21.06.2015

# Lecture Notes, weeks 6, 7 and 8
# Topology SS 2015 (Weiss)

### 3.1. The E. H. Brown representation theorem

Let $\mathcal{C}$ be the category of based connected CW-spaces. (There is an understanding that the base point in a based CW-space is always a 0-cell.) Let $\mathcal{H}o\mathcal{C}$ be the corresponding homotopy category. We look for a characterization of the *representable* contravariant functors from $\mathcal{H}o\mathcal{C}$ to sets.

**Definition 3.1.** A contravariant functor $F$ from $\mathcal{H}o\mathcal{C}$ to sets is *half-exact* if it satisfies the conditions below.

(i) If $X$ in $\mathcal{C}$ is the union of two based CW-subspaces $A$ and $B$ such that $A, B$ and $A \cap B$ are again in $\mathcal{C}$, then for any $s \in F(A)$ and $t \in F(B)$ which determine the same element of $F(A \cap B)$, there exists $u \in F(X)$ restricting to $s \in F(A)$ and to $t \in F(B)$.

(ii) (*Wedge axiom.*) If $X$ in $\mathcal{C}$ is a wedge, $X = \bigvee X_\alpha$, then the maps $F(X) \to F(X_\alpha)$ induced by the inclusions of $X_\alpha$ in $X$ determine a bijection

$$F(X) \longrightarrow \prod_\alpha F(X_\alpha).$$

In (ii), the case of an empty indexing set is allowed. In that case the condition means that $F(\star)$ has exactly one element.

**Remark 3.2.** If $F$ is half-exact, then for every $X$ in $\mathcal{C}$ the set $F(X)$ has a distinguished "zero" element. This is $g^*(z) \in F(X)$ for the unique map $g\colon X \to \star$ and the unique element $z \in F(\star)$.

**Remark 3.3.** Let $g\colon X \to Y$ be a based cellular map, where $X$ and $Y$ are in $\mathcal{C}$. Let cone$(g)$ be the *reduced* mapping cone. (The reduced mapping cone is the quotient space obtained from the standard mapping cone by collapsing the copy of $[0, 1] \times \star$ in the standard mapping cone to a single point. The reduced mapping cone of a based cellular map of CW-spaces is again a based CW-space in a preferred way.) Then the sequence

$$F(\text{cone}(g)) \to F(Y) \xrightarrow{g^*} F(X)$$

(first arrow determined by the inclusion of $Y$ in cone$(g)$) is *exact*, i.e., an element of $F(Y)$ comes from $F(\text{cone}(g))$ if and only if it maps to the zero element of $F(X)$. (*Exercise.*)

**Example 3.4.** Let $Y$ be a based connected CW-space and let $F_Y$ be the contravariant functor defined by $F_Y(X) = [X, Y]_\star$ (set of based homotopy classes of based maps from $X$ to $Y$) for $X$ in $\mathcal{C}$ or $\mathcal{H}o\mathcal{C}$. This $F_Y$ is half-exact. Property (ii) in definition 3.1 is obviously satisfied. For property (i), the idea is as follows. Take $X = A \cup B$ etc.; replace $X$ by $X^\sharp$ which is the quotient of

$$A \sqcup \big([0,1] \times (A \cap B)\big) \sqcup B$$

by the relations $(0,x) \sim x \in A$ and $(1,x) \sim x \in B$ for $x \in A \cap B$, as well as $(t,\star) \sim (0,\star)$ for all $t$ (where $\star$ is the base point in $A \cap B$). There is a projection map

$$X^\sharp \longrightarrow X$$

which forgets the extra coordinate in $[0,1]$ where applicable. It is an exercise to show that this projection map is a homotopy equivalence. (Use the HEP, homotopy extension property, for the inclusion of $A \cap B$ in $A$.) Now suppose given a map $f_0 \colon A \to Y$ and a map $f_1 \colon B \to Y$ such that $f_0|_{A\cap B} \simeq f_1|_{A\cap B}$. Choose a homotopy $(h_t)_{t\in[0,1]}$ from $f_0|_{A\cap B}$ to $f_1|_{A\cap B}$. Define $g \colon X^\sharp \to Y$ so that $g$ agrees with $f_0$ on the copy of $A$ in $X^\sharp$, agrees with $f_1$ on the copy of $B$ in $X^\sharp$, and agrees with $h_t$ on the copy of $\{t\} \times (A \cap B)$ in $X^\sharp$. Then the homotopy class of $g$, viewed as an element of $[X,Y]_\star = F_Y(X)$, satisfies $[g]|_A = [f_0]$ and $[g]|_B = [f_1]$. This confirms property (i) for $F_Y$, but I must apologize for changing labels: $s \rightsquigarrow f_0$, $t \rightsquigarrow f_1$, $u \rightsquigarrow g$.

More exotic examples will be given later.

Let $F$ be half-exact and suppose that $X$ in $\mathcal{C}$ is the monotone union of a sequence of CW-subspaces $A_j$ where $j = 0, 1, 2, \dots$; so

$$X = \bigcup_j A_j$$

and $A_j \subset A_{j+1}$ for all $j$. Let $(v_j \in F(A_j))_{j\geq 0}$ be a sequence such that the map $F(A_{j+1}) \to F(A_j)$ induced by $A_j \hookrightarrow A_{j+1}$ takes $v_{j+1}$ to $v_j$, for all $j \geq 0$.

**Lemma 3.5.** *Then there exists $v_\infty \in F(X)$ which is taken to $v_j$ under the map $F(X) \to F(A_j)$ induced by $A_j \hookrightarrow X$, for all $j \geq 0$.*

*Proof.* Let's use the notation $\mathrm{cyl}(Y \to Z)$ for the mapping cylinder of a map $e \colon Y \to Z$. For the moment this is the unreduced mapping cylinder. In the standard description of the mapping cylinder we have a projection $\mathrm{cyl}(Y \to Z) \to [0,1]$ which takes the standard copy of $Z$ to $0$ and the standard copy of $Y$ to $1$. Here we parameterize this differently so that there is a projection $\mathrm{cyl}(Y \to Z) \to [0,2]$ which takes $Y$ to $0$ and $Z$ to $2$. More precisely, let $\mathrm{cyl}(Y \to Z)$ be (in this proof) the quotient of $\big([0,2] \times Y\big) \cup Z$ obtained by making identifications $(2,y) \sim e(y) \in Z$. If $e$ is a based cellular map of based CW-spaces, then $\mathrm{cyl}(e) = \mathrm{cyl}(Y \to Z)$ is a CW-space in a preferred way, but here we use the CW-structure on $[0,2]$ with three 0-cells:

the elements $0, 1, 2$ of $[0, 2]$. The cylinder $\mathrm{cyl}(e)$ is not a based CW-space, but for later use we not that there is a copy of $[0, 2] = \mathrm{cyl}(\star \to \star)$ inside $\mathrm{cyl}(e)$. This is clearly asking to be collapsed to a single point, but we have to delay that. — The *mapping telescope* of the diagram

$$A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow A_3 \hookrightarrow A_4 \hookrightarrow \cdots$$

is the union

$$\mathrm{cyl}(A_0 \to A_1) \cup_{A_1} \mathrm{cyl}(A_1 \to A_2) \cup_{A_2} \mathrm{cyl}(A_2 \to A_3) \cup_{A_3} \cdots$$

in self-explanatory notation. It is again a CW-space in an obvious way. Let's write $\mathrm{tel}(A_j \mid j \geq 0)$ for the telescope, just to have an abbreviation. There is a projection map

$$\mathrm{tel}(A_j \mid j \geq 0) \longrightarrow X$$

which on the piece $\mathrm{cyl}(A_j \to A_{j+1})$ agrees with the cylinder projection to $A_{j+1}$, followed by the inclusion $A_{j+1} \to X$. *Exercise*: show that this map is a homotopy equivalence.
There is a very useful continuous projection map

$$q\colon \mathrm{tel}(A_j \mid j \geq 0) \longrightarrow [0, 2] \cup [2, 4] \cup [4, 6] \cup \cdots$$

which projects the piece $\mathrm{cyl}(A_j \to A_{j+1})$ to the interval $[2j, 2j + 2]$ in the obvious way. Now let $T_0$ be the preimage under $q$ of $[0, 1] \cup [3, 5] \cup [7, 9] \cup \ldots$ and let $T_1$ be the preimage under $q$ of $[1, 3] \cup [5, 7] \cup \ldots$. These are CW-subspaces of the telescope $\mathrm{tel}(A_j \mid j \geq 0)$ by construction. Then clearly

$$T_0 \simeq A_0 \sqcup A_2 \sqcup A_4 \sqcup A_6 \vee \ldots, \qquad T_1 \simeq A_1 \sqcup A_3 \sqcup A_5 \sqcup \ldots$$

whereas

$$T_0 \cap T_1 \cong A_0 \sqcup A_1 \sqcup A_2 \sqcup A_3 \sqcup \cdots \ .$$

But at this point we need to see reduced versions of these constructions. Let

$$\mathrm{tel}^\rho(A_j \mid j \geq 0) := \frac{\mathrm{tel}(A_j \mid j \geq 0)}{\mathrm{tel}(\star \to \star \to \star \to \cdots)}$$

and let $T_0^\rho$, $T_1^\rho$ be the images of $T_0$ and $T_1$ in $\mathrm{tel}^\rho(A_j \mid j \geq 0)$. Then we have

$$T_0^\rho \simeq A_0 \vee A_2 \vee A_4 \vee A_6 \vee \ldots, \qquad T_1^\rho \simeq A_1 \vee A_3 \vee A_5 \vee \ldots$$

whereas

$$T_0^\rho \cap T_1^\rho \cong A_0 \vee A_1 \vee A_2 \vee A_3 \vee \cdots \ .$$

By the wedge axiom, $(v_0, v_2, v_4, \ldots)$ defines an element in $F(T_0)$ and similarly $(v_1, v_3, v_5, \ldots)$ defines an element in $F(T_1)$. By assumption on the sequence $(v_j)_{j\geq 0}$, these two elements determine the same element

$$(v_0, v_1, v_2, \ldots) \in F(T_0^\rho \cap T_1^\rho)$$

under the restriction maps $F(T_0^\rho) \to F(T_0^\rho \cap T_1^\rho)$ and $F(T_1^\rho) \to F(T_0^\rho \cap T_1^\rho)$. Therefore by half-exactness, there exists

$$w \in F(\mathrm{tel}^\rho(A_j \mid j \geq 0)) \cong F(\mathrm{tel}(A_j \mid j \geq 0)) \cong F(X)$$

which extends $(\nu_0, \nu_2, \nu_4, \dots) \in F(T_0^\rho)$ and $(\nu_1, \nu_3, \nu_5, \dots) \in F(T_1^\rho)$. The element $w$, viewed as an element of $F(X)$, is the answer to our prayers. □

