02.05.2015

Lecture Notes, weeks 3, 4 and 5
Topology SS 2015 (Weiss)

2.1. Homotopy groups of pairs

A *pair of spaces* (X, A) means a space X with a distinguished subspace A. If a base point in A has been selected, then we speak of a pair of based spaces. The base point in A also serves as base point in X.

An important example of a pair of spaces is (D^n, S^{n-1}). The preferred base point for me is probably $(-1, 0, 0, \dots)$.

A map of pairs from (X, A) to (Y, B) is a map $f\colon X \to Y$ such that $f(A)$ is contained in B. We write $f\colon (X, A) \to (Y, B)$ in this situation. The map is *based* if f of the base point is the base point (assuming that we are talking about pairs with base point).

Two maps $f\colon (X, A) \to (Y, B)$ and $g\colon (X, A) \to (Y, B)$ are *homotopic* as maps of pairs if there exists a map of pairs $h\colon (X \times [0, 1], A \times [0, 1]) \to (Y, B)$ such that $h_0 = f$ and $h_1 = g$, where $h_t(x) := h(x, t)$ for $t \in [0, 1]$. Such an h is called a *homotopy* from f to g. If f and g are based maps, and each h_t is also a based map, then we call h a based homotopy (between based maps of pairs). Homotopy (based homotopy) in this sense is an equivalence relation on the set of (based) maps from (X, A) to (Y, B).

Definition 2.1. For $n > 0$, the n-th homotopy set $\pi_n(X, A, \star)$ of the pair (X, A) with base point $\star \in A$ is the set of based homotopy classes of based maps from the pair (D^n, S^{n-1}) to (X, A). For $n = 0$ we define $\pi_0(X, A, \star)$ to be the quotient of $\pi_0(X, \star)$ by the image of $\pi_0(A, \star)$.

It is already routine to verify that $\pi_n(X, A, \star)$ is an abelian group for $n \geq 3$, and still a group for $n = 2$. To define the group structure we use a map of pairs

$$\bar{\kappa}\colon (D^n, S^{n-1}) \longrightarrow (D^n \vee D^n, S^{n-1} \vee S^{n-1})$$

which extends the map $\kappa\colon S^{n-1} \to S^{n-1} \vee S^{n-1}$ which we used previously to define the group structure in $\pi_{n-1}(A, \star)$. In more detail: let $I = [0, 1]$ and $\partial I = \{0, 1\}$ and use a homeomorphism of your choice to identify the pair (D^n, S^{n-1}) with the pair $(I^n/K, \partial I^n/K)$, where

> ∂I^n consists of all points in I^n which have at least one of their n coordinates in ∂I;
>
> $K \subset \partial I^n$ consists of all points in I^n which have at least one of the first $(n-1)$ coordinates in ∂I, or the n-th coordinate equal to 0. (In the case $n = 2$ this looks like $\sqcup$, the union of three edges of the square $\square = \partial I^2$.)

Then define $\bar{\kappa}$ by

$$(x_1,\ldots,x_n) \mapsto \begin{cases} (2x_1, x_2, \ldots, x_n) & \text{if } 2x_1 \leq 1 \\ (2x_1 - 1, x_2, \ldots, x_n) & \text{if } 2x_1 \geq 1. \end{cases}$$

This description makes the proof of the following statement mechanical.

Proposition 2.2. *There is a forgetful map* $\partial\colon \pi_n(X, A, \star) \to \pi_{n-1}(A, \star)$ *which is a homomorphism of groups for* $n \geq 2$*.*

Even more obvious: a based map of pairs $f\colon (X, A) \to (Y, B)$ induces a map $\pi_n(X, A, \star) \to \pi_n(Y, B, \star)$ which is a homomorphism of groups for $n \geq 2$. In particular the inclusion of $(X, \star)$ in (X, A) induces a map from $\pi_n(X, \star, \star) = \pi_n(X, \star)$ to $\pi_n(X, A, \star)$ which is a homomorphism for $n \geq 2$.

For $n \leq 1$ we can only say (in general) that $\pi_n(X, A, \star)$ is a set with a distinguished base point: the class of the constant map with value $\star$ from $(D^n, \partial D^n)$ to X.

Theorem 2.3. *For a based pair of spaces* (X, A)*, the sequence*

$$\cdots \longrightarrow \pi_n(X,\star) \longrightarrow \pi_n(X, A, \star) \xrightarrow{\partial} \pi_{n-1}(A, \star) \longrightarrow \pi_{n-1}(X, \star) \longrightarrow \cdots$$

is exact.

While the proof is not very exciting, the interpretation of exactness for low values of n is interesting. We agree that a sequence of based sets and based maps $\cdots \to \cdot \to \cdot \to \cdots$ is *exact* if for each map in the sequence, the preimage of the base point is equal to the image of the previous map. If the based sets happen to be abelian groups (with the zero element as base point) then this definition agrees with our standard concept of exactness. *Example*: the exactness theorem above implies that the image of $\pi_2(X, \star)$ in $\pi_2(X, A, \star)$ is a normal subgroup of $\pi_2(X, A, \star)$. *Example*: the sequence in the theorem is supposed to end with the term $\pi_0(X, A, \star)$. Consequently it is claimed that the map from $\pi_0(X, \star)$ to $\pi_0(X, A, \star)$ is onto and the preimage of the base element under that map is the image of $\pi_0(A, \star)$ in $\pi_0(X, \star)$ ... this is obviously correct by the very definition of $\pi_0(X, A, \star)$.

Proof. Let's prove exactness at $\pi_n(X, A, \star)$, assuming $n \geq 1$. It is clear that the composition of the two arrows (with that target/source) is zero. Suppose that $\alpha\colon (D^n, S^{n-1}) \longrightarrow (X, A)$ is a based map of pairs representing an element $[\alpha] \in \pi_n(X, A, \star)$. If $\partial[\alpha] = 0 \in \pi_{n-1}(A, \star)$, then we know that the restriction of α to S^{n-1} is based nullhomotopic as a based map from S^{n-1} to A. By the homotopy extension property for the inclusion $S^{n-1} \to D^n$, once we choose such a homotopy $h = (h_t)_{t\in[0,1]}$ we can also extend it to a homotopy $(\bar{h}_t)$ from α to another map $\beta\colon D^n \longrightarrow X$. Each $\bar{h}_t$ is automatically a based map of pairs from (D^n, S^{n-1}) to (X, A), since $\bar{h}_t$

agrees with h_t on A. Therefore α is based homotopic as a map of pairs to $\beta = \bar{h}_1$. But since $\beta(A) = \star$ we can say that $[\beta]$ is in the image of the map from $\pi_n(X, \star)$ to $\pi_n(X, A, \star)$.
Next, let's prove exactness at $\pi_n(X, \star)$, assuming $n \geq 1$. The composition of the two maps is the zero map because there is a commutative square

