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Lecture Notes, week 14
Topology WS 2014/15 (Weiss)

14.1. Higher homotopy groups

Definition 14.1. Let X be a space with base point $\star$ and let n be a non-negative integer. Write $\pi_n(X,\star)$ for the set $[S^n,X]_*$ (based homotopy classes of based maps from S^n to X). It is clear that π_n is a covariant functor from $\mathcal{H}o\mathcal{T}op_\star$ (the homotopy category of based spaces) to sets.

The case $n=1$ has already been looked at in detail and we saw that $\pi_1(X,\star)$ is a group in a natural way.

The case $n=0$ is also useful. Namely, $\pi_0(X,\star)$ is just the set of path components of X. Indeed, a based map $f\colon S^0\to X$ must send the base point -1 of S^0 to the base point of X. So the only interesting feature it has is the value $f(1)\in X$. And if we pass to homotopy classes, only the path component of $f(1)$ remains.
There is no point in trying to put a natural group structure on $\pi_0(X,\star)$. We must accept that it is in most cases just a set. (There are exceptions: if X has the structure of a topological group, then $\pi_0(X)$ also has the structure of a group in an obvious way, and that can be useful.)

Definition 14.2. *For $n\geq 2$, the set $\pi_n(X,\star)$ has the structure of an abelian group in a natural way. In other words we can equip $\pi_n(X,\star)$ with a structure of abelian group in such a way that, for every based map $f\colon X\to Y$, the induced map of sets*

$$\pi_n(X,\star)\to\pi_n(Y,\star)$$

becomes a homomorphism of abelian groups. The neutral element of $\pi_n(X,\star)$ is represented by the unique constant based map from S^n to X.

For the proof, we note first that

$$\pi_n(X,\star)\times\pi_n(X,\star)=[S^n,X]_\star\times[S^n,X]_\star\cong[S^n\vee S^n,X]_\star$$

(where $\cong$ is used for an obvious bijection). Therefore it is reasonable to try to construct a multiplication map

$$\mu\colon\pi_n(X,\star)\times\pi_n(X,\star)\to\pi_n(X,\star)$$

by writing this in the form $\mu\colon[S^n\vee S^n,X]_\star\longrightarrow[S^n,X]_\star$ and defining it as pre-composition with some fixed element $\kappa\in[S^n,S^n\wedge S^n]_*$.

Elementary description of κ. Think of S^n as the quotient space of $[0,1]^n$ obtained by collapsing the subspace consisting of all points which have some coordinate equal to 0 or 1. Think of $S^n\vee S^n$ as the quotient space of

$[0,2] \times [0,1]^{n-1}$ obtained by collapsing all points which have some coordinate equal to 0 or 1, or first coordinate 2. Then κ can be defined by $\kappa(x_1, x_2, \ldots, x_n) := (2x_1, x_2, \ldots, x_n)$, where $x_1, x_2, \ldots, x_n \in [0,1]$. It is easy to verify the following directly: the compositions

$$S^n \xrightarrow{\ \kappa\ } S^n \vee S^n \xrightarrow{\mathrm{id}\vee\kappa} S^n \vee (S^n \vee S^n)$$

and

$$S^n \xrightarrow{\ \kappa\ } S^n \vee S^n \xrightarrow{\kappa\vee\mathrm{id}} (S^n \vee S^n) \vee S^n$$

are based homotopic. This implies that our formula for the multiplication μ on $[S^n, X]_\star$ is *associative*. Next, it is easy to verify the following directly: the composition

$$S^n \xrightarrow{\kappa} S^n \vee S^n \xrightarrow{\text{permute summands}} S^n \vee S^n$$

is based homotopic to κ. (Here we need $n > 1$.) This implies that our formula for the multiplication μ on $[S^n, X]_\star$ is *commutative*. Furthermore, it is easy to verify directly that the constant based map $S^n \to X$ is a two-sided neutral element for the multiplication μ. (In cubical coordinates for S^n, multiplication with the constant map has the effect of replacing a based map

$$f \colon \frac{[0,1]}{\sim} \longrightarrow X$$

by the based map g where $g(x_1, \ldots, x_n) = f(2x_1, x_2, \ldots, x_n)$ when $2x_1 \le 1$ and $g(x_1, \ldots, x_n) = \star \in X$ when $2x_1 \ge 1$. So the task is to show that f is based homotopic to g ... and that is easy.) Next, it is easy to verify directly that an element $[f] \in [S^n, X]_\star$ has an inverse given by $[f \circ \eta]$ where $\eta \colon S^n \to S^n$ is given in cubical coordinates by $(x_1, x_2, \ldots, x_n) \mapsto (1 - x_1, x_2, \ldots, x_n)$. (In cubical coordinates for S^n, the product of $[f]$ and $[f \circ \eta]$ is given by g where $g(x_1, \ldots, x_n) = f(2x_1, x_2, \ldots, x_n)$ when $2x_1 \le 1$ and $g(x_1, \ldots, x_n) = f(2 - 2x_1, x_2, \ldots, x_n)$ when $2x_1 \ge 1$.)

