \pi_k(BG'(n)_{A,\pm 1})$$

for $k \geq 2$ (one induced by the inclusion, the other given by $x \mapsto x \otimes 1$) are isomorphisms. This is clear from lemma 6.2.2 and proposition 6.2.3 if we rewrite them in the form

$$\pi_{k-1}(G'(n)) \otimes_{\mathbb{Z}} A \longrightarrow \pi_{k-1}(G'(n)_{A,\pm 1}) \otimes_{\mathbb{Z}} A \longleftarrow \pi_{k-1} G'(n)_{A,\pm 1}.$$

(The case $k = 2$ requires a separate verification: like example 6.2.1.) □

Let $G'_{A,\pm 1}$ be the union $\bigcup_n G'(n)_{A,\pm 1}$.

Corollary 6.3.2. *The inclusion $BG' \to BG'_{A,\pm 1}$ is an A-localization.*

Proof. It suffices to note that $\otimes_{\mathbb{Z}} A$ commutes with the passage to a (sequential) direct limit of abelian groups. $\square$

For a compact connected based CW-space X we may also write

$$\tilde{K}_{F,A,\pm 1}(X) := [X, BG'_{A,\pm 1}]_*$$

and we can view this as the set of stable fiberwise homotopy equivalence classes of fibrations with fiber $\simeq S_A^{n-1}$ for some n, possible large. (Because of the tilde and all that, we should be a little more careful: fibrations on X with fiber $\simeq S_A^{n-1}$ and a specific choice of homotopy equivalence of the fiber over the base point with S_A^{n-1}. This specific choice must be taken into account in the precise definition of *stable equivalence* between two such fibrations on X. See also remark 6.3.6 below.) Whitney sum (=fiberwise join) of such fibrations then defines an abelian group structure on

$$\tilde{K}_{F,A,\pm 1}(X).$$

It is wonderful how this leads us inexorably to the following exercise:

Exercise 6.3.3. Show that $S_A^{m-1} * S_A^{n-1} \simeq S_A^{m+n-1}$.

Corollary 6.3.4. *For X as above, the homomorphism*

$$\tilde{K}_F(X) \longrightarrow \tilde{K}_{F,A,\pm 1}(X)$$

induced by the inclusion $BG' \longrightarrow BG'_{A,\pm 1}$ induces an isomorphism

$$\tilde{K}_F(X) \otimes_{\mathbb{Z}} A \longrightarrow \tilde{K}_{F,A,\pm 1}(X) \otimes_{\mathbb{Z}} A.$$

Proof. In the case where X is a sphere S^ℓ, we have two competing abelian group structures on

$$\tilde{K}_F(S^\ell) = [S^\ell, BG']_* = \pi_\ell(BG').$$

One of these is the Whitney sum addition, the other is the ordinary addition in a homotopy group. But it is well known, and easy to verify, that these two abelian group structures agree. The same can be said for the two competing abelian group structures on

$$\tilde{K}_{F,A,\pm 1}(S^\ell).$$

Therefore in the case where X is a sphere S^ℓ, the claim is that the homomorphism of homotopy groups

$$\pi_\ell(BG') \longrightarrow \pi_\ell(BG'_{A,\pm 1})$$

becomes an isomorphism after tensoring with A. This is correct for $\ell \geq 2$ by corollary 6.3.2. It is also correct for $\ell = 1$, and we could invoke corollary 6.3.2 for that again, but a better argument is that

$$\pi_1(BG') \to \pi_1(BG'_{A,\pm 1})$$

is already an isomorphism before tensoring with A, by construction. (The two fundamental groups are both isomorphic to $\mathbb{Z}/2 = \{\pm 1\}$.)

In the case of arbitrary X, we now proceed by induction on the number of cells. Suppose that X has dimension ℓ and let $Y \subset X$ be a CW-subspace obtained be deleting an ℓ-cell. (We can suppose that X has only one 0-cell, the base point; therefore it is alright to assume $\ell > 0$.) Then we have a commutative diagram

$$\begin{tikzcd}
\tilde{K}_F(S^1 \wedge Y) \arrow[r] \arrow[d] & \tilde{K}_F(S^\ell) \arrow[r] \arrow[d] & \tilde{K}_F(X) \arrow[r] \arrow[d] & \tilde{K}_F(Y) \arrow[r] \arrow[d] & \tilde{K}_F(S^{\ell-1}) \arrow[d] \\
\tilde{K}_{F,A,\pm1}(S^1 \wedge Y) \arrow[r] & \tilde{K}_{F,A,\pm1}(S^\ell) \arrow[r] & \tilde{K}_{F,A,\pm1}(X) \arrow[r] & \tilde{K}_{F,A,\pm1}(Y) \arrow[r] & \tilde{K}_{F,A,\pm1}(S^{\ell-1})
\end{tikzcd}$$

with exact rows. (*Perhaps more details later ...*) By inductive assumption the vertical arrows number 1 and 4, counting from the left, become isomorphisms after tensoring with A. By the separate verification for S^ℓ, vertical arrows number 1 and 5 became isomorphisms after tensoring with A. (The case where $\ell = 1$, hence $\ell - 1 = 0$, is a little special but in that case the terms in column 5 are all zero.) Therefore the middle vertical arrow becomes an isomorphism after tensoring with A. (This is a special property of A: tensor product $\otimes_{\mathbb{Z}} A$ has good exactness properties; A is a *flat* module over $\mathbb{Z}$.) $\square$