$$\begin{tikzcd} \pi_n(A,\star) \arrow[r] \arrow[d] & \pi_n(X,\star) \arrow[d] \\ \pi_n(A,A,\star) \arrow[r] & \pi_n(X,A,\star) \end{tikzcd}$$

where the term $\pi_n(A, A, \star)$ is trivial (a based set with only one element). If α is a based map from $S^n \cong I^n/\partial I^n$ to X such that the class of α in $\pi_n(X, A, \star)$ is zero, then α is based nullhomotopic as a map of pairs from $(I^n/K, \partial I^n/K)$ to (X, A). Let h be such a homotopy. So h is a map from I^{n+1} to X which takes $2n$ of the $2n+2$ faces of that cube to the base point. The exceptional faces are $I^n \times 0$, where h agrees with α, and $I^{n-1} \times 1 \times I$ which is mapped to A. They intersect in $I^{n-1} \times 1 \times 0$ which is mapped to $\star \in A \subset X$. Parametrizing this somewhat differently (details left to you, gentle reader) we see that h is a homotopy from the restriction of h to one of the two exceptional faces to the restriction of h to the other exceptional face. One of these restricted maps, viewed as a based map from $I^n/\partial I^n$ to X, is just α. The other restricted map can be viewed as a based map from $I^n/\partial I^n$ to A.
Exactness at $\pi_n(A, \star)$ is straightforward and left to the reader. □

2.2. Homotopy groups and homotopy equivalences

Theorem 2.4. (J.H.C. Whitehead) *Let $f \colon X \to Y$ be a map between nonempty CW-spaces such that, for all $x_0 \in X$ and all $n \geq 0$, the map*

$$f_* \colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

is an isomorphism (bijection for $n = 0$). *Then f is a homotopy equivalence.*

As a preparation for the proof we make a few observations.

It is not a serious restriction to assume that f is cellular. In any case we know that f is homotopic to a cellular map (call it g for now), and if $f_* \colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism for all $x_0 \in X$ and $n \geq 0$, then $g_* \colon \pi_n(X, x_0) \to \pi_n(Y, g(x_0))$ will also be an isomorphism for all x_0 and $n \geq 0$. (Here we need to remind ourselves how higher homotopy groups depend on base points. A homotopy from f to g determines a path γ from $f(x_0)$ to $g(x_0)$ and that path determines an isomorphism (bijection) ι_γ from $\pi_n(Y, f(x_0))$ to $\pi_n(Y, g(x_0))$. It is easy to see from the definition of ι_γ

that $g_*\colon \pi_n(X,x_0) \to \pi_n(Y,g(x_0))$ is the composition of f_* from $\pi_n(X,x_0)$ to $\pi_n(Y,f(x_0))$ with ι_γ.)

Next, if we assume that f is cellular then we can easily reduce to the case where it is the inclusion of a CW-subspace. For that reduction step we replace Y by the mapping cylinder cyl(f). This is defined as

$$\frac{Y \sqcup [0,1] \times X}{\sim}$$

where $\sim$ means that we identify $(0,x)$ with $f(x) \in Y$, for all $x \in X$. (Note that cyl(f) contains a copy of $X \cong \{1\} \times X$, and more obviously a copy of Y, and we have cyl$(f)/X = $ cone(f).) There is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\text{inclusion}"] \arrow[d, equal] & \text{cone}(f) \arrow[d, "\simeq"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Moreover cyl(f) has a preferred CW structure. (The k-skeleton of that is the union of Y^k and the image of the k-skeleton of $[0,1] \times X$.) With that preferred CW-structure, the inclusion of $X \cong \{1\} \times X$ in cyl(f) is the inclusion of a CW-subspace. If f has the property that

$$f_*\colon \pi_n(X,x_0) \to \pi_n(Y,f(x_0))$$

is a bijection for all x_0 and $n \geq 0$, then it follows easily that the inclusion $X \to$ cyl(f) has the analogous property. (Use the commutative square just above.)

Next, suppose that $f\colon X \to Y$ is the inclusion of a CW-subspace and that $f_*\colon \pi_n(X,x_0) \to \pi_n(Y,f(x_0))$ is a bijection for all x_0 and $n \geq 0$. Then we can use the exact sequence of theorem 2.3 to deduce that $\pi_n(Y,X,x_0)$ is trivial for all x_0 and $n \geq 0$.

Lemma 2.5. *Let (Y,X) be a pair of nonempty spaces and let $n \geq 0$ be an integer such that $\pi_n(Y,X,x_0)$ is trivial for all $x_0 \in X$. Then for every map*

$$g\colon (D^n, S^{n-1}) \to (Y,X)$$

there exists a homotopy $(h_t\colon D^n \to Y)_{t\in[0,1]}$ such that $h_0 = g$ and $h_1(D^n)$ is contained in X, and h is stationary on S^{n-1} (meaning $h_t(z) = g(z)$ for all $z \in S^{n-1}$ and all t).

Proof. In the important case $n = 0$, the claim is that every point in Y can be connected by a path in Y to a point in the subspace X. Once we select a base point $x_0 \in X$, this is equivalent to saying that the map from $\pi_0(X,x_0)$ to $\pi_0(Y,x_0)$ induced by the inclusion $X \to Y$ is onto. And that is equivalent to saying that $\pi_0(Y, X, x_0)$ is trivial, by our definition of $\pi_0(Y, X, x_0)$. — Now assume $n > 0$. For a map g as in the statement, let $x_0 = g(\star)$ where $\star \in S^{n-1} \subset D^n$ is the base point. Since $\pi_n(Y, X, x_0)$ is trivial and g represents an element in it, we know that there exists a map of pairs

$$G\colon (D^n \times [0,1], S^n \times [0,1]) \longrightarrow (Y, X)$$

such that $G(z, 0) = g(z)$ and $G(z, 1) = x_0$ for all $z \in D^n$. Let L be the union of $S^n \times [0,1]$ and $D^n \times \{1\}$ in $D^n \times [0,1]$. Then $G(L)$ is contained in the subspace X of Y. Now it is easy to construct a homotopy

$$(k_t\colon D^n \to D^n \times [0,1])_{t\in[0,1]}$$

such that $k_0(z) = (z, 0)$ and $k_1(z) \in L$ for all $z \in D$, and k_t is stationary on S^{n-1}, so that $k_t(z) = k_0(z)$ for all $z \in S^{n-1}$ and $t \in [0,1]$. Define $h_t := G \circ k_t$ for $t \in [0,1]$. This gives the required homotopy. □

Proof of theorem 2.4. In view of the above observations it suffices to show the following. Suppose that Y is a CW-space with a CW-subspace X. Suppose that for every $n \geq 0$ and every map of pairs

$$g\colon (D^n, S^{n-1}) \to (Y, X)$$

there exists a homotopy $(h_t\colon D^n \to Y)_{t\in[0,1]}$ such that $h_0 = g$ and $h_1(D^n)$ is contained in X, and the homotopy is stationary on S^{n-1}. *Then the inclusion* $X \to Y$ *is a homotopy equivalence.*

Indeed we are going to construct a homotopy $(F_t\colon Y \to Y)_{t\in[0,\infty]}$ such that $F_0 = \mathrm{id}_Y$ and (F_t) is stationary on X and $F_\infty(Y) \subset X$. This is clearly enough.[1] It turns out to be convenient for induction purposes to parameterize the homotopy by a compact interval of the form $[0, \infty]$; think of this as the one-point compactification of $\{x \in \mathbb{R} \mid x \geq 0\}$. (We have used this idea before to establish the HEP for inclusions of CW-subspaces; lecture notes WS 2014-2015.)

