17.04.2015

Lecture Notes, weeks 1 and 2
Topology SS 2015 (Weiss)

1.1. Higher homotopy groups

Definition 1.1. Let X be a space with base point $\star$ and let n be a non-negative integer. Write $\pi_n(X,\star)$ for the set $[S^n, X]_*$ (based homotopy classes of based maps from S^n to X). It is clear that π_n is a covariant functor from $\mathcal{H}o\mathcal{T}op_\star$ (the homotopy category of based spaces) to sets.

The case $n = 1$ has already been looked at in detail and we saw that $\pi_1(X,\star)$ is a group in a natural way.

The case $n = 0$ is also useful. Namely, $\pi_0(X,\star)$ is just the set of path components of X. Indeed, a based map $f\colon S^0 \to X$ must send the base point -1 of S^0 to the base point of X. So the only interesting feature it has is the value $f(1) \in X$. And if we pass to homotopy classes, only the path component of $f(1)$ remains.
There is no point in trying to put a natural group structure on $\pi_0(X,\star)$. We must accept that it is in most cases just a set. (There are exceptions: if X has the structure of a topological group, then $\pi_0(X)$ also has the structure of a group in an obvious way, and that can be useful.)

Definition 1.2. *For $n \geq 2$, the set $\pi_n(X,\star)$ has the structure of an abelian group in a natural way. In other words we can equip $\pi_n(X,\star)$ with a structure of abelian group in such a way that, for every based map $f\colon X \to Y$, the induced map of sets*

$$\pi_n(X,\star) \to \pi_n(Y,\star)$$

becomes a homomorphism of abelian groups. The neutral element of $\pi_n(X,\star)$ is represented by the unique constant based map from S^n to X.

For the proof, we note first that

$$\pi_n(X,\star) \times \pi_n(X,\star) = [S^n, X]_\star \times [S^n, X]_\star \cong [S^n \vee S^n, X]_\star$$

(where $\cong$ is used for an obvious bijection). Therefore it is reasonable to try to construct a multiplication map

$$\mu\colon \pi_n(X,\star) \times \pi_n(X,\star) \to \pi_n(X,\star)$$

by writing this in the form $\mu\colon [S^n \vee S^n, X]_\star \longrightarrow [S^n, X]_\star$ and defining it as pre-composition with some fixed element $\kappa \in [S^n, S^n \vee S^n]_*$.

Elementary description of κ. Think of S^n as the quotient space of $[0,1]^n$ obtained by collapsing the subspace consisting of all points which have some coordinate equal to 0 or 1. Think of $S^n \vee S^n$ as the quotient space of

$[0,2] \times [0,1]^{n-1}$ obtained by collapsing all points which have some coordinate equal to 0 or 1, or first coordinate 2. Then κ can be defined by $\kappa(x_1, x_2, \ldots, x_n) := (2x_1, x_2, \ldots, x_n)$, where $x_1, x_2, \ldots, x_n \in [0,1]$. It is easy to verify the following directly: the compositions

$$S^n \xrightarrow{\kappa} S^n \vee S^n \xrightarrow{\mathrm{id} \vee \kappa} S^n \vee (S^n \vee S^n)$$

and

$$S^n \xrightarrow{\kappa} S^n \vee S^n \xrightarrow{\kappa \vee \mathrm{id}} (S^n \vee S^n) \vee S^n$$

are based homotopic. This implies that our formula for the multiplication μ on $[S^n, X]_\star$ is *associative*. Next, it is easy to verify the following directly: the composition

$$S^n \xrightarrow{\kappa} S^n \vee S^n \xrightarrow{\text{permute summands}} S^n \vee S^n$$

is based homotopic to κ. (Here we need $n > 1$.) This implies that our formula for the multiplication μ on $[S^n, X]_\star$ is *commutative*. Furthermore, it is easy to verify directly that the constant based map $S^n \to X$ is a two-sided neutral element for the multiplication μ. (In cubical coordinates for S^n, multiplication with the constant map has the effect of replacing a based map

$$f\colon \frac{[0,1]^n}{\sim} \quad \longrightarrow \quad X$$

by the based map g where $g(x_1, \ldots, x_n) = f(2x_1, x_2, \ldots, x_n)$ when $2x_1 \leq 1$ and $g(x_1, \ldots, x_n) = \star \in X$ when $2x_1 \geq 1$. So the task is to show that f is based homotopic to g ... and that is easy.) Next, it is easy to verify directly that an element $[f] \in [S^n, X]_\star$ has an inverse given by $[f \circ \eta]$ where $\eta\colon S^n \to S^n$ is given in cubical coordinates by $(x_1, x_2, \ldots, x_n) \mapsto (1 - x_1, x_2, \ldots, x_n)$. (In cubical coordinates for S^n, the product of $[f]$ and $[f \circ \eta]$ is given by g where $g(x_1, \ldots, x_n) = f(2x_1, x_2, \ldots, x_n)$ when $2x_1 \leq 1$ and $g(x_1, \ldots, x_n) = f(2 - 2x_1, x_2, \ldots, x_n)$ when $2x_1 \geq 1$.)