**Lemma 3.6.** *Let* $E$ *be a half-exact functor, let* $X$ *be an object of* $\mathcal{C}$ *and let* $t \in F(X)$*. There exist a based connected CW-space* $Y$ *containing* $X$ *as a CW-subspace, and an element* $u \in E(Y)$ *such that* $u|_X = t$ *and the map* $\pi_k(Y) \to E(S^k)$ *taking* $[f \colon S^k \to Y]$ *to* $f^*(u)$ *is bijective for all* $k > 0$*.*

As a preparation for the proof we take a closer look at the sets $E(S^k)$ for a half-exact functor $E$ and $k \geq 1$. It turns out that these sets have a preferred group structure (abelian if $k \geq 2$). The reason is that $S^k$ comes with a distinguished map $\kappa \colon S^k \to S^k \vee S^k$ which we used previously to define the group structure in homotopy groups $\pi_k$. Here we can use it to define a map

$$E(S^k) \times E(S^k) \longrightarrow E(S^k)$$

by writing $E(S^k) \times E(S^k) \cong E(S^k \vee S^k)$, wedge axiom for $E$, and then using $\kappa^* \colon E(S^k \vee S^k) \to E(S^k)$. This map makes $E(S^k)$ into a group (abelian if $k \geq 2$) because $\kappa$ has the corresponding properties (which can be, should be and have been expressed in the homotopy category of based spaces). By the same reasoning, for $X$ in $\mathcal{C}$ and $t \in F(X)$ the maps $\pi_k(X) \to E(S^k)$ taking $[f \colon S^k \to X]$ to $f^*(t)$ are group homomorphisms. (Here and in the following we write $\pi_k(X)$ instead of $\pi_k(X)$ on the understanding that $X$ has a preferred base point.)

*Proof of lemma 3.6.* We construct $X \cup Y^n$ by induction on $n$, together with elements $u_n \in E(X \cup Y^n)$ such that the restriction of $u_n$ to $X \cup Y^{n-1}$ is $u_{n-1}$. (The notation is a little informal; there is an understanding that $X \cap Y^n$ is $X^n$.) For the induction beginning set $u_0 := t$ and $X \cup Y_0 := X$, that is, $Y^0 = X^0$. For the first induction step, from $n = 0$ to $n = 1$, choose generators $\mu$ for the entire group $E(S^1)$. Define

$$X \cup Y^1 := X \vee \bigvee_\mu S^1 .$$

By the wedge axiom for $E$, we have

$$E(X \cup Y^1) = E(X) \times \prod_\mu E(S^1)$$

and we determine $u_1 \in E(X \cup Y^1)$ in such a way that the coordinate in $E(X)$ is $u_0 = t$, while the coordinate in the factor $E(S^1$ corresponding to $\mu \in E(S^1)$ is exactly $\mu$. By construction, the map $\pi_1(X \cup Y^1) \to E(S^1)$ taking $[f]$ to

$f^*(u_1)$ is surjective. — For the remaining induction steps, suppose that $X \cup Y^n$ and $u_n \in E(X \cup Y^n)$ have already been constructed for a particular $n \geq 1$. Suppose that the homomorphisms $\pi_k(X \cup Y^n) \to E(S^k)$ taking $[f]$ to $f^*(u_n)$ are bijective for $1 \leq k < n$ and surjective for $k = n$. (This is part of the induction load.) Choose generators $\lambda$ for the kernel of the homomorphism $\pi_n(X \cup Y^n) \to E(S^n)$, and for each $\lambda$, a based cellular map $f_\lambda \colon S^n \to X \cup Y^n$ in that homotopy class. Also choose generators $\mu$ for the entire group $E(S^{n+1})$. Define $X \cup Y^{n+1}$ to be

$$\text{cone}\Big( \bigvee_\lambda g_\lambda \colon \bigvee S^n \longrightarrow X \cup Y^n \Big) \quad \vee \quad \bigvee_\mu S^{n+1}.$$

By half-exactness of $E$ (see also remark 3.3) there is an element $u_{n+1}$ of $E(X \cup Y^{n+1})$ such that the restriction to $X \cup Y^n$ is $u_n$ and the restriction to the wedge summand $S^{n+1}$ with label $\mu$ is precisely $\mu \in E(S^{n+1})$.
Now we have to show that the homomorphisms $\pi_k(X \cup Y^{n+1}) \to E(S^k)$ taking $[f]$ to $f^*(u_{n+1})$ are bijective for $1 \leq k \leq n$ and surjective for $k = n+1$. Surjectivity for $k = n+1$ is obvious from the construction. For the cases $k \leq n$ we look at the composition

$$\pi_k(X \cup Y^n) \to \pi_k(X \cup Y^{n+1}) \to E(S^k)$$

where the first arrow is induced by the inclusion $X \cup Y^n \to X \cup Y^{n+1}$. The first arrow is an isomorphism for $k < n$ by cellular approximation and the composition is an isomorphism for $k < n$ by inductive assumption, so the second arrow is also an isomorphism for $k < n$. For $k = n$ the first arrow is onto by cellular approximation, while the composite arrow is onto by inductive assumption and its kernel is contained in the kernel of the first arrow by construction. Therefore these two kernels must coincide as subgroups of $\pi_k(X \cup Y^n)$. It follows that the second arrow is again an isomorphism.
Now we have constructed $X \cup Y^n$ and $u_n$ for all $n$. Let $Y$ be the union (direct limit or colimit is a better expression) of the $X \cup Y^n$ for all $n$. By lemma 3.5, there exists $u \in E(Y)$ such that $u$ restricted to $X \cup Y^n$ is $u_n$ , for all $n \geq 0$. In particular, we have $u|_X = u_0 = t$ . □

**Theorem 3.7.** (The Brown representation theorem.) *Any half-exact functor $F$ from $\mathcal{H}o\mathcal{C}$ to sets is representable, i.e., there exist $Y$ and $u \in F(Y)$ such that the map from $[X, Y]_\star$ to $F(X)$ given by $[f] \mapsto f^*(u)$ is bijective for every $X$ in $\mathcal{C}$.*

*Proof.* By lemma 3.6 we can construct a based connected CW-space $Y$ and an element $u \in E(Y)$ such that $u|_X = t$ and the map $\pi_k(Y) \to E(S^k)$ taking $[f]$ to $f^*(u)$ is bijective for all $k > 0$. We are going to show that the map

$$\alpha_X \colon [X, Y]_\star \longrightarrow F(X) \; ; \; [f] \mapsto f^*(u) \in F(X)$$

is bijective for every $X$ in $\mathcal{C}$.
The idea is to construct a natural inverse $\beta_X\colon F(X) \to [X,Y]_\star$ for $\alpha_X$ using lemma 3.6 and the JHC Whitehead theorem. For $t \in F(X)$ we apply lemma 3.6 to the element $(t,u) \in F(X) \times F(Y) \cong F(X \vee Y)$. The outcome is that there is a CW-space $Y'$ containing $X \vee Y$ as a CW-subspace and an element $u' \in F(Y')$ which extends $(t,u) \in F(X \vee Y)$ and has the property that the homomorphisms $\pi_k(Y') \to E(S^k)$ given by $[f] \mapsto f^*(u')$ are isomorphisms for all $k > 0$. Since the homomorphisms

$$\pi_k(Y) \to E(S^k)$$

given by $[f] \mapsto f^*(u)$ are also isomorphisms for all $k > 0$, it follows that the inclusion $Y \to Y'$ induces an isomorphism $\pi_k(Y) \to \pi_k(Y')$ for all $k > 0$. Therefore the JHC Whitehead theorem tells us that $Y \to Y'$ is a homotopy equivalence. We attempt to define $\beta_X(t) \in [X,Y]_*$ as the homotopy class of $X \hookrightarrow Y'$ followed by a based homotopy inverse for $Y \hookrightarrow Y'$. Now it remains to show (a) that $\beta_X$ is well defined, (b) that $\alpha_X\beta_X = \mathrm{id}$ and (c) that $\beta_X\alpha_X = \mathrm{id}$.

(a) Suppose that we have selected $Y'$ containing $X \vee Y$ and $u' \in F(Y')$ extending $(t,u) \in F(X \vee Y) \cong F(X) \times F(Y)$. Suppose that we have also selected $Y''$ containing $X \vee Y$ and $u'' \in F(Y'')$ extending $(t,u) \in F(X \vee Y) \cong F(X) \times F(Y)$. We are assuming that the homomorphisms $\pi_k(Y') \to E(S^k)$ and $\pi_k(Y'') \to E(S^k)$ given by $[f] \mapsto f^*(u')$ and $[f] \mapsto f^*(u'')$ respectively are isomorphisms. Then we can find a CW-space $Y'''$ containing the union $Y' \cup_{X\vee Y} Y''$ (better described as a pushout) and an element $u''' \in F(Y''')$ such that the homomorphisms $\pi_k(Y''') \to E(S^k)$ given by $[f] \mapsto f^*(u''')$ are isomorphisms. Now we have three definitions of $\beta_X(t)$ , corresponding to the selections $Y'$, $Y''$ and $Y'''$. But it is clear that the first agrees with the third and the second agrees with the third, since $Y' \subset Y'''$ and $Y'' \subset Y'''$. So the first must agree with the second, as was to be shown.

(b) Let $t \in F(X)$ and suppose $Y'$ containing $X \vee Y$ as well as $u' \in F(Y')$ extending $(t,u) \in F(X \vee Y)$ has been selected so that the composition $Y \hookrightarrow X \vee Y \hookrightarrow Y'$ is a homotopy equivalence. Then $\beta_X(t) \in [X,Y]_*$ is the composition of $X \hookrightarrow X \vee Y \hookrightarrow Y'$ with a homotopy inverse for $Y \to Y'$. Since that homotopy inverse $Y' \to Y$ will take $u \in F(Y)$ to $u' \in F(Y')$, it follows that $\alpha_X(\beta_X(t))$ is the restriction of $u' \in F(Y')$ to $X \subset Y'$. But that is $t \in F(X)$ by construction of $u'$.

