5.1. Spectral Sequences

Spectral sequences were invented by Jean Leray (mid- to late 1940's), and it is said that Jean-Pierre Serre made them prominent. They are not as bad as you have been told. They usually arise in connection with a filtration of a space by subspaces, or a filtration of a chain complex by chain subcomplexes. Let's focus on chain complexes (of abelian groups) for simplicity. A *filtration* of a chain complex C is an ascending sequence of chain subcomplexes

$$\dots \mathsf{C}(-2) \subset \mathsf{C}(-1) \subset \mathsf{C}(0) \subset \mathsf{C}(1) \subset \mathsf{C}(2) \subset \mathsf{C}(3) \subset \dots$$

with the properties

$$\bigcup_s \mathsf{C}(s) = \mathsf{C}\ , \qquad \mathsf{C}(s) = 0 \text{ for some } s$$

(usually $\mathsf{C}(s)$ is zero for $s < 0$). The task is, roughly speaking, to express the homology groups of C in terms of the homology groups of the subquotients $\mathsf{C}(s)/\mathsf{C}(s-1)$. That is what spectral sequences are good for.

Notation 5.1. $\mathsf{C}(s,t) := \mathsf{C}(s)/\mathsf{C}(t)$ for $t \le s$.

More precisely, we are dealing with two families of abelian groups. The first of these consists of the groups

$$\mathcal{E}^1_{s,t} := \mathsf{H}_{s+t}\mathsf{C}(s,s-1) \qquad s,t \in \mathbb{Z}$$

and we pretend that we know it. The second family consists of the groups

$$\mathcal{E}^\infty_{s,t} := \frac{\operatorname{im}\big(\mathsf{H}_{s+t}\mathsf{C}(s) \to \mathsf{H}_{s+t}\mathsf{C}\big)}{\operatorname{im}\big(\mathsf{H}_{s+t}\mathsf{C}(s-1) \to \mathsf{H}_{s+t}\mathsf{C}\big)}$$

(where the arrows are induced by inclusion). These are subquotients of the homology groups of C. We pretend that we want to know them. If we did, we would know $\mathsf{H}_*\mathsf{C}$ up to "extension problems". To repeat—the task is

express (all) the groups $\mathcal{E}^\infty_{s,t}$ in terms of (all) the groups $\mathcal{E}^1_{s,t}$.

We shall introduce further families of abelian groups denoted

$$\mathcal{E}^2_{s,t}\,,\ \mathcal{E}^3_{s,t}\,,\ \mathcal{E}^4_{s,t}\,,\ \dots$$

and depending on $s,t \in \mathbb{Z}$. They will serve as stepping stones.

Notation 5.2. $\mathcal{E}^r_{s,t}$ is the group of those elements in $\mathsf{H}_{s+t}\mathsf{C}(s,s-1)$ which come from $\mathsf{H}_{s+t}\mathsf{C}(s,s-r)$, modulo the group of those which go to zero in $\mathsf{H}_{s+t}\mathsf{C}(s+r-1,s-1)$.

Exercise 5.3. *How do we know that elements in* $\mathsf{H}_{s+t}(\mathsf{C}(s,s-1)$ *which go to zero in* $\mathsf{H}_{s+t}\mathsf{C}(s+r-1,s-1)$ *come from* $\mathsf{H}_{s+t}\mathsf{C}(s,s-r)$ *? }*

Exercise 5.4. Write $C(\infty) := C$, $C(-\infty) = 0$. Prove that $\mathcal{E}^{\infty}_{s,t}$ has a description very similar to that of $\mathcal{E}^{r}_{s,t}$ just given, namely: *The group of all elements in* $H_{s+t}C(s, s-1)$ *which come from* $H_{s+t}C(s, -\infty)$, *modulo the subgroup of those which go to zero in* $H_{s+t}C(\infty, s-1)$.

Exercise 5.4 suggests that some kind of "convergence" takes place: $\mathcal{E}^{r}_{s,t}$ goes to $\mathcal{E}^{\infty}_{s,t}$ as $r \to \infty$. We return to this point below.

Our problem right now is: How can we make the "step" from $\mathcal{E}^{r}_{s,t}$ to $\mathcal{E}^{r+1}_{s,t}$? We can make it by introducing certain homomorphisms

$$d : \mathcal{E}^{r}_{s,t} \longrightarrow \mathcal{E}^{r}_{s-r,t+r-1}$$

(for all $r > 0$ and $s, t \in \mathbb{Z}$) and verifying two facts:

- the composition $\mathcal{E}^{r}_{s+r,\, t-r+1} \xrightarrow{\ d\ } \mathcal{E}^{r}_{s,t} \xrightarrow{\ d\ } \mathcal{E}^{r}_{s-r,\, t+r-1}$ is zero ;
- the quotient group

$$\frac{\ker\left(\mathcal{E}^{r}_{s,t} \xrightarrow{\ d\ } \mathcal{E}^{r}_{s-r,\, t+r-1}\right)}{\operatorname{im}\left(\mathcal{E}^{r}_{s+r,\, t-r+1} \xrightarrow{\ d\ } \mathcal{E}^{r}_{s,t}\right)}$$

  is isomorphic to $\mathcal{E}^{r+1}_{s,t}$.

Briefly: each $\mathcal{E}^{r}_{**}$ comes equipped with a *differential*, and $\mathcal{E}^{r+1}_{**}$ is simply the *homology* of the differential on $\mathcal{E}^{r}_{**}$. But this is too brief — one should also know the "direction" of the differential. Picture:

$$\begin{array}{rccccccccccccccccccc}
\mathcal{E}^{1}_{**}: & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet \\
 & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet \\
 & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet \\
 & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet \\
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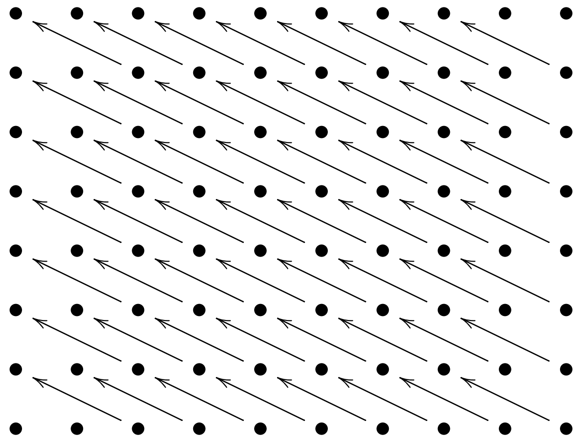
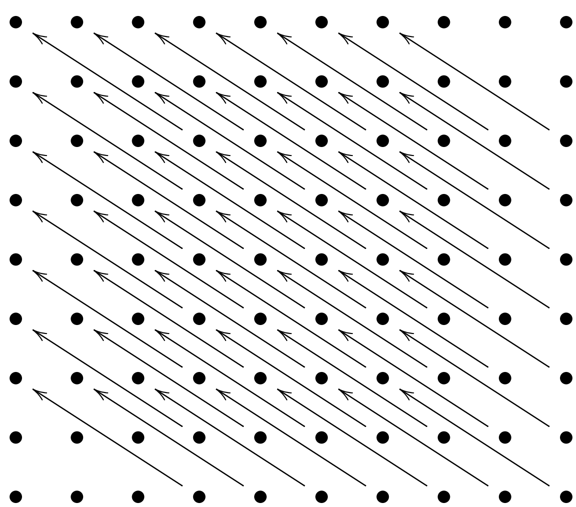
 & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet \\
 & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet
\end{array}$$

