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# Lecture Notes, week 10
# Topology SS 2015 (Weiss)

**4.1. Blakers-Massey homotopy excision theorem (special form)**

Situation: based CW-space $X$, two distinct cells attached (of dimensions $m$ and $n$, respectively) to give CW-spaces $Y_1, Y_2 \supset X$:

$$\begin{array}{ccc} X & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ Y_2 & \longrightarrow & Y_1 \cup Y_2 \end{array}$$

Therefore we have an inclusion $(Y_1, X) \to (Y_1 \cup Y_2, Y_2)$ of pairs. We want to look at the induced map

$$\pi_k(Y_1, X) \longrightarrow \pi_k(Y_1 \cup Y_2, Y_2) \tag{4.1}$$

(where $k > 0$) and we ask whether it is surjective, injective etc.

The plan here is to use general position arguments to answer this. There are three integer variables $m, n, k$ in the question and the answer should be formulated in terms of them. Let $E_1 = Y_1 \smallsetminus X$ be that $m$-cell and let $E_2 \subset Y_2 \smallsetminus X$ be that $n$-cell, so $E_1$ open in $Y_1$ and $E_2$ open in $Y_2$. Choose $z_1 \in E_1$ and $z_2 \in E_2$ . (These choices can be reconsidered in the following.) Let's try to show surjectivity in (4.1) first. Therefore we begin with

$$f \colon (D^k, S^{k-1}) \to (Y_1 \cup Y_2, Y_2). \tag{4.2}$$

Think of $D^k$ as unit disk in $\mathbb{R}^k$ with $(1, 0, \dots, 0)$ as center, so that $(0, ..., 0)$ can take the role of the base point.

We can assume that $f$ is smooth in the open subset $f^{-1}(U_1)$ and $f^{-1}(U_2)$ of $D^k$, where $U_1$ and $U_2$ are open neighborhoods of $z_1$ and $z_2$ in $E_1$ and $E_2$ , respectively. Then we can also assume that $z_1$ is a regular value for $f$ (as in Sard's theorem, and by Sard's theorem). It follows that $A_1 := f^{-1}(z_1)$ is a smooth compact submanifold of $D^k \smallsetminus S^{k-1}$, without boundary. Let $A_2 = f^{-1}(z_2)$. It is important that $A_1 \cap A_2 = \emptyset$. It is important that $\dim(A_i) = k - m$. Neither $A_1$ nor $A_2$ contain the base point $(0, ..., 0)$ of $D^k$. In particular, since $A_2$ is compact, we can choose $\varepsilon > 0$ so that the small disk $\varepsilon D^k$ has empty intersection with $A_2$.

Now we try to move $A_1$ into the small disk $\varepsilon D^k$ without disturbing $A_2$. As a first attempt we try the homotopy (*isotopy* is a better word here)

$$\big(\varphi_t \colon A_1 \to D^k\big)_{t \in [\varepsilon, 1]}$$

given by $\varphi_t(x) := (1+\varepsilon-t)x \in D^k$. *Isotopy* means that $\varphi_t\colon A_1 \to D^k$ is a smooth embedding for every $t \in [\varepsilon, 1]$. We have $\varphi_\varepsilon =$ inclusion and $\varphi_1(A_1)$ is contained in $\varepsilon D^k$ minus boundary. We can also think of $(\varphi_t)$ as a single smooth map

$$\varphi\colon A_1 \times [\varepsilon, 1] \longrightarrow D^k.$$

Since the plan was not to disturb $A_2$, we ask whether $\mathrm{im}(\varphi)$ intersects $A_2$. Better way to ask the question: whether $z_2$ is in the image of $f\varphi$. We can assume that $z_2$ is a regular value for the smooth map $f\varphi$. Since $\dim(A_1 \times [\varepsilon, 1]) = k - m + 1$ and the dimension of the target of $f\varphi$, as far as it is of interest here, is $n$, it follows that $z_2$ is not in the image of $f\varphi$ if

$$\boxed{k - m + 1 < n}.$$

This condition turns out to be the decisive one for surjectivity in (4.1).

**Lemma 4.3.** (Special case of Thom's isotopy extension theorem.) *There exists a diffeotopy*

$$\big(\Phi_t\colon D^k \to D^k\big)_{t\in[\varepsilon,1]}\big)$$

*which extends the isotopy* $(\varphi_t)$. *More precisely: each* $\Phi_t\colon D^k \to D^k$ *is a diffeomorphism and* $\Phi_t$ *agrees with the identity map on a neighborhood of* $S^{k-1} \cup A_2$ *, whereas* $\Phi_t$ *agrees with* $\varphi_t$ *on* $A_1$. (This is the precise meaning of: moving $A_1$ into $\varepsilon D^k$ without disturbing $A_2$.)

We postpone the proof of the lemma, but we use it right away to finish the proof of surjectivity in (4.1). First we note that our map $f$ in (4.2) is homotopic, as a map of pairs, to $g := f\Phi_1^{-1}$. The map $g$ has the following convenient properties: $g^{-1}(z_1) \subset \varepsilon D^k$ whereas $g^{-1}(z_2) \cap \varepsilon D^k = \emptyset$. We now try the homotopy

$$g_t\colon D^k \to Y_1 \cup Y_2$$

where $t \in [\varepsilon, 1]$ and $g_t(z) = g((1+\varepsilon-t)z)$. It is not guaranteed that $g_t(S^{k-1}) \subset Y_2$ but it is guaranteed that $g_t(S^{k-1}) \subset Y_1 \cup Y_2 \smallsetminus \{z_1\}$, and this is good enough for us since the inclusion

$$Y_2 \longrightarrow Y_1 \cup Y_2 \smallsetminus \{z_1\}$$

is a homotopy equivalence. Similarly, although it is not guaranteed that $g_1(D^k) \subset Y_1$ , it is guaranteed that $g_1(D^k) \subset Y_1 \cup Y_2 \smallsetminus \{z_2\}$, and this is again good enough for us. Therefore we can conclude that $g = g_\varepsilon$ is homotopic, as a map of pairs from $(D^k, S^{k-1})$ to $(Y_1 \cup Y_2, Y_2)$, to a map from $(D^k, S^{k-1})$ to $(Y_1, X)$. This establishes surjectivity in (4.1) under the condition $k + 1 < m + n$.