Although the homotopy groups π_n have a great deal of theoretical importance, they are very hard to compute in general, especially for large n. Recently I read in an article about homotopy theory: *not a single compact connected CW-space X is known for which we have a formula describing $\pi_n(X)$ for all $n > 0$*, except for two types:

- the totally uninteresting case where X is contractible (so that $\pi_n(X)$ is the trivial group for all $n > 0$);
- the more interesting case where $\pi_1(X)$ is nontrivial but the universal covering of X is contractible (in which case we can say that $\pi_n(X)$ is the trivial group for all $n > 1$). Examples of this type are $X = S^1$, or $X =$ oriented surface of any positive genus.

In particular nobody has a really convincing formula for $\pi_n(S^2)$, for all $n \geq 1$ (although there are some deep results which describe these abelian groups in algebraic/combinatorial terms ... but not in such a way that we can easily read off how many elements they have). But there are many partial results, especially about $\pi_n(S^m)$. For example, we know that $\pi_n(S^m)$ is always a finitely generated abelian group $(m, n > 1)$. It is known that $\pi_n(S^m)$ is the trivial group if $n < m$ and that $\pi_n(S^m) \cong \mathbb{Z}$ if $n = m$; see theorem 1.3 below. It is known that $\pi_n(S^m)$ is infinite if and only if m is even and $n = m$ or $n = 2m - 1$. An example of that is $\pi_3(S^2) \cong \mathbb{Z}$. Recall that $\pi_3(S^2)$ is not trivial according to example 2.5.3, lecture notes WS 2014-2015. (This was conditional at the time; we needed to know that S^2 is not contractible. Later we did learn that S^2 is not contractible since $H_2(S^2) \cong \mathbb{Z}$.)

1.2. Homotopy groups of spheres: the easy cases

Theorem 1.3. *For $0 < n < m$, the group $\pi_n(S^m)$ is trivial. For all $n > 0$, the group $\pi_n(S^n)$ is isomorphic to $\mathbb{Z}$, with* [id] *as the generator.*

Proof. The proof is fiddly, but it is an important result. The case $n < m$ is an easy consequence of cellular approximation. By remark 11.5.2 in the lecture notes for WS 2014-2015, any based map from S^n to S^m is based homotopic to a cellular map. But a cellular map from S^n to S^m must be constant. (Use the CW structure on S^m which has one 0-cell and one m-cell.)

For the case $m = n$, it suffices to show that $\pi_n(S^n)$ is generated by the element [id]. Indeed, this gives us an upper bound on the size of $\pi_n(S^n)$. A lower bound comes from the map $\pi_n(S^n) \to H_n(S^n)$ which takes the homotopy class of a map f to the class of the mapping cycle f. It is an exercise to show that this is a homomorphism.[1] Since $[\mathrm{id}] \in \pi_n(S^n)$ maps to a generator of $H_n(S^n)$, this homomomorphism $\pi_n(S^n) \to H_n(S^n)$ is onto.

With that in mind, the most important tool is Sard's theorem. (We used this earlier in connection with approximation of maps by cellular maps). This states that for a smooth map $f\colon U \to \mathbb{R}^m$ where U is open in $\mathbb{R}^n$, the set of critical values of f is a set of Lebesgue measure zero (in $\mathbb{R}^m$). An element $y \in \mathbb{R}^m$ is a *critical value* of f if there exists $x \in U$ such that $f(x) = y$ and the derivative $f'(x)$, viewed as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, is not surjective. We can also assume $n > 1$ since $\pi_1(S^1, \star)$ is well understood. We need a few observations.

(i) Any based map $S^n \to S^n$ can be written in the form of a map

$$f\colon \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\},$$

[1] Hint: you need to say what $\kappa\colon S^n \vee S^n \to S^n$ does in homology.

and after a homotopy we can assume that f is smooth in a neighborhood U of the compact set $f^{-1}(D^n)$.

(ii) In the situation of (i), if $f^{-1}(0)$ contains exactly one element $x \in \mathbb{R}^n$ and the derivative $f'(x)$ is an invertible linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, then f is based homotopic either to the identity map or to the map

$$\eta\colon (x_1, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$$

from $\mathbb{R}^n \cup \{\infty\}$ to itself.

(iii) The inclusion of the wedge $S^n \vee S^n$ into the product $S^n \times S^n$ induces an isomorphism from $\pi_n(S^n \vee S^n)$ to $\pi_n(S^n \times S^n) \cong \pi_n(S^n) \times \pi_n(S^n)$.

(iv) Let $\alpha\colon S^n \to S^n \vee S^n$ be any based map. Let $\varphi\colon S^n \vee S^n \to S^n$ be the fold map (which is the identity on the first summand S^n and also on the second summand S^n). Then we have

$$[\varphi\alpha] = [\varphi q_1\alpha] + [\varphi q_2\alpha] \in \pi_n(S^n),$$

writing $+$ for the multiplication in $\pi_n(S^n)$ and $q_i\colon S^n \vee S^n \to S^n \vee S^n$ for the map which is the identity on summand i and takes the other summand to the base point.