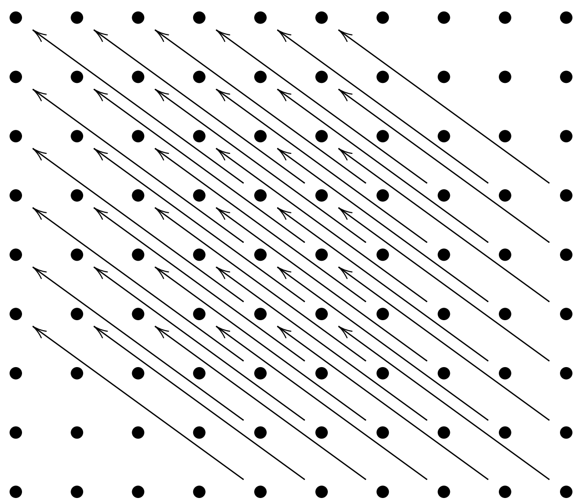
$\mathcal{E}^2_{**}:$



$\mathcal{E}^3_{**}:$



$\mathcal{E}^4_{**}$:



Definition 5.5. The differentials

$$d : \mathcal{E}^r_{s,t} \longrightarrow \mathcal{E}^r_{s-r,t+r-1}$$

are defined as follows. Each $x \in \mathcal{E}^r_{s,t}$ is the image of some

$$\bar{x} \in H_{s+t}C(s, s-r)\,.$$

We have the boundary operator

$$\partial : H_{s+t}C(s, s-r) \longrightarrow H_{s+t-1}C(s-r)\,.$$

Let $d(x)$ be the coset of $\partial(\bar{x})$ in $\mathcal{E}^r_{s-r,\, t+r-1}$.
To check that this is well defined, we need to show that $\partial(\bar{x}) \equiv 0$ in $\mathcal{E}^r_{s-r,\, t+r-1}$ if $x = 0$ in $\mathcal{E}^r_{s,t}$. If $x = 0$, then $\bar{x}$ goes to 0 in the homology group

$$H_{s+t}C(s+r-1, s-1).$$

(Remember notation 5.2.) From the commutative diagram (horizontal arrows induced by various inclusions)

$$\begin{tikzcd}
H_{s+t}C(s, s-r) \arrow[r] \arrow[d, "\partial"'] & H_{s+t}C(s+r-1, s-1) \arrow[d, "\partial"'] \\
H_{s+t-1}C(s-r) \arrow[r] & H_{s+t-1}C(s-1)
\end{tikzcd}$$

we conclude that $\partial(\bar{x})$ goes to zero in $H_{s+t-1}C(s-1)$. Then it also goes to zero in $H_{s+t}C(s-1, s-r-1)$, and then $\partial(\bar{x}) \equiv 0$ in in $\mathcal{E}^r_{s-r,\, t+r-1}$ by notation 5.2, as required.

Verification 5.6. It is easy to verify the first of the two "facts": $dd = 0$ (on each $\mathcal{E}^r_{**}$). To describe the isomorphism

$$\mathcal{E}^{r+1}_{s,t} \quad \cong \quad \frac{\ker\big(d\colon \mathcal{E}^r_{s,t} \to \mathcal{E}^r_{s-r,\,t+r-1}\big)}{\mathrm{im}\big(d\colon \mathcal{E}^r_{s+r,\,t-r+1} \to \mathcal{E}^r_{s,t}\big)}$$

we introduce some notation:

$$\begin{aligned} Z^r_{s,t} :&= \mathrm{im}\big(H_{s+t}C(s,s-r) \to H_{s+t}C(s,s-1)\big)\,, \\ B^r_{s,t} :&= \ker\big(H_{s+t}C(s,s-1) \to H_{s+t}C(s+r-1,s-1)\big) \\ &= \mathrm{im}\big(\partial : H_{s+t+1}C(s+r-1,s) \to H_{s+t}C(s,s-1)\big) \end{aligned}$$

for $r > 0$ (actually, allow $r = \infty$ also). Observe that

$$\cdots \subset B^r_{s,t} \subset B^{r+1}_{s,t} \cdots \subset B^\infty_{s,t} \subset Z^\infty_{s,t} \cdots \subset Z^{r+1}_{s,t} \subset Z^r_{s,t} \subset \dots$$

$$\mathcal{E}^r_{s,t} = Z^r_{s,t}/B^r_{s,t}\,, \qquad \mathcal{E}^{r+1}_{s,t} = Z^{r+1}_{s,t}/B^{r+1}_{s,t}\,.$$

We shall check that

$$\ker\big(d\colon \mathcal{E}^r_{s,t} \to \mathcal{E}^r_{s-r,\,t+r-1}\big) = Z^{r+1}_{s,t}/B^r_{s,t}\,,$$

$$\mathrm{im}\big(d\colon \mathcal{E}^r_{s+r,\,t-r+1} \to \mathcal{E}^r_{s,t}\big) = B^{r+1}_{s,t}/B^r_{s,t}.$$

(These are equalities, not random isomorphisms, between subgroups of $\mathcal{E}^r_{s,t}$.) This is equivalent to saying that the map

$$\begin{array}{rcl} u\colon Z^r_{s,t}/Z^{r+1}_{s,t} & \longrightarrow & B^{r+1}_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1} \\ x + Z^{r+1}_{s,t} & \mapsto & \partial(\bar{x}) + B^r_{s-r,\,t+r-1} \end{array}$$

