16.07.2015

5.1. Spectral Sequences: definition and some general facts

Spectral sequences were invented by Jean Leray (mid- to late 1940s), and it is said that Jean-Pierre Serre made them prominent. They are not as bad as you have been told. It is not clear that the notion of *spectral sequence* comes with a genuine definition. But here is an attempt.

Definition 5.1. A *spectral sequence* is a sequence $\mathcal{E}^1, \mathcal{E}^2, \mathcal{E}^3, \ldots$ of chain complexes (graded over $\mathbb{Z}$) together with specified isomorphisms

$$\nu^r_j \colon H_j(\mathcal{E}^r) \cong \mathcal{E}^{r+1}_j$$

for all $j \in \mathbb{Z}$ and $r \geq 1$.

(Notation: $\mathcal{E}^r_j$ is the j-th chain group of the chain complex $\mathcal{E}^r$.) In words: the homology of the chain complex $\mathcal{E}^r$ is identified with the graded abelian group obtained from the chain complex $\mathcal{E}^{r+1}$ by forgetting the differential. We will see that this definition does not give a sufficiently detailed picture. In most examples the chain complex $\mathcal{E}^r$ is a direct sum (again indexed by the integers) of chain subcomplexes and the isomorphisms ν^r_j set up a complicated relationship between the preferred splitting of $\mathcal{E}^r$ and the preferred splitting of $\mathcal{E}^{r+1}$. But as a starting point, definition 5.1 is not all bad.

Spectral sequences usually arise in connection with a filtration of a space by subspaces, or a filtration of a chain complex by chain subcomplexes. Let's focus on chain complexes (of abelian groups) to begin with. A *filtration* of a chain complex C is an ascending sequence of chain subcomplexes

$$\ldots C(-2) \subset C(-1) \subset C(0) \subset C(1) \subset C(2) \subset C(3) \subset \ldots$$

with the properties

$$\bigcup_s C(s) = C\ , \qquad C(s) = 0 \text{ for some } s$$

(usually $C(s)$ is zero for $s < 0$). The task is, roughly speaking, to express the homology groups of C in terms of the homology groups of the subquotients $C(s)/C(s-1)$. That is what spectral sequences are good for.

Notation 5.2. $C(s,t) := C(s)/C(t)$ for $t \leq s$.

More precisely, we are dealing with two families of abelian groups. The first of these consists of the groups

$$\mathcal{E}^1_{s,t} := H_{s+t}C(s, s-1) \qquad s, t \in \mathbb{Z}$$

and we pretend that we know it. The second family consists of the groups

$$\mathcal{E}^\infty_{s,t} := \frac{\mathrm{im}\big(H_{s+t}C(s) \to H_{s+t}C\big)}{\mathrm{im}\big(H_{s+t}C(s-1) \to H_{s+t}C\big)}$$

(where the arrows are induced by inclusion). These are subquotients of the homology groups of C. We pretend that we want to know them. If we did, we would know H_*C up to "extension problems". To repeat—the task is

express all of the groups $\mathcal{E}^{\infty}_{s,t}$ *in terms of all of the groups* $\mathcal{E}^{1}_{s,t}$.

We introduce some notation for some subgroups of $H_{s+t}C(s,s-1)=\mathcal{E}^1_{s,t}$:

$$\begin{aligned} Z^r_{s,t}: &= \mathrm{im}\big(H_{s+t}C(s,s-r)\to H_{s+t}C(s,s-1)\big)\,,\\ B^r_{s,t}: &= \ker\big(H_{s+t}C(s,s-1)\to H_{s+t}C(s+r-1,s-1)\big)\\ &= \mathrm{im}\big(\partial: H_{s+t+1}C(s+r-1,s)\to H_{s+t}C(s,s-1)\big)\end{aligned}$$

for $r>0$ (allow $r=\infty$ also). Important exercise: show that

$$\cdots\subset B^r_{s,t}\subset B^{r+1}_{s,t}\cdots\subset B^{\infty}_{s,t}\subset Z^{\infty}_{s,t}\cdots\subset Z^{r+1}_{s,t}\subset Z^r_{s,t}\subset\dots$$

Lemma 5.3. *There are preferred isomorphisms*

$$u\colon Z^r_{s,t}/Z^{r+1}_{s,t}\longrightarrow B^{r+1}_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1}\,.$$

Proof. The idea is that we represent an element x of $Z^r_{s,t}/Z^{r+1}_{s,t}$ by an element x_1 of $Z^r_{s,t}$ which in turn we represent by an element $\bar{x}_1$ of $H_{s+t}C(s,s-r)$; see the definition of $Z^r_{s,t}$. Then we apply the boundary operator

$$\partial\colon H_{s+t}C(s,s-r)\longrightarrow H_{s+t-1}C(s-r,s-r-1) \tag{5.4}$$

associated with the short exact sequence of chain complexes

$$C(s-r,s-r-1)\to C(s,s-r-1)\to C(s,s-r). \tag{5.5}$$

We get $\partial\bar{x}_1\in\mathrm{im}(\partial)$. Now we note that this $\mathrm{im}(\partial)$ is exactly $B^{r+1}_{s-r,\,t+r-1}$. In more detail, the source in (5.4) can also be written in the form

$$H_{(s-r)+(t+r-1)+1}C((s-r)+r,s-r)$$

and the target can be written as $H_{(s-r)+(t+r-1)}C(s-r,s-r-1)$. Therefore $\partial\bar{x}_1$ in $\mathrm{im}(\partial)=B^{r+1}_{s-r,\,t+r-1}$ represents an element

$$u(x):=[\partial\bar{x}_1]\in B^{r+1}_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1}.$$