Observation (iii) is a good exercise in cellular approximation; $n > 1$ is important. Observation (iv) follows from observation (iii). Namely, (iii) shows that α is homotopic to a based map obtained by composing $\kappa\colon S^n \to S^n \vee S^n$ with a map $S^n \vee S^n \to S^n \vee S^n$ which agrees with $q_1\alpha$ on the first wedge summand S^n and with $q_2\alpha$ on the second.

We had observation (ii) as an exercise (sheet 5 of WS14-15) but it did not find many friends. It is easy to reduce to the situation[2] where $x = 0 \in \mathbb{R}^n$. Then $f^{-1}(0) = \{0\}$ and $f'(0)$ is an invertible linear map. The next idea is to show that f is based homotopic to the map $g\colon \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\}$ where g is the linear map $f'(0)$ (except for $g(\infty) = \infty$). A based homotopy is given by

$$(h_t\colon \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\})$$

where $h_t(v) = t^{-1}f(tv)$ for $v \in \mathbb{R}^n$ and t runs from 1 to 0. To be more precise, h_1 is of course f and h_0 is of course not really defined by our formula for h_t , but if you (re)define $h_0 = g$ then it ought to make a good homotopy, by definition of differentiability. The next idea is to note that the space of linear isomorphisms from $\mathbb{R}^n \to \mathbb{R}^n$, also known as $GL_n(\mathbb{R})$, is a space with exactly two path components. One of these path components contains the identity matrix and the other one contains the diagonal matrix with -1 in row one, column one and $+1$ in the other diagonal positions. Therefore our (linear) map

$$g\colon \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\}$$

[2] In the lecture on 10.04. I forgot this step ...

is based homotopic (by a homotopy through invertible linear maps) to either $\mathrm{id}\colon \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\}$ or to the map η from $\mathbb{R}^n \cup \{\infty\}$ to itself. This proves observation (ii).

Now let's turn to the proof of this theorem, properly speaking. We start with f as in (i). We want to show that $[f] \in \pi_n(S^n)$ is in the subgroup generated by $[\mathrm{id}]$. By Sard, we know that f has a regular value arbitrarily close to 0 and it is easy to reduce to the case where 0 itself is regular value (by composing with a translation of $\mathbb{R}^n$). The preimage $f^{-1}(0)$ is compact and discrete with the subspace topology (since $f'(x)$ is invertible for any $x \in f^{-1}(0)$ ... use the inverse function theorem). Therefore $f^{-1}(0)$ is a finite set. Assume that it has k distinct elements $x^{(1)}, \dots, x^{(k)}$. We want to argue by induction on k. The case $k = 1$ has already been settled in observation (ii). We can therefore assume $k > 1$.

Choose a small open ball B_ε of radius ε about the origin $0 \in \mathbb{R}^n$ such that $f^{-1}(B_\varepsilon)$ is a *disjoint* union of k open sets $U_1, \dots, U_k$ (so that $x^{(i)} \in U_i$) in such a way that f restricts to a diffeomorphism from U_i to B_ε. (This is possible by the inverse function theorem.) Choose a map

$$e\colon \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\}$$

which maps B_ε diffeomorphically to all of $\mathbb{R}^n$ and maps the complement of B_ε to ∞ and has $e'(0)$ equal to the identity (matrix). Then we know that $e \simeq \mathrm{id}$ and so $ef \simeq f$. But ef can also be written as a composition

$$S^n \xrightarrow{\ \gamma\ } S^n \vee S^n \xrightarrow{\ \varphi\ } S^n$$

where $S^n = \mathbb{R}^n \cup \{\infty\}$, the first map takes U_1 to the first wedge summand S^n by ef and takes $\bigcup_{i>1} U_i$ to the second wedge summand by ef, and takes all remaining points to the base point ∞ of the wedge. Then by (iv) we have

$$[f] = [ef] = [\varphi\gamma] = [\varphi q_1 \gamma] + [\varphi q_2 \gamma]$$

where $\varphi q_1 \gamma$ and $\varphi q_2 \gamma$ are maps as in (i) for which $0 \in \mathbb{R}^n \cup \{\infty\}$ is a regular value with fewer than k preimage points. By inductive assumption, $[\varphi q_1 \gamma]$ and $[\varphi q_2 \gamma]$ are in the subgroup of $\pi_n(S^n)$ generated by $[\mathrm{id}]$ and therefore $[f]$ is also in that subgroup. $\square$

1.2. Change of base point

Proposition 1.4. *Let X be a space, $x_0, x_1 \in X$ and $n \geq 2$. If x_0, x_1 are in the same path component of X, then $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic as abelian groups.*

More precisely, any path γ in X from x_0 to x_1 determines a group isomorphism ι_γ from $\pi_n(X, x_0)$ to $\pi_n(X, x_1)$. The isomorphism ι_γ depends only on the homotopy class of γ with start- and endpoints fixed.