(c) Let $[g] \in [X,Y]_\star$, so that $\alpha_X([g]) = g^*(u)$. To find out what $\beta_X(g^*(u))$ is we should construct $Y'$ containing $X \vee Y$ and $u' \in F(Y')$ such that $u'|_X$ is $g^*(u)$. Then $\beta_X(g^*(u))$ is the composition $X \hookrightarrow X \vee Y \hookrightarrow Y' \simeq Y$. But we can take $Y' = \mathrm{cyl}(g\colon X \to Y)$. This contains a copy of $X \vee Y$. The cylinder projection $Y' \to Y$ is an explicit homotopy inverse for the inclusion $Y \to Y'$. For $u' \in F(Y')$ we can (must) take the unique element which restricts to

$u \in F(Y)$; fortunately it is obvious that $u'|_X = g^*(u)$ for this choice of $u'$. Moreover the composition of $X \hookrightarrow X \vee Y \hookrightarrow Y'$ with the cylinder projection $Y' \to Y$ is exactly $g$. □

### 3.2. Eilenberg-MacLane spaces (*draft*)

Representing spaces for reduced cohomology $H^n(-)$, fixed $n > 0$, can be obtained as special cases of the Brown representation theorem.

Generalizations: use cohomology with abelian coefficient group $G$. Three variants: (co-)homology based on mapping cycles, singular (co)chain complexes, cellular cochain complexes. *Universal coefficient theorem.* For $G = \mathbb{Z}$, the cohomology $H^n(-; G)$ is the cohomology that we are used to.
*Warning*: In the mapping cycle setting, composition of mapping cycles only survives the introduction of coefficient groups $G$ if $G$ has the additional structure of a (commutative?) ring. There are also (in all three settings) cup-like products of the form $H^m(X; G_1) \times H^n(Y; G_2) \longrightarrow H^{m+n}(X \times Y; G_1 \otimes G_2)$. So if $G$ is the underlying abelian group of a (commutative) ring, then there are cup products

$$H^m(X; G) \times H^n(Y; G) \longrightarrow H^{m+n}(X \times Y; G \otimes G) \longrightarrow H^{m+n}(X \times Y; G).$$

Then we get representing spaces for cohomology $H^n(-; G)$ from the Brown representation theorem. *Observation*: all these representing (based) spaces have $\pi_k$ trivial except for one particular $k$, namely, $k = n$.

**Definition 3.8.** Let $n$ be a positive integer and let $G$ be a group (abelian if $n > 1$). It is customary to write $K(G, n)$ for a connected based CW-space $X$ which has $\pi_k(X)$ trivial for positive $k \neq n$ and $\pi_n(X)$ equipped with a group isomorphism to $G$. These spaces are also called *Eilenberg-MacLane spaces.*

This convention suggests that if two based CW-spaces are both entitled to the name $K(G, n)$, then they are homotopy equivalent. Let's state and prove something more systematic. Let $X$ and $Y$ be based CW-spaces; suppose that $X$ is a $K(G, m)$ and $Y$ is a $K(J, n)$.

**Proposition 3.9.** *If $m > n$, then $[X, Y]_\star$ has only one element, the homotopy class of the constant map. If $m = n$, then the evaluation map*

$$[X, Y]_\star \longrightarrow \hom(\pi_m(X), \pi_m(Y)) = \hom(G, J)$$

*is bijective.* (For $m = n = 1$ the right-hand side must be read as the *set* of group homomorphisms from $G$ to $J$; for $m = n > 1$ it is the abelian group of homomorphisms from $G$ to $J$.)

Note that this doesn't say much about the case $m < n$. Indeed that case is more difficult. There is some similarity with homotopy groups of spheres. We have $\pi_m(S^n)$ trivial if $m < n$, isomorphic to $\mathbb{Z}$ if $m = n \geq 1$, and difficult in the remaining cases. But note the difference: $\pi_m(S^n)$ trivial if $m < n$, whereas $[X, Y]_\star$ trivial if $m > n$.

*Proof.* We can assume that $X^k = \star$ for $k < m$ and $X^m = \bigvee_{\lambda \in \Lambda_m} S^m$ , and $X^{k+1}$ for $k \geq m$ is the mapping cone of a based map

$$a_k \colon \bigvee_{\lambda \in \Lambda_{k+1}} S^k \longrightarrow X^k \,.$$

The more interesting case is $m = n$ and we assume this to begin with. Without being specific about $m$, we show first that if $m \geq n$ then the restriction map

$$[X, Y]_\star \longrightarrow [X^m, Y]_\star$$

is injective and that its image consists of those $[f] \in [X^m, Y]_\star$ with the property that $[fa_m]$ is the trivial element of $[\bigvee_{\lambda \in \Lambda_{m+1}} S^m, Y]_*$. To put it more elegantly: there is an exact sequence of pointed sets

$$(\maltese) \qquad [X, Y]_* \xrightarrow{\text{res.}} [\,X^m, Y]_* \xrightarrow{a_m^*} [\,\bigvee_{\lambda \in \Lambda_{m+1}} S^m, Y]_*$$

where the arrow on the left is injective. — To show this, suppose that $f, g$ are two based maps $X \to Y$ whose restrictions to $X^m$ are based homotopic. Let $(h_{t,m} \colon X^m \to Y)_{t \in [0,1]}$ be such a based homotopy, so that $h_{0,m}$ is $f$ restricted to $X^m$ and $h_{1,m}$ is $g$ restricted to $X^m$. We try to extend the homotopy to a homotopy

$$(h_{t,m+1} \colon X^{m+1} \to Y)_{t \in [0,1]}$$

where $h_{0,m+1}$ is $f$ restricted to $X^{m+1}$ and $h_{1,m+1}$ is $g$ restricted to $X^{m+1}$. This boils down to constructing a map $B_m \to Y$ which is prescribed on $A_m$ , where $B_m = \text{cone}(\text{source}(a_m)) \times [0,1]$ and $A_m$ is the subspace

$$\text{cone}(\text{source}(a_m)) \times \{0,1\} \bigcup \text{source}(a_m) \times [0,1] \,.$$

(Let $e_{m+1}$ be the standard map from cone(source($a_m$)) to cone($a_m$) $=$ $X^{m+1} \subset X$. The prescribed map $A_m \to Y$ is given by $fe_{m+1}$ on the subspace cone(source($a_m$)) $\times \{0\}$, by $ge_{m+1}$ on cone(source($a_m$)) $\times \{1\}$ and by $h_t a_m$ on source($a_m$) $\times \{t\}$.) Since the pair $(B_m, A_m)$ can be written as a union (indexed by $\lambda \in \Lambda_{m+1}$) of pairs

$$(D^{m+1} \times [0,1]\,,\, D^{m+1} \times \{0,1\} \cup S^m \times [0,1]\,)$$

and since the latter pairs are each homeomorphic to $(D^{m+2}, S^{m+1})$, and since $\pi_{m+1}(Y)$ is trivial by assumption, it is easy to see that the required extension exists. A similar argument shows that we can also extend the homotopy

$(h_{t,m+1}\colon X^{m+1} \to Y)_{t\in[0,1]})$ to a homotopy $(h_{t,m+2}\colon X^{m+2} \to Y)_{t\in[0,1]})$ from $f$ restricted to $X^{m+2}$ to $g$ restricted to $X^{m+2}$ etc.; in short we can extend all the way to a homotopy from $f$ to $g$. This proves the injectivity of the restriction map $[X,Y]_\star \to [X^m,Y]_\star$.

Next, if $f_m\colon X^m \to Y$ is a based map such that $[f_m] \in [X^m,Y]_\star$ is in the image of the restriction map $[X,Y]_\star \to [X^m,Y]_\star$, then it is also in the image of the restriction map $[X^{m+1},Y]_\star \to [X^m,Y]_\star$ and so $f_m a_m$ must be based nullhomotopic (since $X^{m+1} = \text{cone}(a_m)$). Conversely, if $f_m a_m$ is based nullhomotopic then $f_m$ extends to a based map $f_{m+1}\colon X^{m+1} \to Y$. Furthermore $f_{m+1}$ extends to a based map $f_{m+2}\colon X^{m+2} \to Y$ since $f_{m+1}a_{m+1}$ is nullhomotopic (no conditions). And so on; it follows that $f_m$ extends all the way to a based map $X \to Y$, as was to be shown. Therefore the claims about (✠) are now established.

It is easy to make the terms in (✠) explicit. Suppose first that $m = n = 1$. Then the terms in the middle and on the right have the form $\text{hom}(Q,\pi_1(Y))$ and $\text{hom}(P,\pi_1(Y))$ where $P$ and $Q$ are certain free groups. The map $\text{hom}(Q,\pi_1(Y)) \to \text{hom}(P,\pi_1(Y))$ in (✠) is induced by a homomorphism $P \to Q$. This identifies the term on the left of (✠) with $\text{hom}(R,\pi_1(Y))$, where $R$ is the quotient of $Q$ by the smallest normal subgroup of $Q$ containing the image of $P \to Q$. But $R$ is $\pi_1(X) = G$ and $\pi_1(Y)$ is $J$; so we have identified the term on the left with $\text{hom}(G,J)$, which is essentially what we had to do. — Suppose next that $m = n > 1$. Then the terms in the middle and on the right have the form $\text{hom}(Q,\pi_1(Y))$ and $\text{hom}(P,\pi_1(Y))$ where $P$ and $Q$ are certain free abelian groups. The map $\text{hom}(Q,\pi_1(Y)) \to \text{hom}(P,\pi_1(Y))$ in (✠) is induced by a homomorphism $P \to Q$. This identifies the term on the left of (✠) with $\text{hom}(R,\pi_1(Y))$, where $R$ is the quotient of $Q$ by the the image of $P \to Q$. But $R$ is $H_m(X) \cong \pi_m(X) = G$ and $\pi_m(Y)$ is $J$; so we have identified the term on the left with $\text{hom}(G,J)$, which is essentially what we had to do. Therefore we have now taken care of the cases $m = n > 0$.