Now let us show that $u(x):=[\partial\bar{x}_1]$ is well defined. By linearity of the construction, it is enough to show that if

$$x_1\in Z^{r+1}_{s,t}\subset Z^r_{s,t}$$

then

$$\partial\bar{x}_1\in B^r_{s-r,\,t+r-1}\subset B^{r+1}_{s-r,\,t+r-1}$$

as long as we follow the instructions for choosing $\bar{x}_1$. Indeed, $x_1\in Z^{r+1}_{s,t}$ implies that

$$\bar{x}_1=y+z\in H_{s+t}C(s,s-r)$$

where y comes from $H_{s+t}C(s,s-r-1)$ and z maps to zero in $H_{s+t}C(s,s-1)$, and therefore comes from $H_{s+t}C(s-1,s-r)$. The homomorphism (5.4) takes

y to zero, by exactness of the long exact sequence associated with (5.5). The homomorphism (5.4) takes z to an element of

$$\mathrm{im}\big(\partial\colon H_{s+t}C(s-1,s-r) \longrightarrow H_{s+t-1}C(s-r,s-r-1)\big) = B^r_{s-r,\,t+r-1}\,.$$

This convinces us that u is well defined! Injectivity of u: if $\partial\bar{x}_1$ belongs to $B^r_{s-r,\,t+r-1}$, then it is in the image of

$$\partial\colon H_{s+t}C(s-1,s-r) \longrightarrow H_{s+t-1}C(s-r,s-r-1)$$

and so $\bar{x}_1 = y + z \in H_{s+t}C(s,s-r)$ where z comes from $H_{s+t}C(s-1,s-r)$ and y comes from $H_{s+t}C(s,s-r-1)$. Passing from $\bar{x}_1$ to $x_1 \in Z^r_{s,t}$, we see that the contribution of z is zero and the contribution of y lands in the subgroup $Z^{r+1}_{s,t}$. Therefore when we pass from x_1 to x, the element y also contributes zero. This establishes the injectivity of u. Surjectivity is an exercise. □

Definition 5.6. Put $\mathcal{E}^r_{s,t} = Z^r_{s,t}/B^r_{s,t}$. The differential $d = d^r$ on $\mathcal{E}^r$ has the form $\mathcal{E}^r_{s,t} \longrightarrow \mathcal{E}^r_{s-r,t+r-1}$ and is defined as a composition

$$\begin{tikzcd}
Z^r_{s,t}/B^r_{s,t} \arrow[r, "\mathrm{proj}"] & Z^r_{s,t}/Z^{r+1}_{s,t} \arrow[d, "u", "\cong"'] & \\
& B^{r+1}_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1} \arrow[r, "\mathrm{incl.}"] & Z^r_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1}
\end{tikzcd}$$

Note that the first arrow in this composition of three is *surjective* and the other two are injective, so that the kernel of the composition is the kernel of the first arrow:

$$\ker\Big(d\colon Z^r_{s,t}/B^r_{s,t} \longrightarrow Z^r_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1}\Big) = Z^{r+1}_{s,t}/B^r_{s,t}.$$

Similarly, the last arrow in the composition of three is injective and the other two are surjective, so that the image of the composition is the image of the last arrow:

$$\mathrm{im}\Big(d\colon Z^r_{s,t}/B^r_{s,t} \longrightarrow Z^r_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1}\Big) = B^{r+1}_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1}.$$

On the basis of these little observations it is easy to verify that $d^r d^r = 0$, that is, the composition of

$$\mathcal{E}^r_{s+r,t-r+1} \xrightarrow{\ d\ } \mathcal{E}^r_{s,t} \xrightarrow{\ d\ } \mathcal{E}^r_{s-r,t+r-1} \tag{5.7}$$

is zero (because the image of the first arrow in this composition of two is contained in the kernel of the second arrow). Moreover, we can see immediately that kernel of the second arrow modulo image of the first arrow in (5.7) becomes

$$\frac{Z^{r+1}_{s,t}/B^r_{s,t}}{B^{r+1}_{s,t}/B^r_{s,t}} \;\cong\; Z^{r+1}_{s,t}/B^{r+1}_{s,t}. \tag{5.8}$$

This makes good on the promise expressed in definition 5.1 that the homology of $\mathcal{E}^r$ (with differential $d = d^r$) should be identified with $\mathcal{E}^{r+1}$ (without differential).

Note also that we now have two definitions of $\mathcal{E}^{\infty}_{s,t}$, one given in the first sentence of 5.6 and one given earlier, right after 5.2. They are however isomorphic (exercise). This will be very important in the coming paragraph.

Finally, it is clear from the definitions that $\mathcal{E}^{r+1}_{s,t}$ is a subquotient (quotient of subgroup) of $\mathcal{E}^r_{s,t}$, but what is the exact relationship between $\mathcal{E}^r_{s,t}$ and $\mathcal{E}^{\infty}_{s,t}$? From the definitions, $Z^r_{s,t}$ becomes independent of r for large r, in which case $\mathcal{E}^{r+1}_{s,t}$ is simply a quotient of $\mathcal{E}^r_{s,t}$. Furthermore $B^{\infty}_{r,t}$ is the union of the increasing sequence of abelian groups $B^r_{s,t}$. It follows that we can think of $\mathcal{E}^{\infty}_{s,t}$ as the direct limit (in the sense of category theory) of a sequence of surjective homomorphisms of abelian groups

$$\mathcal{E}^r_{s,t} \to \mathcal{E}^{r+1}_{s,t} \to \mathcal{E}^{r+2}_{s,t} \to \cdots$$

where r should be taken big enough so that $C(s-r) = 0$. This is enough justification for saying that the spectral sequence *converges* to the homology C. A standard (but informal) way of writing this would be

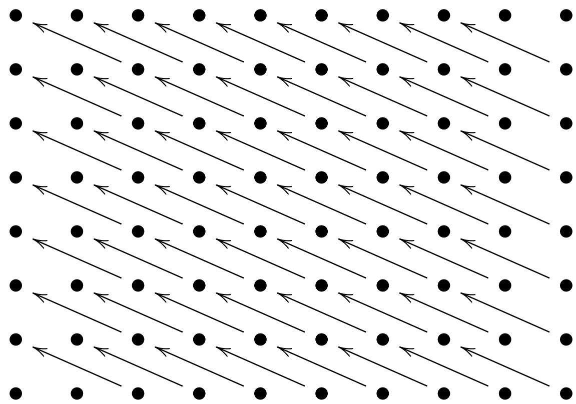
$$\mathcal{E}^1_{s,t} = H_{s+t}C(s, s-1) \;\Rightarrow\; H_{s+t}C\,.$$

In our main example, $Z^r_{s,t}$ is independent of r if $r > s$ and $B^r_{s,t}$ is also independent of r as soon as $r > t+1$. In this situation we can say briefly that $\mathcal{E}^r_{s,t} = Z^r_{s,t}/B^r_{s,t}$ is *the same* as $\mathcal{E}^{\infty}_{s,t}$ for $r > \max\{s, t+1\}$. That is a much easier kind of convergence.