Proof. The definition of $\iota_\gamma\colon \pi_n(X,x_0) \longrightarrow \pi_n(X,x_1)$ is as follows. Suppose that $\gamma\colon [0,1] \to X$ has $\gamma(0)=x_0$ and $\gamma(1)=x_1$. Let $\alpha\colon S^n \to X$ be a map such that $\alpha(\star)=x_0$ where $\star \in S^n$ is the base point. Choose a homotopy

$$\big(h_t\colon S^n \to X\big)_{t\in[0,1]}$$

such that $h_0=\alpha$ and $h_t(\star)=\gamma(t)$. This is possible because the inclusion $\star \to S^n$ is a cofibration. Let $\iota_\gamma[\alpha] \in \pi_n(X,x_1)$ be the based homotopy class of h_1 (a map from S^n to X taking $\star$ to x_1).
We need to show that ι_γ is well defined. Suppose that $\alpha'\colon S^n \to X$ is another map such that $\alpha'(\star)=x_0$ and $[\alpha]=[\alpha'] \in \pi_n(X,x_0)$. Suppose that

$$\big(h'_t\colon S^n \to X\big)_{t\in[0,1]}$$

is a homotopy such that $h'_0=\alpha'$ and $h'_t(\star)=\gamma(t)$. We need to show that $[h_1]=[h'_1] \in \pi_n(X,x_1)$. Choose a based homotopy $(g_t)_{t\in[0,1]}$ from α to α'. Since the inclusion of $S^n \times \{0,1\}$ union $\star \times [0,1]$ in $S^n \times [0,1]$ is a cofibration, we can construct a homotopy

$$\big(H_t\colon S^n \times [0,1] \to X\big)_{t\in[0,1]}$$

in such a way that $H_0(x,s)=g_s(x)$ for all $x \in S^n$ and

$$H_t(x,0)=h_t(x), \quad H_t(x,1)=h'_t(x)$$

for all $x \in S^n$ and $t \in [0,1]$, and $H_t(\star,s)=\gamma(t)$ for all $s,t \in [0,1]$. Then H_1 is the required homotopy showing that $[h_1]=[h'_1] \in \pi_n(X,x_1)$.
Next we need to ask whether ι_γ is a homomorphism. In fact this is true by inspection. In slightly more detail, if we have $\alpha,\beta\colon S^n \to X$ such that $\alpha(\star)=x_0=\beta(\star)$ and homotopies

$$\big(h^\alpha_t\colon S^n \to X\big)_{t\in[0,1]}\,, \qquad \big(h^\beta_t\colon S^n \to X\big)_{t\in[0,1]}$$

as above, satisfying $h^\alpha_0=\alpha$ and $h^\beta_0=\beta$ and $h^\alpha_t(\star)=\gamma(t)=h^\beta_t(\star)$, then the homotopy

$$\Big((h^\alpha_t \vee h^\beta_t)\circ\kappa\Big)_{t\in[0,1]}$$

demonstrates that $\iota_\gamma(\alpha+\beta)=\iota_\gamma(\alpha)+\iota_\gamma(\beta)$, where we use "+" for the group operation in π_n. (Recall that κ is a based map from S^n to $S^n \vee S^n$ which we have used to define the group structure in π_n.)
Next we need to show that ι_γ is bijective. From the definition of ι_γ, it is clear that an inverse is given by $\iota_{\bar\gamma}$ where $\bar\gamma(t)=\gamma(1-t)$ as usual.
Next we need to show that ι_γ depends only on the homotopy class (start- and endpoints fixed) of γ. So let $\Gamma\colon [0,1]\times[0,1] \to X$ be a map such that

$\Gamma(s,0)=x_0$ for all s and $\Gamma(s,1)=x_1$ for all s. Let $\alpha\colon S^n\to X$ be a map such that $\alpha(\star)=x_0$. We need to show that

$$\iota_{\Gamma_0}=\iota_{\Gamma_1}$$

where $\Gamma_0(t):=\Gamma(0,t)$ and $\Gamma_1(t):=\Gamma(1,t)$. Since the inclusion of $\star\times[0,1]$ in $S^n\times[0,1]$ is a cofibration, we can construct a homotopy

$$\big(H_t\colon S^n\times[0,1]\to X\big)_{t\in[0,1]}$$

in such a way that $H_0(x,s)=\alpha(x)$ for all $x\in S^n$ and $H_t(\star,s)=\Gamma(s,t)$ for all $s,t\in[0,1]$. Then H_1 is the required homotopy showing that ι_{Γ_0} and ι_{Γ_1} take the same value on $[\alpha]$. □

Remark 1.5. Suppose that $\beta,\gamma\colon[0,1]\to X$ are paths such that $\beta(1)=\gamma(0)$. Then the concatenated path $\gamma\circ\beta$ is defined. (It is parameterized by the interval $[0,2]$; you can re-parameterize if you wish.) We have

$$\iota_{\gamma\circ\beta}=\iota_\gamma\circ\iota_\beta$$

where both sides of the equation describe isomorphisms from $\pi_n(X,\beta(0))$ to $\pi_n(X,\gamma(1))$. This should be clear from the construction.