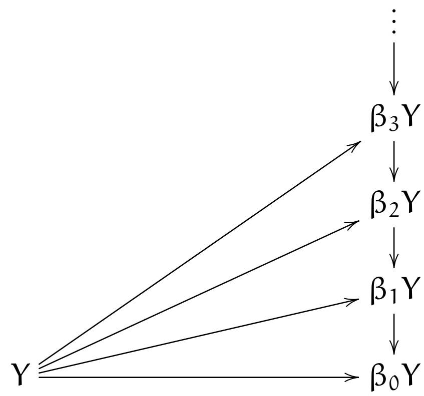
The reasoning in the cases $m > n > 0$ is much easier. Let $f\colon X \to Y$ be any based map. Construct a nullhomotopy for $f$ restricted to $X^k$ by induction on $k$, starting with $k = m$. The details are omitted. □

**Corollary 3.10.** *Up to homotopy equivalence, there is a unique Eilenberg-MacLane space* $K(G,n)$. *It is a representing space for cohomology* $H^n(-;G)$.

*Proof.* If $X$ and $Y$ both satisfy the conditions for being called $K(G,n)$, then by the above proposition there is a based map $X \to Y$ inducing an isomorphism $\pi_n(X) \to \pi_n(Y)$. That map is a homotopy equivalence by the JHC Whitehead theorem. Therefore $X$ is homotopy equivalent to $Y$. This proves the first statement. The other statement is then clear since we know that there is a representing space for cohomology $H^n(-;G)$. □

### 3.3. Postnikov tower and Postnikov-Moore factorization

Let $Y$ be a based connected CW-space. The Postnikov tower of $Y$ is a diagram of spaces




where the map $Y \to \beta_k Y$ has the following property: isomorphism on $\pi_m$ for all $m \le k$, whereas $\pi_m(\beta_k Y)$ is trivial for $m > k$. We also say that $\beta_k Y$ is obtained from $Y$ by killing homotopy groups in dimensions $> k$. From the point of view of the Brown representation theorem, the spaces $\beta_k Y$ have a very pleasant definition. (A weakness of this point of view: it only constructs $\beta_k Y$ in the homotopy category and it only constructs the Postnikov tower as a diagram in the homotopy category.)

**Definition 3.11.** (in the style of Brown's representation theorem.) The space $\beta_k Y$ is a representing space for the half-exact functor

$$X \mapsto \mathrm{im}\big[\, [X^{k+1}, Y]_\star \xrightarrow{\text{res}} [X^k, Y]_\star \,\big].$$

This definition is rather terse. You can read it as follows: a homotopy class of based maps from $X$ (variable) to $\beta_k Y$ is the same as a homotopy class of based maps $X^k \to Y$ which can be extended to (a homotopy class of) based maps $X^{k+1} \longrightarrow Y$. (The extension $X^{k+1} \to Y$ is not specified; it is only required to exist.)

The biggest puzzle with definition 3.11 is that it does not obviously describe a (contravariant) *functor* on $\mathcal{H}o\mathcal{C}$. We need to show that a homotopy class of based maps $g\colon W \to X$ determines a a map

$$\begin{array}{c} \mathrm{im}\big[[X^{k+1}, Y]_\star \xrightarrow{\text{res}} [X^k, Y]_\star\big] \\ \big\downarrow \\ \mathrm{im}\big[[W^{k+1}, Y]_\star \xrightarrow{\text{res}} [W^k, Y]_\star\big]. \end{array}$$

If $g\colon W \to X$ is cellular, then it is clear how we can use it to define the dotted arrow: by precomposition with $g$, restricted to appropriate skeletons. But suppose that $f, g\colon W \to X$ are two cellular maps in the same homotopy class. Choose a cellular homotopy. The cellular homotopy restricts to a map from $W^k \times [0,1]$ to $X^{k+1}$. Using that observation, it is easy to see that it does not matter whether we define the dotted arrow using precomposition with $g$, or precomposition with $f$.

We should also verify that the functor described in definition 3.11 is half-exact as claimed. But the verification is unexciting. It is obvious that the functor satisfies the strong wedge axiom. Suppose now that $X = A \cup B$, where $A$ and $B$ are based connected CW-subspaces of the based CW-space $X$, and $A \cap B$ is also connected. We can replace $X$ by $X^\sharp$ as in example 3.4. Suppose given based maps $f\colon A^k \to Y$ and $g\colon B^k \to Y$ such that the restrictions of $f$ and $g$ to $A^k \cap B^k$ are homotopic by a homotopy $(h_t)_{t \in [0,1]}$, and such that $f$ extends to a map $\bar{f}\colon A^{k+1} \to Y$ while $g$ extends to a map $\bar{g}\colon B^{k+1} \to Y$. Then we have a map from the $(k+1)$-skeleton of $X^\sharp$ to $Y$ given by $\bar{f}$ on the copy of $A^{k+1}$, by $\bar{g}$ on the copy of $B^{k+1}$, and by $h_t$ on the copy of $(A^k \cap B^k) \times \{t\}$. This map and its restriction to the $k$-skeleton of $X^\sharp$ constitute the solution to our problem.

**Definition 3.12.** (in the style of Postnikov and Moore). The CW-space $\beta_k Y$ contains $Y$ as a CW-subspace and is obtained from $Y$ by attaching cells of dimension $> k+1$ to kill the homotopy groups in dimensions $> k$.

This calls for some explanations, too. There is the following more systematic definition, followed by an existence statement and a uniqueness statement.

**Definition 3.13.** Let $f\colon X \to Y$ be a based map of based connected CW-spaces. A *Postnikov-Moore* $k$*-factorization* of $f$ consists of a connected CW-space $X'$ containing $X$ as a CW-subspace and a based map $f_k\colon X' \to Y$ which extends $f$ and has the following properties. The inclusion $X \to X'$ induces an isomorphism in $\pi_j$ for $j \le k$ and a surjection for $j = k+1$, while $f_k$ induces an injection in $\pi_j$ for $j = k+1$ and an isomorphism in $\pi_j$ for all $j > k+1$. (The homotopy groups of $X'$ are then determined as follows: $\pi_j(X')$ is isomorphic to $\pi_j(X)$ when $j \le k$, isomorphic to the image of $f_*\colon \pi_j(X) \to \pi_j(Y)$ when $j = k+1$ and isomorphic to $\pi_j(Y)$ when $j \ge k+2$.)

I like to write $\beta_{f,k}X$ for $X'$ , in view of existence and uniqueness statements below, but this is probably not standard notation.

**Proposition 3.14.** *Let* $f\colon X \to Y$ *be a based map of based connected CW-spaces. A Postnikov-Moore* $k$*-factorization of* $f$ *exists.*

*Proof.* We construct inductively $\beta_{f,k,\ell}X$, a CW-space containing $X$, and a map

$$f_{k,\ell}\colon \beta_{f,k,\ell}X \to Y$$

extending $f$, such that the inclusion $X \to \beta_{f,k,\ell}X$ induces an isomorphism in $\pi_j$ for $j \leq k$ and a surjection for $j = k+1$, while $f_{k,\ell}$ induces an injection in $\pi_j$ for $j = k+1$, an isomorphism in $\pi_j$ for all $j$ such that $k+2 \leq j < \ell$ and a surjection for $j = \ell$. These conditions just start to make sense when $\ell = k+2$ and so the work begins with the construction of $f_{k,k+2}$ and $\beta_{f,k,k+2}X$. To construct $\beta_{f,k,k+2}X$ from $X$ we choose a based map

$$u\colon \bigvee_{\lambda\in\Lambda} S^{k+1} \longrightarrow X$$

such that the image of the homomorphism in $\pi_{k+1}$ determined by $u$ is the kernel of the homomorphism in $\pi_{k+1}$ determined by $f$. Choose also an extension of $fu$ to a map

$$v\colon \bigvee_{\lambda\in\Lambda} D^{k+2} \longrightarrow Y$$

(such an extension exists by the construction of $u$). Choose also a based map

$$w\colon \bigvee_{\tau} S^{k+2} \longrightarrow Y$$

which induces a surjection in $\pi_{k+2}$. Let

$$\beta_{f,k,k+2}X := \mathrm{cone}(u) \vee \bigvee_{\tau} S^{k+2}$$

and define $f_{k,k+2}$ so that it agrees with $w$ on the wedge of $(k+2)$-spheres, with $f$ on the copy of $X$ and with $v$ when composed with the standard map from cone(source($u$)) to cone($u$). Note that the inclusion of $X$ in the cone of $u$ induces a surjection in $\pi_{k+1}$ (by cellular approximation) whose kernel contains the kernel of $f_*\colon \pi_{k+1}(X) \to \pi_{k+1}(Y)$ by construction, but cannot be bigger (say why), so that cone($u$) already has the correct $\pi_{k+1}$. By taking the wedge with many $S^{k+2}$-spheres we can make $\pi_{k+2}$ bigger without changing $\pi_j$ for $j \leq k+1$. We do this in order to end up with a surjection from $\pi_{k+2}(\beta_{f,k,k+2}X)$ to $\pi_{k+2}(Y)$, as required.
The subsequent induction steps are like the first one. More precisely, if

$$f_{k,\ell}\colon \beta_{f,k,\ell}X \longrightarrow Y$$

has already been constructed, then we can declare $\beta_{f,k,\ell}X$ to be the new $X$ and $f_{k,\ell}$ to be the new $f$ and $\ell-1$ to be the new $k$, and repeat the procedure above.