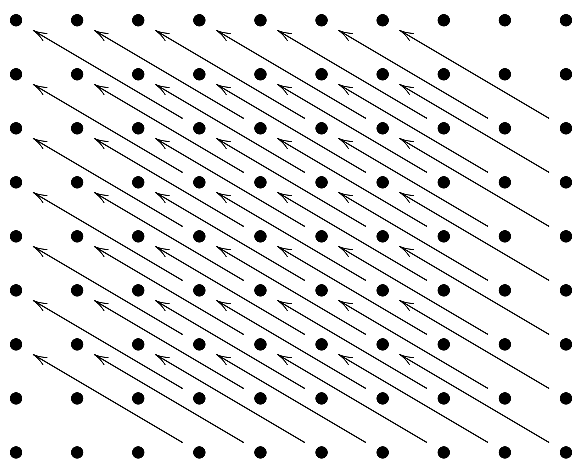
Time for some pictures:

$$\begin{array}{rl}
\mathcal{E}^1_{**}: & \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \\
 & \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \\
 & \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \\
 & \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \\
 & \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \\
 & \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \\
 & \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \\
 & \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet
\end{array}$$

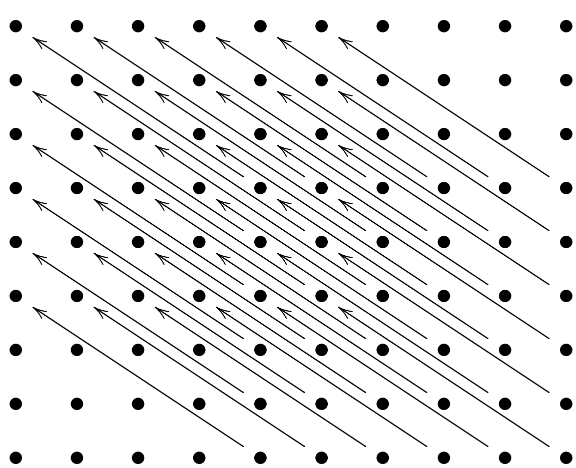
$\mathcal{E}^2_{**}:$



$\mathcal{E}^3_{**}:$



$\mathcal{E}^4_{**}:$



The following problem is important because it shows that what we have seen so far in this section is an enhanced version of the long exact homology sequence of a pair of chain complexes.

Exercise 5.9. Suppose that the filtration of C has only two *stages*; i.e., suppose $C(-1) = 0$ and $C(s) = C(1)$ for all $s \geq 1$. Then all we have is a chain complex $C(1)$ and a chain subcomplex $C(0) \subset C(1)$. What is $\mathcal{E}^1_{**}$, what is $\mathcal{E}^\infty_{**}$, what is the differential on $\mathcal{E}^1_{**}$, what is $\mathcal{E}^2_{**}$, what is the differential on $\mathcal{E}^2_{**}$, etc. ?

Unfortunately I just found out that the above discussion (spectral sequence of a filtered chain complex) is not abstract enough to be really useful for us, and so I have to generalize it slightly. A more general way to obtain a spectral sequence is to start with an *exact couple*. This is a very clever definition due to W. Massey. It's going to get a slightly sketchy treatment here, but it deserves better!

Definition 5.10. An *exact couple* is a diagram of abelian groups

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

(not intended to be particularly commutative) which is *exact* at each vertex.

Example 5.11. The standard example that one should have in mind is one that we know very well, as follows. For a filtered chain complex $C = C(\infty)$ with chain subcomplexes $C(s)$, as above, we set

$$A_{s,t} := H_{s+t}C(s), \quad E_{s,t} := \mathcal{E}^1_{s,t} = H_{s+t}C(s, s-1).$$

Let $A = \bigoplus_{s,t} A_{s,t}$ and $E := \bigoplus_{s,t} E_{s,t}$. Let $i\colon A \to A$ be given on the summand $A_{s,t}$ by the inclusion-induced map

$$A_{s,t} = H_{s+t}C(s) \to H_{s+t}C(s+1) = A_{s+1,t-1} \hookrightarrow A\ ,$$

let $j\colon A \to E$ be the map induced by the projections $C(s) \to C(s, s-1)$ and let k be the map induced by the boundary operators

$$\partial\colon H_*C(s, s-1) \to H_{*-1}C(s-1).$$

Returning to definition 5.10, we make the following observations. Firstly, there is a homomorphism $E \to E$ given by jk. This is a *differential* in the sense of $(jk)(jk) = 0$. Therefore E becomes a *differental abelian group* (like a chain complex without the grading). Next, an exact couple has a *derived* exact couple

$$\begin{tikzcd} A^\delta \arrow[rr, "i^\delta"] & & A^\delta \arrow[dl, "j^\delta"] \\ & E^\delta \arrow[ul, "k^\delta"] & \end{tikzcd}$$

where E^δ is the homology of E with differential jk, that is, $\ker(jk)/\mathrm{im}(jk)$, and $A^\delta = \mathrm{im}(i\colon A \to A)$. The new arrow k^δ is fairly obviously determined by the old k, but beware, the new j^δ is less obviously defined so that $j^\delta(i(a)) = j(a)$ (check that this is well defined). Showing that this is again a derived exact couple is an exercise. Therefore we can repeat this process as many times as we like. Writing $\mathcal{E}^1$ instead of E, then $\mathcal{E}^2$ instead of E^δ, then $\mathcal{E}^3$ instead of $(E^\delta)^\delta$ etc., we get a sequence $\mathcal{E}^1$, $\mathcal{E}^2$, $\mathcal{E}^3$ of differential abelian groups such that $\mathcal{E}^{r+1}$ is (exactly) the homology of $\mathcal{E}^r$. This is therefore a spectral sequence as in definition 5.1, except for the absence of a grading. (But we can add gradings as needed.)