Corollary 1.6. *For a space X with base point x_0 and $n\geq 2$, the abelian group $\pi_n(X,x_0)$ is a* module *over the fundamental group $\pi_1(X,x_0)$; that is to say, the group $\pi_1(X,x_0)$ acts on $\pi_n(X,x_0)$ by group automorphisms.*[3]

Proof. A formula for the action is $[\gamma]\cdot[\alpha]=\iota_\gamma[\alpha]$, where $[\gamma]\in\pi_1(X,x_0)$ and $[\alpha]\in\pi_n(X,x_0)$. Note that since $\gamma(0)=\gamma(1)=x_0$, the isomorphism ι_γ is an automorphism of $\pi_n(X,x_0)$. □

In many cases this action of π_1 on π_n also has another neat description using universal covering spaces. To set this up we start with a proposition about higher homotopy groups of covering spaces.
Let $q\colon E\to X$ be a covering space, alias fiber bundle with discrete fibers. Suppose also E and X are based spaces, with base points $\star_E$ and $\star_X=q(\star_E)$, so that q is a based map.

Proposition 1.7. *Then $q_*\colon\pi_n(E,\star_E)\to\pi_n(X,\star_X)$ is an isomorphism for all $n\geq 2$.*

[3]For a group G, a G-*module* is understood to be an abelian group A with a homomorphism from G to the group of automorphisms of the abelian group A. This terminology is not completely absurd because the group G determines a group ring $\mathbb{Z}[G]$ whose elements are finite formal linear combinations $\Sigma_{g\in G}n_g\cdot g$ where the coefficients n_g are integers. It is easy to see that a G-module A is the same thing as a module over the ring $\mathbb{Z}[G]$.

Proof. This is a consequence of the lifting lemma, section A.10 in the cumulative lecture notes WS2013-14, WS2014-15. According to that, for any based map $f\colon S^n \to X$, there exists a unique based map $g\colon S^n \to E$ such that $f = qg$ (assuming $n \geq 2$ to ensure that $\pi_1(S^n, \star)$ is trivial). This argument applies also with $S^n \times [0,1]$ instead of S^n, so that q induces a bijection $[S^n, E]_* \to [S^n, X]_*$. □

Now suppose that X is path connected and locally path connected, with base point $\star$, and that it has a universal covering space

$$q\colon \tilde{X} \longrightarrow X.$$

In other words, the action of $\pi_1(X, \star)$ on the set $q^{-1}(\star)$ (given by path lifting) is free and transitive. We can make this q unique up to unique isomorphism (of covering spaces of X) by specifying a base point $\star_1 \in q^{-1}(\star)$ for $\tilde{X}$. (That is to say, if two universal coverings of X are given, both with a base point in the fiber over $\star \in X$, then there exists a unique based homeomorphism between them which respects the maps to X.) Now we make a few observations.

- Proposition 1.7 is applicable to this covering space q (set $E := \tilde{X}$).
- Since $\tilde{X}$ is path connected and $\pi_1(\tilde{X}, \star_1)$ is trivial, proposition 1.4 tells us that $\pi_n(\tilde{X}, y)$ is totally independent of the choice of base point y, and we can therefore write $\pi_n(\tilde{X})$. *Little exercise*: the forgetful map from $\pi_n(\tilde{X}, y)$ to $[S^n, X]$ is a bijection ... where $[S^n, X]$ is the set of unbased homotopy classes of maps from S^n to X.
- The translation action of $\pi_1(X, \star)$ on $\tilde{X}$ therefore induces an action of $\pi_1(X, \star)$ on $\pi_n(\tilde{X})$.

(This translation action on $\tilde{X}$ is a confusing theme. Let $G = \pi_1(X, \star)$. We know already that an automorphism of the covering space q is determined by the induced permutation of the set $q^{-1}(\star)$. This permutation is a G-map and as such it can be any G-map we like. We constructed q in such a way that $q^{-1}(\star)$ is a free G-orbit $G \cdot \star_1$. What are the automorphisms of $G \cdot \star_1$ as a G-set? They are given by multiplication with elements of G on the *right*; i.e., for fixed $h \in G$ the map $\rho_h\colon G \cdot \star_1 \to G \cdot \star_1$ given by $g\star_1 \mapsto gh\star_1$ is a G-map. Indeed $\rho_h(fg\star_1) = fgh\star_1 = f\rho_h(g\star_1)$ for $f, g \in G$. Unfortunately $h \mapsto \rho_h$ is not a homomorphism, but an antihomomorphism:

$$\rho_{h_1 h_2} = \rho_{h_2}\rho_{h_1} \; .$$

Therefore the translation action mentioned above is best defined as follows: an element $h \in G$ determines a G-set automorphism $\rho_{h^{-1}} = (\rho_h)^{-1}$ ... which extends uniquely to an automorphism of the covering space q.)