(details as in definition 5.5) is well defined and isomorphic, for all s, t and $r > 0$. (To show *equivalent*, use the commutative diagram

$$\begin{tikzcd} Z^r_{s,t}/B^r_{s,t} \arrow[r, "d"] \arrow[d, "p"] & Z^{r+1}_{s-r,t+r-1}/B^r_{s-r,t+r-1} \\ Z^r_{s,t}/Z^{r+1}_{s,t} \arrow[r, dotted, "u"] & B^{r+1}_{s-r,t+r-1}/B^r_{s-r,t+r-1} \arrow[u, "j"'] \end{tikzcd}$$

where p is a surjection and j is an injection.) The proof of the claim concerning u is a diagram chase: Use the diagram

$$\begin{tikzcd} & H_{s+t}C(s,s-r-1) \arrow[r, "j_1"] \arrow[d] & H_{s+t}C(s,s-1) \arrow[d, equal] \\ H_{s+t}C(s-1,s-r) \arrow[r] \arrow[d, "\partial_1"] & H_{s+t}C(s,s-r) \arrow[r, "j_2"] \arrow[d, "\partial_2"] & H_{s+t}C(s,s-1) \\ H_{s+t-1}C(s-r,s-r-1) \arrow[r, equal] & H_{s+t-1}C(s-r,s-r-1) & \end{tikzcd}$$

with exact middle row and middle column; note that

$$Z^r_{s,t} = \mathrm{im}(j_2)\,, \quad Z^{r+1}_{s,t} = \mathrm{im}(j_1)\,,$$
$$B^{r+1}_{s-r,\,t+r-1} = \mathrm{im}(\partial_2)\,, \quad B^r_{s-r,\,t+r-1} = \mathrm{im}(\partial_1)\,. \quad \square$$

Remark 5.7. The assumption $C(s_0) = 0$ for some s_0 implies that $Z^r_{s,t}$ becomes stationary for fixed s, t and $r \to \infty$. Thus for sufficiently large r (where "sufficient" depends on s, t) the group $\mathcal{E}^{r+1}_{s,t}$ is a quotient of $\mathcal{E}^r_{s,t}$. Therefore

$$\mathcal{E}^\infty_{s,t} \cong \mathrm{colim}_r\ \mathcal{E}^r_{s,t}$$

(if colimits alias direct limits of systems of groups mean anything to you).

The following problem is important because it shows that what we have seen so far in this section is an enhanced version of the long exact homology sequence of a pair of chain complexes.

Exercise 5.8. Suppose that the filtration of C has only two *stages*; i.e., suppose $C(-1) = 0$ and $C(s) = C(1)$ for all $s \geq 1$. Then all we have is a chain complex $C(1)$ and a chain subcomplex $C(0) \subset C(1)$. What is $\mathcal{E}^1_{**}$, what is $\mathcal{E}^\infty_{**}$, what is the differential on $\mathcal{E}^1_{**}$, what is $\mathcal{E}^2_{**}$, what is the differential on $\mathcal{E}^2_{**}$, etc. ?

Of course, when people write papers and books on spectral sequences, they don't write about enhanced versions of the long exact homology sequence of a pair of chain complexes. What these people write about is

- cleverly designed filtrations of certain chain complexes (or similar objects) of general interest ;
- the meaning, interpretation etc. of the first "terms" of the spectral sequence (in practice $\mathcal{E}^1_{**}$ is very "big" and depends on certain choices, whereas $\mathcal{E}^2_{**}$ is not and does not);
- their experiences with the differentials in the spectral sequence.

The last item is the depressing aspect of spectral sequence theory. Quite often the differentials are kind enough to vanish on $\mathcal{E}^r_{**}$ for all $r \geq 2$, but if they do not the prospects are bleak.

The next example/exercise illustrates some of this (not the depressing aspect), and you should recognize it as something familiar.

Exercise 5.9. Let C be the singular chain complex of a CW-space X, and let $C(s)$ be the singular chain complex of the s-skeleton X^s. This defines a filtration on C. Describe the resulting spectral sequence in detail.

We now come to the first (and, for this course, last) serious example of a spectral sequence: the *Leray-Serre spectral sequence of a fibration*. Let $p\colon E \to B$ be a fibration, and assume that B is a simply connected CW-space.

We make no special assumptions on the fibers. Let C be the *singular* chain complex of the total space E, and let $C(s)$ be the singular chain complex of $p^{-1}(B^s)$, where B^s is the s-skeleton of the CW-space B. Then

$$0 = C(-1) \subset C(0) \subset C(1) \subset C(2) \subset C(3) \subset \dots$$

and $C = \cup_s C(s)$ (prove this). This looks like a cleverly designed filtration of C. Let's find out what the E^1_{**} term of the associated spectral sequence is. This amounts to calculating the homology of $C(s, s-1)$ for all s.

To this end, choose a point z_i in each s-cell $V_i \subset B^s$. Let

$$U = B^s \smallsetminus \{z_i\}, \quad V = \bigcup V_i, \quad \bar{U} = p^{-1}(U), \quad \bar{V}_i = p^{-1}(V_i), \quad \bar{V} = \bigcup \bar{V}_i .$$

Then $U \cup V = B^s$ and $\bar{U} \cup \bar{V} = p^{-1}(B^s)$. Since $\bar{U}$ and $\bar{V}$ are open in $p^{-1}(B^s)$, we have excision:

$$H_*(\bar{V}, \bar{U} \cap \bar{V}) \xrightarrow{\cong} H_*(\bar{U} \cup \bar{V}, \bar{U}) .$$

Further, U deforms[1] into B^{s-1}, and therefore $\bar{U}$ deforms into $p^{-1}(B^{s-1})$. Therefore

$$H_*C(s, s-1) = H_*(p^{-1}(B^s), p^{-1}(B^{s-1})) \xrightarrow{\cong} H_*(\bar{U} \cup \bar{V}, \bar{U}) .$$

This shows

(a) $$H_*C(s, s-1) \cong H_*(\bar{V}, \bar{U} \cap \bar{V}) .$$

Next, let F_i be the fiber of p over z_i. We find

(b) $$H_*(\bar{V}, \bar{U} \cap \bar{V}) \cong \bigoplus_i H_*(\bar{V}_i, \bar{V}_i \smallsetminus F_i) .$$

Now each V_i is a cell, hence contractible, and so each $p\colon \bar{V}_i \longrightarrow V_i$ is homotopy equivalent over V_i to a trivial fibration. This gives a (well defined) isomorphism

(c) $$H_*(\bar{V}_i, \bar{V}_i \smallsetminus F_i) \cong H_*\left(V_i \times F_i, (V_i \smallsetminus \{z_i\}) \times F_i\right) \cong H_{*-s}(F_i)$$

(the last isomorphism is essentially the suspension isomorphism). Combining (a), (b) and (c), we see that

(d) $$H_{s+t}C(s, s-1) \cong \bigoplus_i H_t(F_i)$$

where the direct sum is indexed by the set of s-cells of B. We can simplify this further in two ways.

[1] Terminology: Space X *deforms* into subspace A iff A is a deformation retract of X (iff $\exists$ a homtopy $(h_t\colon X \to X)_{t \in [0,1]}$ such that $h_0 = \mathrm{id}$ and $h_1(X) \subset A$ and $h_t|_A = \mathrm{id}_A$ for all t).