If you take the exact couple of example 5.11, then you have a bi-grading on A and E and the spectral sequence that you get from the exact couple turns out to be identical with the spectral sequence that we constructed previously, more by hand. There are many little exercises concealed in this claim! In particular, we are led to the following definitions by comparing the exact couple construction of a spectral sequence with the earlier pedestrian construction for a filtered chain complex. Given an exact couple as in definition 5.10, let Z^r be the subgroup of E which is the pre-image under k of the image of the $(r-1)$-fold iteration of i (so that, for example, $Z^1 = E$). And let B^r be the subgroup of E which is the image under j of the kernel of the $(r-1)$-fold iteration of i (so that, for example, $B^1 = 0$). These definitions should be in agreement with our earlier definitions of $Z^r_{s,t}$ and $B^r_{s,t}$ ... and in particular it should be true, in the general setting of exact couples, that

$$\mathcal{E}^r \cong Z^r/B^r\,.$$

5.2. The spectral sequence associated with a fibration

We now come to the first serious example (and for this course, also the last) of a spectral sequence: the *Serre spectral sequence of a fibration*. There are actually two variants, one for homology and one for cohomology, but we begin with the homology variant. So let $p\colon X \to B$ be a fibration, and assume that B is a simply connected based CW-space. We make no special assumptions on the fibers. If you know something about singular homology: let C be the *singular* chain complex of the total space X, and let $C(s)$ be the singular chain complex of $p^{-1}(B^s)$, where B^s is the s-skeleton of the CW-space B. So you have a filtered chain complex, and you can put this into the machine which makes a spectral sequence out of a filtered chain complex. If you have another definition of homology in your head, not based on chain complexes, then you can proceed as follows: set up an exact couple (with bigrading) where $A_{s,t}$ is $H_{s+t}(p^{-1}(B^s))$ and $E_{s,t}$ is

$$\tilde{H}_{s+t}(p^{-1}(B^s)/\!\!/p^{-1}(B^{s-1})).$$

(Remember that $/\!\!/$ is our notation for the mapping cone of an inclusion.)

Let's find out what the $\mathcal{E}^1_{**}$ term of this spectral sequence is. This amounts to calculating the homology of $p^{-1}(B^s)/\!\!/p^{-1}(B^{s-1})$ for all s. Let $F = p^{-1}(\star)$. Fix s and choose characteristic maps

$$\varphi_\lambda \colon (D^s, \partial D^s) \longrightarrow (B^s, B^{s-1})$$

for the s-cells of B. Let $z_\lambda = \varphi_\lambda(0)$ and $F_z = p^{-1}(z)$. Any choice of path $\gamma\colon [0,1] \to B$ from z_λ to the base point determines an invertible element of $[F_z, F]$. Indeed the homotopy lifting property guarantees that there is a homotopy $(g_t\colon F_z \to X)_{t\in[0,1]}$ such that g_0 is the inclusion and pg_t is constant with value $\gamma(t)$, for all $t \in [0,1]$. Therefore g_1 is essentially a map from F_z to F. (*Exercise*: show that this is well defined, i.e., depends only on γ but not on the lift (g_t).) Then, since B is simply connected, the element of $[F_z, F]$ so constructed does not depend on the path γ either. In the same way, we can use the HLP for the fibration $\varphi^*_\lambda X \to D^s$ (with contractible base space D^s) to show that the pair

$$(\varphi^*_\lambda X, \varphi^*_\lambda X|_{\partial D^s})$$

is homotopy equivalent to $(D^s \times F, \partial D^s \times F)$. By excision for mapping cones, we have

$$p^{-1}(B^s)/\!\!/p^{-1}(B^{s-1}) \simeq \bigvee_\lambda \varphi^*_\lambda X/\!\!/\varphi^*_\lambda X|_{\partial D^s} \simeq \bigvee_\lambda (D^s\times F)/\!\!/(\partial D^s\times F) \simeq \bigvee_\lambda \frac{S^s \times F}{\star \times F}.$$

Therefore the term $\mathcal{E}^1_{s,t}$ is identified with

$$\tilde{H}_{s+t}\left(\bigvee_\lambda \frac{S^s \times F}{\star \times F}\right) \cong \bigoplus_\lambda \tilde{H}_{s+t}\left(\frac{S^s \times F}{\star \times F}\right) \cong \bigoplus_\lambda H_t(F)\,.$$

This proves the following:

Lemma 5.12. *In the Serre spectral sequence for the fibration* $p\colon X \to B$*, the term* $\mathcal{E}^1_{s,t}$ *is identified with* $C(B)_s \otimes H_t(F)$*, where* $C(B)$ *is the* cellular *chain complex of* B*.*

This leads to a guess for the differential d^1.

Lemma 5.13. *In the Serre spectral sequence for the fibration* $p\colon X \to B$, *the differential*

$$d^1\colon E^1_{s,t} \longrightarrow E^1_{s-1,t}$$

agrees with the standard differential $C(B)_s \otimes H_t(F) \to C(B)_{s-1} \otimes H_t(F)$, *i.e., the differential in the cellular chain complex of* B *tensored with* $H_t(F)$.