The idea is to construct (F_t) in steps corresponding to conditions $t \in [k, k+1]$ where k runs through the non-negative integers. This will be done in such a way that F_{k+1} takes the k-skeleton Y^k of Y to X and $F_t(y) = F_{k+1}(y)$ whenever $y \in Y^k$ and $t \geq k+1$. In words, the homotopy (F_t) is stationary on the k-skeleton of Y for $t \geq k+1$.

Suppose then that F_t has already been constructed for $t \in [0, k]$ where k is a positive integer, and that $F_k(Y^{k-1}) \subset X$. Let

$$\varphi\colon (D^k, S^{k-1}) \to (Y^k, Y^{k-1})$$

[1] Such a homotopy is called a strong deformation retraction of Y onto X. We can say that X is a strong deformation retract of Y.

be a characteristic map for a k-cell E of Y. Then $F_k \circ \varphi$ is a map of pairs from (D^k, S^{k-1}) to (Y, X). By our assumption on the pair (Y, X) there exists a homotopy $(h_t\colon D^k \to Y)_{t\in[0,1]}$ which is stationary on S^{k-1} and has $h_0 = F_k \circ \varphi$ and $h_1(D^k) \subset X$. We want to define F_t for $t \in [k, k+1]$ in such a way that $F_t \circ \varphi = h_{t-k}$. This seems to define F_t only on $Y^{k-1} \cup E$. (Note that F_t for $t \in [k, k+1]$ is already defined on Y^{k-1} because it is supposed to agree there with F_k .) But we can proceed in the same way to define F_t for $t \in [k, k+1]$ on all other k-cells of Y, so that it is defined on all of Y^k. By the definition of a CW-space, specialized here to Y, there is no problem with that in regard to continuity. Then we use the homotopy extension theorem for the inclusion $Y^k \to Y$ to extend to a homotopy, parameterized by a time interval $[k, k+1]$, of maps from Y to Y, beginning with $F_k\colon Y \to Y$ which is already given.
This induction process has a trivial beginning, $F_0 = \mathrm{id}_Y$, but it has a slightly nontrivial end. We decree $F_\infty(y) = F_{k+1}(y)$ if $y \in Y^k$. Since every $y \in Y$ is contained in Y^k for some k, this takes care of all $y \in Y$. By construction of F_t for $t < \infty$, this definition of $F_\infty(y)$ is unambiguous. By the definition of a CW-space, there is no problem whatsoever in regard to continuity. $\square$

2.3. Homotopy and homology

We have already seen the *Hurewicz homomorphism*. It is a map

$$\pi_n(X, \star) \longrightarrow H_n(X)$$

defined by $[\alpha] \mapsto \alpha_*(1)$. Here $\alpha\colon S^n \to X$ is a based map and $1 \in H_n(S^n) \cong \mathbb{Z}$ is informal notation for the standard generator (also known as the standard fundamental class of the sphere as an oriented manifold). I am assuming $n \geq 1$. The Hurewicz homomorphism is indeed a homomorphism of groups. In the case $n = 1$, when X is path connected, it is surjective and its kernel is the commutator subgroup (the smallest normal subgroup of $\pi_n(X, \star)$ with a commutative quotient). See lecture notes WS 2014-2015, chapter about fundamental groups. This section is about similar statements for higher n under strong conditions on X.

Theorem 2.6. (Hurewicz.) *Let X be a based CW-space such that $\pi_k(X, \star)$ is trivial for $k = 0, 1, 2, \ldots, n-1$, where $n \geq 2$. Then the Hurewicz homomorphism $\pi_n(X, \star) \to H_n(X)$ is an isomorphism.*

The theorem is an easy consequence of what we already know about homology of CW-spaces, modulo the following lemma.

Lemma 2.7. *If X is a based connected CW-space such that $\pi_k(X, \star)$ is trivial for $k = 1, 2, \ldots, n-1$, where $n \geq 1$, then X is based homotopy equivalent to a CW-space Y such that Y^{n-1} is a single point, which is the base point.*

We postpone the proof of lemma 2.7.

Proof of theorem modulo lemma. We can assume that X in the theorem has X^{n-1} equal to a single point, the base point. Choose characteristic maps for all cells of X. We know already that the homomorphism

$$\pi_n(X^n,\star) \to \pi_n(X,\star)$$

is onto. Moreover X^n is a wedge of (possibly many) n-spheres, say $X^n = \bigvee_{\lambda\in\Lambda} S^n$ where $n \geq 2$. We know (see remark 2.8 below) that the inclusion of these wedge summands in X^n induces an isomorphism from

$$\bigoplus_\lambda \mathbb{Z} \;\cong\; \bigoplus_\lambda \pi_n(S^n,\star)$$

to $\pi_n(X^n)$. Now let $\psi\colon S^n \to X^n$ be an attaching map for an $(n+1)$-cell of X. Then

$$[\psi] \in [S^n, X^n] \cong [S^n, X^n]_* = \pi_n(X^n,\star) \;\cong\; \bigoplus_\lambda \mathbb{Z}$$

goes to zero in

$$[S^n, X^{n+1}] = [S^n, X] = [S^n, X]_* = \pi_n(X,\star)$$

because ψ extends to a map from D^{n+1} to X^n (the characteristic map for that n-cell). Therefore the element of $\bigoplus_\lambda \mathbb{Z}$ determined by $[\psi]$ is in the kernel of the surjective map from $\bigoplus_\lambda \mathbb{Z} \cong \pi_n(X^n,\star)$ to $\pi_n(X,\star)$. This reasoning, carried out for all $(n+1)$-cells of X, gives us a *lower bound* on that kernel (a subgroup contained in the kernel). We do not need more because of the following commutative diagram:

$$\begin{tikzcd}
\bigoplus_\lambda \mathbb{Z} \cong \pi_n(X^n,\star) \arrow[r] \arrow[d, "\text{Hurew.}"', "\cong"] & \pi_n(X^{n+1},\star) \cong \pi_n(X,\star) \arrow[d, "\text{Hurew.}"] \\
\bigoplus_\lambda \mathbb{Z} \cong H_n(X^n) \arrow[r] & H_n(X^{n+1}) \cong H_n(X)
\end{tikzcd}$$