Firstly, any choice of path $\omega : [0,1] \to B$ from z_i to z_j determines inclusions

$$F_i \hookrightarrow \omega^* E \hookleftarrow F_j$$

which are homotopy equivalences. Then

$$H_*(F_i) \cong H_*(\omega^* E) \cong H_*(F_j)\,.$$

The isomorphism seems to depend on the choice of ω. But we are assuming that B is simply connected, so it does not. In more detail: if λ is another path connecting z_i with z_j, then λ is homotopic to ω with endpoints fixed, and by continuity, λ must give rise to the same isomorphism from $H_*(F_i)$ to $H_*(F_j)$. We see that all the $H_*(F_i)$ are canonically isomorphic, so we can do away with the indices altogether and write $H_*(F)$ for all of them without being too ambiguous.

Secondly, we obtain from (d) that

$$H_{s+t}C(s, s-1) \cong \bigoplus_i H_t(F_i) \cong \bigoplus_i H_t(F) \cong W_s(B) \otimes H_t(F)\,,$$

where $W_i(B)$ is the free abelian group generated by the s-cells of B. Note that $W_s(B)$ is the group of s-chains in the *cellular* chain complex of B. Summarizing:

Proposition 5.10. *The E^1_{**} term of the Leray-Serre spectral sequence is*

$$E^1_{s,t} \cong W_s(B) \otimes H_t(F)\,.$$

This is not the final result yet. What do the differentials on E^1_{**} look like? With the identifications of proposition 5.10, they take the form

$$d : W_s(B) \otimes H_t(F) \longrightarrow W_{s-1}(B) \otimes H_t(F)\,.$$

Going back to definition 5.5, you can *perhaps* see that they agree with

$$\partial \otimes \mathrm{id} : W_s(B) \otimes H_t(F) \longrightarrow W_{s-1}(B) \otimes H_t(F)$$

where ∂ is the "usual" boundary operator in the cellular chain complex $W_*(B)$. Let's take this "on trust" for now. (The proof will be given in the next section, right after the proof of proposition 5.15.) Then the resulting homology groups must be the homology groups of B with coefficients in the abelian group $H_t(F)$. This leads to the main theorem of this section:

Theorem 5.11. *The $\mathcal{E}^2_{**}$ term of the Leray-Serre spectral sequence of the fibration $p : E \to B$ is*

$$\mathcal{E}^2_{s,t} \cong H_s(B; H_t(F))$$

and the spectral sequence converges to $H_*(E)$.

The last statement about "convergence" just means that $\mathcal{E}^\infty_{**}$ of the spectral sequence is a piecemeal version of $H_*(E)$, which we already know from the abstract theory.

Exercise 5.12. Investigate the Leray-Serre spectral sequences of the following fibrations or fiber bundles:

- the Hopf fiber bundles $\mathbb{S}^3 \to \mathbb{S}^2$, $\mathbb{S}^7 \to \mathbb{S}^4$, $\mathbb{S}^{15} \to \mathbb{S}^8$;
- for $n \geq 2$, the fibration $p : E \to \mathbb{S}^n$ where E is the space of paths ω in $\mathbb{S}^n$ such that $\omega(0)$ is the base point, and $p(\omega) = \omega(1)$.
- for $n \geq 2$, the fibration $p : E \longrightarrow \mathbb{S}^n \vee \mathbb{S}^n$ where E is the space of paths ω in $\mathbb{S}^n \vee \mathbb{S}^n$ such that $\omega(0)$ is the base point, and $p(\omega) = \omega(1)$.

You should be able to use the second of these items for a calculation of $H_*(\Omega\mathbb{S}^n)$. In the third item, you should also be able to calculate $H_*(\Omega(\mathbb{S}^n \vee \mathbb{S}^n))$. Conclude that the obvious inclusion $\Omega\mathbb{S}^n \vee \Omega\mathbb{S}^n \hookrightarrow \Omega(\mathbb{S}^n \vee \mathbb{S}^n)$ is *not* a homotopy equivalence.

Example 5.13. (See also Fuks-Fomenko-Gutenmacher, *Homotopic Topology.*) For $n > 0$, let's try to calculate the homology of $U(n)$ (the topological group of complex linear automorphisms of $\mathbb{C}^n$ preserving the standard hermitian inner product). First observe that the determinant is a continuous homomorphism

$$\det : U(n) \longrightarrow \mathbb{S}^1$$

with kernel $SU(n)$ (that's a definition). For $n = 1$ the determinant is also a *homeo*morphism, $U(1) \cong \mathbb{S}^1$, and for $n > 1$ the composition

$$U(1) \hookrightarrow U(n) \xrightarrow{\det} \mathbb{S}^1$$

is an isomorphism of topological groups. Thus $U(n)$ is a semidirect product, $U(n) \cong SU(n) \rtimes \mathbb{S}^1$, and therefore

$$U(n) \cong SU(n) \times \mathbb{S}^1 \qquad \text{in the category of spaces}$$

(but not in the category of topological groups). So it should be enough to calculate the homology of $SU(n)$ for $n > 1$. For this we observe that the evaluation map

$$p_n : SU(n) \longrightarrow \mathbb{S}^{2n-1} \quad ; \quad p(A) = Ae_1 \in \mathbb{S}^{2n-1} \subset \mathbb{C}^n$$

(where e_1 is the well-known standard basis vector) is a fiber bundle with fibers homeomorphic to $SU(n-1)$. (Proving this is about an exercise. But it is clear that the fibers are as claimed: for $v \in \mathbb{S}^{2n-1}$, the fiber $p^{-1}(v)$ consists of all unitary $n \times n$ matrices of determinant 1 sending e_1 to v.)

We now try to use our spectral sequence and induction. The fibers of the fibration p_2 are homeomorphic to $SU(1)$, which is a point, so

$$SU(2) \cong \mathbb{S}^3$$

which in particular calculates the homology. Next we have

$$p_3 : SU(3) \longrightarrow \mathbb{S}^5$$

with fibers homeomorphic to $SU(2) \cong \mathbb{S}^3$. This means that the $\mathcal{E}^2_{**}$ term of the Leray-Serre spectral sequence for this fibration looks like this:

$$\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

with the nonzero terms in positions $(0,0)$, $(0,3)$, $(5,0)$, $(5,3)$. It follows immediately that the differentials on $\mathcal{E}^2_{**}$ as well as those on $\mathcal{E}^3_{**}$, $\mathcal{E}^4_{**}$ etc. are zero, so that

$$\mathcal{E}^2_{**} \cong \mathcal{E}^\infty_{**}$$

(the spectral sequence *collapses*). We conclude that

$$H_*(SU(3)) \cong H_*(\mathbb{S}^3 \times \mathbb{S}^5)$$

(but it is not claimed that $SU(3) \simeq \mathbb{S}^3 \times \mathbb{S}^5$). Next we have

$$p_4 : SU(4) \longrightarrow \mathbb{S}^7$$

with fibers homeomorphic to $\mathsf{SU}(3)$. This means that the $\mathcal{E}^2_{**}$ term of the Leray-Serre spectral sequence for this fibration looks like this:

$$
\begin{array}{ccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

with the nonzero terms in positions $(0,0)$, $(0,3)$, $(0,5)$, $(0,8)$, $(7,0)$, $(7,3)$, $(7,5)$, $(7,8)$. Again you can easily convince yourself that none of the differentials on $\mathcal{E}^2_{**}$, $\mathcal{E}^3_{**}$, $\mathcal{E}^4_{**}$ etc. has a chance to be nonzero. Therefore

$$\mathsf{H}_*(\mathsf{SU}(4)) \cong \mathsf{H}_*(\mathsf{SU}(3) \times \mathbb{S}^7) \cong \mathsf{H}_*(\mathbb{S}^3 \times \mathbb{S}^5 \times \mathbb{S}^7)\,.$$