Sketch proof. Choose a q-cell in B and a characteristic map

$$\varphi\colon (D^q, \partial D^q) \to (B^q, B^{q-1})$$

for that cell. Then we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \emptyset \arrow[r] \arrow[d] & \varphi^*X|_{\partial D^q} \arrow[r] \arrow[d] & \varphi^*X \arrow[r, "="] \arrow[d] & \varphi^*E \arrow[r, "="] \arrow[d] & {} \\
\cdots \arrow[r] & p^{-1}(B^{q-2}) \arrow[r] & p^{-1}(B^{q-1}) \arrow[r] & p^{-1}(B^q) \arrow[r] & p^{-1}(B^{q+1}) \arrow[r] & \cdots
\end{tikzcd}$$

In the lower row we see the filtration of X which we used to make the Serre spectral sequence for $p\colon X \to B$. In the upper row we see a filtration of φ^*X which can also use to make a spectral sequence; let's write

$$(\mathcal{D}^r_{s,t})_{r,s,t}$$

for that. Here we clearly have $\mathcal{D}^1_{s,t} = 0$ except when $s = q$ or $s = q-1$; and we have

$$\mathcal{D}^1_{q,t} \cong H_t(F), \qquad \mathcal{D}^1_{q-1,t} \cong H_{q-1+t}(S^{q-1} \times F) \cong H_t(F) \oplus H_{q-1+t}(F)\,.$$

It is not hard to see that the d^1 differential

$$\mathcal{D}^1_{q,t} \longrightarrow \mathcal{D}^1_{q-1,t}$$

is given by the inclusion of $H_t(F)$ in the sum $H_t(F) \bigoplus H_{q-1+t}(F)$. By naturality, the diagram above induces a morphism of spectral sequences

$$(\mathcal{D}^r_{s,t})_{r,s,t} \longrightarrow (\mathcal{E}^r_{s,t})_{r,s,t}\;.$$

This takes $\mathcal{D}^1_{q,t} \cong H_t(F)$ isomorphically to the summand of

$$\mathcal{E}^1_{q,t} \cong C(B)_q \otimes H_t(F)$$

which corresponds to the cell λ. Therefore we can read off what the differential d^1 on $\mathcal{E}^1_{q,t}$ does on that summand. □

Corollary 5.14. *In the Serre spectral sequence for a fibration* $p\colon X \to B$, *the* $\mathcal{E}^2$*-term is given by*

$$\mathcal{E}^2_{s,t} \cong H_s(B; H_t(F)).$$

Example 5.15. (See also Fuks-Fomenko-Gutenmacher, *Homotopic Topology*.) For $n > 0$, let's try to calculate the homology of $SU(n)$ (the topological group of unitary complex $n \times n$-matrices with determinant 1). For this we observe that the evaluation map

$$p_n : SU(n) \longrightarrow S^{2n-1} \quad ; \quad p(A) = Ae_1 \in S^{2n-1} \subset \mathbb{C}^n$$

(where e_1 is the well-known standard basis vector) is a fiber bundle with fibers homeomorphic to $SU(n-1)$. (Proving this is an exercise. But it is clear that the fibers are as claimed: for $v \in S^{2n-1}$, the fiber $p^{-1}(v)$ consists of all unitary $n \times n$ matrices of determinant 1 sending e_1 to v.) We now try to use our spectral sequence and induction. The fibers of the fibration p_2 are homeomorphic to $SU(1)$, which is a point, so

$$SU(2) \cong \mathbb{S}^3$$

which in particular calculates the homology. Next we have

$$p_3 : SU(3) \longrightarrow S^5$$

with fibers homeomorphic to $SU(2) \cong S^3$. This means that the $\mathcal{E}^2_{**}$ term of the Leray-Serre spectral sequence for this fibration looks like this:

$$\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}$$

with the nonzero terms in positions $(0,0)$, $(0,3)$, $(5,0)$, $(5,3)$. It follows immediately that the differentials on $\mathcal{E}^2_{**}$ as well as those on $\mathcal{E}^3_{**}$, $\mathcal{E}^4_{**}$ etc. are zero, so that

$$\mathcal{E}^2_{**} \cong \mathcal{E}^\infty_{**}$$

(the spectral sequence *collapses*). We conclude that

$$H_*(SU(3)) \cong H_*(S^3 \times S^5)$$

(but it is not claimed that $SU(3) \simeq S^3 \times S^5$). Next we have

$$p_4 : SU(4) \longrightarrow S^7$$

with fibers homeomorphic to $\mathsf{SU}(3)$. This means that the $\mathcal{E}^2_{**}$ term of the Leray-Serre spectral sequence for this fibration looks like this:

$$
\begin{array}{ccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&\mathbb{Z}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&\mathbb{Z}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&\mathbb{Z}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&\mathbb{Z}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}
$$

with the nonzero terms in positions $(0,0)$, $(0,3)$, $(0,5)$, $(0,8)$, $(7,0)$, $(7,3)$, $(7,5)$, $(7,8)$. Again you can easily convince yourself that none of the differentials on $\mathcal{E}^2_{**}$, $\mathcal{E}^3_{**}$, $\mathcal{E}^4_{**}$ etc. has a chance to be nonzero. Therefore

$$\mathsf{H}_*(\mathsf{SU}(4)) \cong \mathsf{H}_*(\mathsf{SU}(3) \times \mathsf{S}^7) \cong \mathsf{H}_*(\mathsf{S}^3 \times \mathsf{S}^5 \times \mathsf{S}^7)\,.$$

One might hope that this will go on forever. Let's try one more time: The $\mathcal{E}^2_{**}$ term of the spectral sequence for $\mathfrak{p}_5$ looks like

$$
\begin{array}{cccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}
$$

and we have a problem. Namely, there are two differentials in the spectral sequence which could be nonzero: they would be in the $\mathcal{E}^9_{**}$ term, from position $(9,0)$ to position $(0,8)$ and from position $(9,7)$ to position $(0,15)$. So our argument breaks down. All we know is that

$$H_*(U(n) \cong H_*(S^1 \times S^3 \times \cdots \times S^{2n-1}) \qquad \text{for } n \leq 4\,.$$

For the cases $n > 4$, we need better equipment.

5.3. Some remarks on filtered spaces

As a preparation for the cohomology version of the Serre spectral sequence, we need to develop the elementary theory of filtered spaces. (I leave out some proofs for lack of time.) Generally speaking, if we have a definition of homology/cohomology (also generalized forms) in mind which does not rely very much on chain complexes, then it is no longer appropriate to pretend that spectral sequences arise *mainly* in connection with filtered chain complexes. Instead we can take the view that spectral sequences arise *mainly* in connection with filtered spaces and homology or cohomology theories.