The outcome is $\beta_{f,k,\ell+1}X$ containing $\beta_{f,k,\ell}X$ and $f_{k,\ell+1}$ extending $f_{k,\ell}$. (Let me explain this in more detail. Following the instructions we get isomorphisms

$$\pi_j(\beta_{f,k,\ell}X) \longrightarrow \pi_j(\beta_{f,k,\ell+1}X)$$

induced by inclusion for $j \leq \ell - 1$, a surjection

$$(*) \qquad \pi_\ell(\beta_{f,k,\ell}X) \longrightarrow \pi_\ell(\beta_{f,k,\ell}X),$$

induced by inclusion, an injection

$$(**) \qquad \pi_\ell(\beta_{f,k,\ell+1}X) \longrightarrow \pi_\ell(Y)$$

induced by $f_{k,\ell+1}$ and a surjection

$$\pi_{\ell+1}(\beta_{f,k,k+3}X) \longrightarrow \pi_{\ell+1}(Y)$$

induced by $f_{k,\ell+1}$. But since the composition of $(**)$ and $(*)$ is surjective by construction (of $f_{k,\ell}$), it follows that $(**)$ is an isomorphism, which is what we want for $f_{k,\ell+1}$ . ) — When the induction is finished we define

$$\beta_{f,k}X := \bigcup_{\ell=k}^{\infty} \beta_{f,k,\ell}X$$

and we define $f_k$ so that it agrees with $f_{k,\ell}$ on $\beta_{f,k,\ell}$. ☐

**Remark 3.15.** The Postnikov-Moore $k$-factorization of a based map

$$f\colon X \to Y$$

of based CW-spaces has a uniqueness property. Suppose that

$$X \hookrightarrow X' \to Y, \qquad X \hookrightarrow X'' \to Y$$

are two Postnikov-Moore $k$-factorizations of $f$. That is, both compositions are equal to $f$, and we suppose that the inclusions $X \to X'$ and $X \to X''$ induce isomorphisms in $\pi_j$ for $j \leq k$ and a surjection in $\pi_{k+1}$, and that the maps $X' \to Y$ and $X'' \to Y$ induce an injection in $\pi_{k+1}$ and an isomorphism in $\pi_j$ for $j \geq k+2$. Form

$$X''' := X' \sqcup_X X'',$$

the union of $X'$ and $X''$ along their common CW-subspace $X$ (strictly speaking: the pushout or colimit of the diagram $X' \hookleftarrow X \hookrightarrow X''$). The maps $X' \to Y$ and $X'' \to Y$ that we started with agree on $X$ and so define a map $X''' \to Y$. Make a Moore-Postnikov $k$-factorization for that:

$$X''' \hookrightarrow X'''' \to Y\,.$$

*Exercise*: Show that $X \hookrightarrow X'''' \to Y$ is also a Postnikov-Moore $k$-factorization of $f$. (This can be said to contain the other two that we started with. It follows that the inclusions $X' \hookrightarrow X''''$ and $X'' \hookrightarrow X''''$ are homotopy equivalences.)

**Remark 3.16.** Taking $Y = \star$, we have a unique $f\colon X \to Y$. We can define $\beta_k X$ to be the $X'$ in a Postnikov-Moore $k$-factorization $X \hookrightarrow X' \to Y$ of this $f$, to make the connection with definition 3.11 at last.

**Remark 3.17.** Taking $X = \star$ is also a good idea! Let $\star \hookrightarrow X' \to Y$ be a Moore-Postnikov $k$-factorization of $f\colon \star \to Y$. In this case $\pi_j(X')$ is trivial for $j \le k+1$ and and $f_k\colon X' \to Y$ induces an isomorphism in $\pi_j$ for $j \ge k+2$. In particular when $k = 0$ the map $f_k$ has the homotopical properties of the universal covering of $Y$. More to the point, the diagram

$$\star \hookrightarrow \tilde{Y} \to Y$$

(where $\tilde{Y} \to Y$ is the universal covering) is an instance of a Postnikov-Moore $0$-factorization.

**Remark 3.18.** Let $f\colon X \to Y$ be a based map of based connected CW-spaces, let

$$X \xrightarrow{e} X' \xrightarrow{f_k} Y$$

be a Postnikov-Moore $k$-factorization for $f$ and let

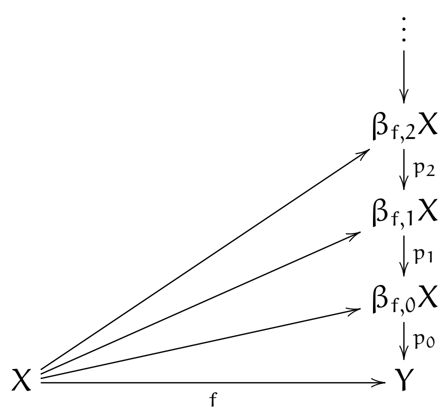
$$X \xrightarrow{d} X'' \xrightarrow{e_\ell} X'$$

be a Postnikov-Moore $\ell$-factorization for $e\colon X \to X'$, where $\ell > k$. Then

$$X \xrightarrow{d} X'' \xrightarrow{f_k e_\ell} Y$$

is a Postnikov-Moore $\ell$-factorization for $f$. (*Exercise.*)

**Definition 3.19.** Let $f\colon X \to Y$ be a based map of based connected CW-spaces. The Postnikov-Moore tower (or decomposition) of $f$ is a commutative diagram




where the lowest triangle is obtained by choosing a Postnikov-Moore $0$-factorization of $f$, the triangle above that is obtained by choosing a Postnikov-Moore $1$-factorization for $X \to \beta_{f,0}X$, the triangle above that is obtained by choosing a Postnikov-Moore 2-factorization for $X \to \beta_{f,1}X$, and so on. The naming of terms in the right-hand column is justified by remark 3.18; in particular

$$\begin{array}{ccc} & & \beta_{f,k}X \\ & \nearrow & \downarrow{\scriptstyle p_0p_1\cdots p_k} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a Postnikov-Moore $k$-factorization of $f$.

**Remark 3.20.** In the case where $Y$ is a point we obtain the Postnikov tower of $X$. In that case it does not hurt to delete $Y$ from the diagram and to write $\beta_k X$ instead of $\beta_{f,k}X$.

We continue with $f\colon X \to Y$ as in definition 3.19. Recall that the *relative homotopy groups of* $f$ are defined as the homotopy groups of the pair $(\mathrm{cyl}(f), X)$ where $\mathrm{cyl}(f)$ is the reduced mapping cylinder of $f$. Since $\mathrm{cyl}(f) \simeq Y$, we can write the long exact sequence of homotopy groups of that pair in the form

$$\ldots \xrightarrow{\partial} \pi_m(X) \xrightarrow{f_*} \pi_m(Y) \longrightarrow \pi_m(f) \xrightarrow{\partial} \pi_{m-1}(X) \longrightarrow \ldots$$

The Postnikov-Moore factorization

$$X \longrightarrow \beta_{f,k}X \xrightarrow{f_k = p_0p_1\cdots p_k} Y$$

of $f$ leads to a map of pairs $(\mathrm{cyl}(f), X) \to (\mathrm{cyl}(f_k), X)$. That map of pairs leads to a commutative diagram

$$\begin{array}{ccccccccc} \ldots \xrightarrow{\partial} & \pi_m(X) & \xrightarrow{f_*} & \pi_m(Y) & \longrightarrow & \pi_m(f) & \xrightarrow{\partial} & \pi_{m-1}(X) & \longrightarrow \ldots \\ & \downarrow & & \| & & \downarrow & & \downarrow & \\ \ldots \xrightarrow{\partial} & \pi_m(\beta_{f,k}X) & \xrightarrow{(f_k)_*} & \pi_m(Y) & \longrightarrow & \pi_m(f_k) & \xrightarrow{\partial} & \pi_{m-1}(\beta_{f,k}X) & \longrightarrow \ldots \end{array}$$

¿From that we can deduce:

**Lemma 3.21.** *The arrow* $\pi_m(f) \to \pi_m(f_k)$ *is an isomorphism for* $m \leq k+1$ *while* $\pi_m(f_k)$ *is trivial for* $m > k+1$. □

In the case where $Y = \star$ we have, rather obviously, $\pi_m(f) \cong \pi_{m-1}(X)$ and $\pi_m(f_k) \cong \pi_{m-1}(\beta_k X)$. So the lemma reduces in that case to something we already know: $\pi_{m-1}(\beta_k X) \cong \pi_{m-1}(X)$ for $m-1 \leq k$ and $\pi_{m-1}(\beta_k X)$ is trivial for $m-1 > k$.

**Corollary 3.22.** *For the map* $p_k\colon \beta_{f,k}X \to \beta_{f,k-1}X$ *we have*

$$\pi_{k+1}(p_k) \cong \pi_{k+1}(f_k) \cong \pi_{k+1}(f)$$

*and* $\pi_m(p_k)$ *is trivial for all* $m \neq k+1$.

*Proof.* If we take $m = k+1$ in the commutative diagram

$$\begin{array}{ccccccccc}
\dots \xrightarrow{\partial} & \pi_m(\beta_{f,k}X) & \xrightarrow{(p_k)_*} & \pi_m(\beta_{f,k-1}X) & \longrightarrow & \pi_m(p_k) & \xrightarrow{\partial} & \pi_{m-1}(\beta_{f,k}X) & \longrightarrow \dots \\
 & \| & & \downarrow & & \downarrow{\scriptstyle a} & & \| & \\
\dots \xrightarrow{\partial} & \pi_m(\beta_{f,k}X) & \xrightarrow{(f_k)_*} & \pi_m(Y) & \longrightarrow & \pi_m(f_k) & \xrightarrow{\partial} & \pi_{m-1}(\beta_{f,k}X) & \longrightarrow \dots
\end{array}$$

then the five lemma tells us that the arrow labeled $a$ is an isomorphism. (More precisely the two vertical arrows to the left of $a$ are bijective, the one to the right of $a$ is obviously bijective and the next one to the right is injective. There is enough group structure to go around, though perhaps not as much as we assume in the standard formulation of the five lemma; we can assume $k \geq 1$ and therefore $m \geq 2$.) If $m < k+1$ then $(p_k)_*$ is surjective in degree $m$, bijective in degree $m-1$, so that $\pi_m(p_k)$ is trivial by exactness of the top row. If $m > k+1$ then $(p_k)_*$ is bijective in degree $m$, injective in degree $m-1$, so that $\pi_m(p_k)$ is again trivial by exactness of the top row. $\square$

## 3.4. One-step obstruction theory

This section is mostly about the problem indicated in the following diagram.

$$\begin{array}{ccc}
 & & E \\
 & \overset{g}{\nearrow} & \downarrow{\scriptstyle p} \\
X & \xrightarrow[f]{} & Y
\end{array}$$

Here $X, Y, E$ are based connected CW-spaces (until declared otherwise) and $f, p$ are based maps, all given in advance. The problem is to find $g$. The diagram is meant to be commutative up to a specified homotopy $h = (h_t)$ from $pg$ to $f$. It is convenient to formulate the problem in this generality, but we will get the most significant results when $p$ satisfies a strong condition: *there is an integer* $\ell \geq 1$ *such that* $\pi_k(p)$ *is trivial whenever* $k \neq \ell$.