Showing that the two descriptions of the action of $\pi_1(X, \star)$ on $\pi_n(X, \star)$ agree:

let $h \in \pi_1(X, \star)$ be represented by a path $\gamma\colon [0,1] \to X$ from $\star$ to $\star$. Let

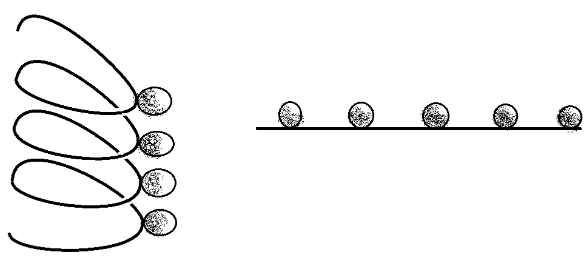
$$\beta\colon [0,1] \to \tilde{X}$$

be a path in $\tilde{X}$ which covers γ, begins at $h^{-1}\star_1$ and so ends at $\star_1$.

$$\begin{array}{ccccc}
\pi_n(X,\star) & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \pi_n(X,\star) \\
\uparrow{\scriptstyle q_* \,\cong} & & & & \uparrow{\scriptstyle q_* \,\cong} \\
\pi_n(\tilde{X},\star_1) & \xrightarrow{(\text{action of } h^{-1} \text{ on } \tilde{X})_*} & \pi_n(\tilde{X}, h^{-1}\cdot\star_1) & \xrightarrow{\iota_\beta} & \pi_n(\tilde{X},\star_1)
\end{array}$$

In the lower row, if we identify $\pi_n(\tilde{X}, \star_1)$ and $\pi_n(\tilde{X}, h^{-1}\cdot\star_1)$ forgetfully with $[S^n, \tilde{X}]$, then the left-hand arrow is the interesting one; the other one, labeled ι_β , is the identity! To make the diagram commutative, the dotted arrow has to be ι_γ .

Example 1.8. Let's look at $\pi_2(X, \star)$ where X is $S^2 \vee S^1$ with the standard base point. The following picture gives two ways of drawing $\tilde{X}$:



From the picture or otherwise, we get that

$$\tilde{X} \simeq \bigvee_{k\in\mathbb{Z}} S^2 ,$$

a wedge of spheres S^2 indexed by the integers. In this description the action of $\ell \in \mathbb{Z} \cong \pi_1(X, \star)$ takes the summand S^2 with label k to the summand S^2 with label $k - \ell$ in the obvious way. An argument like observation (iii) in the proof of theorem 1.3 then shows that

$$\pi_2(X, \star) \cong \pi_2(\tilde{X}, \star_1) \cong \bigoplus_{k\in\mathbb{Z}} \mathbb{Z}\,.$$

The action of $\ell \in \mathbb{Z} \cong \pi_1(X, \star)$ takes the summand $\mathbb{Z} \subset \pi_2(X, \star)$ with label k to the summand $\mathbb{Z}$ with label $k - \ell$ in the obvious way. As an abelian group, $\pi_2(X, \star)$ is obviously not finitely generated. But as a module over

the group ring $\mathbb{Z}[\pi_1(X,\star)] = \mathbb{Z}[\mathbb{Z}]$ it is free on one generator, and therefore certainly finitely generated.
This raises the question: if X is a compact CW-space with base point $\star$, and $n \geq 2$, is $\pi_n(X,\star)$ always finitely generated as a module over $\mathbb{Z}[\pi_1(X,\star)]$? See exercises.

1.3. Cup product in cohomology and homotopy groups

Let X be a based path connected space and $f\colon S^n \to X$ a based map, where $n \geq 1$. We form $Y = \text{cone}(f)$, the mapping cone of f. Often by taking a hard look at Y, we can show that $[f]$ is not the trivial element of $\pi_n(X,\star)$. This is based on the following observation.

Lemma 1.9. *Let* $u, v\colon A \to B$ *be any maps. If* u *is homotopic to* v*, then* $\text{cone}(u)$ *is homotopy equivalent to* $\text{cone}(v)$*.*

Proof. Exercise. (As an exercise in WS2014-15, this did not find many friends, but the formulation was more complicated at the time. I hope that it will find more friends this time.) But we can make a stronger statement. There exists a homotopy commutative diagram of the shape

$$\begin{array}{ccccc} B & \xrightarrow{\text{incl.}} & \text{cone}(u) & \longrightarrow & \Sigma A = \text{cone}(u)/B \\ \downarrow{=} & & \downarrow{\simeq} & & \downarrow{=} \\ B & \xrightarrow{\text{incl.}} & \text{cone}(v) & \longrightarrow & \Sigma A = \text{cone}(v)/B \end{array}$$

where the horizontal maps are the usual ones. □

Now let's return to the based map $f\colon S^n \to X$ and $Y = \text{cone}(f)$ and the quotient map from Y to $Y/X = S^{n+1}$.