Definition 5.16. A *filtered space* is a space X with a sequence of subspaces $X(s)$, where $s \in \mathbb{Z}$, such that $X(s) \subset X(s+1)$ for all s.
Let us say that a filtered space X (with distinguished subspaces $X(s)$ for $s \in \mathbb{Z}$) is *well-filtered* if the following conditions are satisfied:

- there is some $s \in \mathbb{Z}$ such that $X(s) = \emptyset$;
- the inclusion $X(s) \to X(s+1)$ is a cofibration, for all $s \in \mathbb{Z}$;
- $X = \bigcup_s X(s)$ and X has the direct limit topology with respect to the subspaces $X(s)$, so that a subset V of X is open in X if and only if $V \cap X(s)$ is open in $X(s)$ for every s.

Example 5.17. If X and Y are filtered spaces, with distinguished subspaces $X(s)$ and $Y(s)$ for $s \in \mathbb{Z}$, then $X \times Y$ has a preferred structure of a filtered space in the following way:

$$(X \times Y)(s) := \bigcup_{p+q \leq s} X(p) \times Y(q)\,.$$

Suppose now that X and Y are well-filtered. Does it follow that $X \times Y$ with this preferred filtration structure is also well-filtered? That would be nice but I don't know.

Definition 5.18. Let X and Y be filtered spaces. A *morphism* (or *filtered map*) from X to Y is a continuous map f from X to Y such that

$$f(X(s)) \subset Y(s)$$

for all $s \in \mathbb{Z}$. Two filtered maps $f, g\colon X \to Y$ are *filtered homotopic* if there exists a homotopy $(h_t\colon X \to Y)_{t \in [0,1]}$ such that $h_0 = f$, $h_1 = g$ and each h_t

is a filtered map from X to Y. A filtered map $f\colon X \to Y$ is a *filtered homotopy equivalence* if there exists a filtered map $g\colon Y \to X$ such that gf and fg are filtered homotopic to id_X and id_Y , respectively.

Example 5.19. Let $p\colon X \to B$ be a fibration, where B is a CW-space. Let B_T be the *telescope* associated with B and the filtration by skeletons. This is the space

$$[0,1] \times B^0 \cup [1,2] \times B^1 \cup [2,3] \times B^2 ...$$

which is also a CW-space in an obvious way. There is a projection $B_T \to B$. It is an exercise to show that this is a homotopy equivalence (for example by showing first that it is a weak homotopy equivalence). But the situation is a little better. Let q_B from B_T to $[0,\infty[$ be the obvious projection. Then B_T is filtered by subspaces $B_T(s)$, the preimage(s) of $[0,s]$ under q_B , where $s = 0,1,2,\dots$. The map $B_T \to B$ is then a *filtered* homotopy equivalence of *filtered* spaces. Similarly, let X_T be the telescope associated with X and the filtration by subspaces $X(s)$. This is the space

$$[0,1] \times X(0) \cup [1,2] \times X(1) \cup [2,3] \times X(2) ...$$

where a subset U is considered open if its intersection with $[k,k+1] \times X(k)$ is open, for every k. Let q_X from X_T to $[0,\infty[$ be the obvious projection. Then X_T is filtered by subspaces $X_T(s)$, the preimage(s) of $[0,s]$ under q_X^{-1}. Moreover X_T is clearly a *well-filtered* space. I believe that the commutative square

$$\begin{tikzcd} X_T \arrow[r] \arrow[d] & X \arrow[d] \\ B_T \arrow[r] & B \end{tikzcd}$$

is a pullback square in the category of topological spaces. Therefore, using the homotopy lifting property, and the fact that the lower horizontal arrow is a homotopy equivalence of filtered spaces, we can easily deduce that the upper horizontal arrow is also a homotopy equivalence of *filtered* spaces. Conclusion: although there is not much evidence that X, with the filtration by subspaces $X(s) = p^{-1}(B^s)$, is well-filtered, we can say that X is filtered homotopy equivalent to X_T , a well-filtered space.

5.4. Cohomology version of the Serre spectral sequence

Let us now look at the cohomology version of the Serre spectral sequence for a fibration $p\colon X \to B$. As in the homology case, we take the view that it is about a space with a filtration to begin with, the space X with subspaces $X(s) = p^{-1}(B^s)$. Roughly as before, we set ourselves the task to compute $H^*(X)$ or something close to it, and we pretend that we know

$$H^*(X(s)/\!\!/X(s-1))$$

for all s. More precisely, we are interested in the filtration subquotients of the filtration of $H^q(X)$ given by subgroups $\mathrm{im}(H^q(X/\!\!/X(s-1)) \to H^q(X))$. Still more precisely:

$$\mathcal{E}_\infty^{s,t} = \frac{\mathrm{im}\big(H^{s+t}(X/\!\!/X(s-1)) \to H^{s+t}(X)\big)}{\mathrm{im}\big(H^{s+t}(X/\!\!/X(s)) \to H^{s+t}(X)\big)} = \frac{\ker\big(H^{s+t}(X) \to H^{s+t}(X(s-1))\big)}{\ker\big(H^{s+t}(X) \to H^{s+t}(X(s))\big)}$$

We can set up an exact couple

(5.20)
$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

as follows. Let $E^{s,t} = H^{s+t}(X(s)/\!\!/X(s-1))$ and put $E = \bigoplus_{s,t} E^{s,t}$. Let $A^{s,t} = H^{s+t}(X/\!\!/X(s-1))$ and put $A = \bigoplus A^{s,t}$. Note that this is a little surprising; you might have expected that we take $A^{s,t}$ to be $H^{s+t}(X(s))$, but no. The map i is the map

$$A^{s+1,t-1} = H^{s+t}(X/\!\!/X(s)) \longrightarrow H^{s+t}(X/\!\!/X(s-1)) = A^{s,t}$$

induced by the inclusion of mapping cones $X/\!\!/X(s-1) \to X/\!\!/X(s)$, and the map j is given by restrictions

$$A^{s,t} = H^{s+t}(X/\!\!/X(s-1)) \longrightarrow H^{s+t}(X(s)/\!\!/X(s-1)) = E^{s,t}.$$

The map k is a boundary map given by

$$\partial\colon E^{s,t} = H^{s+t}(X(s)/\!\!/X(s-1)) \longrightarrow H^{s+t+1}(X/\!\!/X(s)) = A^{s+1,t}$$

induced by $X/\!\!/X(s) \to \star/\!\!/X(s) \cong S^1 \wedge X(s) \hookrightarrow S^1 \wedge (X(s)/\!\!/X(s-1))$. Beware that the grading behavior is somewhat different from what we have seen before.