Let's begin with technical considerations. In the problem just formulated, the data $f$ and $p$ are given and by a *solution* of the problem we mean a pair $(g, h)$ consisting of $g\colon X \to E$ and a based homotopy $h$ from $pg$ to $f$. The homotopy $h = (h_t)_{t\in[0,1]}$ is a map from $X \times I$ to $Y$, where $I := [0,1]$. It seems clear that we need to organize these pairs $(g, h)$ into a *space*. More precisely we should first define $\text{map}_\star(X, E)$, the space of based maps from $X$

to $E$, and also $\mathrm{map}_\star(X,Y)$, the space of based maps from $X$ to $Y$. Then the space of solutions for our problem is

$$L(f;p) := \{(g,h) \in \mathrm{map}_\star(X,E) \times \mathrm{map}_\star(X \ltimes I, Y) \mid h_0 = pg,\ h_1 = f\},$$

where $X \ltimes I$ is improvised notation for the quotient of $X \times I$ by the subspace $\star \times I$. It is a subspace of $\mathrm{map}_\star(X,E) \times \mathrm{map}_\star(X \ltimes I, Y)$. The letter $L$ is used to suggest *lift*; the solutions $(g,h)$ are considered to be *lifts* of $f$ (up to a specified homotopy).

**Definition 3.23.** Let $V$ be a CW-space and let $W$ be any space. Let $\mathrm{map}(V,W)$ be the set of all continuous maps from $V$ to $W$. We use the compact-open topology to regard $\mathrm{map}(V,W)$ as a space. That is to say, a subset $Q$ of $\mathrm{map}(V,W)$ is considered to be open if for every $e \in \mathrm{map}(V,W)$ there exists a non-negative integer $\ell$, compact subsets $K_1,\dots,K_\ell \subset V$ and open subsets $U_1,\dots,U_\ell \subset W$ such that

- $e(K_j) \subset U_j$ for $j = 1,\dots,\ell$;
- all $e' \in \mathrm{map}(V,W)$ which satisfy $e'(K_j) \subset U_j$ for $j = 1,\dots,\ell$ also belong to $Q$.

**Remark 3.24.** Let $P$ be a *compact* CW-space. There is a map (of sets)

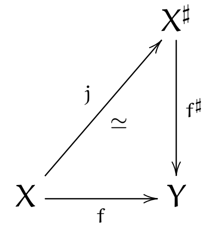
$$\mathrm{map}(P \times V, W) \longrightarrow \mathrm{map}(P, \mathrm{map}(V,W))$$

given by adjunction. I hope it is an exercise to show that the map is a homeomorphism. This is good enough for many purposes. We are very interested in the cases where $P = S^n$ or $P = S^n \times I$ for some $n$.
The condition that $P$ be compact can be dropped at a price: we must redefine the topology on $P \times V$ in such a way that a subset $C$ of $P \times V$ is closed if and only if $C \cap (P' \times V')$ is closed for every choice of compact CW-subspaces $P' \subset P$ and $V' \subset V$. We might write $P \times_{CW} V$ for this modified product and call it the CW-product, etc. It does have the universal property of a product *in the category of CW-spaces and continuous maps.*

**Remark 3.25.** It is known that if $V$ and $W$ are CW-spaces, then $\mathrm{map}(V,W)$ is homotopy equivalent to a CW-space. This is shown in an old paper by John Milnor, *On spaces having the homotopy type of a CW-complex.* Beautifully written and highly recommended reading. But I will try not to use this fact.
In that connection, let's also keep the following in mind. If $Z$ is *any* space, then there exists a CW-space $Z^\natural$ and a map $Z^\natural \to Z$ which is a weak equivalence. This is an easy consequence of the Brown representation theorem. (For any choice of base point in $Z$ we have a half-exact functor $[-,Z]_\star$. The different path components of $Z$ should be treated separately, and a base point should be selected in each of them.) We say that $Z^\natural$ is a CW-replacement

for $Z$. This notion is of course applicable with $Z = \mathrm{map}(V, W)$, even if we don't know or don't want to know that $\mathrm{map}(V, W)$ is homotopy equivalent to a CW-space.

**Example 3.26.** *Serre construction*: every map $f\colon X \to Y$ has a factorization




where $f^\sharp$ is a fibration, $j$ is "often" a cofibration and, more importantly, $j$ is always a homotopy equivalence. Definition of $X^\sharp$: it is the space of all pairs $(x, \omega)$ where $x \in X$ and $\omega\colon I \to Y$ is a path such that $\omega(0) = f(x)$. It is a subspace of $X \times \mathrm{map}(I, Y)$. Definition of $f^\sharp$: let $f^\sharp(x, \omega) = \omega(1) \in Y$. Definition of $j$: let $j(x) := (x, \omega)$ where $\omega$ is the constant path in $Y$ at $f(x)$.

*Showing that* $f^\sharp\colon X^\sharp \to Y$ *is a fibration.* Let's first solve the path lifting problem. For a path $\gamma\colon I \to Y$ and a choice of element $(x, \omega) \in X^\sharp$ such that $f^\sharp(x, \omega) = \gamma(0)$, define $\bar{\gamma}\colon I \to X^\sharp$ as follows. The first coordinate of $\bar{\gamma}(t)$ is $x$. The second coordinate is the path in $Y$ obtained by concatenating $\omega$ with $\gamma|_{[0,t]}$ and reparameterizing, $s \mapsto (1 + t)s$. This works because $\omega(1) = f^\sharp(x, \omega) = \gamma(0)$. Now $\bar{\gamma}$ satisfies $f^\sharp \circ \bar{\gamma} = \gamma$ and $\bar{\gamma}(0) = (x, \omega)$. So $\bar{\gamma}$ is a solution for this particular path lifting problem. — Since this solution depends very nicely (continuously) on the problem, we can use it to solve the homotopy lifting problem in general. Let $P$ be any space, let

$$(g_t\colon P \to Y)_{t \in [0,1]}$$

be a homotopy and let $G\colon P \to X^\sharp$ be a map such that $f^\sharp G = g_0$. Then for each $p \in P$ we obtain a path $\gamma = \gamma_p$ in $Y$ by $\gamma_p(t) = g_t(p)$, and an element $G(p)$ in $X^\sharp$ such that $f^\sharp(G(p)) = \gamma_p(0)$. This path lifting problem has a solution $\bar{\gamma}_p$ , constructed exactly as above. Then the map $(p, t) \mapsto \bar{\gamma}_p(t)$ from $P \times I$ to $X^\sharp$ is continuous, and it solves the homotopy lifting problem consisting of $(g_t)$ and $G$.

*Showing that* $j$ *is a homotopy equivalence.* We start with the observation that $j(X) \subset X^\sharp$ consists of all $(x, \omega) \in X^\sharp$ where the path $\omega$ is constant. There is a projection $X^\sharp \to X$ given by $(x, \omega) \mapsto x$. Restricting this to $j(X)$, we see that $j$ is a homeomorphism onto its image, $j(X)$, and that $j(X)$ is closed in $X^\sharp$. That's a good start. Now, for $t \in I$, let $\mu_t\colon I \to I$ be the map $s \mapsto st$. A homotopy $(h_t\colon X^\sharp \to X^\sharp)_{t \in [0,1]}$ is defined by $h_t(x, \omega) := (x, \omega \circ \mu_t)$. This homotopy is a deformation retraction to $j(X)$; that is, $h_0 = \mathrm{id}$, the image of $h_1$ is $j(X)$, and $h_t$ restricted to $j(X)$ is the identity, for all $t$. (*End of proof showing that* $j$ *is a homotopy equivalence.*)

For $z \in Y$, the fiber of $f^\sharp$ over $z$ is called the *homotopy fiber of* $f$ *over* $z$; if $Y$ has a preferred base point and $z$ is that base point, then simply the *homotopy fiber of* $f$. Notation: $\mathrm{hofiber}_z(f\colon X \to Y)$ or $\mathrm{hofiber}(f\colon X \to Y)$.

If $f\colon X \to Y$ is the inclusion of a subspace (and the subspace $X$ has a base point $\star$), then there is a canonical bijection (isomorphism of groups for $k \geq 1$)

$$\pi_{m-1}(\mathrm{hofiber}_\star(f)) \longrightarrow \pi_m(Y, X, \star) = \pi_m(Y, X).$$

This can be seen by writing the pair $(D^m, S^{m-1})$, which we use in the definition of $m$-th homotopy group of pairs, as $(\mathrm{cone}(S^{m-1}, S^{m-1})$, where $\mathrm{cone}(...)$ is the *reduced* cone. A map of pairs from $(\mathrm{cone}(S^{m-1}, S^{m-1})$ to $(Y, X)$ is "the same" (by a form of adjunction) as a map from $S^{m-1}$ to $\mathrm{hofiber}_\star(X \hookrightarrow Y)$. In this way, we understand at last that the long exact sequence

$$\dots \xrightarrow{\partial} \pi_m(X) \xrightarrow{f_*} \pi_m(Y) \longrightarrow \pi_m(Y, X) \xrightarrow{\partial} \pi_{m-1}(X) \longrightarrow \dots$$

of the pair $(Y, X)$ is (isomorphic to) the long exact sequence of homotopy groups associated with the fibration $f^\sharp\colon X^\sharp \to Y$. More generally, if $f\colon X \to Y$ is any map of based spaces, then the standard projection $\mathrm{cyl}(f) \to Y$ induces a map

$$\mathrm{hofiber}_\star(X \hookrightarrow \mathrm{cyl}(f)) \quad \longrightarrow \quad \mathrm{hofiber}_\star(f\colon X \to Y).$$

It is unfortunately an exercise to show that this is a weak equivalence. Therefore $\pi_m(f) = \pi_m(\mathrm{cyl}(f), X)$ is isomorphic to $\pi_{m-1}(\mathrm{hofiber}_\star(f))$.