Next, we think about injectivity in (4.1). Therefore we should start by expressing the following situation: we have two elements in $\pi_k(Y_1, X)$ which determine the same element of $\pi_k(Y_1 \cup Y_2, Y_2)$. Note that this formulation does not assume group structures in $\pi_k$ , with a view to the possibility that $k \leq 2$. A very straightforward way to express the situation is then to say that we have a map

$$F\colon D^k \times [0,1] \to Y_1 \cup Y_2$$

such that $F(D^k \times 0) \cup F(D^k \times 1) \subset Y_1$ and $F(S^{k-1} \times [0,1]) \subset Y_2$. This takes $(0,...,0) \times [0,1]$ to the base point of $Y_1 \cup Y_2$. We now reason with this $F$ as we reasoned with $f$ in (4.2) before.

We can assume that $F$ is smooth in $F^{-1}(U_1)$ and $F^{-1}(U_2)$ and that $z_1$ is a regular value for $F$. Then we have a new $A_1 := F^{-1}(z_1)$, smooth submanifold of $D^k \times [0,1]$. This has empty intersection with $S^{k-1} \times [0,1]$ but it can have nonempty intersection with $D^k \times \partial[0,1]$. More precisely, $A_1$ is a smooth submanifold with boundary of $D^k \times [0,1]$ avoiding $S^{k-1} \times [0,1]$, of dimension $k+1-m$, and the boundary $\partial A_1$ is the transverse intersection of $A_1$ and $D^k \times \partial[0,1]$. We put $A_2 := F^{-1}(z_2)$. We choose $\varepsilon > 0$ in such a way that $\varepsilon D^k \times [0,1]$ has empty intersection with $A_2$. Define a smooth isotopy

$$\big(\varphi_t\colon A_1 \to D^k \times [0,1]\big)_{t\in[\varepsilon,1]}$$

by $\varphi_t(x,s) := ((1+\varepsilon-t)x, s) \in D^k \times [0,1]$. If

$$\boxed{k+1-m+1 < n}$$

then by Sard's theorem (general position) we may assume or arrange that $z_2$ is a regular value of $F\varphi$ (where $\varphi\colon A_1 \times [\varepsilon,1] \to D^k \times [0,1]$ is $(\varphi_t)_{t\in[\varepsilon,1]}$ reorganized). Then the image of each $\varphi_t$ has empty intersection with $A_2$ .

**Lemma 4.4.** *There exists a diffeotopy*

$$\big(\Phi_t\colon D^k \times [0,1] \to D^k \times [0,1])_{t\in[\varepsilon,1]}\big)$$

*which extends the isotopy* $(\varphi_t)$*. More precisely: each*

$$\Phi_t\colon D^k \times [0,1] \to D^k \times [0,1]$$

*is a diffeomorphism and* $\Phi_t$ *agrees with the identity map on a neighborhood of* $S^{k-1} \times [0,1] \ \cup A_2$ *, whereas* $\Phi_t$ *agrees with* $\varphi_t$ *on* $A_1$. (Moreover $\Phi_t$ takes $D^k \times \{0\}$ to itself and takes $D^k \times \{1\}$ to itself ... but this is automatic).

Again we postpone the proof (it is much like the postponed proof of lemma 4.3) and use the lemma to finish the proof of injectivity in (4.1). First we can replace $F = F\Phi_0$ by $G := F\Phi_1^{-1}$, since $(F\Phi_t^{-1})_{t\in[\varepsilon,1]}$ is a homotopy from $F$ to $G$ respecting all the essential features. After that, we try the homotopy

$$G_t\colon D^k \times [0,1] \to Y_1 \cup Y_2$$

where $t \in [\varepsilon,1]$ and $G_t(z,s) = G((1+\varepsilon-t)z, s)$. The details are as in the proof of surjectivity. Therefore we have shown:

**Proposition 4.5.** *The map* (4.1) *is surjective if* $k < m+n-1$ *and bijective if* $k < m+n-2$ (in addition to $k > 0$).

Let's remark that surjectivity of (4.1) was already known to us in case $k < n$ (by cellular approximation) and also in case $k < m$ (because then both groups/sets are trivial, again by cellular approximation). So the interesting cases of the proposition, as far as surjectivity is concerned, are the cases where $\max\{m, n\} \le k < m+n-1$. Similarly injectivity of (4.1) was already known to us in case $k+1 < n$ and in case $k < m$. So the interesting cases of the proposition, as far as injectivity is concerned, are the cases where $\max\{m, n+1\} \le k < m+n-2$.

*Proof of lemma 4.3. Under construction ... with regrets about wrong ideas I may have given in the Friday lecture.* □

## 4.2. The Freudenthal theorem

This is a special case of proposition 4.5. We take $X = S^{m-1}$ and $Y_1$ equal to the (closed) upper hemisphere $S^m_+$ of $S^m$, while $Y_2$ is equal to the closed lower hemisphere $S^m_-$ of $S^m$.