Although the homotopy groups π_n have a great deal of theoretical importance, they are very hard to compute in general, especially for large n. Recently I read in an article about homotopy theory: *not a single compact connected CW-space* X *is known for which we have a formula describing* $\pi_n(X)$ *for all* $n > 0$, except for two types:

- the totally uninteresting case where X is contractible (so that $\pi_n(X)$ is the trivial group for all $n > 0$);
- the more interesting case where $\pi_1(X)$ is nontrivial but the universal covering of X is contractible (in which case we can say that $\pi_n(X)$ is the trivial group for all $n > 0$). Examples of this type are $X = S^1$, or $X =$ oriented surface of any positive genus.

In particular nobody has a really convincing formula for $\pi_n(S^2)$, for all $n \geq 1$ (although there are some deep results which describe these abelian groups in algebraic/combinatorial terms ... but not in such a way that we can easily read off how many elements). But there are many partial results, especially about $\pi_n(S^m)$. For example, we know that $\pi_n(S^m)$ is always a finitely generated abelian group ($m, n > 1$). It is known that $\pi_n(S^m)$ is the trivial group if $n < m$ and that $\pi_n(S^m) \cong \mathbb{Z}$ if $n = m$; see theorem 14.3 below. It is known that $\pi_n(S^m)$ is infinite if and only if m is even and $n = m$ or $n = 2m - 1$. An example of that is $\pi_3(S^2) \cong \mathbb{Z}$. Recall that $\pi_3(S^2)$ is not trivial according to example 2.5.3, cumulative lecture notes. (This was conditional at the time; we needed to know that S^2 is not contractible, but later we learned that S^2 is not contractible since $H_2(S^2) \cong \mathbb{Z}$.)

14.2. Some homotopy groups of spheres

Theorem 14.3. *For $0 < n < m$, the group $\pi_n(S^m)$ is trivial. For all $n > 0$, the group $\pi_n(S^n)$ is isomorphic to $\mathbb{Z}$, with* [id] *as the generator.*

Proof. The proof is fiddly, but it is an important result. The case $n < m$ is an easy consequence of cellular approximation. By remark 11.5.2 in the cumulative lecture notes, any based map from S^n to S^m is based homotopic to a cellular map. But a cellular map from S^n to S^m must be constant. (Use the CW structure on S^m which has one 0-cell and one m-cell.)
For the case $m = n$, it suffices to show that $\pi_n(S^n)$ is generated by the element [id]. Indeed, this gives us an upper bound on the size of $\pi_n(S^n)$. A lower bound comes from the map $\pi_n(S^n) \to H_n(S^n)$ which takes the homotopy class of a map f to the class of the mapping cycle f. It is an exercise to show that this is a homomorphism! (Hint: you need to say what κ does in homology.)
With that in mind, the most important tool is Sard's theorem (which we also used in connection with approximation of maps by cellular maps). This states that for a smooth map $f\colon U \to \mathbb{R}^m$ where U is open in $\mathbb{R}^n$, the set of critical values of f is a set of Lebesgue measure zero (in $\mathbb{R}^m$). An element $y \in \mathbb{R}^m$ is a *critical value* of f if there exists $x \in U$ such that $f(x) = y$ and the derivative $f'(x)$, which I view as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, is not surjective. We can also assume $n > 1$ since $\pi_1(S^1, \star)$ is well understood. We need a few observations.

(i) Any based map $S^n \to S^n$ can be written in the form of a map

$$f\colon \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\},$$

and after a homotopy we can assume that f is smooth in a neighborhood U of the compact set $f^{-1}(D^n)$.

(ii) In the situation of (i), if $f^{-1}(0)$ contains exactly one element $x \in \mathbb{R}^n$ and the derivative $f'(x)$ is an invertible linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, then f is based homotopic either to the identity map or to the map

$$\eta\colon (x_1, \ldots, x_n) \mapsto (-x_1, x_2, \ldots, x_n)$$

from $\mathbb{R}^n \cup \{\infty\}$ to itself.

(iii) The inclusion of the wedge $S^n \vee S^n$ into the product $S^n \times S^n$ induces an isomorphism from $\pi_n(S^n \vee S^n)$ to $\pi_n(S^n \times S^n) \cong \pi_n(S^n) \times \pi_n(S^n)$.