Remark 5.21. If B is simply connected, then we have a preferred homotopy class of homotopy equivalences from $X(s)/\!\!/X(s-1)$ to $\bigvee_\lambda (S^s \times F)/(\star \times F)$, as noted before. It is also very useful to note that

$$H^{s+t}(X/\!\!/X(s-1)) = 0 \text{ if } t < 0\,.$$

To show this replace the filtered space X by X_T as in example 5.19. Given an element in $H^{s+t}(X_T/\!\!/X_T(s-1)) \cong H^{s+t}(X_T/X_T(s-1))$, represent by a mapping cycle (for example) and try to construct a nullhomotopy for it. Construct this on $X_T(s+k)/X_T(s-1)$, by induction on k. The obstruction in each step is an element of

$$\tilde{H}^{s+t}(X_T(s+k)/X_T(s+k-1)) \cong \tilde{H}^{s+t}\Big(\bigvee_\lambda \frac{S^s \times F}{\star \times F}\Big) \cong \prod_\lambda H^t(F) = 0,$$

so that there is no obstruction. (Most important point here: when the induction is completed, the partial nullhomotopies defined on $X_T(s+k)$ for all k define a nullhomotopy on X_T.)

Now we are in good shape for a discussion of convergence of the spectral sequence. Let $Z_r = k^{-1}(\mathrm{im}(i^{r-1})) \subset E$ as before (except for the positioning of the r in Z_r) and $B_r = j(\ker(i^{r-1})) \subset E$ as before ... and write $\mathcal{E}_r = Z_r/B_r$. Superscripts s, t can be added as needed. Since $X(s) = \emptyset$ for $s < 0$, it follows that $B_r^{s,t}$ is independent of r as soon as $r > s$, so that $\mathcal{E}_{r+1}^{s,t}$ is a subgroup of $\mathcal{E}_r^{s,t}$ for $r > s$. Since $H^{s+t}(X(s)/\!\!/X(s-1)) = 0$ for $t < 0$, it follows that $Z_r^{s,t} = E_{s,t} \cap \ker(k)$ for $r > t+1$, which is also independent of r. Therefore we can say that $\mathcal{E}_\infty^{s,t} = \mathcal{E}_r^{s,t}$ for $r > \max\{r, t+1\}$.

Now there is an additional problem: we want products. Massey wrote a paper on this (Annals of Mathematics, 1954), explaining what kind of additional structure we need on an exact couple to get a spectral sequence with products. His paper is about *internal* products, but I understand it better with *external* products. So I have adapted his arguments just a little. Let

$$\begin{tikzcd} A(\rho) \arrow[rr, "i"] & & A(\rho) \arrow[dl, "j"] \\ & E(\rho) \arrow[ul, "k"] & \end{tikzcd}$$

be an exact couple, where $\rho = 1, 2, 3$. The goal is to say what we mean by an external multiplication from exact couple number 1 times exact couple number 2 to exact couple number 3. To start with, the multiplication only relates $E(\rho)$ for $\rho = 1, 2, 3$. This is enough to give us external products relating the three associated spectral sequences. We assume a bi-grading in each of the three exact couples as in our example (5.20). We assume bilinear (bi-additive) maps

$$E(1)^{p,q} \times E(2)^{s,t} \longrightarrow E(3)^{p+s,q+t} \tag{5.22}$$

for which we write $(x, y) \mapsto x \cdot y$ where possible. Massey asks: what condition should we impose on these products to ensure that these bilinear maps induce similar maps on the derived exact couples? Here is his condition.

Definition 5.23. The product (5.22) satisfies *condition* μ_n if, for $x \in E(1)^{p,q}$ and $y \in E(2)^{s,t}$ and $a \in A(1)^{p+n+1,q-n}$ and $b \in A(2)^{s+n+1,t-n}$ such that $k(x) = i^n(a)$ and $k(y) = i^n(b)$, there is $c \in A(3)^{p+s+n+1,q+t-n}$ such that

$$k(x \cdot y) = i^n(c) \quad \text{ and } \quad j(c) = j(a) \cdot y + (-1)^{p+q} x \cdot j(b).$$

(Note: i^n is the n-fold iteration of the map i in the exact couples.) The product (5.22) is said to satisfy *condition* μ if it satisfies μ_n for all $n \geq 0$.

The case $n = 0$ is special. For $n = 0$ the letters a, b, c are superfluous: $a = k(x)$, $b = k(y)$ and $c = k(x \cdot y)$. So condition μ_0 just means

$$(jk)(x \cdot y) = (jk)(x) \cdot y + (-1)^{p+q} x \cdot (jk)(y).$$

In other words, condition μ_0 means that the *differential* jk in $E(3)$ behaves like a derivation for the product.

If the product satisfies condition μ_0 , then we can pass to homology, ker(jk)/im(jk), to get a similar product on the derived exact couples:

$$\mathcal{E}(1)_2^{p,q} \times \mathcal{E}(2)_2^{s,t} \longrightarrow \mathcal{E}(3)_2^{p+s,q+t}. \tag{5.24}$$

(This is in the curly notation so that $\mathcal{E}(\rho)_r$ is the $(r-1)$-fold derived exact couple of $E(\rho) = \mathcal{E}(\rho)_1$.) Under these circumstances, if I understand him correctly, Massey claims that the product (5.22) satisfies μ_n if and only if the product (5.24) satisfies μ_{n-1} . And of course he claims that it is an exercise. Let's believe that. Therefore, if (5.22) satisfies μ, then (5.24) satisfies μ.