Corollary 1.10. *If* f *is nullhomotopic, then there exists a graded ring homomorphism* $H^*(Y) \to H^*(S^{n+1})$ *such that the composition*

$$H^*(S^{n+1}) \xrightarrow{\text{induced by quot. map}} H^*(Y) \longrightarrow H^*(S^{n+1})$$

is the identity.

Proof. If f is nullhomotopic, then we can assume (by the lemma) that it is the map which sends every point of S^n to the base point of X. Then Y is $X \vee S^{n+1}$. The inclusion $S^{n+1} \to Y$ of the wedge summand induces a homomorphism in cohomology which has the stated property. □

Example 1.11. Let $f\colon S^3 \to S^2$ be the Hopf map. (Write $S^2 = \mathbb{C}P^1 = S^3/\sim$ where $S^3 \subset \mathbb{C}^2$; the equivalence relation is $(z_1, z_2) \sim (uz_1, uz_2)$ for $u \in S^1 \subset \mathbb{C}$ and $z_1, z_2 \in \mathbb{C}$ with $|z_1|^2 + |z_2|^2 = 1$. Let f be the quotient map.) Here $X = S^2$ and Y can be identified with $\mathbb{C}P^2$. (To put it differently: $\mathbb{C}P^2$

has a well-known CW structure with one 0-cell, one 2-cell and one 4-cell; the attaching map for the 4-cell happens to be the Hopf map $S^3 \to S^2$.) The cohomology ring $H^*(Y) = H^*(\mathbb{C}P^2)$ is well known: it is the graded ring $\mathbb{Z}[x]/(x^3)$ where x lives in degree 2. It follows that a graded ring homomorphism from $H^*(\mathbb{C}P^2)$ to $H^*(S^4)$ can never be surjective (because it must take x to 0). Therefore f is not nullhomotopic. (We have already seen other proofs of this fact.)

More generally, let $f\colon S^{2n-1} \to \mathbb{C}P^{n-1}$ be the usual quotient map (where S^{2n-1} is viewed as the unit sphere in $\mathbb{C}^n$). Then $X = \mathbb{C}P^{n-1}$ and Y can be identified with $\mathbb{C}P^n$. The cohomology ring $H^*(Y) = H^*(\mathbb{C}P^n)$ is well known:[4] it is the graded ring $\mathbb{Z}[x]/(x^{n+1})$ where x lives in degree 2. It follows that a graded ring homomorphism from $H^*(\mathbb{C}P^n)$ to $H^*(S^{2n})$ can never be surjective. Therefore f is not nullhomotopic.

Definition 1.12. The *Hopf* invariant of a based map $f\colon S^{4n-1} \to S^{2n}$, where $n \geq 1$, is defined as follows. Form $Y = \text{cone}(f)$, a CW-space with three cells: a 0-cell, a $2n$-cell and a $4n$-cell. (The 0-cell and the $2n$-cell together make up S^{2n}.) The cohomology $H^*(Y)$ as a graded group is then given by

$$H^r(Y) = \begin{cases} \mathbb{Z} & \text{if } r = 0, 2n, 4n \\ 0 & \text{otherwise.} \end{cases}$$

Let $x_{2n} \in H^{2n}(Y)$ and $x_{4n} \in H^{4n}(Y)$ be the preferred generators of these infinite cyclic groups. We have

$$(x_{2n})^2 = a \cdot x_{4n}$$

for some $a \in \mathbb{Z}$, inevitably. This integer a determines the ring structure in $H^*(Y)$. It is the Hopf invariant of f. (By corollary 1.10, if the Hopf invariant of f is $\neq 0$, then f is not nullhomotopic.)

Example 1.13. The Hopf invariant of the Hopf map $S^3 \to S^2$ is 1, as we have seen. There are similar maps $S^7 \to S^4$ (constructed using the Hamilton Quaternions instead of $\mathbb{C}$) and $S^{15} \to S^8$ (constructed using the Cayley Octonions). These, too, have Hopf invariant 1. It is a theorem (J.F. Adams 1961) that there is no map $S^{4n-1} \to S^{2n}$ of odd Hopf invariant except in the cases $n = 1, 2, 4$. The original proof by Adams was very difficult, but an easier proof using K-theory (a *generalized* form of cohomology) became available a few years later. — But there are maps $S^{4n-1} \to S^{2n}$ of Hopf invariant 2 for any $n \geq 1$. We shall return to this in a little while.

Example 1.14. To see more applications of corollary 1.10 it is a good idea to work backwards, i.e., to begin with Y. So take $Y = S^m \times S^n$ where $m, n \geq 1$.

[4] Although well known, this is not easy. We came very close to it in WS 2014/15 with problems 3,4,5 on exercise sheet 11.