We know the kernel of the lower horizontal arrow and we therefore get an *upper bound* on the kernel of the upper horizontal arrow (a subgroup containing that kernel) using the commutativity of the diagram. But we already had a lower bound for it, and the upper bound agrees with the lower bound. Therefore, since the horizontal arrows are both surjective, the right-hand vertical arrow must be an isomorphism. $\square$

Remark 2.8. For a wedge of spheres $\bigvee_{\lambda\in\Lambda} S^n$, where $n \geq 2$ is fixed, the inclusions of the wedge summands induce an isomorphism

$$\bigoplus_{\lambda\in\Lambda} \pi_n(S^n,\star) \to \pi_n(\bigvee_{\lambda\in\Lambda} S^n,\star)\,.$$

This was already mentioned in example 1.8 (lecture notes weeks 1 and 2). There are two steps to the proof. Firstly, it is clear that every element of

$\pi_n(\bigvee_{\lambda\in\Lambda} S^n,\star)$ comes from $\pi_n(\bigvee_{\lambda\in\Lambda'} S^n,\star)$ for some finite subset $\Lambda'\in\Lambda$. Moreover $\pi_n(\bigvee_{\lambda\in\Lambda'} S^n,\star)$ injects into $\pi_n(\bigvee_{\lambda\in\Lambda} S^n,\star)$ as a direct summand (since $\bigvee_{\lambda\in\Lambda'} S^n$ is a *retract* of $\bigvee_{\lambda\in\Lambda} S^n$). Secondly, as we saw in lecture notes weeks 1 and 2, the inclusion

$$\bigvee_{\lambda\in\Lambda'} S^n \longrightarrow \prod_{\lambda\in\Lambda'} S^n$$

induces an isomorphism

$$\pi_n(\bigvee_{\lambda\in\Lambda'} S^n,\star) \;\longrightarrow\; \pi_n(\prod_{\lambda\in\Lambda'} S^n,\star) \cong \prod_{\lambda\in\Lambda'} \pi_n(S^n,\star) = \bigoplus_{\lambda\in\Lambda'} \pi_n(S^n,\star).$$

Corollary 2.9. *Let X be a path connected CW-space X with base point such that $\pi_1(X,\star)$ is trivial and the homology groups $H_n(X)$ are trivial for $n\geq 1$. Then X is contractible.*

Proof. Suppose for a contradiction that X is not contractible. Then by the JHC Whitehead theorem 2.4, there is $n\geq 1$ such that $\pi_n(X,\star)$ is nontrivial. Find the minimal such n. It is >1 by assumption. Then $\pi_n(X,\star)\cong H_n(X)$ for this n, by the Hurewicz theorem 2.6. So $H_n(X)$ is also nontrivial. Contradiction. □

2.4. Homotopy of pairs and homology

Theorem 2.10. (Hurewicz.) *Let (Y,X) be a pair of based connected CW-spaces such that the fundamental group $\pi_1(X,\star)$ is trivial and $\pi_k(Y,X,\star)$ is trivial for $k=0,1,2,\dots,n-1$, where $n\geq 2$. Then the composition*

$$\pi_n(Y,X,\star) \xrightarrow{\text{ind. by quot. map}} \pi_n(Y/X,\star) \xrightarrow{\text{Hurewicz homom.}} H_n(Y/X)$$

is an isomorphism.

For the proof we need a lemma similar to lemma 2.7 but slightly more general. (And once again we postpone the proof.)

Lemma 2.11. *Let (Z,X) be a pair of based connected CW-spaces such that $\pi_k(Z,X,\star)$ is trivial for $k=0,1,2,\dots,n-1$, where $n\geq 1$. Then there exists a CW-space Y containing X as a CW-subspace, and a map $Y\to Z$ which is the identity on X, such that*

- *the map $Y\to Z$ is a homotopy equivalence*
- *$Y^{n-1}=X^{n-1}$ (i.e., all cells in $Y\smallsetminus X$ have dimension $\geq n$).*

Remark 2.12. Let (Y,X) be a pair of based spaces where X is path connected, and let $x_0,x_1\in X$. Then for $n\geq 2$ the groups $\pi_n(Y,X,x_0)$ and $\pi_n(Y,X,x_1)$ are isomorphic; in fact a choice of path in X from x_0 to x_1 determines an isomorphism ι_γ between the two. This can also be used to define an action (by group automorphisms) of $\pi_1(X, x_0)$ on $\pi_n(Y, X, x_0)$. The forgetful map from $\pi_n(Y, X, x_0)$ to the set of homotopy classes of unbased maps from (D^n, S^{n-1}) to (Y, X) is always surjective; two elements of $\pi_n(Y, X, x_0)$ determine the same unbased homotopy class if and only if they are in the same orbit of the action of $\pi_1(X, x_0)$. (The proof of these statements is an exercise.)

We need one more lemma. Assuming lemma 2.11, we can restate the assumptions of the Hurewicz theorem 2.10 as saying that we have a pair of CW-spaces (Y, X) where X is based, connected and has trivial fundamental group, and $Y^{n-1} = X^{n-1}$, where $n \geq 2$ is fixed. Choose characteristic maps

$$\varphi_j \colon (D^n, S^{n-1}) \longrightarrow (Y^n, Y^{n-1}) = (Y^n, X^{n-1})$$

for the n-cells of Y not contained in X. These maps need not take base point to base point, but for each j we can choose a path γ_j in X^n from φ_j of the base point (of S^{n-1}) to $\star$, the base point of X^n. Together the φ_j and the γ_j define a based map of pairs

$$f \colon \big((\textstyle\coprod_j D^n)/\!\!/J\,, (\coprod_j S^{n-1})/\!\!/J\big) \longrightarrow (Y^n, X^n).$$

Here $/\!\!/$ stands for the mapping cone construction and J is the collection of the base points (one in each copy of $S^{n-1} \subset D^n$). The cone point c serves as the base point in $(\coprod_j S^{n-1})/\!\!/J$. Here is a budding artist's impression of $(\coprod_j D^n)/\!\!/J$ and $(\coprod_j S^{n-1})/\!\!/J$.

Lemma 2.13. *The map* $\pi_n((\coprod_j D^n)/\!\!/J\,, (\coprod_j S^{n-1})/\!\!/J\,, c) \to \pi_n(Y^n, X^n, \star)$ *induced by* f *is surjective.*

Proof. By remark 2.12, when we represent elements of $\pi_n(Y^n, X^n, \star)$ by maps of pairs there is no need to pay attention to base points. In addition we like a rectangular representation in this proof, so we begin with

$$g \colon (I^n, \partial I^n) \to (Y^n, X^n)$$

where $I = [0, 1]$. The goal is to show that g is unbased homotopic, as a map of pairs, to a map in the form of a composition

$$(D^n, S^{n-1}) \dashrightarrow \big((\textstyle\coprod_j D^n)/\!\!/J\,, (\coprod_j S^{n-1})/\!\!/J\big) \xrightarrow{f} (Y^n, X^n).$$

(It is not important whether the broken arrow is a based map or not — it will automatically be homotopic to a based map since $(\coprod_j S^{n-1})/\!\!/J$ is path connected.) — By smooth approximation and Sard's theorem, we can assume that the sets $S_j := g^{-1}(\varphi_j(0))$ are finite and that for each $z \in S_j$ there is a small cube K_z inside I^n, centered at z, such that $\varphi_j^{-1} g$ maps a neighborhood of K_z smoothly and diffeomorphically to a neighborhood of 0 in $D^n \smallsetminus S^{n-1}$.