One might hope that this will go on forever. Let's try one more time: The $\mathcal{E}^2_{**}$ term of the spectral sequence for p_5 looks like

$$\begin{array}{cccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}$$

and we have a problem. Namely, there are two differentials in the spectral sequence which could be nonzero: they would be in the $\mathcal{E}^9_{**}$ term, from position $(9,0)$ to position $(0,8)$ and from position $(9,7)$ to position $(0,15)$. So our argument breaks down. All we know is that

$$H_*(U(n) \cong H_*(\mathbb{S}^1 \times \mathbb{S}^3 \times \cdots \times \mathbb{S}^{2n-1}) \qquad \text{for } n \leq 4\,.$$

For the cases $n > 4$, we need better equipment.

Exercise 5.14. Let $p : E \to \mathbb{S}^n$ be a fibration, where $n > 1$. Let F be the fiber of p over the base point. Show that there exists a long exact sequence of the form

$$\cdots \to H_{k-n+1}(F) \longrightarrow H_k(F) \xrightarrow{i} H_k(E) \longrightarrow H_{k-n}(F) \longrightarrow H_{k-1}(F) \to \dots\ .$$

This is called the *Wang sequence*.

5.2. Naturality properties of the Leray-Serre Spectral Sequence

In the previous section, we started with a filtered chain complex $C = \cup_s C(s)$, where $C(s) \subset C(s+1)$ for all s, and $C(s) = 0$ for some s. Then we constructed families of abelian groups $\mathcal{E}^1_{**}$, $\mathcal{E}^2_{**}$, $\mathcal{E}^3_{**}$ and differentials on each of these. (For more precision, I shall write $\mathcal{E}^r_{**}(C)$ instead of just $\mathcal{E}^r_{**}$.) It is obvious that all this depends functorially on the filtered chain complex C. Although it is obvious, it may be worth saying it in detail: Let $D = \cup_s D(s)$ be another filtered chain complex (so $D(s) \subset D(s+1)$ for all s, and $D(s) = 0$ for some s). Let $f : C \to D$ be a chain map taking $C(s)$ to $D(s)$, for all s. Then f induces maps

$$f^r_* : \mathcal{E}^r_{s,t}(C) \longrightarrow \mathcal{E}^r_{s,t}(D) \qquad \forall s,t \in \mathbb{Z}$$

for any $r > 0$, commuting with the differentials on $\mathcal{E}^r_{**}(C)$ and $\mathcal{E}^r_{**}(D)$. *Moreover*, if we make the identifications

$$\mathcal{E}^{r+1}_{**}(C) = H(\mathcal{E}^r_{**}(C)), \qquad \mathcal{E}^{r+1}_{**}(D) = H(\mathcal{E}^r_{**}(D))$$

where H means "homology", then f^{r+1}_* is simply the map of homology groups induced by f^r, and this holds for all $r > 0$. Briefly: f induces a *morphism of spectral sequences.*

Let's apply this observation to the situation where we have a commutative diagram of spaces and maps

$$\begin{array}{ccc} E & \xrightarrow{\bar{g}} & E_1 \\ {\scriptstyle p}\downarrow & & {\scriptstyle p_1}\downarrow \\ B & \xrightarrow{g} & B_1 \end{array}$$

where p and p_1 are fibrations, B and B_1 are simply connected CW-spaces, and g is cellular. We made the Leray-Serre spectral sequence of p using the filtration of the singular chain complex $C(E)$ by subcomplexes $C(p^{-1}(B^s))$. We would of course also make the Leray-Serre spectral sequence of p_1 by using the filtration of $C(E_1)$ by subcomplexes $C(p_1^{-1}(B_1^s))$. But $\bar{g}$ takes $p^{-1}(B^s)$ to $p_1^{-1}(B_1^s)$, for all s, so induces a map $C(E) \to C(E_1)$ respecting the filtrations. By the previous observation, this will lead to a morphism of spectral sequences. Call it $(g, \bar{g})_*$. What does it do to the $\mathcal{E}^1_{**}$ terms? There it will take the form

$$(g, \bar{g})_* : W_s(B) \otimes H_t(F) \longrightarrow W_s(B_1) \otimes H_t(F_1) \quad s,t \in \mathbb{Z}$$

(use proposition 5.10, write F for the fiber of p over some $x \in B$, and write F_1 for the fiber of p_1 over $g(x) \in B_1$).

Proposition 5.15. *On $\mathcal{E}^1_{**}$ terms, $(g,\bar{g})_*$ is the tensor product of the chain map*

$$W_*(B) \longrightarrow W_*(B_1)$$

induced by g with the homomorphism

$$H_*(F) \longrightarrow H_*(F_1)$$

induced by the restriction of $\bar{g}$ to F.

Corollary 5.16. *On E^2_{**} terms,*

$$(g,\bar{g})_* : H_s(B; H_t(F)) \longrightarrow H_s(B_1; H_t(F_1)) \qquad (s,t \in \mathbb{Z})$$

agrees with the homomorphism induced by $g : B \to B_1$ and the homomorphism of coefficient groups $H_t(F) \to H_t(F_1)$ induced by the restriction of $\bar{g}$ to F.