Example 5.25. Let $X \to B(1)$ and $Y \to B(2)$ be fibrations. Then we have a fibration $Z \to B(3)$, where $Z = X \times Y$ and $B(3) = B(1) \times B(2)$. We assume that $B(1)$ and $B(2)$ are CW-spaces. For simplicity, assume that the number of cells in $B(1)$ and $B(2)$ is finite or countable. Then $B(3)$ with the product topology is also a CW-space. (This was mentioned, but not proved in full generality or detail, in the chapters on CW-spaces.) For each of the three fibrations, we obtain a cohomology spectral sequence as in (5.20), with the interpretation where for example $X(s) \subset X$ is the preimage of the skeleton $B(1)^s \subset B(1)$. Therefore our external product needs have the form

$$\begin{array}{c} \tilde{H}^{p+q}(X(p)/\!\!/X(p-1)) \times \tilde{H}^{s+t}(Y(s)/\!\!/Y(s-1)) \\ \big\downarrow \\ \tilde{H}^{p+s+q+t}(Z(p+s)/\!\!/Z(p+s-1)). \end{array}$$

By example 5.19 there are homotopy equivalences

$$X(p)/\!\!/X(p-1) \simeq X(p)/X(p-1),$$

and similarly for Y and Z. Then the external product that we need to invent can be given the alternative form

$$\begin{array}{c} \tilde{H}^{p+q}\Big(\dfrac{X(p)}{X(p-1)}\Big) \times \tilde{H}^{s+t}\Big(\dfrac{Y(s)}{Y(s-1)}\Big) \\ \big\downarrow \\ \tilde{H}^{p+s+q+t}\Big(\dfrac{Z(p+s)}{Z(p+s-1)}\Big). \end{array}$$

Now it emerges what it ought to be: the composition

$$\begin{array}{c}\tilde{H}^{p+q}\Big(\dfrac{X(p)}{X(p-1)}\Big)\times\tilde{H}^{s+t}\Big(\dfrac{Y(s)}{Y(s-1)}\Big)\\ \Big\downarrow \\ \tilde{H}^{p+s+q+t}\Big(\dfrac{X(p)}{X(p-1)}\wedge\dfrac{Y(s)}{Y(s-1)}\Big)\\ \Big\downarrow \\ \tilde{H}^{p+s+q+t}\Big(\dfrac{Z(p+s)}{Z(p+s-1)}\Big)\end{array}$$

where the first arrow is a standard external product in cohomology[1] and the other is induced by an obvious quotient map from $Z(p+s)/Z(p+s-1)$ to $X(p)/X(p-1)\wedge Y(s)/Y(s-1)$. — Now we should verify that these external products satisfy Massey's conditions μ_n for all $n\geq 0$. Here is a sketch of an argument. Suppose that we have $x\in E(1)^{p,q}$ and $y\in E(2)^{s,t}$ and $a\in A(1)^{p+n+1,q-n}$ and $b\in A(2)^{s+n+1,t-n}$ such that $k(x)=i^n(a)$ and $k(y)=i^n(b)$. What does it mean? It means

$$x\in\tilde{H}^{p+q}\Big(\frac{X(p)}{X(p-1)}\Big),\quad y\in\tilde{H}^{s+t}\Big(\frac{Y(s)}{Y(s-1)}\Big)$$

and

$$a\in\tilde{H}^{p+q+1}\Big(\frac{X}{X(p+n)}\Big),\quad b\in\tilde{H}^{s+t+1}\Big(\frac{Y}{Y(s+n)}\Big)$$

such that

$$a\mapsto\partial x\in\tilde{H}^{p+q+1}\Big(\frac{X}{X(p)}\Big),\quad b\mapsto\partial y\in H^{s+t+1}\Big(\frac{Y}{Y(s)}\Big).$$

Then there exist

$$\bar{x}\in\tilde{H}^{p+q}\Big(\frac{X(p+n)}{X(p-1)}\Big),\quad \bar{y}\in\tilde{H}^{s+t}\Big(\frac{Y(s+n)}{Y(s-1)}\Big)$$

such that $\bar{x}$ extends x and $\partial(\bar{x})=a$, and $\bar{y}$ extends y and $\partial(\bar{y})=b$. (Proof: ... I would say, Mayer-Vietoris.) Then we can form

$$\bar{x}\cdot\bar{y}\in\tilde{H}^{p+q+s+t}\Big(\frac{X(p+n)}{X(p-1)}\wedge\frac{Y(s+n)}{Y(s-1)}\Big)$$

[1] Warning: here I am using the mapping cycle interpretation of cohomology. Readers who prefer the singular homology interpretation should probably not work with the filtered space X but instead with the singular chain complex C of X, filtered by chain subcomplexes $C(s)$ corresponding to $X(s)$, and with the chain complex $\hom(C,\mathbb{Z})$ etc. etc.

and we can move from there to

$$\tilde{H}^{p+q+s+t}\Big(\frac{(X\times Y)(p+s+n)}{(X\times Y)(p+s+n-1)}\Big)$$

using a map

$$\begin{array}{c} \dfrac{(X\times Y)(p+s+n)}{(X\times Y)(p+s+n-1)} \\ \Big\downarrow \\ \dfrac{X(p+n)}{X(p-1)} \wedge \dfrac{Y(s+n)}{Y(s-1)} \end{array}$$

which I hope is obvious! Therefore I take the liberty to write

$$\bar{x}\cdot\bar{y} \in \tilde{H}^{p+q+s+t}\Big(\frac{(X\times Y)(p+s+n)}{(X\times Y)(p+s+n-1)}\Big).$$

With that in mind we can write

$$c := \partial(\bar{x}\cdot\bar{y}) \in H^{p+q+s+t+1}\Big(\frac{(X\times Y)}{(X\times Y)(p+s+n)}\Big).$$