The cubes K_z for $z \in \bigcup_j S_j$ are pairwise disjoint and we can also suppose that their images under the first projection $p_1 \colon I^n \to I$ are pairwise disjoint. (If not, pre-compose g with a suitable perturbation, alias diffeomorphism $I^n \to I^n$ which is the identity in a neighborhood of the boundary.) This gives us a way to number the $z \in \bigcup S_j$ consecutively by comparing their first coordinates; so we write $z(1), z(2), \ldots, z(r)$. Now draw a straight line segment $L(1)$ from the right-hand face (maximal first coordinate) of $K_{z(1)}$ to the left-hand face of $K_{z(2)}$; next a straight line segment $L(2)$ from the right-hand face of $K_{z(2)}$ to the left-hand face of $K_{z(3)}$, etc. Let Q be the union of the little cubes $K_{z(1)}$, $K_{z(2)}$, $K_{z(3)}$, ..., $K_{z(r)}$ and the segments $L_{z(1)}$, $L_{z(2)}$, $L_{z(3)}$, ... $L_{z(r-1)}$. *Claim*:

$$\partial Q := \partial K_{z(1)} \cup L(1) \cup \partial K_{z(2)} \cup L(2) \cup \cdots \cup L(q-1) \cup K_{z(q)}$$

is a strong deformation retract of

$$I^n \smallsetminus \mathrm{int}(Q) \;=\; I^n \smallsetminus \mathrm{int}\left(K_{z(1)} \cup K_{z(2)} \cup \cdots \cup K_{z(q)}\right).$$

Meaning: there exists a homotopy $(h_s \colon I^n \smallsetminus \mathrm{int}(Q) \to I^n \smallsetminus \mathrm{int}(Q))_{s\in[0,1]}$ which is stationary on ∂Q and such that $h_0 = \mathrm{id}$ whereas $h_1(I^n \smallsetminus \mathrm{int}(Q)) \subset \partial Q$. The proof of the claim is left to the gentle reader, but the budding artist is back trying to help us visualize the inclusion of ∂Q into $I^n \smallsetminus \mathrm{int}(Q)$: Let $h^e_t \colon I^n \to I^n$ be defined like h_t on $I^n \smallsetminus Q$ and like the identity on Q. The homotopy $(gh^e_t)_{t\in[0,1]}$ can be viewed as an unbased homotopy of maps with source $(I^n, \partial I^n)$ and target $(Y^n, Y^n \smallsetminus U)$ where U is a tiny standard neighborhood of the collection of points $\varphi_j(0)$, the center points in each n-cell of $Y^n \smallsetminus X^n$. The inclusion of (Y^n, X^n) in $(Y^n, Y^n \smallsetminus U)$ is a homotopy equivalence of pairs, so we can in fact replace (Y^n, X^n) by $(Y^n, Y^n \smallsetminus U)$ without loss of essential information. The homotopy $(gh^e_t)_{t\in[0,1]}$ begins with $gh^e_0 = g$ and ends with gh^e_1. But gh^e_1 is the composition of h^e_1 from $(I^n, \partial I^n)$ to $(Q, \partial Q)$ with

$$g|_Q \colon (Q, \partial Q) \longrightarrow (Y^n, Y^n \smallsetminus U).$$

Therefore it only remains to show that $g|_Q \colon (Q, \partial Q) \to (Y^n, Y^n \smallsetminus U)$ is homotopic to a composition

$$(Q, \partial Q) \dashrightarrow \left((\textstyle\coprod_j D^n)/\!\!/J\,, (\coprod_j S^{n-1})/\!\!/J\right) \xrightarrow{\;f\;} (Y^n, Y^n \smallsetminus U).$$

Here it is convenient to replace S^{n-1} by $D^n \smallsetminus V$, the complement in D^n of an open ball of small radius about 0. (It should be done in such a way that $f^{-1}(U) = \coprod_j V \subset \coprod_j D^n$.) So now we are hoping to show that $g|_Q \colon (Q, \partial Q) \to (Y^n, Y^n \smallsetminus U)$ is homotopic to a composition

$$(Q, \partial Q) \overset{\bar g}{\dashrightarrow} \left((\textstyle\coprod_j D^n)/\!\!/J\,, (\coprod_j D^n \smallsetminus V)/\!\!/J\right) \xrightarrow{\;f\;} (Y^n, Y^n \smallsetminus U).$$

But this is easy. Define $\bar{g}$ on $K_z \subset Q$ so that it agrees with $\varphi_j^{-1}g$, where $g(z) = \varphi_j(0)$. Then $\bar{g}(\partial K_z)$ is contained in $(\coprod_j D^n \smallsetminus V)/\!\!/J$ because V is small enough. Now try to extend the definition of $\bar{g}$ to the segments $L(i)$. Beware that they have to be mapped to $(\coprod_j D^n \smallsetminus V)/\!\!/J$. This extension problem has a solution because $(\coprod_j D^n \smallsetminus V)/\!\!/J$ is path connected. In this way $\bar{g}$ can be defined on all of Q. On the little cubes K_z we have agreement of $f\bar{g}$ with g. Therefore it suffices to show that $f\bar{g}$ restricted to a segment $L(i)$ is homotopic to g restricted to the segment $L(i)$, by a homotopy (of maps to $Y^n \smallsetminus U$) which is stationary on the boundary points of the segment. This is clear since the fundamental group of $Y^n \smallsetminus U \simeq X^n$ is trivial. $\square$

Proof of theorem 2.10 modulo lemma 2.11. We can assume that $Y^{n-1} = X^{n-1}$. By analogy with the proof of theorem 2.6 we start with a commutative diagram

$$\begin{array}{ccc} \pi_n(Y^n, X^n) & \longrightarrow & \pi_n(Y, X) \\ \downarrow & & \downarrow \\ \bigoplus_j \mathbb{Z} \cong H_n(Y^n/X^n) & \longrightarrow & H_n(Y/X) \end{array}$$

where j runs through a set of labels for the n-cells of $Y \smallsetminus X$. (Base points have been suppressed.) The horizontal arrows are induced by the inclusion of the n-skeleton, $Y^n \to Y$, and are known to be surjective (the upper horizontal arrow by cellular approximation). By lemma 2.13, there is a surjective homomorphism

$$f_*\colon \pi_n((\textstyle\coprod_j D^n)/\!\!/J\,, (\coprod_j S^{n-1})/\!\!/J) \longrightarrow \pi_n(Y^n, X^n).$$