Corollary 5.17. *If $g : B \to B_1$ is a homotopy equivalence, and the restriction of $\bar{g}$ to F is a homotopy equivalence $F \to F_1$, then $(g,\bar{g})_*$ is an isomorphism of the $\mathcal{E}^r_{**}$ terms for any $r \geq 2$.*

Sketch proof of proposition 5.15. Let $V_i \subset B$ and $V'_j \subset B_1$ be s-cells, with orientations and chosen "center" points $z_i \in V_i$ and $z'_j \in V'_j$. Let's throw in the following slightly technical assumption:

$$g^{-1}(z'_j) \cap V_i$$

is a finite set $\{x_1, x_2, \ldots, x_k\}$. This can be justified or arranged using transversality. Let $D_i \subset V_i$ be a (compact) disk containing all the x_r as well as z_i, and let U_r be a small contractible neighborhood of x_r in V_i. Using bars over letters as a shorthand for a prefix p^{-1}, we find that the (i,j)-entry of $(g,\bar{g})_*$ on the first pages, in bidegree (s,t), is the composition

$$\begin{array}{c}
H_{s+t}(\bar{V}_i, \bar{V}_i \smallsetminus \bar{z}_i) \cong H_t(F) \\
\text{inclusion-induced}\ \Big\uparrow \cong \\
H_{s+t}(\bar{V}_i, \bar{V}_i \smallsetminus \bar{D}_i) \cong H_t(F) \\
\text{diagonal}\ \Big\downarrow \\
\bigoplus_{r=1}^k H_{s+t}(\bar{V}_i, \bar{V}_i \smallsetminus \bar{x}_r) \cong \bigoplus_{r=1}^k H_t(F) \\
\text{inclusion-induced}\ \Big\uparrow \cong \\
\bigoplus_{r=1}^k H_{s+t}(\bar{U}_r, \bar{U}_r \smallsetminus \bar{x}_r) \cong \bigoplus_{r=1}^k H_t(F) \\
\Big\downarrow \bar{g}_* \\
H_{s+t}(\bar{V}'_j, \bar{V}'_j \smallsetminus \bar{z}'_j) \cong H_t(F_1).
\end{array}$$

Here only the last arrow is slighly mysterious, but after some inspection one finds that it is given on the r-th summand by *local degree of* $g|U_i$ *at* x_r times $(\bar{g}|F)_*$. (The inspection should be easy at this stage because we are now looking at a commutative square

$$\begin{array}{ccc} E|U_r & \xrightarrow{\text{restr of } \bar{g}} & E_1|V'_j \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p_1}\big\downarrow \\ U_r & \xrightarrow{\text{restr of } g} & V'_j \end{array}$$

where U_r and V'_j are contractible. Since the vertical arrows are fibrations, they must be "trivial", i.e., fiberwise homotopy equivalent to trivial bundles. The local degree of $g|U_r$ at x_r is the value at 1 of the homomorphism

$$\mathbb{Z} \cong H_s(U_r, U_r \smallsetminus x_r) \longrightarrow H_s(V'_j, V'_j \smallsetminus z'_j) \cong \mathbb{Z}$$

induced by $g|U_r$.) Therefore the composition, top to bottom, is the (i,j)-entry of $g_*\colon W_s(B) \to W_s(B_1)$ times $(\bar{g}|F)_*$. That was what we wanted to show. ☐

Sketch proof of theorem 5.11. We fix an s-cell V in B with an orientation, and choose a map of pairs $g\colon (D^s, S^{s-1}) \to (B^s, B^{s-1})$ which, as a map of pairs, is homotopic to a characteristic map[2] for the cell V. At the same time we want this to be cellular, with the cell decomposition $S^{s-1} = e^0 \cup e^{s-1}$ (two cells) and $D^s = e^0 \cup e^{s-1} \cup e^s$ (three cells). Then the induced map of cellular chain complexes

$$W_*(D^s) \longrightarrow W_*(B)$$

takes $[e^s] \in W_s(D^s)$ to $[V] \in W_s(B)$, assuming that orientations have been chosen compatibly. We now have a commutative diagram

$$\begin{array}{ccc} g^*E & \xrightarrow{\bar{g}} & E \\ {\scriptstyle p_0}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ D^s & \xrightarrow{g} & B \end{array}$$

where p_0 and $\bar{g}$ are the projections from g^*E to D^s and E. This gives us a way to understand the differential in $\mathcal{E}^1_{s,t}(p)$ on the summand $[V] \otimes H_t(F) \cong H_t(F)$, because we have the commutative diagram

$$\begin{array}{ccc} H_t(F) \cong \mathcal{E}^1_{s,t}(p_0) & \longrightarrow & \mathcal{E}^1_{s,t}(p) \cong W_s(B) \otimes H_t(F) \\ {\scriptstyle d}\big\downarrow & & {\scriptstyle d}\big\downarrow \\ H_t(F) \cong \mathcal{E}^1_{s-1,t}(p_0) & \longrightarrow & \mathcal{E}^1_{s-1,t}(p) \cong W_{s-1}(B) \otimes H_t(F) \end{array}$$

[2]If this causes confusion, recall definition of a CW-space.

where we know the horizontal arrows from proposition 5.15. Of course we also know the left-hand vertical arrow, because p_0 is fiberwise homotopy equivalent to a trivial bundle (since its base space D^s is contractible). As a map from $H_t(F)$ to $H_t(F)$, it is the identity. □

The proof of corollary 5.17 is by induction on r starting with $r = 2$. □

One consequence of corollary 5.17 that one should certainly be aware of is that the Leray-Serre spectral sequence of a fibration, from the $\mathcal{E}^2_{**}$ term onwards, does not depend on the CW-structure of the base space chosen. In fact, it is enough to assume that the base space is homotopy equivalent to a CW-space. To see why, let $q : D \to A$ be a fibration, where A is simply connected and homotopy equivalent to a CW-space. How can we set up a Leray-Serre spectral sequence for calculating $H_*(D)$? We can choose a homotopy equivalence $e_0 : A_0 \to A$, where A_0 is an honest CW-space. Then we have the "usual" commutative square

$$\begin{tikzcd} e_0^*D \arrow[r, "\bar{e}_0"] \arrow[d, "e_0^*q"'] & D \arrow[d, "q"] \\ A_0 \arrow[r, "e_0"] & A \end{tikzcd}$$

and it turns out that $\bar{e}_0$ is also a homotopy equivalence, like e_0 (exercises 5.18 and 5.19 below). Then

$$H_*(e_0^*D) \cong H_*(D)$$

and for $H_*(e_0^*D)$ we have the Leray-Serre spectral of the fibration e_0^*q. Forgetting about its $\mathcal{E}^1_{**}$ term, we declare this to be the Leray-Serre spectral sequence of the fibration q also. It certainly converges to the right thing. We need to check that it is sufficiently well defined. Before we do so, let's observe that the $\mathcal{E}^2_{**}$ term is

$$\mathcal{E}^2_{s,t} \cong H_s(A_0; H_t(F)) \cong H_s(A; H_t(F))$$

where F is any fiber of e_0^*q (homeomorphic to some fiber of q, automatically). It follows that the $\mathcal{E}^2_{**}$ term at least is well defined (independent of the choices A_0 and e_0). Suppose now that $e_1 : A_1 \to A$ is another homotopy equivalence from a CW-space to A. Then we can find a third homotopy equivalence $e_I : A_I \to A$ from a CW-space to A, and *cellular* maps $j_0 : A_0 \to A_I$, $j_1 : A_1 \to A_I$ making the diagram