Remark 6.3.5. The case $X = S^1$ shows us that $\tilde{K}_{F,A,\pm1}(X)$ in corollary 6.3.4 need not be an A-module. This is a little disappointing. The reason is clearly that we were too afraid to touch π_1 in the localization process. Here is an observation which sheds more light on this. We can write

$$\tilde{K}_{F,A,\pm1}(X) \cong \tilde{K}_{F,A,+1}(X) \times H^1(X; \mathbb{Z}/2)$$

where $\tilde{K}_{F,A,+1}(X)$ is, after all, an A-module. Namely, for a fibration $E \to X$ with fibers S_A^{n-1} , classified stably by a map

$$X \to BG'_{A,\pm1} \ ,$$

there is a homomorphism from $\pi_1(X)$ to $\mathbb{Z}/2 = \{\pm 1\}$ which encodes the orientation behavior of $E \to X$. (For a based map $\gamma \colon S^1 \to X$, fiber transport in E along γ gives a map of degree ± 1 from S_A^{n-1} to itself. In this way we get a homomorphism $\pi_1(X) \to \{\pm 1\}$. This is worth as much as an element of H^1 of X with coefficients in $\mathbb{Z}/2$.) If the orientation behavior of E is nontrivial, then we can form the Whitney sum (fiberwise join) of E with an appropriate spherical fibration with fibers S^0 in such a way that the Whitney sum is an *oriented* spherical fibration, classified by a map from X to $BG'_{A,+1}$ and giving an element of

$$\tilde{K}_{F,A,+1}(X).$$

Showing that $\tilde{K}_{F,A,+1}(X)$ is an A-module follows the usual inductive pattern; the important new feature is that S^1 does not present a problem because

$$\tilde{K}_{F,A,+1}(S^1) = 0.$$

Proof of proposition 4.5.4. At last. Let $E \to X$ and $E' \to X$ be fibrations on X with fiber S^{n-1} and assume that there exists a map $f\colon E \to E'$ over X which, on the fibers, has degree $\pm k$. Let

$$g\colon S^2_A \to S^2_A$$

be a map of degree k^{-1}, which we can also view as a map between fibrations over a point. Taking the fiberwise (external) join of E and E' with S^2_A , and of f with e, we obtain a map which, for lack of helpful notation, I describe in words as the fiberwise join of f and g. It is a map (over X) of fiberwise degree $\pm k \cdot k^{-1} = \pm 1$ from a stabilization (and fiberwise localization) of E to a stabilization (and fiberwise localization) of E'. (We stabilized and simultaneously localized by taking fiberwise join with a localized 2-sphere.) It follows that E and E' represent the same element of

$$\tilde{K}_{F,A,\pm 1}(X).$$

Now corollary 6.3.4 tells us that $[E] = [E'] \in \tilde{K}_F(X) \otimes_{\mathbb{Z}} A$. $\square$

Remark 6.3.6. In chapter 1 we encountered spaces of homotopy automorphisms Q_T , where T is a compact CW-space and $Q_T \subset \mathrm{map}(T, T)$ is the space (and group-like topological monoid) of homotopy invertible maps from T to T. We constructed BQ_T , a classifying space for fibrations with fibers homotopy equivalent to T.

In this chapter I may have given the impression (so far) that we can take a localized sphere S^{n-1}_A for T. Unfortunately these localized spheres are typically not compact (and typically not even homotopy equivalent to compact CW-spaces). Therefore it is necessary to proceed somewhat differently. For example, for an arbitrary CW-space T, we can re-define Q_T as the geometric realization of the *simplicial set* of maps from T to T. (Then a k-simplex in this simplicial set is a map from $\Delta^k \times T$ to T; we may even require this to be cellular.) With such a definition, Q_T is a CW-space and a group-like topological monoid. Then we can construct BQ_T.

(This calls for a further explanation. A CW-space is a *compactly generated Hausdorff space*, that is to say, it is a Hausdorff space in which a subset is open if and only if its intersection with every compact subset is open. There is such a thing as the category of compactly generated Hausdorff spaces. It has products, but the product of two compactly generated Hausdorff spaces X and Y does not quite agree with the product of X and Y in the category of all spaces. It can be constructed by forming the ordinary product topology $\mathcal{U}$

on $\mathsf{X}\times\mathsf{Y}$ first, then redefining a subset of $\mathsf{X}\times\mathsf{Y}$ to be open if its intersection with every compact subset of $\mathsf{X}\times\mathsf{Y}$, according to the topology $\mathcal{U}$, is open. This gives a new topology $\mathcal{U}'$ on $\mathsf{X}\times\mathsf{Y}$ which certainly contains $\mathcal{U}$ but which, in some cases, is strictly bigger. — The product of two CW-spaces is again a CW-space (with an obvious cell decomposition, so that cells in the product are products of cells in the factors) if we interpret *product* as the product in the category of compactly generated Hausdorff spaces. So when I say that the CW-space Q_T is a group-like topological monoid, I have in mind a map $\mathsf{Q}_\mathsf{T}\times\mathsf{Q}_\mathsf{T}\to\mathsf{Q}_\mathsf{T}$ where the product $\mathsf{Q}_\mathsf{T}\times\mathsf{Q}_\mathsf{T}$ is formed in the category of compactly generated spaces ... and in that situation $\mathsf{Q}_\mathsf{T}\times\mathsf{Q}_\mathsf{T}$ is again a CW-space. Furthermore, a construction like $\mathsf{B}\mathsf{Q}_\mathsf{T}$ should also be done in the category of compactly generated Hausdorff spaces, for consistency.)