If $n \geq 3$, source and target of this homomorphism are abelian and the source group is isomorphic to $\bigoplus_j \mathbb{Z}$, from the long exact sequence of homotopy groups of the pair $(\coprod_j D^n)/\!\!/J\,, (\coprod_j S^{n-1})/\!\!/J)$. If $n = 2$, the target group is abelian, from the long exact sequence of homotopy groups of the pair (Y^n, X^n). The source group is a free group with generators corresponding to the labels j; this follows again from the long exact sequence of homotopy groups of the pair $(\coprod_j D^n)/\!\!/J\,, (\coprod_j S^{n-1})/\!\!/J)$. Therefore, in all cases, we have a surjection from $\bigoplus_j \mathbb{Z}$ to the abelian group $\pi_n(Y^n, X^n)$. Using this, it follows that the left-hand vertical arrow in the little square above is an isomorphism (of abelian groups). Therefore, as in the proof of theorem 2.6, it suffices to show that the kernel of the upper horizontal arrow "contains" the kernel of the lower horizontal arrow. (Quotation marks apply because the two kernels are subgroups of two different abelian groups, which are however related by a preferred isomorphism.) This is easy to establish by looking at the element of $\pi_n(Y^n, X^n)$ defined by the attaching map $\alpha\colon S^n \to Y^n$ for an $(n+1)$-cell of Y not in X. That element is in the kernel of the upper horizontal arrow. $\square$

2.5. Trading cells

Definition 2.14. A map of spaces $f\colon X \to Y$ is 0-*connected* if it induces a surjection of path components, $f_*\colon \pi_0(X) \to \pi_0(Y)$. (The sets $\pi_0(X)$ and $\pi_0(Y)$ do not really depend on base points, so none has been specified.)
A map of spaces $f\colon X \to Y$ is n-*connected*, where n is a positive integer, if it is 0-connected and for every $x_0 \in X$ the map

$$f_*\colon \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$$

is bijective for $k = 0, 1, 2, \dots, n-1$ and surjective for $k = n$.
A map of spaces $f\colon X \to Y$ which is n-connected for all $n \geq 0$ is also called a *weak equivalence*.

Example 2.15. (i) Let X be a CW-space. We have seen that the inclusion $X^n \to X$ is n-connected.
(ii) Let (Y, X) be a pair of nonempty spaces. The long exact sequence of homotopy groups (homotopy sets) of the pair (Y, X) implies that the inclusion $X \to Y$ is n-connected if and only if $\pi_k(Y, X, x_0)$ is trivial (has just one element) for $k = 0, 1, \dots, n$.
(iii) If $f\colon X \to Y$ is n-connected, where $n > 0$, then it is also $(n-1)$-connected.
(iv) If $f\colon X \to Y$ is a map of CW-spaces which is a weak equivalence, then it is a homotopy equivalence according to JHC Whitehead's theorem.

Proof of lemma 2.7. Given a CW-space X with the stated properties, we construct a CW-space Y such that $Y^{n-1} = \star$ and a map $g\colon Y \to X$ which is a weak equivalence (and therefore a homotopy equivalence). The plan is to construct Y^k and a map $g^k\colon Y^k \to X$ simultaneously, by induction on k, so that g^k is k-connected and g^k agrees with $g^{(k+1)}$ on $Y^k \subset Y^{k+1}$. The induction begins with $Y^{n-1} = \star$ and $g^{n-1}\colon Y^{n-1} \to X$ equal to the inclusion of the base point. By our assumptions on X, this is indeed an $(n-1)$-connected map. Now assume that $g^k\colon Y^k \to X$ has already been constructed and is k-connected, where $k \geq n-1$ is fixed. We distinguish two cases.
(i) If $n = 1$ and $k = n - 1 = 0$, then $Y^0 = \star$. The map from $\pi_0(Y^0, \star)$ to $\pi_0(X, \star)$ determined by g^0 is a bijection, but the map from $\pi_1(Y^0, \star)$ to $\pi_1(X, \star)$ determined by g^0 need not be surjective. Choose based maps $\gamma_i\colon S^1 \to X$ such that the classes $[\gamma_i] \in \pi_1(X, \star)$ form a generating set for that group. Define Y^1 to be the wedge $\bigvee_i S^1$ of as many circles and define g^1 so that it agrees with γ_i on the circle (wedge summand) with label i. Then $g^1\colon Y^1 \to X$ is 1-connected.

(ii) Otherwise start by observing that the map $\pi_k(Y^k,\star) \to \pi_k(X,\star)$ determined by g^k is a surjective homomorphism of groups. Choose based maps $\alpha_i\colon S^k \to X$ such that the classes $[\alpha_i]$ generate the kernel of that homomorphism, and for each i choose a map $\beta_i\colon D^{k+1} \to X$ which extends $g^k \circ \alpha_i$. Choose based maps $\gamma_j\colon S^{k+1} \to X$ such that the classes $[\gamma_j]$ generate the group $\pi_{k+1}(X,\star)$. Define Y^{k+1} to be

$$\Big(Y^k \cup_{\bigvee \alpha_i} \bigvee_i D^{k+1}\Big) \vee \bigvee_j S^{k+1}.$$

In words: Y^{k+1} is the $(k+1)$-dimensional CW-space obtained from the k-dimensional CW-space Y^k by first using the maps α_i as attaching maps for so many $(k+1)$-cells, and then taking the wedge with so many spheres S^{k+1}. Define g^{k+1} so that the composition

$$\bigvee_i D^{k+1} \longrightarrow Y^{k+1} \xrightarrow{g^{k+1}} X$$

agrees with $\bigvee_i \beta_i$ and so that the composition

$$\bigvee_j S^{k+1} \longrightarrow Y^{k+1} \xrightarrow{g^{k+1}} X$$

agrees with $\bigvee_i \gamma_i$. — Finally let $Y = \bigcup Y^k$ and define g on Y so that it agrees with g^k on Y^k. □

Proof of lemma 2.11. This is very similar to the proof of lemma 2.7. Given a CW-pair (Z,X) with the stated properties, we construct a CW-space Y containing X as a CW-subspace such that $Y^{n-1} = X^{n-1}$ and a map $g\colon Y \to Z$ which is a weak equivalence (and therefore a homotopy equivalence). The plan is to construct Y^k and a map $g^k\colon Y^k \to Z$ simultaneously, by induction on k, so that g^k is k-connected and g^k agrees with $g^{(k+1)}$ on $Y^k \subset Y^{k+1}$. The induction begins with $Y^{n-1} = X^{n-1}$ and $g^{n-1}\colon Y^{n-1} \to Z$ equal to the inclusion of X^{n-1} in Z. By our assumptions on the pair (Z,X), this is indeed an $(n-1)$-connected map. Now assume that $g^k\colon Y^k \to Z$ has already been constructed and is k-connected, where $k \geq n-1$ is fixed. We distinguish two cases.