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, "e_0"'] & A_I \arrow[d, "e_I"] & A_1 \arrow[l] \arrow[d, "e_1"] \\ A \arrow[r, "="] & A \arrow[r, "="] & A \end{tikzcd}$$

commutative (exercise 5.20 below). As above, the map j_0 gives rise to a morphism from the spectral sequence of e_0^*q to the spectral sequence of e_1^*q which is an isomorphism on $\mathcal{E}^2_{**}$ terms. The same can be said of j_1. It follows that the spectral sequences of e_0^*q and e_1^*q are isomorphic from the $\mathcal{E}^2_{**}$ term onwards. The isomorphism that we constructed extends the identification of $\mathcal{E}^2_{**}$ terms that we found previously: all $\mathcal{E}^2_{**}$ terms in sight look like

$$E^2_{s,t} \cong H_s(A; H_t(F)) .$$

(It follows that the isomorphism we constructed is unique: it does not depend on the choice of A_I, e_I, j_0, j_1, because it does not depend on anything as far as $\mathcal{E}^2_{**}$ terms go.)

Exercise 5.18. The *mapping cylinder* of a continuous map $f : A \to B$ is the space

$$Z = \big(A \times [0,1]\big) \amalg B/\!\sim$$

where $\sim$ identifies $(a, 1)$ with $f(a)$ for all $a \in A$. Prove that Z deforms into the subspace B. Prove that Z also deforms into the subspace $A \cong A \times \{0\}$ if f is a homotopy equivalence.

Exercise 5.19. Let $q : D \to A$ be a fibration, and let $e_0 : A_0 \to A$ be a homotopy equivalence. Prove that the projection map $(x, y) \mapsto y$ from $e_0^*D \subset A_0 \times D$ to D is a homotopy equivalence. (You may want to use exercise 5.18. Try to reduce to the situation where e_0 is the inclusion of a subspace A_0 such that A deforms into A_0.)

Exercise 5.20. Let $e_0 : A_0 \to A$ and $e_1 : A_1 \to A$ be homotopy equivalences, where A_0 and A_1 are CW-spaces. Show that there exists another homotopy equivalence $e_I : A_I \to A$, and maps $j_0 : A_0 \to A$, $j_1 : A_1 \to A$, such that A_I is a CW-space, j_0, j_1 are cellular, and $e_I j_0 = e_0$, $e_I j_1 = e_1$. (See the commutative diagram just above).

Back to filtrations and spectral sequences: Let C be a chain complex with filtration

$$\dots C(-2) \subset C(-1) \subset C(0) \subset C(1) \subset C(2) \subset C(3) \subset \dots$$

such that $C = \cup_s C(s)$. This time asssume

$$C(s) = 0 \text{ for some } s\ , \qquad C(t) = C \text{ for some } t\ ,$$

and this time let's see what the filtration tells us about the *co*homology of C. This will only require minor changes. Thus let $D = \hom(C, \mathbb{Z})$, and grade it by giving degree $-n$ to the elements in $\hom(C_n, \mathbb{Z})$. This is unusual, but it means that the differential in D lowers degree by one as usual. Let $D(s) := \hom(C/C(-s-1), \mathbb{Z})$ (same grading conventions). Then

$$\dots D(-2) \subset D(-1) \subset D(0) \subset D(1) \subset D(2) \subset D(3) \subset \dots$$

and $D(s) = D$ for some s, $D(t) = 0$ for some t. So we are in the same situation as before, and we can make a spectral sequence converging to $H_*(D) = H^{-*}(C)$, with $\mathcal{E}^1_{**}$ term

$$\mathcal{E}^1_{s,t} = H_{s+t}D(s, s-1)\,.$$

Now throw in another assumption, namely: the inclusion homomorphisms from $C(s)_n$ to $C(s+1)_n$ are *split* injective in each degree n (for all s). Then there is a short exact sequence of chain complexes

$$D(s-1) \longrightarrow D(s) \longrightarrow \hom(C(-s, -s-1), \mathbb{Z})\ ,$$

the second arrow given by evaluation of homomorphisms to $\mathbb{Z}$ on $C(-s)$. Therefore

$$\mathcal{E}^1_{s,t} = H^{-s-t}C(-s, -s-1).$$

Example 5.21. Let $p : E \to B$ be a fibration and assume that B is a simply connected *compact* CW-space. Let C be the singular chain complex of E and let $C(s)$ be the singular chain complex of $p^{-1}(B^s)$, where B^s is the s-skeleton. Then the cohomology Leray-Serre spectral sequence has

$$\mathcal{E}^1_{s,t} \cong H^{-s-t}C(-s, -s-1) \cong \prod_{s-\text{cells}} H^{-t}(F) \cong \hom\big(W_s, H^{-t}(F)\big)$$

(notation and proof as for proposition 5.10). The $\mathcal{E}^2_{**}$ term then becomes

$$\mathcal{E}^2_{s,t} \cong H^{-s}(B; H^{-t}(F)\,.$$

Here is some disturbing news: it is customary to switch the top and bottom indices as well as some of the signs, so that

$$\mathcal{E}^{s,t}_r := \mathcal{E}^r_{-s,-t}$$

which for the Leray-Serre cohomology spectral sequence means

$$\mathcal{E}^{s,t}_2 \cong H^s(B; H^t(F))\,.$$

This is nice because it means that the nonzero terms of the spectral sequence are all in the first quadrant ($s, t \geq 0$), but of course it also means that all the differentials go in the wrong direction. For example, in $\mathcal{E}^{**}_1$, differentials move one to the right and zero down, in $\mathcal{E}^{**}_2$ they move two to the right and one down, in $\mathcal{E}^{**}_r$ they move r to the right and $r-1$ down.

The spectral sequence converges to $H^*(E)$, of course, but to be more precise we would have to say that the $\mathcal{E}^{**}_\infty$ term is as follows:
$\mathcal{E}^{s,t}_\infty$ is a subquotient of $H^{s+t}(E)$, namely,

$$\frac{\text{kernel of restriction from } H^{s+t}(E) \text{ to } H^{s+t}(p^{-1}(B^{s-1}))}{\text{kernel of restriction from } H^{s+t}(E) \text{ to } H^{s+t}(p^{-1}(B^{s}))}.$$

At this point we can see the possibility of introducing products. For example, on the $\mathcal{E}_\infty^{**}$ term just described, we can use the cup product to get a bilinear map

$$\mathcal{E}_\infty^{s,t} \times \mathcal{E}_\infty^{m,n} \longrightarrow \mathcal{E}_\infty^{s+m,t+n}\,.$$