*Showing that* $j\colon X \to X^\sharp$ *is a cofibration when* $Y$ *is a CW-space.* (Not very important and not very reliable.) I believe that every closed subset in a CW-space is the intersection of a countable decreasing sequence of open subsets. Suggested proof: induction on skeleta. Apply this to the diagonal $\Delta Y$ as a subspace of the CW-space $Y \times_{CW} Y$. It follows (a variant of the Tietze-Urysohn extension lemma, applicable since $Y \times_{CW} Y$ is a normal space) that there exists a continuous function $q\colon Y \times_{CW} Y \to I$ such that $q^{-1}(0) = \Delta Y$. Define a function $\bar{q}\colon X^\sharp$ to $I$ by $\bar{q}(x, \omega) := \max\{q(\omega(0), \omega(t)) \mid t \in I\}$. I hope that this is continuous. The preimage of $0$ under $\bar{q}$ is exactly the closed subset $j(X)$ of $X^\sharp$. Make a map $r$ from $X^\sharp \times I$ to itself by

$$r((x, \omega), t) := ((x, \gamma), s)$$

where

- if $t \geq \bar{q}(x, \omega)$, then $\gamma$ is constant with value $\omega(0)$, and $s$ is the difference $t - \bar{q}(x, \omega)$;
- if $t = (1 - u) \cdot \bar{q}(x, \omega)$ where $u \in I$, then $\gamma$ is $\omega \circ \mu_u$ where $\mu_u$ is multiplication by $u$, and $s$ is $0$.

Now I hope that $rr = r$ and that the image of $r$ is exactly the union of $X^\sharp \times \{0\}$ and $j(X) \times I$. By a well-known criterion, this would prove that $j$ is a cofibration.

**Example 3.27.** Our definition of the space of lifts $L(f;p)$ for a diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

can be reformulated as follows:

$$L(f;p) = \mathrm{hofiber}_f(\mathrm{map}_\star(X,E) \xrightarrow{p\circ} \mathrm{map}_\star(X,Y)).$$

And here we define $\mathrm{map}_\star(X,Y)$ etc. as a subspace of $\mathrm{map}(X,Y)$.

**Definition 3.28.** The *homotopy pullback* of a diagram

$$\begin{array}{ccc} & & B \\ & & \downarrow g \\ C & \xrightarrow{v} & D \end{array}$$

is the space $\{(x,\omega,y) \in B \times \mathrm{map}(I,D) \times C \mid \omega(0) = g(x),\ \omega(1) = v(x)\}$. A commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow f & & \downarrow g \\ C & \xrightarrow{v} & D \end{array}$$

is a *weak homotopy pullback square* if the map from $A$ to the homotopy pullback of $g$ and $v$ defined by $a \mapsto (u(a), \text{const.path}, f(b))$ is a weak homotopy equivalence.

**Remark 3.29.** The homotopy pullback of $g$ and $v$ (notation as above) can also be defined as follows: replace $g$ by a fibration $g^\sharp\colon B^\sharp \to D$ using the Serre construction. Then form the ordinary pullback of $v$ and $g^\sharp$.

*Exercise.* Show that a commutative square of spaces

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow f & & \downarrow g \\ C & \xrightarrow{v} & D \end{array}$$

is a weak homotopy pullback square if and only if, for every $c \in C$, the map $\mathrm{hofiber}_c(f) \to \mathrm{hofiber}_{v(c)}(g)$ induced by the horizontal arrows $u$ and $v$ is a weak homotopy equivalence.

*Exercise.* Let $u\colon Y \to Z$ be a map of spaces. Let $X$ be a CW-space. If $u$ is a weak equivalence, then the map $u\circ : \mathrm{map}(X, Y) \to \mathrm{map}(X, Z)$ is again a weak equivalence.

**Proposition 3.30.** *Let*

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{v} & D \end{array}$$

*be a weak homotopy pullback square of spaces. Let $X$ be a CW-space. Then*

$$\begin{array}{ccc} \mathrm{map}(X, A) & \xrightarrow{u\circ} & \mathrm{map}(X, B) \\ \downarrow{\scriptstyle f\circ} & & \downarrow{\scriptstyle g\circ} \\ \mathrm{map}(X, C) & \xrightarrow{v\circ} & \mathrm{map}(X, D) \end{array}$$

*is again a weak homotopy pullback square.*

*Proof.* Let $P$ be the homotopy pullback of $C \to D \leftarrow B$ and let $Q$ be the homotopy pullback of $\mathrm{map}(X, C) \to \mathrm{map}(X, D) \leftarrow \mathrm{map}(X, B)$. We need to show that the comparison map from $\mathrm{map}(X, A)$ to $Q$ is a weak equivalence. But $Q$ can easily be identified with $\mathrm{map}(X, P)$. With that identification, the comparison map from $\mathrm{map}(X, A)$ to $Q$ becomes the map from $\mathrm{map}(X, A)$ to $\mathrm{map}(X, P)$ induced by the comparison map $A \to P$. It is therefore a weak equivalence by one of the two exercises just above, since the comparison map $A \to P$ is a weak equivalence by assumption. □

**Proposition 3.31.** *Suppose that, in a commutative diagram*

$$\begin{array}{ccccc} & & E_0 & \longrightarrow & E_1 \\ & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p_1} \\ X & \xrightarrow{f} & Y_0 & \xrightarrow{v} & Y_1 \end{array}$$

*of based spaces and based maps, the right-hand square is a weak homotopy pullback square, and $X$ is a connected CW-space. Then the map*

$$L(f; p_0) \longrightarrow L(vf; p_1)$$

*determined by the horizontal arrows in the square is a weak equivalence.*

*Proof.* Follows from proposition 3.30 and the second of the two exercises. □

**Lemma 3.32.** *Let*

$$\begin{array}{c} E \\ \downarrow{\scriptstyle p} \\ Y \end{array}$$

*be a map of based connected CW-spaces. Suppose that* $Y$ *is 1-connected. Suppose that there exists* $\ell \geq 2$ *such that* $\pi_k(p)$ *is trivial for all* $k \neq \ell$*; if* $\ell = 2$*, assume in addition that* $\pi_\ell(p)$ *is abelian. Then* $p$ *is part of a weak homotopy pullback square of based connected CW-spaces*

$$\begin{array}{ccc} E & \longrightarrow & E_1 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_1} \\ Y & \longrightarrow & Y_1 \end{array}$$

*where* $E_1$ *is contractible.*

*Proof.* We begin with the commutative square

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{cone}(E) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ Y & \longrightarrow & \mathrm{cone}(p) \end{array} \tag{3.33}$$

where $q$ is the inclusion $\mathrm{cone}(E) \to Y \sqcup \mathrm{cone}(E)$ followed by the quotient map from $Y \sqcup \mathrm{cone}(E)$ to $\mathrm{cone}(p)$. This square is a first approximation to the solution; the upper right-hand term is already good (because it is contractible), but the lower right-hand term is not. Now compose with the Postnikov approximation

$$u\colon \mathrm{cone}(p) \hookrightarrow \beta_\ell \mathrm{cone}(p).$$

(Recall that $\beta_\ell\mathrm{cone}(p)$ is obtained from $\mathrm{cone}(p)$ by attaching cells of dimension $> \ell + 1$ to kill the homotopy groups of $\mathrm{cone}(p)$ in degrees $> \ell$.) In this way we get

$$\begin{array}{ccccc} E & \longrightarrow & \mathrm{cone}(E) & & \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \\ Y & \longrightarrow & \mathrm{cone}(p) & \xrightarrow{u} & \beta_\ell\mathrm{cone}(p) \end{array}$$

and now deleting $\mathrm{cone}(p)$ gives a commutative square

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{cone}(E) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle uq} \\ Y & \longrightarrow & \beta_\ell\mathrm{cone}(p). \end{array} \tag{3.34}$$

It turns out that this is the solution. In other words, we will now show that (3.34) is a weak homotopy pullback square. Briefly: The Hurewicz theorem (relative case) implies that in the square (3.33), the Hurewicz homomorphism is an isomorphism

$$\pi_m(p) \to H_m(p)$$

for $m \le \ell$; only the case $m = \ell$ is really interesting. (Note, if it wasn't clear, that the homology of a *map* is by definition the reduced homology of its mapping cone.) Similarly the Hurewicz homomorphism

$$\pi_m(q) \to H_m(q)$$

is an isomorphism for $m \le \ell$. The map cone$(p) \to$ cone$(q)$ induced by the horizontal arrows in square (3.33) is a homotopy equivalence (easy exercise) and so the horizontal arrows induce an isomorphism $H_\ell(p) \to H_\ell(q)$. Combining this with what we already know about the Hurewicz homomorphisms, we conclude that the horizontal arrows induce an isomorphism

$$\pi_m(p) \longrightarrow \pi_m(q) \cong \pi_m(\text{cone}(p))$$

when $m \le \ell$. For $m > \ell$, we know that $\pi_m(p) = 0$ and we don't know much about about $\pi_m(\text{cone}(p))$. But this gets much better if we look at square (3.34) instead. For $m \le \ell$ there is no need to distinguish between $\pi_m(q) \cong \pi_m(\text{cone}(p))$ and $\pi_m(uq) \cong \pi_m(\beta_\ell \text{cone}(p))$. Therefore the homomorphism $\pi_m(p) \to \pi_m(uq)$ is still an isomorphism for $m \le \ell$. But for $m > \ell$, the homotopy group $\pi_m(uq) \cong \pi_m(\beta_\ell \text{cone}(p))$ is zero by definition or construction. Therefore the horizontal arrows in the square (3.34) induce an isomorphism

$$\pi_m(p) \longrightarrow \pi_m(uq)$$

for all $m$ (in particular, for $m > \ell$ because any homomorphism between two groups which are zero is an isomorphism). It follows (by one of the exercises) that square (3.34) is a weak homotopy pullback square. □

**Remark 3.35.** The space $Y_1$ in lemma 3.32 is an Eilenberg-MacLane space, with only one possibly nontrivial homotopy group in degree $\ell$. That is so because

$$\pi_m(Y_1) \cong \pi_m(p_1)$$

since $E_1$ is contractible, and

$$\pi_m(p_1) \cong \pi_m(p)$$

since the square in the lemma is meant to be a homotopy pullback square, and $\pi_m(p)$ is trivial except possibly for $m = \ell$.

Let's return to our obstruction theory problem

$$\begin{array}{ccc} & & E \\ & \overset{g}{\nearrow} & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

stated at beginning of section, $\pi_m(p)$ nontrivial only for $m = \ell$, where $\ell$ is fixed. If $Y$ happens to be 1-connected and $\ell \geq 3$ or $\ell = 2$ and $\pi_2(p)$ is abelian, then proposition 3.31 and lemma 3.32 reduce this to the special situation where $E$ is contractible and $Y$ is consequently an Eilenberg-MacLane space. We now assume all that and write out the solution.