(i) If $n = 1$ and $k = n-1 = 0$, then $Y^0 = X^0$. Choose based maps $\gamma_j\colon S^1 \to Z$ such that the classes $[\gamma_j]$ generate the group $\pi_1(Z,\star)$. Let

$$Y^1 \;:=\; X^1 \vee \bigvee_j S^1.$$

Define $g^1\colon Y^1 \to Z$ so that it agrees with the inclusion $X^1 \to Z$ on X^1 and with $\bigvee_j \gamma_j$ on $\bigvee_j S^1$.

(ii) Otherwise start by observing that the map $\pi_k(Y^k,\star) \to \pi_k(Z,\star)$ determined by g^k is a surjective homomorphism of groups. Choose based maps $\alpha_i \colon S^k \to X$ such that the classes $[\alpha_i]$ generate the kernel of that homomorphism, and for each i choose a map $\beta_i \colon D^{k+1} \to X$ which extends $g^k \circ \alpha_i$. Choose based maps $\gamma_j \colon S^{k+1} \to X$ such that the classes $[\gamma_j]$ generate the group $\pi_{k+1}(X,\star)$. Define Y^{k+1} to be ... (continue as in the proof of lemma 2.7). □

2.6. Homotopy equivalences and homology

Theorem 2.16. (G. Whitehead) *Let $f \colon X \to Y$ be a map between path connected CW-spaces which induces an isomorphism $H_k(X) \to H_k(Y)$ for all k. Suppose also that $\pi_1(X,x_0)$ and $\pi_1(Y,y_0)$ are trivial (for some or all $x_0 \in X$, $y_0 \in Y$). Then f is a homotopy equivalence.*

Proof. Without loss of generality, f is the inclusion $X \hookrightarrow Y$ of a CW-subspace. The long exact sequence of homology groups implies that $\tilde{H}_k(Y/X)$ is zero for all k. From our assumptions we also get that $\pi_1(Y,X,\star)$ is trivial for any choice of base point $\star \in X$. The second Hurewicz theorem 2.10 then implies that $\pi_k(Y,X,\star)$ is trivial for every choice of $\star \in X$ and $k \geq 2$. (If not, choose minimal $k \geq 2$ for which $\pi_k(Y,X,\star)$ is nontrivial; note that this $\pi_k(Y,X,\star)$ is isomorphic to $H_k(Y/X)$ which is zero, contradiction.) Since $\pi_k(Y,X,\star)$ is trivial for all $k \geq 1$, it follows that the inclusion $X \to Y$ is a weak equivalence and therefore a homotopy equivalence by JHC Whitehead's theorem 2.4. □

2.7. Related thoughts

Remark 2.17. Under the assumptions of the second Hurewicz theorem 2.10, the space Y/X is path connected and has trivial fundamental group. This follows from lemma 2.11. By that lemma we can pretend that $Y^{n-1} = X^{n-1}$, in which case Y/X has no cells in dimension $< n$ other than the base point. (And n is at least 2.) Therefore by the *first* Hurewicz theorem 2.6, the Hurewicz homomorphism $\pi_n(Y/X,\star) \to H_n(Y/X)$ is an isomorphism. Therefore the *second* Hurewicz theorem is equivalent to the statement that

$$\pi_n(Y,X,\star) \to \pi_n(Y/X,\star)$$

(induced by the quotient map ...) is an isomorphism of groups, under such and such assumptions.
One may ask whether this homomorphism $\pi_n(Y,X,\star) \to \pi_n(Y/X,\star)$ is an isomorphism in more general circumstances. That is what the Blakers-Massey theorem is about. We will probably get to know it later.

Remark 2.18. The G. Whitehead theorem 2.16 becomes false if the condition that $\pi_1(X,\star)$ be trivial is dropped. Specifically, there exist connected based CW-spaces X with nontrivial $\pi_1(X,\star)$ such that the unique map from X to a point induces isomorphisms in homology (that is, X has the homology of a point, $H_0(X) \cong \mathbb{Z}$ and $H_k(X) = 0$ for $k > 0$). Note that this map from X to a point is not a homotopy equivalence because it does not induce an isomorphism of fundamental groups. — See the exercises for more (counter)examples.

Remark 2.19. There are more complicated variants of the second Hurewicz theorem 2.10 in which X is allowed to have a nontrivial fundamental group. I recommend to work around them as follows.

(i) *Case* $n \geq 3$. Let (Y,X) be a pair of based connected CW-spaces with the property that $\pi_k(Y,X,\star)$ is trivial for $k = 0,1,2,\dots,n-1$, where $n \geq 3$. Then the inclusion-induced homomorphism from $\pi_1(X,\star)$ to $\pi_1(Y,\star)$ is an isomorphism. We can pass to the pair of universal covers

$$(\tilde{Y},\tilde{X}).$$

We have $\pi_n(Y,X,\star) \cong \pi_n(\tilde{Y},\tilde{X},\star)$ for very general reasons (as in Prop. 1.7, lecture notes for weeks 1 and 2). But the pair $(\tilde{Y},\tilde{X})$ satisfies the assumptions of theorem 2.10 and so we get

$$\pi_n(Y,X,\star) \cong \pi_n(\tilde{Y},\tilde{X},\star) \cong H_n(\tilde{Y}/\tilde{X})\ .$$

This is quite satisfactory in my opinion. But if you still wish to make a connection with $H_n(Y/X)$, then you are asking how $H_n(\tilde{Y}/\tilde{X})$ is related to $H_n(Y/X)$. This can be answered by comparing the corresponding cellular chain complexes. The result should be that

$$H_n(Y/X) \cong \mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X,\star)]} H_n(\tilde{Y}/\tilde{X})\,.$$

Exercise: make sense of that and prove it.

(ii) *Case* $n = 2$. Let (Y,X) be a pair of based connected CW-spaces with the property that $\pi_1(Y,X,\star)$ is trivial. Then the inclusion-induced homomorphism $\pi_1(X,\star) \to \pi_1(Y,\star)$ is onto. Let $\tilde{Y} \to Y$ be the universal covering of Y and let

$$\tilde{Y}|_X \longrightarrow X$$

be the *connected* covering space of X obtained by restricting that to X. Then we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
\pi_2(\tilde{Y}) \arrow[r] \arrow[d, "\cong"] & \pi_2(\tilde{Y},\tilde{Y}|_X) \arrow[r, "\text{onto}"] \arrow[d] & \pi_1(\tilde{Y}|_X) \arrow[d, "\text{understood}"] \\
H_2(\tilde{Y}) \arrow[r] & H_2(\tilde{Y}/(\tilde{Y}|_X)) \arrow[r, "\text{onto}"] & H_1(\tilde{Y}|_X)
\end{tikzcd}$$

It follows that the vertical arrow in the middle is an *abelianization* like the vertical arrow on the right; i.e., it is onto and the kernel is the smallest normal subgroup of the source with an abelian quotient. This is again satisfactory in my opinion! But if you still wish to make a connection with $H_2(Y/X)$, then you are asking how $H_2(\tilde{Y}/(\tilde{Y}|_X))$ is related to $H_2(Y/X)$. This can be answered by comparing the corresponding cellular chain complexes. The result should be that

$$H_2(Y/X) \cong \mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(Y,\star)]} H_2(\tilde{Y}/\tilde{Y}|_X)\,.$$

Remark 2.20. The ideas in the proofs of lemma 2.7 and lemma 2.11 can also be used to prove the following.