For this we only have to verify that the ordinary cup product $x \cup y$ of two elements in $H^*(E)$ restricts to 0 in $H^*(p^{-1}(B^{s+m}))$ provided x restricts to 0 in $H^*(p^{-1}(B^s))$ and y restricts to zero in $H^*(p^{-1}(B^{m-1}))$. (Verify this.) We can also very easily define products

$$\mathcal{E}_2^{s,t} \times \mathcal{E}_2^{m,n} \longrightarrow \mathcal{E}_2^{s+m,t+n}\,.$$

This amounts to specifying bilinear maps

$$H^s(B;H^t(F)) \times H^m(B;H^n(F)) \longrightarrow H^{s+m}(B;H^{t+n}(F)\,.$$

We take the "usual" cup product

$$H^s(B;H^t(F)) \times H^m(B;H^n(F)) \longrightarrow H^{s+m}(B;H^t(F) \otimes H^n(F))$$

and then apply the other "usual" cup product $H^t(F) \otimes H^n(F) \to H^{t+n}(F)$ to the coefficients.— The cup product on $\mathcal{E}_2^{**}$ is bilinear, associative, and commutative in the graded sense: $\alpha \cup \beta = (-1)^{|\alpha||\beta|}(\beta \cup \alpha)$ where $|\alpha| = s+t$ if $\alpha \in \mathcal{E}^2_{s,t}$. The cup product on $\mathcal{E}_\infty^{**}$ is also bilinear, associative, and commutative in the graded sense.

Summarizing, we have a cup product on the $\mathcal{E}_\infty^{**}$ term which is a piecemeal version of the honest cup product on the total space E of our fibration ; and we have another cup product on the $\mathcal{E}_2^{**}$ term. How are these two related ?

The answer is easy. The differential d on $\mathcal{E}_2^{**}$ is compatible with the cup product, which means that $d(\alpha \cup \beta)$ equals $d(\alpha) \cup \beta + (-1)^{|\alpha|}\alpha \cup d(\beta)$ for all α and β. Using this fact, you can define a cup product on $\mathcal{E}_3^{**}$ by choosing representatives and multiplying them as you would in $\mathcal{E}_2^{**}$. Again, the differential on $\mathcal{E}_3^{**}$ is compatible with the cup product on $\mathcal{E}_3^{**}$; you can use this fact to define a cup product on $\mathcal{E}_4^{**}$, and so on. You end up with a cup product in $\mathcal{E}_\infty^{**}$. But now $\mathcal{E}_\infty^{**}$ already has a cup product, as we saw. Now, as you might guess, agreement is supposed to take place. The exercises just below amount to a proof of these claims (if you solve them).

Exercise 5.22. Let C and C' be chain complexes with filtrations

$$\dots C(-2) \subset C(-1) \subset C(0) \subset C(1) \subset C(2) \subset C(3) \subset \dots$$

$$\dots C'(-2) \subset C'(-1) \subset C'(0) \subset C'(1) \subset C'(2) \subset C'(3) \subset \dots$$

so that $C = \bigcup_s C(s)$ and $C' = \bigcup_s C'(s)$, and $C(s) = 0$, $C'(s) = 0$ for $s \ll 0$. Let $C'' = C \otimes C'$ with the filtration defined by

$$C''(r) = \sum_{s+s'=r} C(s) \otimes C'(s') \qquad \subset C \otimes C'\,.$$

Show that the operation of taking the tensor product of homology cycles induces homomorphisms $\times\colon H_i(C) \otimes H_j(C') \to H_{i+j}(C'')$ and

$$\times_r\colon \mathcal{E}^r_{s,t}(C) \otimes \mathcal{E}^r_{m,n}(C') \longrightarrow \mathcal{E}^r_{s+t,m+n}(C'')$$

for $1 \le r \le \infty$, such that

$$d_r(a \times_r b) = d_r(a) \times_r b + a \times_r (-1)^{s+t} d_r(b)$$

for $a \in \mathcal{E}^r_{s,t}(C)$ and $b \in \mathcal{E}^r_{m,n}(C')$, and $1 \le r < \infty$. Explain how and why $\times\colon H_i(C) \otimes H_j(C') \to H_{i+j}(C'')$ is compatible with $\times_\infty$.

Exercise 5.23. Let X and Y be CW-spaces. The Eilenberg-Zilber theorem leads to a chain map $C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ where C_* denotes the singular chain complexes. That chain map induces in the usual way an external product

$$\times\colon H_i(X) \times H_j(Y) \to H_{i+j}(X \times Y)\,.$$

Use the previous exercise to show that the *cellular* chain complexes of X, Y and $X \times Y$ satisfy the relationship

$$W_*(X) \otimes W_*(Y) \cong W_*(X \times Y) \qquad (\$)$$

Show also that the map $H_i(X) \times H_j(Y) \to H_{i+j}(X \times Y)$ resulting from the isomorphism ($) agrees with the standard external product (of Eilenberg-Zilber).

Exercise 5.24. Do the cohomology version of the previous exercise.

Exercise 5.25. Let $p\colon E \to B$ and $p'\colon E' \to B'$ be fibrations where the base spaces are compact CW-spaces. Let $E'' = E \times E'$ and $B'' = B \times B'$ and let $p''\colon E'' \to B''$ be the product of p and p'.

- Use Eilenberg-Zilber and your solution of problem 5.22 to relate the cohomology Leray-Serre spectral sequences of p, p' and p'' by constructing external product homomorphisms

  $$\times_r\colon \mathcal{E}_r^{s,t}(p) \otimes \mathcal{E}_r^{m,n}(p') \longrightarrow \mathcal{E}_r^{s+t,m+n}(p'')$$

  for $1 \le r \le \infty$, such that

  $$d_r(a \times_r b) = d_r(a) \times_r b + a \times_r (-1)^{s+t} d_r(b)$$

  for $a \in \mathcal{E}_r^{s,t}(p)$ and $b \in \mathcal{E}_r^{m,n}(p')$, and $1 \le r < \infty$.
- Explain how and why the standard (Eilenberg-Zilber) external product $\times\colon H^i(E) \otimes H^j(E') \to H^{i+j}(E'')$ is compatible with $\times_\infty$.
- Describe $\times_1$ explicitly, using the cohomology version of proposition 5.10.
- Describe $\times_2$ explicitly, using the cohomology version of theorem 5.11 and your solution of problem 5.24.

Exercise 5.26. Let $p\colon E \to B$ be a fibration where B is a compact CW-space. Use the diagonal maps $B \to B \times B$ and $E \to E \times E$, as well as the cohomology version of proposition 5.15 to set up cup products in the Leray-Serre spectral sequence of p. Prove

$$d_r(a \cup b) = d_r(a) \cup b + a \cup (-1)^{s+t} d_r(b)$$

for $a \in \mathcal{E}_r^{s,t}(p)$ and $b \in \mathcal{E}_r^{m,n}(p)$, and $1 \leq r < \infty$. Describe what the cup products look like on the $\mathcal{E}_{**}^2$-page and on the $\mathcal{E}_{**}^{\infty}$-page.