A first step is to reduce one more little step to the case where $p\colon E \to Y$ is the inclusion of the base point. (Justify this by applying proposition 3.31.) The map $f$ then corresponds to a cohomology class

$$\kappa_f \in H^\ell(X; G)$$

where $G = \pi_\ell(Y)$. The space $L(f; p)$ is nonempty if and only if $f$ is nullhomotopic, if and only if $\kappa_f = 0 \in H^\ell(X; G)$.

If $L(f; p)$ is nonempty and we select an element $z = (g, h)$ in it, then $g$ is uninteresting and $h$ is a nullhomotopy for $f$. It is not difficult at all to use the nullhomotopy $h$ to make a homotopy equivalence from $L(f; p)$ to $L(e; p)$ where $e\colon X \to Y$ is the zero map.

Now $L(e; p)$ simplifies to $\Omega(\mathrm{map}_\star(X, Y))$. Here $\Omega Z := \mathrm{map}_\star(S^1, Z)$, for a based space $Z$. (Name: *loop space* of $Z$.) In this way, since $Y$ is still an Eilenberg-MacLane space,

$$\begin{array}{rcccl} \pi_m(L(e; p)) & = & [S^m, \Omega(\mathrm{map}_\star(X, Y)]_\star & \cong & [S^1 \wedge S^m, \mathrm{map}_\star(X, Y)]_\star \\ \cong\ [S^{m+1} \wedge X, Y]_\star & \cong & \tilde{H}^\ell(S^{m+1} \wedge X; G) & \cong & \tilde{H}^{\ell-m-1}(X; G). \end{array}$$

In all, we have a fairly good understanding of $L(f; p)$.

**Remark 3.36.** The combined isomorphism

$$\pi_m(L(f; p)) \cong \pi_m(L(e; p)) \cong \tilde{H}^{\ell-m-1}(X; G)$$

which we have constructed depends on a choice of based homotopy $h$ from $f$ to the constant map $e$. Let's call it $J_h$ therefore. If $k$ is another homotopy from $f$ to the constant based map, then the concatenation of $h$ and the reverse of $k$ is a based map from $S^1 \wedge X$ to $Y$, where $S^1$ appears as a quotient of an interval such as $[0, 2]$. This corresponds to an element $\delta(h, k)$ of $H^\ell(S^1 \wedge X; G) \cong H^{\ell-1}(X; G)$. By inspection, the bijective map

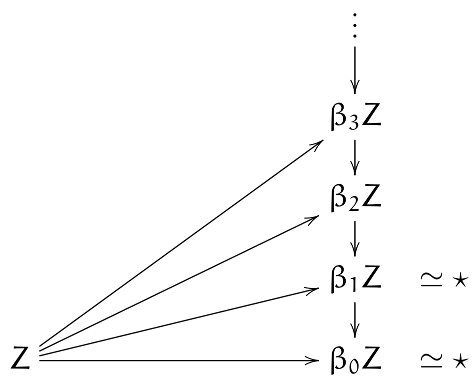
$$\tilde{H}^{\ell-m-1}(X; G) \xrightarrow{J_h^{-1}} \pi_m(L(f; p)) \xrightarrow{J_k} \tilde{H}^{\ell-m-1}(X; G)$$

is the identity when $m > 0$ and is given by addition of $\delta(h, k)$ when $m = 0$.

## Many-step obstruction theory

This section consists only of a few words on how you are supposed to combine the science of the Postnikov tower (or the Postnikov-Moore factorization) with one-step obstruction theory to obtain something more generally applicable.

**Example 3.37.** Suppose that $X$ and $Z$ are based connected CW-spaces. (The letter $Z$ is used to avoid confusion with $Y$ in the previous subsection, but it is not supposed to remind you of the set of integers.) Suppose that $Z$ is simply connected, that is, $\pi_1(Z)$ is trivial. Suppose that $X$ is finite-dimensional, $X = X^m$. We ask: what does the set $[X, Z]_\star$ look like. It is a good moment to remember the Postnikov tower of $Z$:




This determines a diagram of sets (with base point)

$$\begin{array}{c}
[X, Z] \\
\| \\
[X, \beta_m Z]_\star \\
\downarrow \\
[X, \beta_{m-1} Z]_\star \\
\downarrow \\
[X, \beta_{m-2} Z]_\star \\
\downarrow \\
\vdots \\
\downarrow \\
[X, \beta_3 Z]_\star \\
\downarrow \\
[X, \beta_2 Z]_\star \\
\downarrow \\
[X, \beta_1 Z]_\star \\
\| \\
\star
\end{array}$$

At the top of this diagram, we have $[X, Z]_\star = [X, \beta_m Z]_\star$ by the description of $\beta_m Z$ in definition 3.11. The obstruction to climbing from $[X, \beta_{k-1} Z]_\star$ to $[X, \beta_k Z]_\star$ is described by an exact sequence

$$[X, \beta_k Z]_\star \longrightarrow [X, \beta_{k-1} Z]_\star \longrightarrow H^{k+1}(X; \pi_k Z)$$

so that an element in $[X, \beta_{k-1} Z]_\star$ comes from $[X, \beta_k Z]_\star$ if and only if it goes to zero in $H^{k+1}(X; \pi_k Z)$. Also, if $g \colon X \to \beta_k Z$ is any based map, then we have a *surjection*

$$H^k(X; \pi_k Z) \longrightarrow \text{preimage of } [p_k g] \text{ under } [X, \beta_k Z]_\star \to [X, \beta_{k-1} Z]_\star .$$

We get this from the previous (sub)section by substituting $p_k \colon \beta_k Z \to \beta_{k-1} Z$ for what was called $p \colon E \to Y$ there. It is allowed because $p_k \colon \beta_k Z \to \beta_{k-1} Z$ has a single nonzero homotopy group, by the construction of the Postnikov tower. It sits in dimension $k+1$ and is isomorphic to $\pi_k(Z)$. The *surjection* above comes from writing $f := p_k g$ . Then we have a bijection $\pi_0(L(f; p_k)) \cong H^k(X; \pi_k Z)$ as in the previous subsection. It should be clear that there is a forgetful map $\pi_0(L(f; p_k)) \to [X, \beta_k Z]$ whose image is exactly the preimage of the element $[p_k g]$ under the map $[X, \beta_k Z]_\star \to [X, \beta_{k-1} Z]_\star$. Unfortunately this surjective forgetful map is not always *bijective*.

**Example 3.38.** Here is a long story to show that ... *this surjective map is not always bijective.* Maybe I am writing this mainly for my own education. It is a little over the top, to use a good English phrase.

Take $X = \mathbb{C}P^2$ and $Z = S^3$. This is a situation where $[X, Z]_\star$ can be determined without obstruction theory, although some knowledge of homotopy groups of spheres is required. It helps to know that $\pi_4(S^3) \cong \mathbb{Z}/2$ and it helps to know that the suspension homomorphism $\pi_3(S^2) \to \pi_4(S^3)$ is surjective. We will learn some of that in the next section.

Use the standard CW-structure on $X = \mathbb{C}P^2$ where $X^0 = X^1$ is a point, $X^2 = X^3 = \mathbb{C}P^1$ and $X^4 = \mathbb{C}P^2$. Use the standard structure on $Z = S^3$, too, where $Z^0 = Z^1 = Z^2$ is a point and $Z^3 = S^3$. Any based map from $X$ to $Z$ is homotopic to a cellular one, and a cellular map must have the form of a composition $X \to X/X^2 \to Z$, where $X \to X/X^2$ is the quotient map. Since $X/X^2 \cong S^4$, and since $[S^4, S^3]_\star = \pi_4(S^3)$ has only two elements, this means that $[X, Z]_\star$ has at most two elements. We get the two candidates by using representatives $S^4 \to S^3$ for the two elements of $\pi_4(S^3)$, and pre-composing with the quotient map $X \to X/X^2 = S^4$. But it turns out that the result is in both cases nullhomotopic as a based map $X \to Z$. (Idea for that: there is an exact sequence of pointed sets

$$[S^1 \wedge X^2, Z]_\star \longrightarrow [X/X^2, Z]_\star \longrightarrow [X, Z]_\star$$

where the map on the right is the one we have seen, induced by the quotient map $X \to X/X^2$. The other map is determined by the inclusion of $X/X^2$ in cone($X \to X/X^2$) and uses the observation cone($X \to X/X^2$) $\simeq S^1 \wedge X^2$, a special case of a general fact. With the identifications $X/X^2 \cong S^4$ and $S^1 \wedge X^2 \cong S^3$ that map $X/X^2 \to S^1 \wedge X^2$ becomes the suspension of the attaching map $S^3 \to S^2 = X^2$ for the unique 4-cell of $X$; again a special case of a general fact. That attaching map is the Hopf map and its suspension is therefore *the* nontrivial element in $\pi_4(S^3)$. It follows that the left-hand arrow in the above "exact sequence" is onto and the other one is therefore trivial by exactness. The exact sequence is of course also a special case of a more general construction, called the *Barratt-Puppe* sequence.) Summarizing, $[X, Z]_\star$ has only one element, represented by the constant based map $X \to Z$.

But now let us try to compute $[X, Z]_\star$ using obstruction theory. We can begin with $[X, \beta_3 Z]_\star$. The space $\beta_3 Z$ is an Eilenberg-MacLane space $K(\mathbb{Z}, 3)$, so $[X, \beta_3 Z]_\star$ must be in bijection with $H^3(X; \mathbb{Z})$ which is however $= 0$ as a group. So $[X, \beta_3 Z]_\star$ has only one element. So the preimage $P$ of that one element under the standard map

$$[X, \beta_4 Z]_\star \longrightarrow [X, \beta_3 Z]_\star$$

is all of $[X, \beta_4 Z]_\star$ , and that means, it is all of $[X, Z]_\star$ since $X$ is 4-dimensional. So that preimage $P$ has only one element by the above calculation of $[X, Z]_\star$. BUT obstruction theory gives us a surjection from $H^4(X; \pi_4(Z))$ to $P$, and now $H^4(X; \pi_4(Z)) = H^4(X; \mathbb{Z}/2) \cong \mathbb{Z}/2$ has two elements. So that surjective map is not bijective.