(iv) Let $\alpha\colon S^n \to S^n \vee S^n$ be any based map. Let $\varphi\colon S^n \vee S^n \to S^n$ be the fold map (which is the identity on the first summand S^n and also on the second summand S^n). Then we have

$$[\varphi\alpha] = [\varphi q_1 \alpha] + [\varphi q_2 \alpha] \in \pi_n(S^n),$$

writing $+$ for the multiplication in $\pi_n(S^n)$ and $q_i\colon S^n \vee S^n \to S^n \vee S^n$ for the map which is the identity on summand i and takes the other summand to the base point.

Observation (iii) is a good exercise in cellular approximation; $n > 1$ is important. Observation (iv) follows from observation (iii). Namely, (iii) shows that α is homotopic to a based map obtained by composing $\kappa\colon S^n \to S^n \vee S^n$ with a map $S^n \vee S^n \to S^n \vee S^n$ which agrees with $q_1\alpha$ on the first wedge summand S^n and with $q_2\alpha$ on the second.

We had observation (ii) as an exercise (sheet 5) but it did not find many friends. It is easy to reduce to the situation where $x = 0 \in \mathbb{R}^n$. Then $f^{-1}(0) = \{0\}$ and $f'(0)$ is an invertible linear map. The next idea is to show that f is based homotopic to the map $g\colon \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\}$ where g is the linear map $f'(0)$ (except for $g(\infty) = \infty$). A based homotopy is given by

$$(h_t\colon \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\})$$

where $h_t(v) = t^{-1}f(tv)$ for $v \in \mathbb{R}^n$ and t runs from 1 to 0. To be more precise, h_1 is of course f and h_0 is of course not really defined by our formula for h_t , but if you (re)define $h_0 = g$ then it ought to make a good homotopy, by definition of differentiability. The next idea is to note that the space of linear isomorphisms from $\mathbb{R}^n \to \mathbb{R}^n$, also known as $\mathrm{GL}_n(\mathbb{R})$, is a space with exactly two path components. One of these path components contains the identity matrix and the other one contains the diagonal matrix with -1 in row one, column one and $+1$ in the other diagonal positions. Therefore our (linear) map

$$g\colon \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\}$$

is based homotopic (by a homotopy through invertible linear maps) to either $\mathrm{id}\colon \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\}$ or to the map η from $\mathbb{R}^n \cup \{\infty\}$ to itself.

Now we start with f as in (i). We want to show that $[f] \in \pi_n(S^n)$ is in the subgroup generated by $[\mathrm{id}]$. By Sard, we know that f has a regular value

arbitrarily close to 0 and it is easy to reduce to the case where 0 itself is regular value (by composing with a translation $\mathbb{R}^n$). The preimage $f^{-1}(0)$ is compact and discrete with the subspace topology (since $f'(x)$ is invertible for any $x \in f^{-1}(0)$ ... use the inverse function theorem). Therefore $f^{-1}(0)$ is a finite set. Assume that it has k distinct elements $x^{(1)}, \dots, x^{(k)}$. We want to argue by induction on k. The case $k = 1$ has already been settled in (ii) and we can assume $k > 1$.
Choose a small open ball B_ε of radius ε about the origin $0 \in \mathbb{R}^n$ such that $f^{-1}(B_\varepsilon)$ is a *disjoint* union of k open sets $U_1, \dots, U_k$ (so that $x^{(i)} \in U_i$) in such a way that f restricts to a diffeomorphism from U_i to B_ε. (This is possible by the inverse function theorem.) Choose a map

$$e \colon \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\}$$

which maps B_ε diffeomorphically to all of $\mathbb{R}^n$ and maps the complement of B_ε to ∞ and has $e'(0)$ equal to the identity (matrix). Then we know that $e \simeq \mathrm{id}$ and so $ef \simeq f$. But ef can also be written as a composition

$$S^n \xrightarrow{\ \gamma\ } S^n \vee S^n \xrightarrow{\ \varphi\ } S^n$$

where $S^n = \mathbb{R}^n \cup \{\infty\}$, the first map takes U_1 to the first wedge summand S^n by ef and takes $\bigcup_{i>1} U_i$ to the second wedge summand by ef, and takes all remaining points to the base point ∞ of the wedge. Then by (iv) we have

$$[f] = [ef] = [\varphi\gamma] = [\varphi q_1 \gamma] + [\varphi q_2 \gamma]$$

where $\varphi q_1 \gamma$ and $\varphi q_2 \gamma$ are maps as in (i) for which $0 \in \mathbb{R}^n \cup \{\infty\}$ is a regular value with fewer than k preimage points. By inductive assumption, $[\varphi q_1 \gamma]$ and $[\varphi q_2 \gamma]$ are in the subgroup of $\pi_n(S^n)$ generated by $[\mathrm{id}]$ and therefore $[f]$ is also in that subgroup. $\square$