(i) For any space Z there exists a CW-space Y and a map $g\colon Y \to Z$ which is a weak equivalence.

(ii) For any space Z and map f from a CW-space X to Z, there exists a CW-space Y containing X as a CW-subspace and a map $g\colon Y \to Z$ which is a weak equivalence and satisfies $g|_X = f$.

Obviously (ii) implies (i); so here is a sketch proof of (ii). We proceed by induction. Suppose that Y^k has already been constructed and contains X^k; also $g^k\colon Y^k \to Z$ has been constructed and is k-connected, and agrees with f on X^k. (The induction beginning is easy; so assume $k \geq 0$.) Then

$$(g^k \cup f)\colon Y^k \cup X \longrightarrow Z$$

is also k-connected. For every choice of $\star \in Y^0$ and element in the kernel of

$$(g^k \cup f)_*\colon \pi_k(Y^k \cup X, \star) \to \pi_k(Z, g^k(\star)),$$

represent the element by a based cellular map $\alpha_i\colon S^k \to Y^k$ and choose an extension $\beta_i\colon D^{k+1} \to Z$ of $g^k \circ \alpha_i$. For every element of $\pi_{k+1}(Z, g^k(\star))$, represent the element by a based map $\gamma_j\colon S^{k+1} \to Z$. Define Y^{k+1} to be

$$\left((Y^k \cup X^{k+1}) \cup_{\bigvee \alpha_i} \bigvee_i D^{k+1} \right) \vee \bigvee_j S^{k+1}.$$

Define g^{k+1} so that the composition

$$\textstyle\bigvee_i D^{k+1} \longrightarrow Y^{k+1} \xrightarrow{g^{k+1}} Z$$

agrees with $\bigvee_i \beta_i$ and so that the composition

$$\textstyle\bigvee_j S^{k+1} \longrightarrow Y^{k+1} \xrightarrow{g^{k+1}} Z$$

agrees with $\bigvee_i \gamma_i$.

2.8. Homotopy groups and fibrations

Theorem 2.21. *Let* $p\colon E \to B$ *be a fibration* (w.r.t. the class of compact spaces) *which is also a based map of based spaces (base points* $\star_s \in E$ *and* $\star_t \in B$*). Suppose that* B *is path connected. Let* F *be the fiber of* p *over* $\star_t$ *,* $F = p^{-1}(\star_t)$*. Then the map*

$$\pi_n(E, F, \star_s) \to \pi_n(B, \star_t)$$

induced by p *is a bijection for all* $n \geq 0$*.*

Corollary 2.22. *In the circumstances of theorem 2.21 there is a long exact sequence of homotopy groups/sets*

$$\cdots \twoheadrightarrow \pi_n(F, \star_s) \twoheadrightarrow \pi_n(E, \star_s) \twoheadrightarrow \pi_n(B, \star_t) \twoheadrightarrow \pi_{n-1}(F, \star_s) \twoheadrightarrow \pi_{n-1}(E, \star_s) \twoheadrightarrow \cdots$$

ending in $\cdots \twoheadrightarrow \pi_0(E, \star_s) \twoheadrightarrow \pi_0(B, \star_t)$.

Proof. In the long exact sequence of the pair (E, F), replace $\pi_n(E, F, \star_s)$ by $\pi_n(B, \star_t)$ using the bijection of theorem 2.21. $\square$

Proof of theorem 2.21. Case $n = 0$: here the claim is that the inclusion of F in E induces a surjection $\pi_0(F) \to \pi_0(E)$. Proof of this: given $x \in E$, choose a path ω from $p(x)$ to $\star_t$ in B. Use the path lifting property of p to lift this to a path $\tilde{\omega}$ in E from x itself to some point $y \in E$. The $p(y) = \star_t$, so $y \in F$.
Case $n > 0$, surjectivity. Represent an element of $\pi_n(B, \star_t)$ by a map $g\colon D^n \to B$ such that $g(z) = \star_t$ for all $z \in S^{n-1}$. View this as a homotopy $(h_t\colon S^{n-1} \to B)_{t\in[0,1]}$ where

$$h_t(z) = g(z + (1-t)b)$$

for $z \in S^{n-1}$; here $b \in D^n$ is the base point $(-1, 0, 0, \dots, 0)$. The homotopy begins with h_0 which is the constant map with value $\star_t$ and ends with h_1 which is again the constant map with value $\star_t$. By the HLP for the map p, there exists a homotopy $(\bar{h}_t\colon S^{n-1} \to E)_{t\in[0,1]}$ such that $p\bar{h}_t = h_t$ for all t and $\bar{h}_0$ is the constant map with value $\star_s$. The homotopy $(\bar{h}_t)$ can also be viewed as a single map

$$\bar{g}\colon (D^n, S^{n-1}) \to (E, F)$$

determined by $\bar{g}(z + (1-t)b) = \bar{h}_t(z)$ for $z \in S^{n-1}$. Then $p \circ \bar{g} = g$.
Case $n > 0$, injectivity. Represent an element of $\pi_n(E, F, \star_s)$ by a based map $f\colon (D^n, S^{n-1}) \to (E, F)$. Suppose that $p \circ f$ is nullhomotopic as a based map from D^n/S^{n-1} to B. Let

$$(k_t\colon D^n/S^{n-1} \longrightarrow B)_{t\in[0,1]}$$

be a based nullhomotopy, so that $k_0 = p \circ f$ and k_1 is constant with value $\star_t$. We can also write this in the form $(k_t \colon D^n \longrightarrow B)_{t\in[0,1]}$. Use the HLP for p to find a homotopy

$$(\bar{k}_t \colon D^n \to E)_{t\in[0,1]}$$

such that $\bar{k}_0 = f$ and $p \circ \bar{k}_t = k_t$ for all t. Each $\bar{k}_t$ is then a based map of pairs from (D^n, S^{n-1}) to (E, F), and $\bar{k}_1$ is a based map from (D^n, S^{n-1}) to (F, F). Therefore $(\bar{k}_t)$ is not exactly a nullhomotopy for $f = \bar{k}_0$, but it is nevertheless the kind of homotopy that we require because we believe that any based map from (D^n, S^{n-1}) to (F, F) is nullhomotopic as such. $\square$