This has a standard CW-structure with 4 cells: a 0-cell, an m-cell, an n-cell and an $(m+n)$-cell. We allow $m = n$. The graded cohomology *ring* $H^*(Y)$ can be described as $\mathbb{Z}[x,y]/(x^2,y^2)$ where x is in degree m and y is in degree n. (This notation indicates that xy is in degree $m+n$, not zero, and $H^{m+n}(Y)$ is the infinite cyclic group generated by xy. There is also an understanding that $xy = (-1)^{mn}yx$.) In any case we see that any graded ring homomorphism $H^*(Y) \to H^*(S^{m+n})$ must take xy to zero because it will take x and y to zero. So there cannot be a surjective ring homomorphism from $H^*(Y)$ to $H^*(S^{m+n})$. Therefore, if we take $X = S^m \vee S^n$ to be the $(m+n-1)$-skeleton of Y, then the attaching map for the unique $(m+n)$-cell of Y is a map $w\colon S^{m+n-1} \to S^m \vee S^n$ and it is not nullhomotopic. This is called the *Whitehead* map (in honor of JHC Whitehead again).
For an explicit description of w it is best to think of S^{m+n-1} as the boundary of $D^m \times D^n$:

$$S^{m+n-1} \cong \{(y,z) \in D^m \times D^n \mid \|y\| = 1 \text{ or } \|z\| = 1\}.$$

The right-hand expression can be written as $K \cup L$ where $K = D^m \times S^{n-1}$ and $L = S^{m-1} \times D^n$, so that $K \cap L = S^{m-1} \times S^{n-1}$. In these coordinates, w is the map which takes $(y,z) \in K$ to the class of $y \in D^m/S^{m-1} \cong S^m \subset S^m \vee S^n$ and which takes $(y,z) \in L$ to the class of $z \in D^n/S^{n-1} \cong S^n \subset S^m \vee S^n$. Note that this takes $K \cap L$ to the base point. We want to think of w as a based map, so it is probably best to choose the base point of S^{m+n-1} as (y,z) in the above coordinates, where $y = (-1,0,0,\dots) \in D^m$ and $z = (-1,0,0,\dots) \in D^n$.

Definition 1.15. Let X be a based space and $a \in \pi_m(X,\star)$, $b \in \pi_n(X,\star)$, where $m,n \geq 2$. The *Whitehead product* $\lceil a,b \rceil$ of a and b is the element of $\pi_{m+n-1}(X,\star)$ obtained as follows. Choose representatives $\alpha\colon S^m \to X$ and $\beta\colon S^n \to X$ for a and b and let $\lceil a,b \rceil$ be the based homotopy class of the composition of $\alpha \vee \beta$ with the Whitehead map w:

$$S^{m+n-1} \xrightarrow{w} S^m \vee S^n \xrightarrow{\alpha\vee\beta} X$$

(Official notation for the Whitehead product of a and b is $[a,b]$, but since we use the square brackets in so many ways for homotopy classes and sets of homotopy classes, I prefer to write $\lceil a,b \rceil$ instead.)

Example 1.16. Let $\iota = [\mathrm{id}] \in \pi_{2m}(S^{2m},\star)$, where $m \geq 1$. Then the Whitehead product $\lceil \iota,\iota \rceil \in \pi_{4m-1}(S^{2m},\star)$ is $\neq 0$. In fact it is an element of Hopf invariant 2. — To see this let $X = S^{2m} \times S^{2m}$ and $A = S^{2m} \vee S^{2m}$ and let Y be the pushout of

$$X \xleftarrow{\text{incl.}} A \xrightarrow{\varphi} S^{2m}$$

where φ is the fold map. In other words Y is obtained from X by gluing together the two cells of dimension $2m$ in X using the fold map. The ring

$H^*(X)$ is isomorphic to $\mathbb{Z}[s,t]/(s^2,t^2)$ where s and t are in degree $2m$. We view X and Y as CW-spaces with 4 and 3 cells, respectively. The quotient map $X \to Y$ is cellular. Comparing cellular chain complexes, it is therefore easy to see that the graded ring homomorphism $H^*(Y) \to H^*(X)$ determined by the quotient map $X \to Y$ is injective and its image is the graded subring of $H^*(X)$ generated by $u = s+t$ and $v = st$. Since $u^2 = s^2 + 2st + t^2 = 2st = 2v$ in $H^*(X)$, we have $H^*(Y) \cong \mathbb{Z}[u,v]/(u^2 - 2v, uv, v^2)$, where u is in degree $2m$ and v is in degree $4m$. This proves that the attaching map $S^{4m-1} \to S^{2m} = Y^{2m}$ for the $4m$-dimensional cell of Y has Hopf invariant 2. But that attaching map can also be written as the attaching map

$$w\colon S^{4m-1} \to S^{2m} \vee S^{2m} = X^{2m}$$

for the $4m$-dimensional cell of X, followed by the fold map

$$\varphi\colon S^{2m} \vee S^{2m} \longrightarrow S^{2m} .$$

Its homotopy class is therefore $[\iota,\iota]$ by the definition of the Whitehead product in terms of the Whitehead map w.